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Filled band Fermi systems

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Abstract

Extending the results in (B.M.) on one dimensional interacting fermions in a periodic potential we study the infrared behaviour of the two points Schwinger function in the filled band case. If the fermions are spinless such behaviour is completely determined and it depends on the ratio between the amplitude of the gap and the strength of the interaction. If the ratio is large the Schwinger function behaviour is similar to the one in the free non interacting case while if it is small the Schwinger function is deeply modified and it depends on two anomaly indices, in terms of which the occupation number discontinuity and the spectral gap are expressed. A heuristic second order analysis of the spinning case is also performed.

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1 Introduction and statement of the results

The hamiltonian of a system of one dimensional fermions with spin σ which move in a field $-u\partial_{\vec{x}}c(\vec{x})$ and interact via a short range pair potential $\lambda v(\vec{x}-\vec{y})$ is:

$$H = T + uP + \lambda V \tag{1}$$

$$T = \sum_\sigma \int_{-L/2}^{L/2} d\vec{x}\, \psi^+_{\vec{x},\sigma} \left(-\frac{\hbar^2\partial^2_{\vec{x}}}{2m}\right) \psi^-_{\vec{x},\sigma} \qquad P = \sum_\sigma \int_{-L/2}^{L/2} d\vec{x}\, \psi^+_{\vec{x},\sigma} c(\vec{x}) \psi^-_{\vec{x},\sigma}$$

$$V = \sum_{\sigma,\sigma'} \int_{-L/2}^{L/2} d\vec{x} d\vec{y}\; v(\vec{x}-\vec{y})(\psi^+_{\vec{x},\sigma}\psi^-_{\vec{x},\sigma})\,(\psi^+_{\vec{y},\sigma'}\psi^-_{\vec{y},\sigma'})$$

where $\psi^\pm_{\vec{x},\sigma}$ are creation or annihilation fermionic field operators with spin σ on the Fock space of fermions confined in a box $(-L/2, L/2)$ with periodic boundary conditions, obeying to the anticommutation rule $\{\psi^\varepsilon_{\vec{x},\sigma}, \psi^{\varepsilon'}_{\vec{y},\sigma'}\} = \delta_{\varepsilon,-\varepsilon'}\delta_{\sigma,\sigma'}\delta(\vec{x}-\vec{y})$, $m > 0$ is the fermion mass, $c(\vec{x}) = c(\vec{x}+a)$ is a C^∞–smooth periodic potential, with $\int_0^a d\vec{x} c(\vec{x}) = 0$ for definiteness, $v(\vec{r})$ is the spin-independent fermion-fermion interaction assumed bounded C^∞ smooth and with interaction range p_0^{-1}. If $\sigma = 0$ we say that the fermions are *spinless* while if $\sigma = \pm 1/2$ they are *spinning.* We take $L = Na$, N integer. We assume that u and λ are *dimensionless* and $c(\vec{x}) = \frac{\hbar^2}{2ma^2}\tilde{c}(\vec{x}/a)$, $v(\vec{r}) = \frac{\hbar^2}{2ma^2}\tilde{v}(\vec{r}p_0)$, with $\tilde{c}(\vec{x})$, $\tilde{v}(\vec{r})$ also dimensionless; moreover it is not restrictive to assume $u \geq 0$. Finally $\tilde{c}(\vec{x})$ and $\tilde{v}(\vec{r})$ are assumed "rotationally invariant" *i.e.* even; u is called *amplitude* of the periodic potential and λ *strength* of the interaction.

In this paper we study the fermionic system with hamiltonian eq.(1) with $|\lambda| << 1$, *i.e.* weakly interacting fermions ; this allows us to distinguish in the hamiltonian a "free" part given by $T + uP$ and a perturbation λV. When also $u << 1$ the above distinction is ambiguous as we could equally consider T as the "free" part of the hamiltonian and $\lambda V + uP$ as a perturbation.

We write:

$$\psi^\varepsilon_{\vec{x},\sigma} = \frac{1}{\sqrt{L}} \sum_{\vec{k}} \phi(\vec{k}, \varepsilon\vec{x}, u)\psi^\varepsilon_{\vec{k},\sigma} \tag{2}$$

where $\phi(\vec{k}, \vec{x}, u)$ are the solution of:

$$-\frac{\hbar^2\partial^2_{\vec{x}}}{2m}\phi(\vec{k},\vec{x},u) + uc(\vec{x})\phi(\vec{k},\vec{x},u) = \varepsilon(\vec{k},u)\phi(\vec{k},\vec{x},u) \tag{3}$$

with the condition $\phi(\vec{k}, \vec{x}+a, u) = e^{i\vec{k}a}\phi(\vec{k},\vec{x},u)$; the functions $\phi(\vec{k},\vec{x},u)$ are called *Bloch waves* and $\vec{k}$ *crystalline momentum.* From the classical theory of Bloch waves, see for instance (K.), it follows that $\varepsilon(\vec{k},u)$ is not continuous only in correspondence of $\vec{k} = n\pi/a$ for any integer n and it is non decreasing for $\vec{k} \geq 0$. The part of the energy spectrum between $\varepsilon((n\pi/a)^+, u)$ and $\varepsilon(((n+1)\pi/a)^-, u)$ and between $\varepsilon((-(n+1)\pi/a)^+, u)$ and $\varepsilon((-n\pi/a)^-, u)$ is called the *n-th energy band.*

The zero temperature Schwinger functions are defined as

$$\bar{S}(\vec{x}_1,t_1,\varepsilon_1,\sigma_1,\ldots,\vec{x}_n,t_n,\varepsilon_n,\sigma_n) = \lim_{L\to\infty} \langle\Omega|\psi^{\varepsilon_1}_{\vec{x}_1,\sigma_1}\cdot$$
$$e^{(t_2-t_1)\frac{(H-\mu N-\tilde{E}_0)}{\hbar}}\psi^{\varepsilon_2}_{\vec{x}_2,\sigma_2}e^{(t_3-t_2)\frac{(H-\mu N-\tilde{E}_0)}{\hbar}}\ldots e^{(t_n-t_{n-1})\frac{(H-\mu N-\tilde{E}_0)}{\hbar}}\psi^{\varepsilon_n}_{\vec{x}_n,\sigma_n}|\Omega\rangle$$

where $N=\sum_\sigma\int_{-L/2}^{L/2}d\vec{x}\,\psi^+_{\vec{x},\sigma}\psi^-_{\vec{x},\sigma}$ is the *number operator*, μ is the *chemical potential*, $t_1\geq t_2\geq\ldots\geq 0$, $|\Omega\rangle$ is such that $(H-\mu N)|\Omega\rangle=\tilde{E}_0|\Omega\rangle$ and $\tilde{E}_0$ is the minimum eigenvalue of $H-\mu N$. We will write:

$$S(x,y)\equiv\begin{cases} -\bar{S}(\vec{x},x_0,-,s;\vec{y},y_0,+,s) & x_0>y_0\\ \bar{S}(\vec{y},y_0,+,s;\vec{x},x_0,-,s) & x_0\leq y_0\end{cases} \tag{4}$$

where $x=(x_0,\vec{x})$, $y=(y_0,\vec{y})$, and we call it the *two points Schwinger function* which can be written in the following way:

$$S(x,y)\equiv\int dk\phi(\vec{k},\vec{x},u)\phi(\vec{k},-\vec{y},u)e^{ik_0(x_0-y_0)}S(k)\equiv$$
$$\int dk\phi(\vec{k},\vec{x},u)\phi(\vec{k},-\vec{y},u)S(\vec{k},x_0-y_0) \tag{5}$$

where $k=(k_0,\vec{k})$. The physical properties of the model can be expressed in terms of the Schwinger functions; a very important one is the *occupation number* defined as:

$$n_{\vec{k}}=S(\vec{k};0^-). \tag{6}$$

The *Fermi momentum* $p_F(\lambda,u,\mu)>0$ in a $d=1$ fermionic rotation invariant theory is defined by the following two conditions:

1. the occupation number is not regular *i.e.* $n_{\vec{k}}$ or some of its derivatives are singular at $\vec{k}=\pm p_F(\lambda,u,\mu)$.

2. $p_F(\lambda,u,\mu)-p_F(0,u,\mu)$ vanishes for $\lambda=0$.

In order to motivate the above definition note that if $\lambda=0$ the occupation number is analytic everywhere except on two points (see below) which we call $\pm p_F(0,u,\mu)$; in general there could be more than two points where the occupation number is not regular and we choose among them the one which can be continuously parametrized by λ so that it reduces to $p_F(0,u,\mu)$ if $\lambda=0$; if this choice is ambiguous we say that the Fermi momentum is not well defined. The two momenta $\pm p_F(\lambda,u,\mu)$ are called the *Fermi surface*.

An important role in physics is plaid by the amplitude of the discontinuity at the Fermi surface Z^{-1} which in $d=1$ is simply defined by:

$$Z^{-1}=n_{p_F^-}-n_{p_F^+} \tag{7}$$

Other important physical informations can be obtained from $S(k)$. If $|\alpha\rangle$, $\alpha=0,1,..$, is a complete set of eigenstates of $H-\mu N$ and $\tilde{E}_\alpha$ the corresponding eigenvalues, it is easy to check that:

$$S(k)=\sum_\alpha \frac{|\langle\Omega|\psi^-_{\vec{k},\sigma}|\alpha\rangle|^2}{-ik_0-(\tilde{E}_\alpha-\tilde{E}_0)\hbar^{-1}}+\sum_\beta \frac{|\langle\Omega|\psi^+_{\vec{k},\sigma}|\beta\rangle|^2}{-ik_0+(\tilde{E}_\beta-\tilde{E}_0)\hbar^{-1}} \tag{8}$$

where in the first sum only the states with $\bar{n}+1$ particles contribute, if $N|\Omega\rangle=\bar{n}|\Omega\rangle$ and in the second sum there are only the states with $\bar{n}-1$ particles. As a function of k_0 the first addend in the r.h.s of eq.(8) has poles at imaginary points $i\hbar^{-1}k_{0,\alpha}$ with $k_{0,\alpha}=E^{\bar{n}+1}_\alpha-E^{\bar{n}}_0-\mu>0$ if $E^{\bar{n}+1}_\alpha$ are the eigenvalues of H over the states with $\bar{n}+1$ particles, and $H|\Omega\rangle=E^{\bar{n}}_0|\Omega\rangle$; the second addend has poles at $-i\hbar^{-1}k_{0,\beta}$ with $k_{0,\beta}=E^{\bar{n}-1}_\beta-E^{\bar{n}}_0+\mu>0$ if $E^{\bar{n}-1}_\beta$ are the eigenvalues of H over the states with $\bar{n}-1$ particles. The *spectral gap* $\bar{\Delta}$ can be defined as $E^{\bar{n}+1}_0+E^{\bar{n}-1}_0-2E^{\bar{n}}_0$ where $E^{\bar{n}+1}_0$, $E^{\bar{n}-1}_0$ are the minimum values of H over the states respectively with $\bar{n}+1$, $\bar{n}-1$ fermions, so that it can be obtained by the poles of $S(k)$ by $\bar{\Delta}=\min_\beta k_{0,\beta}+\min_\alpha k_{0,\alpha}$.

It is easy to check by an explicit computation that the Schwinger functions of the free $\lambda=0$ particle system with hamiltonian $T+uP$ are given by the Wick's rule in term of the *propagator* $S_0(x,y)$, given by eq.(5) with $\lambda=0$:

$$S_0(x,y)=\int dk\phi(\vec{k},\vec{x},u)\phi(\vec{k},-\vec{y},u)e^{ik_0(x_0-y_0)}\frac{1}{-ik_0-(\varepsilon(\vec{k},u)-\mu)\hbar^{-1}}\equiv \tag{9}$$
$$\equiv\int dk\phi(\vec{k},\vec{x},u)\phi(\vec{k},-\vec{y},u)e^{ik_0(x_0-y_0)}g(k)$$

From eq.(9) it is easy to compute the occupation number $n_{\vec{k}}=\chi\big(\varepsilon(\vec{k},u)-\mu\le 0\big)$ where $\chi(condition)$ is 1 if the condition is verified and 0 otherwise. The Fermi momentum p_F is then defined by the condition $\varepsilon(p_F,u)=\mu$; note that since $\varepsilon(\vec{k},u)$ is not continuous there is not a one to one correspondence between p_F and μ, *i.e.* to $p_F=n\pi/a$ correspond all the values of μ belonging to $[\varepsilon((n\pi/a)^-,u),\varepsilon((n\pi/a)^+,u)]$. The discontinuity at the Fermi surface is $Z^{-1}=1$.

Contrary to the occupation number, the asymptotic behaviour of $g(k)$ for small k_0 and $|\vec{k}|-p_F$ depends on the value of the Fermi momentum:

1. if μ does not belong to the closed set $[\varepsilon((n\pi/a)^-,u),\varepsilon((n\pi/a)^+,u)]$ then $g(k)$ behaves for small k_0 and $|\vec{k}|-p_F$ as $(-ik_0-v_0(|\vec{k}|-p_F))^{-1}$ where $v_0=\frac{1}{\hbar}\frac{\partial\varepsilon(\vec{k},u)}{\partial\vec{k}}|_{\vec{k}=p_F}$ *i.e.* it has the same asymptotic behaviour of the Schwinger function in the $u=0$ case (see eq.(11) below).

2. quite different is the case in which $\mu\in[\varepsilon((n\pi/a)^-,u),\varepsilon((n\pi/a)^+,u)]$ *i.e.* $p_F=n\pi/a$. In this case we have that:

$$\lim_{|\vec{k}|\to(\pi/a)^\pm,k_0\to 0} g(k)=-\frac{\hbar}{\varepsilon((\pi/a)^\pm)-\mu} \tag{10}$$

The above limits are finite unless one chooses $\mu=\varepsilon((n\pi/a)^-)$ or $\mu=\varepsilon((n\pi/a)^+)$, in which cases one of the two limits is singular. Of course the physical properties do not depend on the choice of $\mu \in [\varepsilon((n\pi/a)^-, u), \varepsilon((n\pi/a)^+, u)]$: for instance the spectral gap defined above in the $\lambda = 0$ case is equal to $\varepsilon((n\pi/a)^+, u) - \varepsilon((n\pi/a)^-, u)$ if $p_F = n\pi/a$ for any choice of μ and it is 0 for all the other values of p_F. It is convenient to define the adimensional spectral gap, if $p_F = n\pi/a$, as $\Delta_n(u) = \frac{2ma^2}{\hbar^2}[\varepsilon((n\pi/a)^+, u) - \varepsilon((n\pi/a)^-, u)]$. For small u we have (see Appendix 2), $\Delta_1(u) \equiv \Delta(u) = c_1 u + O(u^2)$ where c_1, the first Fourier coefficient of $\tilde{c}(\vec{x})$, is assumed from now on equal to 1.

If $\lambda = u = 0$ we have that

$$g(k) = \frac{1}{-ik_0 - \frac{\hbar^{-1}}{2m}(\vec{k}^2 - p_F^2)} \tag{11}$$

which is singular for any value of the chemical potential $\mu = p_F^2/2m$; moreover the Fermi momentum is given by $\mu = \frac{\hbar^2 p_F^2}{2m}$, the spectral gaps are zero and the discontinuity at the Fermi surface $Z^{-1} = 1$.

The free Schwinger function $S_0(x, y)$ decay for large values of $|x-y| = \sqrt{(\vec{x} - \vec{y})^2 + v_0^2(x_0 - y_0)^2}$:

1. if $u = 0$ or $u \neq 0$ but $p_F \neq n\pi/a$, as $f(\vec{x}, \vec{y})/|x - y|$, where $f(\vec{x}, \vec{y})$ is a bounded function equal to $\sin(p_F(\vec{x} - \vec{y}))$ if $u = 0$;

2. if $u \neq 0$ and $p_F = n\pi/a$ decays faster than any power for large distances *i.e.* for all N one can find constants C_N, C such that

$$|S_0(x, y)| \leq \frac{C_N \Delta_n a^{-1}}{1 + \Delta_n^N |x - y|^N} \quad \text{for} \quad |x - y| > \Delta_n^{-1} \tag{12}$$

$$|S_0(x, y)| \leq \frac{C a^{-1}}{|x - y|} \quad \text{for} \quad 1 \leq |x - y| \leq \Delta_n^{-1} \quad \text{if} \quad \Delta_n < 1 \tag{13}$$

Note that $S_0(x, y)$ has the dimension of an inverse length,*i.e.* of a^{-1}.

The free $\lambda = 0$ system is called a *metal* if $p_F \neq \frac{n\pi}{a}$ while if $p_F = \frac{n\pi}{a}$ and $u \neq 0$ it is called an *insulator*: we can extend to the $\lambda \neq 0$ case such definitions by saying that a fermionic system in which both $S(0, p_F^+)$ and $S(0, p_F^-)$ are infinite is a metal while if at least one between $S(0, p_F^+)$ or $S(0, p_F^-)$ is finite it is an insulator.

We consider now what happens in presence of an interaction $\lambda \neq 0$. We note first that generally $p_F(\lambda, u, \mu) \neq p_F(0, u, \mu)$, *i.e.* the interaction changes the Fermi momentum at fixed μ.

Since the early works on the theory of Fermi systems, (L.W.), it has been realized that it is more natural to study the properties of weakly interacting fermionic systems when λ is varied at fixed Fermi momentum rather than at fixed chemical potential. Therefore we write the chemical potential in eq.(1) as $\mu = \mu_0 + \nu$ where μ_0 is the chemical potential of a fermionic system with Fermi momentum p_F and hamiltonian $T + uP$.

In the case $p_F = n\pi/a$, μ_0 is chosen at the center of the interval $[\varepsilon((n\pi/a)^-, u), \varepsilon((n\pi/a)^+, u)]$; such choice is rather arbitrary, as one could take any value in that interval and we make it for definiteness. We fix the "counterterm" ν so that $p_F(\lambda, u, \mu_0 + \nu) = p_F(0, u, \mu_0) = p_F$, *i.e. we force the Fermi momentum in the "interacting" $\lambda \neq 0$ theory and "free" $\lambda = 0$ theory to be* *the same.* According to the formal Luttinger theorem (L1.), (B.G.L.), fixing $p_F(\lambda, u, \mu_0 + \nu)$ should be equivalent to fixing the physical density $\rho(\lambda, u, \mu_0 + \nu)$, *i.e.* $p_F(\lambda, u, \mu_0 + \nu)$ should be independent of λ if $\rho(\lambda, u, \mu_0 + \nu)$ is fixed.

Even a very small interaction can modify dramatically the physical properties of a fermionic system with respect to the $\lambda = 0$ case. This is what happens when the fermions are spinless and $p_F \neq n\pi/a$ (not filled band case) in which case $Z = \infty$ and $S(k)$ behaves, for small $k_0, |\vec{k}| - p_F$ as $|k_0^2 + v_0^2(|\vec{k}| - p_F)^2|^{-1/2+\eta/2}$ with $v_0 = \hbar\pi/ma$ and $\eta = O(\lambda^2) > 0$ (see (B.M.)). Such result extends the theory of (B.G.P.S.) for spinless fermions with no periodic potential; if the fermions are spinning similar results hold only if the interaction is repulsive, see (B.M.).

In this paper we study the $p_F = n\pi/a$ case, which is also called *filled band case.* If *the strength of the interaction* λ *is small with respect to the gap amplitude* $\Delta(u)$ *of the free system* one expects from physical arguments that the behaviour of the free system is not different from the behaviour in presence of small interactions.

This is in fact what we find (see the first statement of the theorem below) as we obtain that the Schwinger function is given by $S_0(x, y) + (\lambda/\Delta(u))\tilde{S}(x, y)$ with both $S_0(x, y), \tilde{S}(x, y)$ obeying to the bound eq.(12). We fix the counterterm $\nu = 0$ *but from the proof it will appear that we could equally fix* $\nu = O(\lambda)$; this means that in this interacting system as in the free one, in the filled band case, the chemical potential *is not completely determined* by the Fermi momentum.

More interesting is the case in which *the strength of the interaction* λ *is large with respect to the gap amplitude* $\Delta(u)$ *of the free system*; in this case in fact one expects that the interaction could change in a relevant way the physical properties of the system with respect to the free $\lambda = 0$ case. We are able to treat this case only if λ and u are both small, so that $\Delta(u) = u + O(u^2)$ (see the second statement of the theorem below in which for generality the case of any value of the ratio $\frac{\lambda}{u}$ is discussed).

We find a behaviour described in term of two *anomaly indices*, $\eta_1(\lambda, u)$ and $\eta_2(\lambda, u)$ with $\lambda^{-2}\eta_1(\lambda, u), \lambda^{-1}\eta_2(\lambda, u)$ tending to a positive u-independent limit as $\lambda \to 0$ *i.e.* $\eta_1 \simeq \lambda^2 c_1 + \ldots$, $\eta_2 =\simeq \lambda c_2 + \ldots$ where c_1, c_2 are positive constants. The two point Schwinger function is in fact written as $S_A(x, y) + S_B(x, y)$ with $S_A(x, y)$, $S_B(x, y)$ obeying to similar bounds for large distances (see the theorem below) and $S_A(x, y)$, the "dominant" contribution if λ, u are small, is formally similar to the free Schwinger functions eq.(9) with the difference that the Bloch waves $\phi(\vec{k}, \vec{x}, u)$ and the dispersion relation $\varepsilon(\vec{k}, u)$, defined by eq.(3), are replaced by $\varepsilon(\vec{k}, \hat{u})$ and $\hat{Z}^{-1}\phi(\vec{k}, \vec{x}, \hat{u})$ where $\hat{u} \equiv \hat{u}(k, \lambda, u)$ and $\hat{Z} \equiv \hat{Z}(k, \lambda, u)$ are bounded functions equal respectively to $u(1 + O(\lambda))$ and $1 + O(\lambda)$ for large $|\vec{k}| - \pi/a$ and $|k_0|$ and their limit for $|\vec{k}| \to \pi/a, k_0 \to 0$ is equal respectively to $u^{1-\eta_2}$ and $u^{-\eta_1}$. We can interpret this by saying that the " interacting one-particle wavefunction" in the filled band case are approximately *i.e.* neglecting the corrections, *Bloch waves in which the interaction changes in a momentum dependent way the amplitude of the periodic potential and the normalization.* From physical considerations we expect that the interaction modifies the system properties that depend mainly on momenta near the Fermi surface, so that the momentum dependence of $\hat{u}(k,\lambda,u)$ and $\hat{Z}(k,\lambda,u)$ is not surprising.

The occupation number discontinuity Z^{-1} vanishes for $u \to 0$ as u^{η_1} if $\lambda \neq 0$, while it is 1 if $\lambda \neq 0$ *i.e.* the interaction modifies dramatically the system properties.

Following an usual therminology (see for instance (A.)) a metal with weakly interacting fermions is called a *normal liquid* if the occupation number discontinuity is $O(1)$, while it is called a *Luttinger liquid* if, for small $|\vec{k}|$, $n_{p_F+\vec{k}} - n_{p_F-\vec{k}} \simeq |\vec{k}|^\eta$ with $\eta = O(\lambda^2)$ in general. In analogy with the above definition in the case of filled band we can define a *normal insulator* a Fermi system in which for all u it is $Z^{-1} = O(1)$, while we can define a *Luttinger insulator* a system in which the quantity Z^{-1} vanishes as $u \to 0$ (which is therefore an *anomalous behaviour*). We can then conclude that our filled band Fermi system is a Luttinger insulator.

We are not really able to compute the full analytic structure near the Fermi surface in the complex k-plane (hence the poles) of $S(k)$ so that we do not really know the spectral gap in the interacting case; however our results provide in our opinion a very strong evidence that there is a gap of order $O(u^{1-\eta_2})$ *i.e.* that the interaction changes the amplitude of the gap so that the ratio between the free and the interacting gap is $O(u^{-\eta_2})$ *i.e.* vanishing or diverging as $u \to 0$ depending on the repulsive or attractive nature of the interaction. We think that a more careful study of the analytic structure of $S(k)$ will allow us to prove the above claims about the gap.

An analogous statement was claimed in a similar one dimensional fermionic model by (L.E.1) on the basis of alternative heuristic arguments (and it seems quite difficult to put them in a rigourous form, see (S.)).

Our results are summarized by the following theorem which is formulated in terms of the following quantities:

1. $\mu_0 = (\varepsilon(\pi/a)^-, u) + \varepsilon((\pi/a)^+, u)/2$: *unperturbed chemical potential*
2. $\mu = \mu_0 + \nu$: *chemical potential*
3. $\Delta(u) = \frac{2ma^2}{\hbar^2}[\varepsilon(\pi/a)^+, u) - \varepsilon((\pi/a)^-, u)]$: *adimensional unperturbed gap*
4. $v_0 = \hbar\pi/ma$: *Fermi velocity*
5. $|k| = a\sqrt{v_0^{-2}k_0^2 + (|\vec{k}| - \frac{\pi}{a})^2}$: *adimensional momentum distance from the Fermi surface*
6. $|x| = a^{-1}\sqrt{x_0^2 v_0^2 + \vec{x}^2}$: *adimensional length*

Theorem 1 (main result) *Consider a Fermi system with hamiltonian*

$$\sum_i^n \Big(-\frac{\hbar^2 \partial_{\vec{x}_i}^2}{2m} + uc(\vec{x}_i) - \mu\Big) + 2\lambda \sum_{i<j} v(\vec{x}_i - \vec{y}_j) \tag{14}$$

Constants $\varepsilon, \varepsilon_1$ exists such that, for $|\lambda| \leq \varepsilon$:

case 1 *(small interaction compared to the gap), if u is such that $\frac{|\lambda|}{\Delta(u)} \leq \varepsilon_1$ and $\nu = 0$ then the Schwinger function $S(x,y)$ is given by*

$$S(x,y) = S_0(x,y) + \frac{\lambda}{\Delta(u)}\tilde{S}(x,y)$$

and for all N and $|x-y| > (\Delta(u))^{-1}$

$$|S_0(x,y)|, |\tilde{S}(x,y)| \leq \frac{\Delta(u) C_N a^{-1}}{1+\Delta(u)^N |x-y|^N} \tag{15}$$

where C_N is a suitable constant, while $|S(x,y)| \leq \frac{C}{|x-y|}$ for $1 \leq |x-y| \leq \Delta^{-1}(u)$, if C is a suitable constant and $\Delta(u) < 1$. The occupation number discontinuity eq.(6) is such that $Z^{-1} = 1 + O(\frac{\lambda}{\Delta(u)})$.

case 2 *(λ and the gap small), for $u < \varepsilon$ there are two regular functions $\hat{u}(k,\lambda,u)$ and $\hat{Z}^{-1}(k,\lambda,u)$ such that $|\hat{u}(k,\lambda,u) - u| = O(\lambda)$, $|\hat{Z}^{-1}(k,\lambda,u) - 1| = O(\lambda)$ for $|k| > \pi/2a$ and*

$$\lim_{|k|\to 0} \hat{u}(k,\lambda,u) = u^{1-\eta_2(\lambda,u)} \qquad \lim_{|k|\to 0} \hat{Z}^{-1}(k,\lambda,u) = u^{\eta_1(\lambda,u)}$$

where $\lambda^{-2}\eta_1(\lambda,u)$ and $\lambda^{-1}\eta_2(\lambda,u)$ are both positive and independent of u in the $\lambda \to 0$ limit, such that, for a suitable $\nu = \nu(\lambda,u) = O(\lambda^2)$, the function $S(x,y)$ is given by,

$$S(x,y) = S_A(x,y) + S_B(x,y) \tag{16}$$

with

$$\begin{aligned} S_A(x,y) \;\; = \;\; & \int dk \phi(\vec{x},k,\hat{u}(k,\lambda,u))\phi(-\vec{y},k,\hat{u}(k,\lambda,u)) e^{ik_0(x_0-y_0)} \, . \\ & \frac{1}{\hat{Z}(k,\lambda,u)} \frac{1}{-ik_0 - (\varepsilon(k,\hat{u}(k,\lambda,u)) - \mu_0)\hbar^{-1}} \end{aligned} \tag{17}$$

The occupation number discontinuity eq.(6) is $Z^{-1} = u^{\eta_1(\lambda,u)}(1+O(\lambda))$ and for any N and for $|x-y| > u^{-(1-\eta_2)}$ we have

$$\begin{aligned} |S_A(x,y)| &\leq \frac{Z^{-1} u^{1-\eta_2(\lambda,u)} C_N a^{-1}}{1 + u^{(1-\eta_2(\lambda,u))N} |x-y|^N} \\ |S_B(x,y)| &\leq \mathrm{Max}(|\lambda|, u, u^{1-\eta_1(\lambda,u)}) \frac{Z^{-1} u^{1-\eta_2(\lambda,u)} C_N a^{-1}}{1 + u^{(1-\eta_2(\lambda,u))N} |x-y|^N} \end{aligned} \tag{18}$$

where C_N is a suitable constant, while

$$|S_A(x,y)| \leq C|x-y|^{-1-\eta_3(\lambda,u)} \qquad |S_B(x,y)| \leq C\mathrm{Max}(|\lambda|, u, u^{1-\eta_1(\lambda,u)})|x-y|^{-1-\eta_3(\lambda,u)} \tag{19}$$

for $1 \leq |x-y| \leq u^{-(1-\eta_2(\lambda,u))}$, with $\eta_3(\lambda,u) = \eta_1(\lambda,u)(1-\eta_2(\lambda,u))^{-1}$ and C is a suitable constant

The second statement of the theorem holds for any value of the ratio $\frac{\lambda}{u}$ so that for $|\lambda| < \varepsilon_1 u$ and $|\lambda|, |u| \le \varepsilon$ both the statements holds. Note that the Schwinger functions bounds for large $|x-y|$ in the two statement, respectively

$$S(x,y) \le \frac{\Delta(u) C_N a^{-1}}{1 + \Delta(u)^N |x-y|^N}$$

and

$$S(x,y) \le \frac{Z^{-1} u^{1-\eta_2(\lambda,u)} C_N a^{-1}}{1 + u^{(1-\eta_2)N} |x-y|^N}$$

coincide in this region, in the sense that

$$\frac{Z^{-1} u^{1-\eta_2(\lambda,u)} C_N a^{-1}}{1 + u^{(1-\eta_2)N} |x-y|^N} \le \frac{u \tilde{C}_N a^{-1}}{1 + u^N |x-y|^N}$$

and

$$\frac{\Delta(u) C_N a^{-1}}{1 + \Delta(u)^N |x-y|^N} \le \frac{u \tilde{C}_N a^{-1}}{1 + u^N |x-y|^N}$$

for some constant $\tilde{C}_N$, as it is trivial to check as by the computations of sec.2.3 and the Appendix 1 $\Delta(u) = u + O(u^2)$ and $\hat{u}(k,\lambda,u) = u(1 + \frac{\lambda}{u} c_1))$, $\hat{Z}(k,\lambda,u) = 1 + \frac{\lambda}{u} c_2$ with c_1 and c_2 bounded. Note moreover that the behaviour of the Schwinger function in case 2 for $1 \le |x-y| \le u^{-(1-\eta_2(\lambda,u))}$ is the same anomalous behaviour found in the spinless theory with no periodic potential in (B.G.P.S.), or in the not filled band case of (B.M.), for spinless fermions or spinning with $\lambda > 0$.

The proof of the theorem is performed by using renormalization group techniques, which provide us with an algorithm to express the Schwinger function in the interacting case $S(x-y)$ as a sum of functions decaying faster than any power for large $|x-y|$. In the first case, *i.e.* $|\lambda|$ small with respect to the amplitude of the gap, we are able to write the Schwinger fuction as:

$$S(x,y) = \sum_{1 \ge h > -\infty} [g^h(x,y) + \frac{\lambda}{\Delta(u)} \hat{g}^h(x,y)] \tag{20}$$

where

$$g^h(x,y) = \int dk \phi(\vec{x},k,u) \phi(-\vec{y},k,u) e^{ik_0(x_0-y_0)} \frac{f^h(k_0^2 + (\varepsilon(\vec{k},u) - \mu_0)^2)}{-ik_0 - (\varepsilon(\vec{k},u) - \mu_0)}$$

with $f^1(t)$ is a C^∞ function which is equal to 1 for $|t| \ge \gamma$, $f^h(t)$, for $h \ne 1$, is a C^∞ compact support function with support in $\gamma^h \le |t| \le \gamma^{h+2}$ and $\sum_{1 \le h < -\infty} f^h(t) \equiv 1$ (see the next section). Of course $g^h(x,y) \equiv 0$ for $h \le h^*$ with $\gamma^{h^*} \simeq \Delta(u)$ as $\min_{\vec{k}} |\varepsilon(\vec{k},u) - \mu_0| = \Delta(u)$, *i.e.* the Schwinger function is expressed in terms of a finite sum of functions. In Appendix 1,2 it is shown that, for any N:

$$|g^h(x,y)|, |\hat{g}^h(x,y)| \le \frac{\gamma^h C_N a^{-1}}{1 + \gamma^{hN} |x-y|^N}$$

so that the first statement of the theorem follows.

In the second case, in which the interaction and the gap amplitude are small and $u << |\lambda|$ for simplicity, we have

$$S(x,y) = \sum_{1\geq h > -\infty} [g^h(x,y) + \bar{g}^h(x,y)] \tag{21}$$

with

$$g^h(x,y) = \frac{1}{Z_h}\int dk \phi(\vec{x},k,\sigma_h)\phi(-\vec{y},k,\sigma_h)e^{ik_0(x_0-y_0)}\frac{f^k(k_0^2 + (\varepsilon(\vec{k},\sigma_h)-\mu_0)^2)}{-ik_0 - (\varepsilon(\vec{k},\sigma_h)-\mu_0)}$$

with $Z_h \simeq \gamma^{-\eta_3(\lambda,u)h}$ and $\sigma_h \simeq u\gamma^{-\eta_2(\lambda,u)h}$. Again the Schwinger function is given by a finite sum but now the "last scale" h^* is given by the condition $\gamma^{h^*} \simeq \sigma_{h^*}$ *i.e.* $h^* \simeq (1 - \eta_2(\lambda,u))\log(u)$. Since for all N

$$|g^h(x,y)| \leq \frac{1}{Z_h}\frac{\gamma^h C_N a^{-1}}{1+\sigma_h^N|x-y|^N} \qquad |\bar{g}^h(x,y)| \leq \mathrm{Max}(|\lambda|, u, u^{1-\eta_2(\lambda,u)})\frac{1}{Z_h}\frac{\gamma^h C_N a^{-1}}{1+\sigma_h^N|x-y|^N} \tag{22}$$

the second statement of the theorem follows. Note that $\hat{u}(k,\lambda,u)$ in the theorem is not completely determined as for a small λ $\phi(\vec{k},\vec{x},u(1+\lambda)) = \phi(\vec{k},\vec{x},u) + \lambda\tilde{\phi}(\vec{k},\vec{x},u)$ where $\tilde{\phi}(\vec{k},\vec{x},u)$, $\phi(\vec{k},\vec{x},u)$ and all their derivatives obeys to the same bounds and the same holds for $\varepsilon(k,u)$ (see Appendix 2) so that we can replace $\hat{u}$ in eq.(17) with $\hat{u}(1+O(\lambda))$ by simply replacing $S_B(x,y)$ with a proper $\tilde{S}_B(x,y)$ obeing to the same bound.

If $u \to 0$ than $h^* \to \infty$ and, from eq.(21), we find that $S(x,y)$ asymptotically behaves for $|x-y| \to \infty$ as $\frac{S_0(x-y)}{|x-y|^{\eta_3(\lambda,0)}}(1+\lambda A(\lambda,x-y))$ with $A(\lambda,x-y)$ bounded, so that the results of (B.G.P.S.) are recovered.

At the end of this paper we give some arguments supporting the conjecture that a similar behaviour is found also in the spinning case, *if the interaction is either repulsive or attractive but* $u \geq k_1 e^{-|\lambda|^{-k_2}}$ *for some positive constants* k_1, k_2. This suggests that the $u = 0$ spinning case with attractive interaction generates spontaneously at the Fermi surface a gap $O(e^{-|\lambda|^{-k_2}})$ even in absence of a periodic potential.

For simplicity in the following we use dimensionless quantities defined as:

$$\tilde{x} = x/a \qquad \tilde{t} = \hbar t/2ma^2 \qquad \tilde{H} = 2ma^2 H/\hbar^2 \qquad \tilde{\psi}^\varepsilon_{\vec{x},\sigma} = \sqrt{\frac{1}{a}}\psi^\varepsilon_{\vec{x},\sigma} \tag{23}$$

so that the hamiltonian $\tilde{H}$ is given by eq.(1) with $\bar{h} = a = 1, 2m = 1, \mu = \pi^2$. In the following the tilde's will be omitted and we consider directly dimensionless quantities. Moreover we write simply $\varepsilon(\vec{k})$ for $\varepsilon(\vec{k},u)$ and Δ for $\Delta(u)$ unless explicitely stated.

2 Proof of the theorem

2.1 Multiscale decomposition and the localization operator

In this section we prove the second statement of the theorem, which is the more technically involved one; the proof of the first statement is easier and it is in Appendix 1. We consider a

Grassman algebra, whose elements $\psi^{\varepsilon}_{k,\sigma}$, $\varepsilon = \pm$, verify $\{\psi^{\varepsilon}_{k,\sigma}, \psi^{\varepsilon'}_{k',\sigma'}\} = 0$. The *Euclidean fields* can be defined as $\psi^{\varepsilon}_{x,\sigma} = \int dk e^{i\varepsilon(k_0 x_0 + \vec{k}\vec{x})}\psi^{\varepsilon}_{k,\sigma}$ and $\int dk = \frac{(2\pi)^2}{\beta N}\sum_k$ with $e^{ik_0\beta} = -1, e^{i\vec{k}N} = 1$ and the grassmanian integration is defined on monomials by:

$$\int P(d\psi)\psi^+_{x_1}\cdots\psi^+_{x_n}\psi^-_{y_1}\cdots\psi^-_{y_n} = \sum_\pi (-1)^{\sigma_\pi}\prod_i g(x_i - y_{\pi(i)}) \tag{24}$$

where π is a permutation of the set $(1,\ldots,n)$, σ_π is the parity of the permutation and

$$g(x-y) = \int dk \frac{e^{-ik(x-y)}}{-ik_0 - (\vec{k}^2 - \pi^2)}$$

An application of Trotter's formula allows us, as usual (see (B.G.)), to write the two points Schwinger function in terms of grassmanian integral $P(d\psi)$ as:

$$S(x,y) = \lim_{\substack{\beta\to\infty \\ N\to\infty}} \frac{\int P(d\psi)e^{-\bar{V}(\psi)}\psi^+_x\psi^-_y}{\int P(d\psi)e^{-\bar{V}(\psi)}} \tag{25}$$

where $\bar{V}(\psi)$ is given by $\bar{V}(\psi) = \lambda V + \nu N + uP$ with:

$$\begin{aligned} V &= \sum_{\sigma,\sigma'}\int_{\Lambda\times\Lambda} dx_1 dx_2 v(\vec{x}_1 - \vec{x}_2)\delta(x_{0,1} - x_{0,2})\psi^+_{x_1,\sigma}\psi^+_{x_2,\sigma'}\psi^-_{x_2,\sigma'}\psi^-_{x_1,\sigma} \\ N &= \sum_\sigma \int_\Lambda dx \psi^+_{x,\sigma}\psi^-_{x,\sigma} \qquad P = \sum_\sigma \int_\Lambda dx c(\vec{x})\psi^+_{x,\sigma}\psi^-_{x,\sigma} \end{aligned} \tag{26}$$

with $\Lambda = (-\beta/2, \beta/2)\times(-N/2, N/2)$ and $c(\vec{x}) = c(\vec{x} + a)$. We start by studying the partition function $\mathcal{N}$ *i.e.* the denominator of eq.(25). It is convenient to rewrite it as:

$$\mathcal{N} = \int P(d\psi_{i.r.})e^{-V^0(\psi_{i.r.})} \tag{27}$$

$$e^{-V^0(\psi_{i.r.})} = \int P(d\psi_{u.v.})e^{-\bar{V}(\psi_{i.r.}+\psi_{u.v.})} \tag{28}$$

where $\psi_{u.v.}$, $\psi_{i.r.}$, are grassmanian fields and $P(d\psi_{u.v.})$, $P(d\psi_{i.r.})$ denote respectively the grassmanian integrations with vanishing cross propagator and with propagators $g_{u.v.}, g_{i.r.}$ given by:

$$\begin{aligned} g_{u.v}(x,y) &= \int dk \frac{e^{-ik(x-y)}}{-ik_0 - (\vec{k}^2 - \pi^2)}(1 - h(k_0^2 + (|\vec{k}| - \pi)^2)) \\ g_{i.r}(x,y) &= \int dk \frac{e^{-ik(x-y)}}{-ik_0 - (\vec{k}^2 - \pi^2)}h(k_0^2 + (|\vec{k}| - \pi)^2) \end{aligned} \tag{29}$$

where $h(t)$ is a C^∞ function in its argument t which is identically 0 if $t > \pi^2$ and 1 in a neighbourhood of the origin.

In (B.G.P.S.) it is shown that the ultraviolet part of the propagator can be written as

$$g_{u.v.}(x) = G(x) + R(x) \qquad G(x) = H(\vec{x})H(x_0)\frac{e^{-\frac{\vec{x}^2}{2x_0}}}{\sqrt{2\pi x_0}} \tag{30}$$

where $H(t)$ is a smooth function of compact support such that $H(t) = e^{t^2\pi^2}$ if $|t| \leq 1$ and $H(t) = 0$ if $|t| \geq \gamma > 1$, and $R(x) \leq \frac{C_N}{1+|x|^N}$ for any integer N. A minor adaptation of the ultraviolet problem analysis for the $u = 0$ case in (B.G.P.S) to the $u \neq 0$ case allows to check that there exists an ε such that V^0 can be written, for $|z| \leq \varepsilon$, if $z = (\lambda, \nu, u)$, in the following way:

$$
\begin{aligned}
V^0(\psi) &= \sum_{\sigma,\sigma'} \int dk_1 dk_2 dk_3 dk_4 \hat{v}(\vec{k}_1 - \vec{k}_4) \psi^+_{k_1,\sigma} \psi^+_{k_2,\sigma'} \psi^-_{k_3,\sigma'} \psi^-_{k_4,\sigma} \delta(k_1 + k_2 - k_3 - k_4) + \\
&+ 2 \sum_\sigma \lambda \int dk dp \hat{v}(\vec{p}) \hat{R}(\vec{p} - \vec{k}) \psi^+_{k,\sigma} \psi^-_{k,\sigma} + (\nu - 4\pi\lambda\hat{v}(0)\hat{R}(0)) \sum_\sigma \int dk \psi^+_{k,\sigma} \psi^-_{k,\sigma} + \\
&+ \sum_\sigma \sum_{n=1}^\infty u c_n \int dk [\psi^+_{k+n\pi,\sigma} \psi^-_{k-n\pi,\sigma} + \psi^+_{k-n\pi,\sigma} \psi^-_{k+n\pi,\sigma}] + \\
&+ \sum_{(\sigma_1,k_1),\ldots,(\sigma_{2m},k_{2m})} \sum_{n=0}^\infty \int dk_1 \ldots dk_{2m} W_{m,n}(k_1, \ldots, k_{2m}; z) \psi^+_{k_1,\sigma_1} \ldots \psi^+_{k_m,\sigma_m} \cdot \\
&\psi^-_{k_{m+1},\sigma_{m+1}} \ldots \psi^-_{k_{2m},\sigma_{2m}} \delta(k_1 + \ldots + k_m - k_{m+1} - \ldots - k_{2m} + 2n\pi)
\end{aligned}
\tag{31}
$$

where $c(\vec{x}) = \sum_{n=0}^\infty c_n \cos(2n\pi\vec{x})$, $2n\pi$ is a spatial vector andthe kernels $W_{m,n}(k_1, \ldots, k_m; z)$ are C^∞ bounded functions such that $W_{m,n} = W_{m,-n}$ and

$$
\sup_{k_1,\ldots,k_{2m}} |W_{m,n}(k_1, \ldots, k_{2m}; z)| \leq C^m z^{\max(2,m-1)} \tag{32}
$$

where the sup is over the momenta in the support of $h(k_0^2 + (|\vec{k}| - \pi)^2)$ and C is a suitable constant and of course

$$
\sum_{m,n} \sup_{k_1,\ldots,k_{2m}} |W_{m,n}(k_1, \ldots, k_{2m}; z)| \leq \sum_{m=1}^\infty C^m z^{\max(2,m-1)} \tag{33}
$$

converges for small enough z. We shall not reproduce here the details of the proof of eq.(31) as this would be a word by word repetition of par. 3 of (B.G.P.S.).

In order to perform the "infrared" integration eq.(27) we decompose the grassmanian integration $P(d\psi)$ into a product of independent grassmanian integrations, that is $P(d\psi_{i.r.}) = \prod_{h=-\infty}^0 P(d\psi^h)$. This can be done by setting $g_{i.r.}(k) = \sum_{h=-\infty}^0 g^h(k)$ and by writing $\psi^\pm_{i.r.} = \sum_h \psi^{\pm,h}$, with ψ^h being a family of grassmanian fields with vanishing "cross propagator" (*i.e.* an independent family of variables) and with propagator $\int \psi^h_{k_1,\sigma_1} \psi^h_{k_2,\sigma_2} P(d\psi^h) = \delta_{\sigma_1,\sigma_2} \delta(k_1 - k_2) g^h(k_1)$:

$$
g^h(k) = \frac{f(\gamma^{-2h}(k_0^2 + (|\vec{k}| - \pi)^2))}{-ik_0 - E(\vec{k})} \tag{34}
$$

where $E(\vec{k}) = \vec{k}^2 - \pi^2$, $f(\gamma^{-2h}t^2) = h(\gamma^{-2h-2}t^2) - h(\gamma^{-2h}t^2)$ is a C^∞ function with compact support.

It is convenient to introduce new grassmanian fields $\psi^{\pm,h}_{k,\vec\omega,\sigma}$, called *quasi-particles* Euclidean fields, with propagators $g^h_{\vec\omega}(k)$ and vanishing cross propagators, such that:

$$\psi^{\pm,h}_{k,\sigma}=\sum_{\vec\omega=\pm1}\psi^{\pm,h}_{k,\vec\omega,\sigma}\quad g^h(k)=\sum_{\vec\omega=\pm1}g^h_{\vec\omega}(k)$$

with:

$$g^h_{\vec\omega}(k)=\chi(\vec\omega\vec k)\frac{f(\gamma^{-2h}(k_0^2+(|\vec k|-\pi)^2))}{-ik_0-E(\vec k)}\tag{35}$$

where $\chi(\vec k)+\chi(-\vec k)=1$, $\chi(\vec k)=1$ if $\vec k\geq 0$ and it is 0 otherwise. It will be convenient in the following to write the momentum of the $\vec\omega$ fermion as $\vec k+\vec\omega p_F$, where $\vec k$ is called the *momentum measured from the Fermi surface* and $p_F=\pi$ is the Fermi momentum. It is easy to check, see (B.G.P.S.), that the quasi-particle propagator eq.(35) can be written as:

$$g^h_{\vec\omega}(k+\vec\omega p_F)=\gamma^{-h}g_{\vec\omega}(\gamma^{-h}k)+\bar g^h(\gamma^{-h}k)$$

with $\bar g^h(k)$ regular and weakly dependent on h and:

$$g_{\vec\omega}(k)=\frac{f(k_0^2+\vec k^2)}{-ik_0-\vec\omega\vec k}$$

We can say that $\psi^{h,\pm}_{k+\vec\omega p_F,\vec\omega,\sigma}$ have a distribution which, "up to scaling", is h independent *i.e.* the distribution of $\psi^{h,\pm}_{k+\vec\omega p_F,\vec\omega,\sigma}$ is the same of $\gamma^{-h/2}\psi^{h,\pm}_{\gamma^{-h}k+\vec\omega p_F,\vec\omega,\sigma}$ up to corrections negligible in the $h\to-\infty$ limit.

If we represent $V^0(\psi)$, eq.(31), in terms of quasi-particles fields we obtain a sum of terms like:

$$V^0_m(\psi)=\int\frac{dk_1}{(2\pi)^2}\cdots\frac{dk_m}{(2\pi)^2}f^{m,0}_n(K_m;\Omega_m)\delta(\sum_{i=1}^m(k_i+\vec\omega_ip_F)\varepsilon_i+2n\pi)\prod_{i=1}^m\psi^{\varepsilon_i,0}_{k_i+\vec\omega_ip_F,\vec\omega_i,\sigma_i}\tag{36}$$

where $\varepsilon=\pm$, K_m is the set of variables $\{k_1;\ldots;k_m\}$ and $\Omega_m=\{\vec\omega_1;\ldots;\vec\omega_m\}$ and $f^{m,0}_n(K_m;\Omega_m)$ are the kernels of eq.(31).

From eq.(27) the partition function can be written

$$\mathcal N=\int\prod_{h\leq0}P(d\psi^h)e^{-V^0(\sum_{h\leq0}\psi^h)}\tag{37}$$

and the above equation leads naturally to the definition of *effective potential* on scale γ^h:

$$e^{-V^h(\psi^{\leq h})}=\int P(d\psi^{h+1})\ldots\int P(d\psi^0)e^{-V^0(\psi^{\leq0})}\tag{38}$$

where $\psi^{\pm,\leq h}=\sum_{k\leq h}\psi^{\pm,k}$ and $\int P(d\psi^{\leq h})\psi^{+,\leq h}_k\psi^{-,\leq h}_k=g^{\leq h}(k)$ with:

$$g^{\leq h}(k)=\frac{C_h^{-1}(k)}{\left(-ik_0-(\frac{\vec k^2-\pi^2}{2m})\right)}\tag{39}$$

$$C_h^{-1}(k)=\sum_{k=-\infty}^{h}f(\gamma^{2k}(k_0^2+(|\vec k|-\pi)^2)=\sum_{k=-\infty}^{h}f_h(k_0,\vec k)\tag{40}$$

if $f_h(k_0,\vec k)=f(\gamma^{-2h}(k_0^2+(|\vec k|-\pi)^2))$. Of course $\psi^{\pm}_{i.r.}\equiv\psi^{\pm,\le 0}$ by definition.

As in (B.M.), we can isolate the relevant part of $V^0(\psi)$ by introducing a *localization operator* $\mathcal{L}$ on the Fermi surface acting on $V^0(\psi)$. It will appear clear, after the discussion following eq.(51), that a natural definition for the localization is $\mathcal{L}V^0_m(\psi)=0$ for $m>4$ and to compute the 0-th order, if $m=4$, or 1-st order, if $m=2$, of the Taylor series of $f^{m,0}_m(K_m,;\Omega_m)$ at the Fermi surface *i.e.* for $K_m\equiv 0_m=\{0;\ldots;0\}$. However k cannot assume the value 0 as it has the form $(2\pi n_1/N, 2\pi(n_2+1/2)/\beta)$, with n_1,n_2 integer, because of the antiperiodic boundary temporal conditions , and this leads to the complicated formulae below:

$$\begin{aligned}&\mathcal{L}\int\prod_{i=1}^4 dk_i\sum_n f^{4,0}_n(K_4;\Omega_4)\delta(\sum_{i=1}^4(k_i+\vec\omega_i p_F)\varepsilon_i+2n\pi)\prod_{i=1}^4\psi^{\varepsilon_i}_{k_i+\vec\omega_i p_F,\vec\omega_i,\sigma_i}\\&=\delta_{(\vec\omega_1+\vec\omega_2-\vec\omega_3-\vec\omega_4)p_F+n2\pi,0}f^{4,0}_n(0^\beta_4,\Omega_4)\\&\int\prod_{i=1}^4 dk_i\delta(k_1+k_2-k_3-k_4)\prod_{i=1}^4\psi^{\varepsilon_i,\le 0}_{k_i+\vec\omega_i p_F,\vec\omega_i,\sigma_i}\end{aligned}\tag{41}$$

where $\varepsilon_1=\varepsilon_2=-\varepsilon_3=-\varepsilon_4=+$, $\delta_{a,b}$ is the Kronecher delta, equal to 1 if $a=b$ and zero otherwise and $0^\beta_4=\{(0,\frac{\pi}{\beta});(0,\frac{\pi}{\beta});(0,\frac{\pi}{\beta});(0,\frac{\pi}{\beta})\}$. Moreover

$$\begin{aligned}&\mathcal{L}\int dk_1dk_2\delta(k_1-k_2+(\vec\omega_1-\vec\omega_2)p_F+2n\pi)f^{2,0}_n(K_2;\Omega_2)\psi^{+,\le 0}_{k_1'+\vec\omega_1 p_F,\vec\omega_1,\sigma}\psi^{-,\le 0}_{k_2+\vec\omega_2 p_F,\vec\omega_2,\sigma}\\&=\delta_{(\vec\omega_1-\vec\omega_2)p_F+2n\pi,0}\int dk_1dk_2\delta(k_1-k_2)\\&[f^{2,0,a}_n(\Omega_2)+E(\vec k_1+\vec\omega_1 p_F)\vec\omega_1 f^{2,0,b}_n(\Omega_2)+k_1^0f^{2,0,c}_n(\Omega_2)]\psi^{+,\le 0}_{k_1+\vec\omega_1 p_F,\vec\omega_1,\sigma}\psi^{-,\le 0}_{k_2+\vec\omega_2 p_F,\vec\omega_2,\sigma}\end{aligned}\tag{42}$$

where $\tilde\partial_{\vec k}f(\vec k)$, the discrete derivative, is defined for instance as $\frac{f(\vec k+2\pi/N)-f(\vec k)}{2\pi/N}$, and:

$$\begin{aligned}&f^{2,0,a}_n(\Omega_2)=\frac{1}{2}\sum_{i=1}^2 f^{2,0}_n(0^\beta_{2,i};\Omega_2)\qquad f^{2,0,b}_n(\Omega_2)=\frac{1}{2}\sum_{i=1}^2\tilde\partial_{\vec k_1}f^{2,0}_n(0^\beta_{2,i};\Omega_2)\\&f^{2,0,c}_n(\Omega_2)=\tilde\partial_{k_1^0}f^{2,0}_n(0^\beta_{2,2};\Omega_2).\end{aligned}\tag{43}$$

where $0^\beta_{2,i}=\{(0,(-1)^i\pi/\beta);(0,\pi/\beta)\}$.

Remark: Noting that $E(\vec k_1+\vec\omega p_F)\vec\omega=2\pi\vec k+\vec\omega\vec k^2$, in the limit $\beta\to\infty$ eq.(42) becomes the 1-st term of the Taylor series at the Fermi surface for the kernel $f^{2,0}_n$, plus a term $O(k^2)$. It is crucial for this that $E(\vec k_1+\vec\omega p_F)\vec\omega$ is approximately linear for small k; if in eq.(42) we put $\varepsilon(\vec k,u)$ instead of $\vec k^2$ this would not be true. The reason of the sum over i in the first line of eq.(43) is technical and it is discussed in (B.M.).

The Kroneker delta in the above equations can be satisfied in several ways. If $n=0$ then there are the possibilities $\vec\omega_1=\vec\omega_4=\vec\omega_2=\vec\omega_3$, $\vec\omega_1=\vec\omega_4=-\vec\omega_2=-\vec\omega_3$, $\vec\omega_1=\vec\omega_3=-\vec\omega_2=-\vec\omega_4$. If $|n|=1$ and remembering that $p_F=\pi$ the δ is satiesfied if $\vec\omega_1=\vec\omega_2=\vec\omega_3=-\vec\omega_4$ and if $\vec\omega_1=-\vec\omega_2=\vec\omega_3=\vec\omega_4$. Finally if $|n|=2$ we have $\vec\omega_1=\vec\omega_2=-\vec\omega_3=-\vec\omega_4$. These are all the possibilities in which the effect of the localization operator is not vanishing.

The relevant part of $V^0(\psi)$ in the spinning case is then:

$$\begin{aligned} \mathcal{L}V^0(\psi^{\leq 0}) &= n_0 F_\nu^{\leq 0} + a_0 F_\alpha^{\leq 0} + z_0 F_\zeta^{\leq 0} + s_0 F_\sigma^{\leq 0} + i_0 F_\theta^{\leq 0} + t_0 F_\tau^{\leq 0} + \\ &+ \lambda_{1,0} F_1^{\leq 0} + \lambda_{2,0} F_2^{\leq 0} + \lambda_{3,0} F_3^{\leq 0} + \lambda_{4,0} F_4^{\leq 0} + \lambda_{5,0} F_5^{\leq 0} + \lambda_{6,0} F_6^{\leq 0} \end{aligned} \tag{44}$$

$$\begin{aligned} & F_\nu^{\leq 0} = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega,\sigma} \psi^{-,\leq 0}_{k_2+\vec\omega p_F,\vec\omega,\sigma} \\ & F_\alpha^{\leq 0} = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 E(\vec k_1 + \vec\omega p_F) \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega,\sigma} \psi^{-,\leq 0}_{k_2+\vec\omega p_F,\vec\omega,\sigma} \delta(k_1 - k_2) \\ & F_\sigma^{\leq 0} = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega,\sigma} \psi^{-,\leq 0}_{k_2-\vec\omega p_F,-\vec\omega,\sigma} \delta(k_1 - k_2) \\ & F_\theta^{\leq 0} = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 E(\vec k_1 + \vec\omega p_F) \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega,\sigma} \psi^{-,\leq 0}_{k_2-\vec\omega p_F,-\vec\omega,\sigma} \delta(k_1 - k_2) \\ & F_\zeta^{\leq 0} = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 (-ik_1^0) \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega,\sigma} \psi^{-,\leq 0}_{k_2+\vec\omega p_F,\vec\omega,\sigma} \delta(k_1 - k_2) \\ & F_\tau^{\leq 0} = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 (-ik_1^0) \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega,\sigma} \psi^{-,\leq 0}_{k_2-\vec\omega p_F,-\vec\omega,\sigma} \delta(k_1 - k_2) \\ & F^{\leq 0}(\vec\omega_1,\vec\omega_2,\vec\omega_3,\vec\omega_4) = \\ & \sum_{\sigma,\sigma'} \int \prod_{i=1}^4 dk_i \psi^{+,\leq 0}_{k_1+\vec\omega_1 p_F,\vec\omega_1,\sigma} \psi^{+,\leq 0}_{k_2+\vec\omega_2 p_F,\vec\omega_2,\sigma'} \psi^{-,\leq 0}_{k_3+\vec\omega_3 p_F,\vec\omega_3,\sigma'} \psi^{-,\leq 0}_{k_4+\vec\omega_4 p_F,\vec\omega_4,\sigma} \delta(\sum_i \varepsilon_i k_i) \end{aligned} \tag{45}$$

where $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = 1$ and we call

$$\begin{aligned} F_1^{\leq 0} &= \sum_{\vec\omega} F^{\leq 0}(\vec\omega,-\vec\omega,\vec\omega,-\vec\omega) \qquad & F_2^{\leq 0} &= \sum_{\vec\omega} F^{\leq 0}(\vec\omega,-\vec\omega,-\vec\omega,\vec\omega) \\ F_3^{\leq 0} &= \sum_{\vec\omega} F^{\leq 0}(\vec\omega,\vec\omega,-\vec\omega,-\vec\omega) \qquad & F_4^{\leq 0} &= \sum_{\vec\omega} F(\vec\omega,\vec\omega,\vec\omega,\vec\omega) \\ F_5^{\leq 0} &= \sum_{\vec\omega} F^{\leq 0}(\vec\omega,\vec\omega,-\vec\omega,\vec\omega) \qquad & F_6^{\leq 0} &= \sum_{\vec\omega} F^{\leq 0}(-\vec\omega,\vec\omega,\vec\omega,\vec\omega) \end{aligned} \tag{46}$$

Remark: we can interpret eq.(45) as stating that the relevant part of the interaction between fermions is the one which involves fermions with momenta $k+\vec\omega p_F$ near the Fermi surface. Note moreover that the periodic potential has the effect that the sum of the momenta is not 0 but it is equal to $2n\pi$; this phenomenon is called *Umklapp* and we call the terms in eq.(45) in which the momentum is not conserved *Umklapp terms*. It will be crucial in the following the (trivial) observation that $\underline{s_0 = O(u)}$ and $\underline{i_0, t_0, \lambda_{3,0}, \lambda_{5,0}, \lambda_{6,0} = O(\lambda u)}$.

In the spinless case $\sigma = 0$ and

$$\mathcal{L}V_0 = n_0 F_\nu^{\leq 0} + a_0 F_\alpha^{\leq 0} + z_0 F_\zeta^{\leq 0} + s_0 F_\sigma^{\leq 0} + i_0 F_i^{\leq 0} + t_0 F_\tau^{\leq 0} + \lambda_0 F^{\leq 0} \tag{47}$$

where:

$$F^{\leq 0} = \sum_{\vec\omega} \int \prod_{i=1}^{4} dk_i \psi^{+,\leq 0}_{k_1+\vec\omega p_F,\vec\omega} \psi^{+,\leq 0}_{k_2-\vec\omega p_F,-\vec\omega} \psi^{-,\leq 0}_{k_3-\vec\omega p_F,-\vec\omega} \psi^{-,\leq 0}_{k_4+\vec\omega p_F,\vec\omega} \delta(\sum_i \varepsilon_i k_i) \tag{48}$$

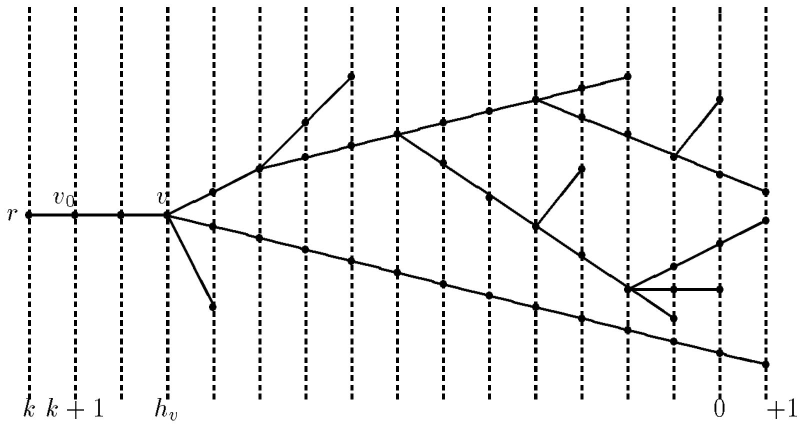
as, in the spinless case, $F_1^{\leq 0} = -F_2^{\leq 0} = F^{\leq 0}$ and $F_3^{\leq 0} = F_4^{\leq 0} = F_5^{\leq 0} = F_6^{\leq 0} = 0$, because of the anticommutation properties of the grassmanian variables so that eq.(47) can be deduced from eq.(45) setting $\lambda_0 = g_{1,0} = -g_{2,0}$.

We perform each integration in eq.(38) by writing

$$V^h(\psi^{\leq h}) = \mathcal{L}V^h(\psi^{\leq h}) + \mathcal{R}V^h(\psi^{\leq h})$$

with $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$ is defined as in eq.(41)(42) with $\psi^{\pm,\leq 0}_{k+\vec\omega p_F,\vec\omega,\sigma}$ replaced by $\psi^{\pm,\leq h}_{k+\vec\omega p_F,\vec\omega,\sigma}$ and $f_n^{m,0}(K_m,\Omega_m)$ by $f_n^{m,h}(K_m,\Omega_m)$. Then $\mathcal{L}V^h(\psi^{\leq h})$ is given by eq.(44) with $n_0, a_0, z_0, i_0, t_0, s_0, \lambda_{i,0}$, $i = 1,\ldots,6$, replaced by $\gamma^h n_h, a_h, z_h, i_h, t_h, \gamma^h s_h, \lambda_{i,h}$; the reason for which we write the terms multipling $F_{\tilde\nu}^{\leq h}$ and $F_{\tilde\sigma}^{\leq h}$ as $\gamma^h s_h$ and $\gamma^h n_h$ instead of simply s_h and n_h will be discussed after eq.(51). The constants $n_h, a_h, z_h, i_h, t_h, s_h \lambda_{i,h}$ will be called *running coupling constants* and will be denoted by $v_{j,h}$.

It is possible to check, see for instance (B.G.1), that the effective potential $V^h(\psi^{\leq h})$ can be written in terms of *trees* as follows.



We call τ_n the set of all the *labeled trees with n end points* $\tau \in \tau_n$ that can be costructed as follows (see also the picture). Draw on the (x,y) plane vertical lines at $x = k, k+1, \ldots, 0, 1$. Let r (*the root*) be a point on the line $x = k$. Starting from r draw an horizontal line leading to a point v_0 on the line $x = k_{v_0} = k+1$. Choose $s_{v_0} \geq 0$ and draw s_{v_0} lines starting from v_0 leading to s_v points $v_1, \ldots, v_{s_v}$ *i.e.* the lines cannot go back. Do the same thing starting with the points v_i and go on recursively. A point v is called an *end point* if $s_v = 0$, *i.e.* if there is no line starting from this point. Moreover a point v is a *trivial vertex* if $s_v = 1$ and a *non trivial vertex* if $s_v \geq 2$. Finally if $h_v = 1$ then v is necessarely an end point. Clearly this process ends when all the reached points are end points. A *cluster* v with frequency h_v is the set of the end-points reacheable from the vertex v with frequency h_v; and the tree provides an organization of the endpoints into a hierarchy of clusters. Each non trivial or trivial vertex bear a label $\mathcal{R}$ except v_0 (see the picture) which can bear either a label $\mathcal{R}$ or a label $\mathcal{L}$. To each tree we associate a term $V^k(\tau, \psi^{\leq k})$ defined recursively as follows. If τ has only one end-point with frequency $k+1$ then $V^k(\tau, \psi^{\leq k})$ is equal to one of the term of eq.(44) with $\vec{v}_k$ instead of $\vec{v}_0$ or, only if $k = 0$, one of the monomial in $\mathcal{R}V^0$. We attach a label to each endpoint of the tree to distinguish among these possibilities. Otherwise

$$V^k(\tau, \psi^{\leq k}) = \mathcal{O}\frac{1}{s_v!}\mathcal{E}^T_{k+1}[V^{k+1}(\tau^1, \psi^{\leq k+1}), \ldots)] \tag{49}$$

where $\mathcal{O}$ is $\mathcal{L}$ or $\mathcal{R}$ if the vertex v bears an $\mathcal{L}$ or $\mathcal{R}$ label, $n \geq 2$, $\tau^1 \ldots \tau^{s_v}$ are the subtrees starting from v and the symbols $\mathcal{E}^T_h$ denote the truncated expectations with respect to an integration with propagator g^h. We have that $\mathcal{O}$ can be equal to $\mathcal{L}$ only if $v = v_0$ and the tree contributes to the local part of the effective potential. We also associate to each field a labels f, $f = 1, \ldots, n_\tau$ where n_τ is the number of the fields associated with all the end-points of the tree. To every field with label f corresponds a momentum $k(f)$ and the indices $\vec{\omega}(f), \sigma(f), \varepsilon(f) = \pm 1$ and, also, the index $s(f) = 0, 1, 2$ allowing us to distinguish the three possibilities $\psi^{\varepsilon(f)}_{k(f)+\vec{\omega}_f p_F, \vec{\omega}(f), \sigma(f)}$, $E(\vec{k})\psi^{\varepsilon(f)}_{k(f)+\vec{\omega}(f)p_F, \vec{\omega}(f), \sigma(f)}$, $-ik_0\psi^{\varepsilon(f)}_{k(f)+\vec{\omega}(f)p_F, \vec{\omega}(f), \sigma(f)}$. We call I_{v_0} the set of f labels.

It is possible to check that the effective potential eq(38) can be written as

$$V^k(\psi^{\leq k}) = \sum_{n=1}^{\infty}\sum_{\tau\in\tau_n} V^k(\tau, \psi^{\leq k}). \tag{50}$$

From eq.(49) we see that each set of running coupling constant $\vec{v}_h = \{v_{h,j}\}$ is determined once that a set $\vec{v}_0$ is given from the relation $\vec{v}_{h-1} = \vec{v}_h + \beta_h(\vec{v}_h, \vec{v}_{h+1}, \ldots \vec{v}_0)$ where β_h, called *beta function*, is a sum over all the trees contributing to the relevant part of the effective potential. We define:

$$V^k(\tau, \psi^{(\leq k)}) = \sum_n \int dk_{v_0} \sum_{P_{v_0}} V^k(\tau, P_{v_0}, k_{v_0}, n)\tilde{\psi}^{\leq k}(P_{v_0}, n)\delta(\sum_{f\in P_{v_0}} \varepsilon(f)k(f) + 2n\pi)$$

where k_{v_0} is the set of all the momentum variables of the n_τ fields, P_{v_0} is a non empty subset of I_{v_0}, $|P_{v_0}|$ are the number of elements of this subset, $\sum_{P_{v_0}}$ is the sum over such subsets and $\tilde{\psi}^{\leq k}(P_{v_0}) = \prod_{f\in P_{v_0}} \psi^{\varepsilon(f), \leq k}_{k(f), \vec{\omega}(f), \sigma(f), s(f)}$

If in eq.(49) we expanded the expectations by Wick's theorem, we could represent the r.h.s. as a sum of *Feynman graphs* (see (B.G.1) for a detailed exposition). A Feynman graph is constructed by symbolizing the fields associated with every end-points of the tree as orientated *half lines* emerging from that point and enclosing the end-points belonging to the cluster v togheter with their half lines into an ideal box. We pair, i.e. *contract*, the half lines in *internal lines* (all but the *external lines* $\tilde{\psi}^{\leq k}(P_{v_0})$) and we associate to each of them a *propagator* g^{h_v}, if the line is contained in the ideal box containing the cluster v and not in any one with higher frequency. Every graph contributes to the effective potential with a term of the form

$$\int dK^{P_{v_0}} f_n^{P_{v_0},h_{v_0}}(K^{P_{v_0}},\Omega^{P_{v_0}})\delta(\sum_{f\in P_{v_0}}\varepsilon(f)(k(f)+\vec{\omega}(f)p_F)+2n\pi)\tilde{\psi}^{\leq k}(P_{v_0})$$

where $K^{P_{v_0}}$ is the set of the variables $k(f)$ with $f\in P_{v_0}$, $\Omega^{P_{v_0}}$ is defined in the same way and $f_n^{P_{v_0},h_{v_0}}$, called *value of the graph*, is the product of the propagators of the graph and of the running couplings or the kernels in eq.(31) associated to the end points, integrated over the momenta of the internal lines.

Furthermore, if G_τ is the set of all Feynman graphs associated with τ, given $g\in G_\tau$, it is natural to associate a *subgraph* g_v to the vertex v enclosing into an ideal box the cluster v and cutting into half lines the lines connecting points in the v cluster with points outside from it. Each g_v is of the form

$$\int dK^{P_v} f_n^{P_v,h_v}(K^{P_v},\Omega^{P_v})\delta(\sum_{f\in P_v}\varepsilon(f)k(f)+2n\pi/a)\tilde{\psi}^{\leq h_v-1}(P_v)$$

where $\tilde{\psi}^{\leq h_v-1}(P_v)$ are the half lines emerging from v before contraction and P_v is defined as P_{v_0}. On all these terms the $\mathcal{R}$ operation acts, if $v\neq v_0$, while if $v\equiv v_0$ the operation $\mathcal{L}$ or $\mathcal{R}$ acts, depending on whether it contributes to the relevant or to the irrelevant part of the effective potential.

We call *scaling dimension* $D(P_{v_0})=-2+\sum_{A\in P_{v_0}}(1/2+\chi_A)$ where $\chi_A=0$ if $A=\psi^h_{k+\vec{\omega}p_F,\vec{\omega},\sigma}$, $\chi_A=1$ if $A=E(k)\psi^h_{k+\vec{\omega}p_F,\vec{\omega},\sigma}$ or $A=-ik_0\psi^h_{k+\vec{\omega}p_F,\vec{\omega},\sigma}$. The *size* of a generic graph associated with a monomial $\tilde{\psi}^{\leq k}(P_{v_0})$ with value $f_n^{P_v,h_v}$ is defined by:

$$||f_n^{P_v,h_v}||=\sup_{k_1\ldots,k_{|P_v|}\in K^{P_v}}\gamma^{hD(P_{v_0})}d_h(k_1)\cdots d_h(k_{|P_v|})f_n^{P_v,h_v}(K^{P_v},\Omega^{P_v}) \tag{51}$$

where $d_h(k)$ is the characteristic function of the support of $f(\gamma^{-2h}(k_0^2+E(\vec{k})^2))$.

In order to motivate our definition of localization suppose for a moment that $\mathcal{R}=I$ where I is the identity operator; by a standard calculation, see (B.G.1), it is possible to prove that the size, eq.(51), of a Feynman graph is bounded by

$$||f_n^{P_v,h_v}||<C^m\varepsilon^m\prod_v\gamma^{-(h_v-h_{v'})D(P_v)}$$

where v' is the vertex preceding v in the tree ordering, m is the number of end-points, $\varepsilon=\max_{j,h}|v_{j,h}|$ and C is a suitable constant. To obtain an estimate of the perturbative contribution of order n to the effective potential, we must sum over trees. In order to have an estimate uniform in β,N it is necessary that $D(P_v)>0$ for all P_v. But we have that $D(P_v)=-1$ if $|P_v|=2$ and $\sum_{A\in P_v}\chi_A=0$, while $D(P_v)=0$ if $|P_v|=4$ and $\sum_{A\in P_v}\chi_A=0$ or $|P_v|=2$ and $\sum_{A\in P_v}\chi_A=1$. Note that $D(P_v)$ depends only on the number of external lines with $\chi=0$ or 1: this is due to the fact that we write the coefficients of $F_\nu^{\leq h},F_\sigma^{\leq h}$ as $\gamma^h n_h$ and $\gamma^h\sigma_h$.

Like in (B.G.) one could define as "relevant part" of the effective potential the sum of its quadratic and quartic parts in the fields. However such definitions would still contain irrelevant terms. This can be easily understood by remarking that for h suitable small the contributions to the effective potential V^h having forms:

$$\int \prod_{i=1}^{4} dk_i f_n^{4,h}(K_4;\Omega_4)\delta(k_1+k_2-k_3-k_4+(\vec{\omega}_1+\vec{\omega}_2-\vec{\omega}_3-\vec{\omega}_4)p_F+2n\pi)$$
$$\psi^{+,\le h}_{k_1+\vec{\omega}_1 p_F,\vec{\omega}_1,\sigma}\psi^{+,\le h}_{k_2+\vec{\omega}_2 p_F,\vec{\omega}_2,\sigma'}\psi^{-,\le h}_{k_3+\vec{\omega}_3 p_F,\vec{\omega}_3,\sigma}\psi^{-,\le h}_{k_4+\vec{\omega}_4 p_F,\vec{\omega}_4,\sigma}$$

or

$$\int dk_1 dk_2 \delta(k_1-k_2+(\vec{\omega}_1-\vec{\omega}_2)p_F+2n\pi) f^2(K_2;\Omega_2)\psi^{+,\le h}_{k_1+\vec{\omega}_1 p_F,\vec{\omega}_1,\sigma}\psi^{-,\le h}_{k_2+\vec{\omega}_2 p_F,\vec{\omega}_2,\sigma} \tag{52}$$

are vanishing unless $(\vec{\omega}_1+\vec{\omega}_2-\vec{\omega}_3-\vec{\omega}_4)p_F+2n\pi=0$ in the first case and $(\vec{\omega}_1-\vec{\omega}_2)p_F+2n\pi=0$ in the second as the delta's in the above equations cannot be satisfied for the support properties of the fiels.

It is possible to check by a standard calculation that, with the definition of $\mathcal{L},\mathcal{R}$ given by eq.(41)(42) the size of the generic Feynman graph contribuing to the effective potential defined above is bounded by:

$$||f_n^{P_v,h_v}|| < C^m \varepsilon^m \prod_v \gamma^{-(h_v-h_{v'})(D(P_v)+z_v)} \tag{53}$$

where $D(P_v)+z_v>0$ (the $\mathcal{R}$ operation was defined in order to make true such an inequality). By repeating the estimates in (B.G.P.S.)(B.M.) it is easy to see that eq.(53) implies that $|V^{(k)}(\tau,\psi^{\le k})|\le \varepsilon^n c^n$ so that if ε is small enough the partition function is analytic in its argument.

The problem is that *we do not expect that* $|v_{h,i}|\le\varepsilon$, and this because a second order computation (in the spinless case for simplicity) shows that $\lambda_{h-1}=\lambda_h$ and

1. $z_{h-1}=z_h+\beta_1\lambda_h^2; \qquad a_{h-1}=a_h+\beta_1\lambda_h^2$
2. $s_{h-1}=\gamma s_h+\gamma s_h\beta_3\lambda_h; \qquad n_{h-1}=\gamma n_h$

with $\beta_3,\beta_1>0$, so that, even if a third order computation showed that $\lambda_h\to_{h\to-\infty}0$ this would mean that $\lambda_h\simeq O(\frac{1}{\sqrt{h}})$ and the couplings a_h,z_h,s_h would be unbounded at this approximation; moreover $n_h=\gamma^{-h}n_0$ *i.e.* it grows as $h\to-\infty$. The unbounded growth of z_h,a_h happens also in the $u=0$ case and it is an indication of the anomalous behaviour of the theory, see (B.G.P.S.), *i.e.* that the Schwinger function has a different asymptotic behaviour for large $|x-y|$ with respect to the $\lambda=0$ case ; the growth of n_h is a signal that the interaction changes the Fermi momentum p_F.

The growth of s_h is, on the other hand, a peculiar problem of the filled band case, as there is not such a running coupling constant if $u=0$ and is an indication that the Schwinger function behaviour for large distances is different in the interacting case respect to the free one, as we will see, or respect the $\lambda \neq 0, u=0$ case of (B.G.P.S.). The idea is then to modify the definition of V^h at each step of the renormalization group generalizing the anomalous scaling procedure of (B.G.) in order to take into account the presence of s_h. Note finally that if $u \neq 0$ but $p_F \neq n\pi$ *i.e.* the band is not filled, there is no σ_h among the running couplings (this is essentially due to the momentum conservation) and the Schwinger function behaviour is similar to the spinless $\lambda \neq 0, u=0$ case, see (B.M.), if the fermions are spinless or spinning but $\lambda > 0$.

2.2 Anomalous scaling

In the preceding section we saw that it is possible to express the partition function as an analytic function of the running coupling constants, if such constants are small enough. However a second order computation suggests that a_h, z_h, s_h are not bounded and we can interpret this as an evidence that the Schwinger function behaviour is different with respect to the free $\lambda = u = 0$ case. This of course is just what we expect as if $\lambda = 0, u \neq 0$ the Schwinger function are explicitely computable and in the $\lambda \neq 0, u = 0$ case their asymptotic behaviour was obtained in (B.G.P.S.) and in both cases it is different with respect to the $\lambda = u = 0$ case (see also the introduction). It is possible to extend the methods so far followed to a more general approach which can take into account a possible Schwinger function behaviour modification respect to the free case. Such approach is called "anomalous scaling" and it was used for the first time in (W.F.) for the infrared problem in the ϕ^4_3 model, and among other applications in (B.G.),(B.G.M.),(B.G.P.S) for the study of a system of $d=1$ Fermi system and in (B.G.1) for the Bose condensation problem.

We write, calling $P(d\psi^{\leq 0}) = P_{Z_0}(d\psi^{\leq 0})$ with $Z_0 = 1$:

$$\mathcal{N} = \int P_{Z_0}(d\psi^{\leq 0}) e^{-V^0(\sqrt{Z_0}\psi^{\leq 0})} = \frac{1}{N_0} \int \prod_{\vec{k},\vec{\omega},\sigma} d\psi^{+,\leq 0}_{k+\vec{\omega}\pi,\vec{\omega},\sigma} d\psi^{-,\leq 0}_{k+\vec{\omega}\pi,\vec{\omega},\sigma} e^{-V^0(\sqrt{Z_0}\psi^{\leq 0})}$$

$$\exp\left\{ -\sum_{\sigma,\vec{\omega}} \int dk Z_0 C_0(k) \frac{-ik_0 - \vec{k}^2 - 2\vec{\omega}\pi\vec{k}}{\chi(\vec{\omega}\vec{k} + p_F)} \psi^{+,\leq 0}_{k+\vec{\omega}\pi,\vec{\omega},\sigma} \psi^{-,\leq 0}_{k+\vec{\omega}\pi,\vec{\omega},\sigma} \right\}$$

where the grassman integration is written formally as an integral over $d\psi^{+,\leq 0}_{k+\vec{\omega}\pi,\vec{\omega},\sigma} d\psi^{-,\leq 0}_{k+\vec{\omega}\pi,\vec{\omega},\sigma}$ times the exponential of the appropriate quadratic form and N_0 is a normalization factor. The idea is now to extract from the effective potential some terms which are difficult to control, *i.e.* $s_0 F^{\leq 0}_{\sigma}$ and $z_0 F^{\leq 0}_{\xi}$ and to put them in the grassmanian integration. In this way it remains in the interaction a term $(a_0 - z_0) F^{\leq 0}_{\alpha}$ which from the second order beta function is easier to control. This is made writing:

$$\mathcal{N} = \int P_{Z_0}(d\psi^{\leq 0}) e^{-V^0(\psi^{\leq 0})} = \int P_{Z_{-1}}(d\psi^{\leq 0}) e^{-\tilde{V}^0(\psi^{\leq 0})}$$

where:

$$P_{Z_{-1}}(d\psi^{\leq 0}) = \frac{1}{N_0} \prod_{\vec{k},\vec{\omega},\sigma} d\psi^{+}_{k+\vec{\omega}\pi,\vec{\omega},\sigma} d\psi^{-}_{k+\vec{\omega}\pi,\vec{\omega},\sigma}$$

$$\exp\Big\{-\sum_{\sigma,\vec\omega}\int dk C_0(k)Z_{-1}(k)\Big[\frac{-ik_0-\vec k^2-2\vec\omega\pi\vec k}{\chi(\vec\omega\vec k+p_F)}\psi^+_{k+\vec\omega\pi,\vec\omega,\sigma}\psi^-_{k+\vec\omega\pi,\vec\omega,\sigma}+$$

$$\sigma_{-1}(k)\psi^+_{k+\vec\omega\pi,\vec\omega,\sigma}\psi^-_{k-\vec\omega\pi,-\vec\omega,\sigma}\Big]\Big\} \tag{54}$$

with $Z_{-1}(k)=Z_0+C_0^{-1}(k)z_0$, $Z_{-1}(k)\sigma_{-1}(k)=C_0^{-1}(k)\sigma_0$ and $\tilde V^0=\mathcal{L}\tilde V^0+(1-\mathcal{L})V^0$ with

$$\mathcal{L}\tilde V^0(\psi^{\le 0})=n_0F_\nu^{\le 0}+(a_0-z_0)F_\alpha^{\le 0}+t_0F_t^{\le 0}+i_0F_\theta^{\le 0}+\sum_{i=1}^{6}\lambda_{i,0}F_i^{\le 0} \tag{55}$$

The Grassman integral $P_{Z_{-1}}(d\psi^{\le 0})$ can also be thougth as an integral over two independent fields adding up to $\psi^{\le 0}=\psi^{\le -1}+\psi^0$ *i.e.*

$$P_{Z_{-1}}(d\psi^{\le 0})=P_{Z_{-1}}(d\psi^{\le -1})P_{Z_{-1}}(d\psi^0)$$

where $P_{Z_{-1}}(d\psi^{\le -1})$, $P_{Z_{-1}}(d\psi^0)$ are equal to $P_{Z_{-1}}(d\psi^{\le 0})$ eq.(54) with $C_0(k)$ replaced by $C_{-1}(k)$ and $f_0(k_0,\vec k)$ respectively. We can write then:

$$\mathcal{N}=\int P_{Z_{-1}}(d\psi^{\le -1})\int P_{Z_{-1}}(d\psi^0)e^{-\hat V^0(\sqrt{Z_{-1}}\psi^{\le 0})} \tag{56}$$

where:

$$\mathcal{L}\hat V^0(\psi^{\le 0})=\nu_0F_\nu^{\le 0}+\delta_0F_\alpha^{\le 0}+\theta_0F_\theta^{\le 0}+\tau_0F_\tau^{\le 0}+\sum_{i=1}^{6}g_{i,0}F_i^{\le 0} \tag{57}$$

$$\nu_0=\frac{Z_0}{Z_{-1}}n_0,\quad \delta_0=\frac{Z_0}{Z_{-1}}(a_0-z_0),\quad \theta_0=\frac{Z_0}{Z_{-1}}i_0,\tau_0=\frac{Z_0}{Z_{-1}}t_0,\quad g_{i,0}=\left(\frac{Z_0}{Z_{-1}}\right)^2\lambda_{i,0}$$

It is convenient to write explicitely the propagator associated to the integration $P_{Z_{-1}}(d\psi^{\le 0})$ *i.e.* $g^0_{\vec\omega,\vec\omega'}(x,y)=\int P_{Z_{-1}}(d\psi^0)\psi^{+,0}_{x,\vec\omega,\sigma}\psi^{-,0}_{y,\vec\omega',\sigma}$; we have that:

$$g^0_{\vec\omega,\vec\omega'}(x,y)=\int dk e^{ik(x-y)}f_0(k_0,\vec k)T_{-1}(k)^{-1}_{\vec\omega,\vec\omega'} \tag{58}$$

where

$$T_{-1}(k)=$$
$$Z_{-1}(k)\begin{pmatrix}\chi(\vec k+p_F)^{-1}\left(-ik_0-\vec k^2-2\pi\vec k\right) & \sigma_{-1}(k)\\ \sigma_{-1}(k) & \chi(-\vec k+p_F)^{-1}\left(-ik_0-\vec k^2+2\pi\vec k\right)\end{pmatrix}$$

which is well defined in the support of $f_0(k_0,\vec k)$ and

$$A_{-1}(k)=Z_{-1}(k)\left(\left(-ik_0+\vec k^2\right)^2-\left(2\pi\vec k\right)^2-\chi(\vec k+p_F)\chi(-\vec k+p_F)\sigma_{-1}(k)^2\right)$$

$$T_{-1}(k)^{-1}=\frac{1}{A_{-1}(k)}\cdot \tag{59}$$
$$\begin{pmatrix}\chi(\vec k+p_F)\left(-ik_0-\vec k^2+2\pi\vec k\right) & -\sigma_{-1}(k)\chi(\vec k+p_F)\chi(-\vec k+p_F)\\ -\sigma_{-1}(k)\chi(\vec k+p_F)\chi(-\vec k+p_F) & \chi(-\vec k+p_F)\left(-ik_0-\vec k^2-2\pi\vec k\right)\end{pmatrix}$$

We perform the integration $\int P_{Z_{-1}}(d\psi^0)e^{-\hat V^0(\sqrt{Z_{-1}(k)}\psi^{\le 0})} = e^{-V^{-1}(\sqrt{Z_{-1}}\psi^{\le -1})}$ where

$$\begin{aligned}\mathcal{L}V^{-1}(\psi^{\le -1}) &= n_{-1}F_\nu^{\le -1} + z_{-1}F_z^{\le -1} + a_{-1}F_\alpha^{\le -1} + s_{-1}F_s^{\le -1} + t_0F_t^{\le -1} + \\ &+ i_{-1}F_\theta^{\le -1} + \sum_{i=1}^6 \lambda_{i,-1}F_i^{\le -1}\end{aligned} \tag{60}$$

and of course the procedure can be iterated.

In general once the fields $\psi^0,\ldots,\psi^{h+1}$ have been integrated proceding as above, we have to evalutate:

$$\mathcal{N} = \int P_{Z_h}(d\psi^{\le h})e^{-V^h(\sqrt{Z_h}\psi^{\le h})}$$

where $\mathcal{L}V^h(\psi^{\le h})$ is defined as

$$\mathcal{L}V^h(\psi^{\le}) = n_hF_\nu^{\le h} + a_hF_\alpha^{\le h} + t_hF_t^{\le h} + i_hF_\theta^{\le h} + s_hF_\sigma^{\le h} + z_hF_\zeta^{\le h} + \sum_{i=1}^6 \lambda_{i,h}F_i^{\le h} \tag{61}$$

and formally:

$$\begin{aligned}&P_{Z_h}(d\psi^{\le h}) = \frac{1}{N_h}\prod_{\vec k,\vec\omega,\sigma} d\psi^{+,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}d\psi^{-,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}\\ &\exp\Big\{-\sum_{\sigma,\vec\omega}\int dk C_h(k)Z_h(k)\Big[\frac{-ik_0-\vec k^2-2\vec\omega\pi\vec k}{\chi(\vec\omega\vec k+p_F)}\psi^{+,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}\psi^{-,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma} -\\ &\sigma_h(k)\psi^{+,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}\psi^{-,\le h}_{k-\vec\omega\pi,-\vec\omega,\sigma}\Big]\Big\}\end{aligned} \tag{62}$$

We include as above in the Grassmanian integration the "dangerous" terms of $\mathcal{L}V^h(\psi)$ *i.e.* $s_hF_\sigma^{\le h}$ and $z_hF_\zeta^{\le h}$:

$$\int P_{Z_h}(d\psi^{\le h})e^{-V^h(\sqrt{Z_h}\psi^{\le h})} = \int P_{Z_{h-1}}(d\psi^{\le h})e^{-\tilde V^h(\sqrt{Z_h}\psi^{\le h})} \tag{63}$$

with

$$\begin{aligned}&P_{Z_{h-1}}(d\psi^{\le h}) = \frac{1}{N_h}\prod_{\vec k,\vec\omega,\sigma} d\psi^{+,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}d\psi^{-,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}\\ &\exp\Big\{-\sum_{\sigma,\vec\omega}\int dk C_h(k)Z_{h-1}(k)\Big[\frac{-ik_0-\vec k^2-2\vec\omega\pi\vec k}{\chi(\vec\omega\vec k+p_F)}\psi^{+,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}\psi^{-,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma} -\\ &\sigma_{h-1}(k)\psi^{+,\le h}_{k+\vec\omega\pi,\vec\omega,\sigma}\psi^{-,\le h}_{k-\vec\omega\pi,-\vec\omega,\sigma}\Big]\Big\}\end{aligned} \tag{64}$$

with $Z_{h-1}(k) = Z_h(k) + C_h^{-1}(k)Z_h(k)z_h$, $Z_{h-1}(k)\sigma_{h-1} = Z_h(k)\sigma_h + Z_h(k)C_h^{-1}(k)s_h$ and $\tilde V^h = \mathcal{L}\tilde V^h + (1-\mathcal{L})V^h$ with

$$\begin{aligned}\mathcal{L}\tilde V^h(\psi^{\le h}) &= n_hF_\nu^{\le h} + (a_h - z_h)F_\alpha + t_hF_t + i_hF_\theta + \\ &+ \lambda_{1,h}F_1 + \lambda_{2,h}F_2 + \lambda_{3,h}F_3 + \lambda_{4,h}F_4 + \lambda_{5,h}F_5 + \lambda_{6,h}F_6\end{aligned} \tag{65}$$

Finall we write:

$$\mathcal{N}=\int P_{Z_{h-1}}(d\psi^{\leq h-1})\int P_{Z_{h-1}}(d\psi^h)e^{-\hat{V}^h(\sqrt{Z_{h-1}}\psi^{\leq h})}$$

with

$$g^h_{\vec\omega,\vec\omega'}(x-y)=\int P_{Z_{h-1}}(d\psi^h)\psi^{+,h}_{x,\vec\omega,\sigma}\psi^{-,h}_{y,\vec\omega',\sigma}=\int dk e^{ik(x-y)}f_h(k_0,\vec k)T_h(k)^{-1}_{\vec\omega,\vec\omega'}\qquad(66)$$

where

$$A_h(k)=Z_h(k)\left(\left(-ik_0+\vec k^2\right)^2-\left(2\pi\vec k\right)^2-\chi(\vec k+p_F)\chi(-\vec k+p_F)\sigma_h(k)^2\right)$$

$$T_h(k)^{-1}=\frac{1}{A_h(k)}\cdot$$
$$\begin{pmatrix}\chi(\vec k+p_F)\left(-ik_0-\vec k^2+2\pi\vec k\right) & -\sigma_h(k)\chi(\vec k+p_F)\chi(-\vec k+p_F)\\ -\sigma_h(k)\chi(\vec k+p_F)\chi(-\vec k+p_F) & \chi(-\vec k+p_F)\left(-ik_0-\vec k^2-2\pi\vec k\right)\end{pmatrix}$$

and

$$\mathcal{L}\hat V^h(\psi)=\gamma^h\nu_hF_\nu+\delta_hF_\alpha+\theta_hF_\theta+\tau_hF_\tau+\sum_{i=1}^6 g_{i,h}F_i\qquad(67)$$

$$\gamma^h\nu_h=\frac{Z_h}{Z_{h-1}}n_h,\quad \delta_h=\frac{Z_h}{Z_{h-1}}(a_h-z_h),\quad \theta_h=\frac{Z_h}{Z_{h-1}}i_h$$
$$\tau_h=\frac{Z_h}{Z_{h-1}}t_h,\quad g_{i,h}=\left(\frac{Z_h}{Z_{h-1}}\right)^2\lambda_{i,0}\qquad(68)$$

In the above formulae we have used that $\sigma_h(k), Z_h(k)$ do not depend on $\vec\omega$: this follows from the "rotation" invariance of the theory, *i.e.* invariance under transformation $\vec x\to-\vec x$, and from the definition of C_h^{-1}.

In the following we call $Z_h=Z_h(\gamma^h)$ and $\sigma_h=\sigma_h(\gamma^h)$. Note moreover that, if k belongs to the support of $f_h(k_0,\vec k)$:

$$Z_h(k)=Z_0\prod_{h+1\geq k>0}(1+C_k^{-1}(k)z_k)=Z_0\prod_{h+1\geq k>h+i}(1+C_k^{-1}(k)z_k)\prod_{h+i\geq k>0}(1+z_k)\qquad(69)$$

where i is a number depending on the support of $f_h(k_0,\vec k)$. A similar formula holds also for $\sigma_h(k)$.

In order to study the renormalization group flow we need estimates on the propagator, which are given by the following lemma that will be proved in appendix 3:

Lemma 2.1 *The propagator $g^h_{\vec\omega,\vec\omega}(x-y)$ can be written as*

$$g^h_{\vec\omega,\vec\omega}(x-y)=g^h_{\vec\omega,L}(x-y)+C_1(x-y)+C_2(x-y)\qquad(70)$$
$$g^h_{\vec\omega,L}(x-y)=\int\frac{dk}{(2\pi)^2}\frac{e^{ikx}}{Z_h}\frac{f(\gamma^{-2h}(k_0^2+\vec k^2))}{-ik_0-\vec\omega\vec k}$$

where $|C_1(x-y)| \leq \frac{\gamma^{2h} C_N}{Z_h(1+(\gamma^h|x-y|)^N)}$ *and* $|C_2(x-y)| \leq \frac{\gamma^h}{Z_h}\left(\frac{\sigma_h}{\gamma^h}\right)^2 \frac{C_N}{(1+(\gamma^h|x-y|)^N}$. *Moreover*

$$|g^h_{\vec{\omega},-\vec{\omega}}(x-y)| \leq \frac{\gamma^h}{Z_h}\frac{\sigma_h}{\gamma^h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$$

We can describe the structure of $\hat{V}^h$ in term of a *tree expansion*. Note in fact that:

$$e^{-V^h(\sqrt{Z_h}\psi^{\leq h})} = \int P_{Z_h}(d\psi^{h+1})e^{-\hat{V}^{h+1}(\sqrt{Z_h}\psi^{\leq h+1})}$$

and at each step $V^h(\sqrt{Z_h}\psi)$ is written as

$$\hat{V}^h(\psi) = \mathcal{L}^* V^h\left(\frac{\sqrt{Z_h}}{\sqrt{Z_{h-1}}}\psi\right) + \mathcal{R}V^h\left(\frac{\sqrt{Z_h}}{\sqrt{Z_{h-1}}}\psi\right)$$

where $\mathcal{L}^* V^h$ differs from $\mathcal{L}V^h$ only because it does not contain anymore the addends F_z and F_σ. We can write

$$\begin{aligned} V^k(\psi^{\leq k}) &= \sum_{n=1}^{\infty}\sum_{\tau\in\tau_n} V^k(\tau,\psi^{\leq k}) \\ V^k(\tau, Z_k^{\frac{1}{2}}\psi^{\leq k}) &= \mathcal{O}\frac{1}{s_{v_0}!}E^T_{k+1}[\hat{V}^{k+1}(\tau^1, Z_{k+1}^{\frac{1}{2}}\psi^{\leq k+1}),\ldots)] \end{aligned} \tag{71}$$

where $n \geq 2$, $\tau^1 \ldots \tau^{s_{v_0}}$ are the subtrees starting from v_0 (the first vertex above the root), the symbols E_h, E^T_h denote the expectations with respect to a grassmanian integration with propagator $g^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}$ and $\mathcal{O}$ is equal to $\mathcal{L}^*$, if the tree contributes to the local part of the potential, or $\mathcal{R}$, if it contributes to the irrelevant part. The trees are defined in an analogous way as in the preceding section.

2.3 The flow of the renormalization group

From the above discussion we have that the set $v_{j,h-1}$ is related to the $v_{j,h'}$ with $h' \geq h$ by the relation:

$$\begin{aligned} \bar{v}_{i,h-1} &= (\frac{Z_h}{Z_{h-1}})^{\alpha_i}[\bar{v}_{i,h} + \beta^h_{g,i}(\{v_h\},\ldots,\{v_0\})] \\ \sigma_{h-1} &= \frac{Z_h}{Z_{h-1}}[\sigma_h + \beta^h_\sigma(\{v_h\},\ldots,\{v_0\})] \\ \nu_{h-1} &= \frac{Z_h}{Z_{h-1}}\gamma[\nu_h + \beta^h_\nu(\{v_h\},\ldots,\{v_0\})] \\ 1 &= \frac{Z_h}{Z_{h-1}}[1 + \beta^h_z(\{v_h\},\ldots,\{v_0\})] \end{aligned} \tag{72}$$

where $v_{h,i} = \{\bar{v}_{h,i}, \nu_h, Z_h, \sigma_h\}$, $\alpha_i = 2$ for the $g_{i,h}$ and $\alpha_i = 1$ otherwise. The function $\beta^h_i(\{v_h\},\ldots,\{v_0\})$ is called *Beta function* and from eq.(71) we know that $\beta^h_i(\{v_h\},\ldots,\{v_0\})$

$= \sum_{n=0}^{\infty} \beta_i^{h,n}(\{v_h\},\ldots,\{v_0\})$. Noting that

$$|g_{\bar\omega,\bar\omega'}(x-y)| \leq \frac{1}{Z_h}\frac{C_N\gamma^h}{1+\gamma^{Nh}|x-y|^N}\left(\frac{\sigma_h}{\gamma^h}\right)^{\alpha} \tag{73}$$

with $\alpha = 0,1,2$ (see lemma 2.1), a slight modification of the proof in (B.G.P.S.)(B.M.), essentially based on the Grahm-Hadamard inequality (Le.), shows that if

$$\max_{i,k\geq h}|\bar v_{i,k}|,|\nu_k| \leq \bar\varepsilon \qquad \max_{\substack{i=0,1,2\\ k\geq h}}\left|\frac{\sigma_k}{\gamma^k}\right|^i \leq \tilde C \qquad \max_{k\geq h}\frac{Z_k}{Z_{k-1}} \leq e^{\beta_1\bar\varepsilon^2} \tag{74}$$

then

$$|\beta_i^{h,n}(\{v_h\},\ldots,\{v_0\})| \leq C^n\tilde C^n\bar\varepsilon^n$$

so that if $\bar\varepsilon$ is small enough *i.e.* if $\bar\varepsilon \leq \hat\varepsilon = \frac{1}{C\tilde C}$ than β_i^h is an analytic function in its arguments.

However it is not obvious at all that that there exists an $\bar\varepsilon$ such that eq.(74) holds for every h.

Given any sequence of $\bar v_h, Z_h, \sigma_h$ such that $\max_{i,k}|\bar v_{i,k}| \leq \bar\varepsilon$, $\max_{\substack{i=0,1,2\\ k}}\left|\frac{\sigma_k}{\gamma^k}\right|^i \leq \tilde C$ and $\max_k \frac{Z_k}{Z_{k-1}} \leq e^{\beta_1\bar\varepsilon^2}$ there is a unique ν_0, analytic in the running coupling constants, such that $|\nu_h| < \bar\varepsilon$ and ν_h converges to 0 for $h \to -\infty$ at the rate $O(\gamma^h)$. The proof of the existence of ν_0 is essentially a version of the unstable manifold theorem, see (B.G.P.S.). This value ν_0 is obtained, given α, λ, by a unique choice of ν.

In all the above consideration the presence of the spin has no importance. However the study of the flow of the running coupling constants in the spinning case is much more involved than in the spinless one, as there are five running constant more in the spinning case, so that we consider from now on the spinless case.

We can write eq.(72) in a more explicit way, calling $\mu_h = (\delta_h, \lambda_h)$ and using the last of eq.(72) in order to eliminate the factors $\frac{Z_h}{Z_{h-1}}$. using also that $\nu_h = O(\gamma^h)$ we get (see also app.4 of (B.G.P.S.)):

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h + G_\lambda^{1,h}(\{\mu_h\};\ldots;\{\mu_0\}) + \\
&+ \sum_{k,k'\geq h}\lambda_{k'}\frac{\sigma_k}{\gamma^k}G^{2,h}_{\lambda,k,k'}(\{\bar v_h,\frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\}) + \\
&+ \gamma^h R^h_\lambda(\{\bar v_h,\frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\},\nu_h) \\
\delta_{h-1} &= \delta_h + G_\delta^{1,h}(\{\mu_h\};\ldots;\{\mu_h\}) + \\
&+ \sum_{k,k'\geq h}\lambda_{k'}\frac{\sigma_k}{\gamma^k}G^{2,h}_{\delta,k,k'}(\{\bar v_h,\frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\}) + \\
&+ \gamma^h R^h_d(\{\bar v_h,\frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\},\nu_h) \\
\tau_{h-1} &= \tau_h + \sum_{k,k'\geq h}\lambda_{k'}\frac{\sigma_k}{\gamma^k}G^{2,h}_{\tau,k,k'}(\{\bar v_h,\frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\},\nu_h) \\
\theta_{h-1} &= \theta_h + \sum_{k,k'\geq h}\lambda_{k'}\frac{\sigma_k}{\gamma^k}G^{2,h}_{\theta,k,k'}(\{\bar v_h,\frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\},\nu_h)
\end{aligned} \tag{75}$$

$$\begin{aligned}
\sigma_{h-1} &= \sigma_h + \sigma_h \lambda_h \beta_3 + \\
&+ \sigma_h \sum_{k\geq h} \lambda_k G^{1,h}_{k,\sigma}(\{\bar v_h, \frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\}) \\
\frac{Z_{h-1}}{Z_h} &= 1 + \lambda_h^2\beta_1 + \lambda_h \sum_{k,k'\geq h} \mu_k \lambda_{k'} G^1_{z,k,k'}(\{\mu_h\};\ldots;\{\mu_0\}) + \\
&+ \sum_{k,k'} \lambda_{k'} \frac{\sigma_k}{\gamma^k} G^{2,h}_{z,k,k'}(\{\bar v_h, \frac{\sigma_h}{\gamma^h}\};\ldots;\{\bar v_0,\sigma_0\}) + \\
&+ \gamma^h R^h_z(\{\bar v_h, \frac{\sigma_h}{\gamma^h}\},\ldots,\{\bar v_0,\sigma_0\},\nu_h)]
\end{aligned}$$

where we write explicitly some lowest order contribution to σ_h, $\frac{Z_{h-1}}{Z_h}$ and $\beta_3, \beta_1 > 0$, and by $\{\bar v_h\}$ we mean a set of the $\bar v_{h,i}$.

In the equations for λ_h, δ_h we have split, using lemma 3.1, the beta function into three parts and the terms $G^{1,h}_\lambda, G^{1,h}_\delta$ are given by sum of integrals of products of the part $g^h_{\vec\omega,L}(x-y)$ of the propagator while in the terms $G^{2,h}_\lambda, G^{2,h}_\delta$ at least a non diagonal propagator or a $C^k_2(x-y)$ one is involved (see lemma 2.2); moreover the second order contribution to the beta function is explicitely written. In the equation for τ_h, θ_h, σ_h we have used that at least a non diagonal propagator has to be involved in the term contributing to the beta function. In the last equation again we have splitted the beta function into three terms as it was done for λ_h, δ_h. Finally note that $G^{2,h}_{\lambda,k,k'} = O(\lambda)$, and $R^h_\lambda = O(\lambda^2)$ while $R^h_i = O(\lambda)$ for $i \neq \lambda$.

A fundamental role is plaied by the following property:

Theorem 2.1 *If we define:*

$$\lim_{h\to-\infty} G^{1,h}_i(\bar\mu;\ldots;\bar\mu) = G^{1,h}_i(\mu)$$

then it is

$$G^h_\lambda(\mu) = G^h_\delta(\mu) \equiv 0 \tag{76}$$

The proof of this theorem was reduced in (B.G.P.S.) to the verification of some technical lemmata proved in (B.M.) using the exact solution of the Luttinger model (M.L.). The Luttinger model describes two spinless fermions with linear dispersion relation. The Hamiltonian is:

$$\begin{aligned}
T_0' + H_I' &= \sum_{\vec\omega} \int d\vec x\, :\psi^+_{\vec x,\vec\omega}(i\vec\omega\vec\partial)\psi^-_{\vec x,\vec\omega}: + && (77)\\
&\sum_{\vec\omega} \int d\vec x d\vec y\, \lambda v(\vec x-\vec y) : (\psi^+_{\vec x,\vec\omega}\psi^-_{\vec x,\vec\omega}) :: (\psi^+_{\vec y,-\vec\omega}\psi^-_{\vec y,-\vec\omega}) : && (78)
\end{aligned}$$

where :: denotes the Wick ordering respect to the ground state of T_0' and $\psi^\pm_{\vec x,\vec\omega,s}$ are creation or annihilation operators of $\vec\omega$-fermions. We can introduce a family of Grassmanian variables $\psi^\varepsilon_{x,\vec\omega}$ and study the Luttinger model by a renormalization group analysis using the anomalous scaling of the preceding section. The propagator is just given by $g^h_{\vec\omega,L}(x-y)$ and the relevant part of the effective potential $\hat V^h$ is given by eq.(67) with $\nu_h = 0$ by the symmetry of the interaction and for the parity of the Luttinger model propagator and $s_h = t_h = i_h = 0$. The Beta function is given by, if $\mu_h = \lambda_h, \delta_h$:

$$\lambda_{h-1} = \lambda_h + B^h_{\lambda,L}(\mu_h;\ldots;\mu_0) + \gamma^h \hat{R}^h_\lambda(\mu_h;\ldots;\underline{\mu}_0)$$
$$\delta_{h-1} = \lambda_h + B^h_{\delta,L}(\mu_h;\ldots;\mu_0) + \gamma^h \hat{R}^h_\delta(\mu_h;\ldots;\underline{\mu}_0)$$

The crucial point is that:

$$B^h_{\lambda,L}(\mu_h;\ldots;\mu_0) = G^{1,h}_\lambda(\mu_h;\ldots;\mu_0) \qquad B^h_{\delta,L}(\mu_h;\ldots;\mu_0) = G^{1,h}_\delta(\mu_h;\ldots;\mu_0)$$

and eq.(76) follows as from the exact solution, see (B.G.P.S.)(B.G.M.), it holds that $B^h_{i,L} = 0$.

From eq.(75)(76) we can obtain a bound on the running coupling:

Lemma 2.2 *If for* $0 \geq k \geq h$ $\max_{i,k\geq h} |\bar{v}_{i,k}| \leq \bar{\varepsilon}$, $\max_{\substack{i=0,1,2\\ k\geq h}} \left|\frac{\sigma_k}{\gamma^k}\right|^i \leq \tilde{C}$, *and* $\max_{k\geq h} \frac{Z_k}{Z_{k-1}} \leq e^{\beta_1 \bar{\varepsilon}^2}$ *then there exists constant* $c_1, c_2, c_3, c_4 > 0$, *depending only on* $\bar{\varepsilon}$ *such that:*

$$-\, c_1 \lambda^2 \quad < |\lambda_{h-1} - \lambda_0| < \quad c_2 \lambda^2 \tag{79}$$
$$-c_1|\lambda| \quad < |\delta_{h-1} - \delta_0| < \quad c_2|\lambda| \tag{80}$$

$$-\, c_1|\lambda| \quad \leq |\tau_{h-1} - \tau_0| \leq \quad c_2|\lambda| \tag{81}$$
$$-c_1|\lambda| \quad \leq |\theta_{h-1} - \theta_0| \leq \quad c_2|\lambda| \tag{82}$$

$$-\, \lambda\beta_3 c_3 h \quad \leq \log\left(\frac{|\sigma_h|}{|\sigma_0|}\right) \leq \quad -\lambda\beta_3 c_4 h \tag{83}$$
$$-\beta_1 c_3 \lambda^2 h \quad \leq \log(|Z_h|) \leq \quad -\beta_1 c_4 \lambda^2 h \tag{84}$$

It is possible to choose λ small enough so that, fixed a generic constant K and a scale h such that $\frac{\sigma_h}{\gamma^h} \geq K$ and $\frac{\sigma_k}{\gamma^k} \leq K$ for any $k > h$ then $\max_{i,k\geq h} |\bar{v}_{i,k}| \leq \hat{\varepsilon}$ and $\max_{k\geq h} \frac{Z_k}{Z_{k-1}} \leq e^{\beta_1 \hat{\varepsilon}^2}$ with $\hat{\varepsilon} \leq \bar{\varepsilon}$. In other words, from lemma 2.2 we know that it is possible to choose λ so that the Beta function β^k_i is analytic in its arguments for $k \geq h$. We can then choose λ so that β^k_i is analytic in its arguments for $k \geq h^*$ where h^* is defined so that $\frac{\sigma_h}{\gamma^h} \leq 1$ for any $h > h^*$ but $\frac{\sigma_{h^*}}{\gamma^{h^*}} > 1$; from lemma 2.2 it is easy to see that:

$$\frac{\log_\gamma u}{1 + |\lambda|\beta_3 c_3} \leq h^* \leq \frac{\log_\gamma u + 1}{1 + |\lambda|\beta_3 c_4} \tag{85}$$

The introduction of the scale h^* is crucial and represents the major difference with the not filled band case, see (B.M.); the "intuitive" meaning of it will be discussed at the end of this section.

From lemma 2.2 and the above considerations it follows that $V^{h^*}(\sqrt{Z_{h^*}}\psi^{\leq h^*})$ is an analytic function of λ for $|\lambda|$ small enough; in order to compute the partition function it remains to perform the integration:

$$\mathcal{N} = \int P_{Z_{h^*}}(d\psi^{\leq h^*}) e^{-V^{h^*}(\sqrt{Z_{h^*}})\psi^{\leq h^*})} \tag{86}$$

The propagator associated with such integration, written in term of particle fields, is given by, from eq.(66):

$$
\begin{aligned}
g^{\leq h^*}(x,y) &= \sum_{\vec{\omega},\vec{\omega}'} \int dk e^{ik(x-y)} C_{h^*}^{-1}(k) T_{h^*}^{-1}(k)_{\vec{\omega},\vec{\omega}'} e^{-i(\omega x - \omega' y)} = \\
&= \int dk \frac{1}{Z_{h^*}(k)} \frac{C_{h^*}^{-1}(k) e^{ik(x-y)}}{(-ik_0 + \vec{k}^2)^2 - (\pi\vec{k})^2 - \sigma_{h^*}(k)^2} \\
&\left[\cos(\pi(x-y))(-ik_0+\vec{k}^2) + i\pi\vec{k}\sin(\pi(x-y)) + \sigma_{h^*}(k)\cos(\pi(x+y))\right]
\end{aligned}
\tag{87}
$$

where the χ functions do not appear becouse they are identically 1 in the support of $C_{h^*}^{-1}(k)$. From a standard integration by parts it follows that:

$$
|g^{\leq h^*}(x,y)| \leq \frac{C_N \gamma^{h^*}}{Z_{h^*}(1 + (|x-y|\gamma^{h^*})^N} \tag{88}
$$

i.e. it obeys to the same bound as $g^h_{\vec{\omega}_1,\vec{\omega}_2}(x-y)$ for $h > h^*$, see eq.(73).

This follows noting that $(-ik_0 + \vec{k}^2)^2 - (\pi\vec{k})^2 - \sigma_{h^*}(k)^2 = \gamma^{2h^*} D(\gamma^{h^*} k)$ and each of the tree term in the numerator can be written as $\gamma^{h^*} N(\gamma^{-h^*} k)$ with $D(k)$, $N(k)$ bounded and $O(1)$ with all their derivatives. Then also the integration eq.(86) is well defined and the analysis of the partition function $\mathcal{N}$ is complete.

The Schwinger function eq.(25) admits a tree expansion similar to the one of the partition function, whose convergence follows from the partition function expansion convergence, see (B.G.P.S.); from brevity we do not report it here but we simply quote the results of the functional integration in eq.(25):

$$
\begin{aligned}
S(x,y) &= \int dk \frac{e^{ik(x-y)}}{(2\pi)^2} \frac{h(k_0^2 + E(\vec{k})^2)}{-ik_0 - E(\vec{k})} + \\
&+ \sum_{h=h^*}^{0} \sum_{\vec{\omega}_1,\vec{\omega}_2} e^{i\pi(\vec{\omega}_1\vec{x} - \vec{\omega}_2\vec{y})} (g^h_{\vec{\omega}_1,\vec{\omega}_2}(x-y) + \bar{g}^h_{\vec{\omega}_1,\vec{\omega}_2}(x-y))
\end{aligned}
\tag{89}
$$

where we call $g^{\leq h^*}(x,y)$ simply $g^{h^*}(x,y)$ and the first addend is bounded by $\frac{C_N}{1+|x-y|^N}$ for all N, $g^h_{\vec{\omega}_1,\vec{\omega}_2}(x-y)$ is given by eq.(66) and $|\bar{g}^h_{\omega_1,\vec{\omega}_2}(x-y)| \leq \max(|\lambda|, u) \frac{1}{Z_h} \frac{C_N \gamma^h}{1+(\gamma^h|x-y|)^N}$.

It is easy to check that we can write, denoting now by $\vec{k}$ the "physical" momentum *i.e.* not the momentum measured from the Fermi surface:

$$
\begin{aligned}
g^h(x-y) &= \sum_{\vec{\omega}_1,\vec{\omega}_2} e^{i\pi(\vec{\omega}_1\vec{x} - \vec{\omega}_2\vec{y})} g^h_{\vec{\omega}_1,\vec{\omega}_2}(x-y) = \\
&\int dk \hat{\phi}(\vec{k}, \vec{x}, \sigma_h)) \hat{\phi}(\vec{k}, -\vec{y}, \sigma_h) e^{ik_0(x_0-y_0)} \frac{f_h(k,k_0)}{Z_h(k)} \frac{1}{-ik_0 - (\hat{\varepsilon}(k,\sigma_h) - \pi^2)}
\end{aligned}
\tag{90}
$$

where:

$$
\hat{\varepsilon}(k,\sigma_h) = \left(|\vec{k}| - \pi\right)^2 + 2\pi \mathrm{sign}\left(|\vec{k}| - \pi\right) \sqrt{\left(|\vec{k}| - \pi\right)^2 + \sigma_h^2} + \pi^2 \tag{91}
$$

and

$$\hat\phi(\vec k,\vec x,\sigma_h)) = e^{i\vec k\vec x}u(\vec k,\vec x,\sigma_h)$$

$$u(\vec k,\vec x,\sigma_h) = e^{-i\text{sign}(\vec k)\pi\vec x}\left[\cos(\pi\vec x)\sqrt{1+\frac{\text{sign}(|\vec k|-\pi)\sigma_h}{\sqrt{(|\vec k|-\pi)^2+\sigma_h^2}}}+\right.$$

$$\left.+i\text{sign}(\vec k)\sin(\pi\vec x)\sqrt{1-\frac{\text{sign}(|\vec k|-\pi)\sigma_h}{\sqrt{(|\vec k|-\pi)^2+\sigma_h^2}}}\right] \tag{92}$$

From the computations in Appendix 2 it follows that:

$$|\hat\phi(\vec k,\vec x,u)-\phi(\vec k,\vec x,u)| = O(u) \qquad |\hat\varepsilon(\vec k,u)-\varepsilon(\vec k,u)| = O(u^2) \tag{93}$$

if $\phi(\vec k,\vec x,u)$ are the Bloch waves *i.e.* the solutions of eq.(3) and $\varepsilon(\vec k,u)$ the dispersion relation. One recognizes in fact, by an elementary calculation, that eq.(92),(91), with $\sigma_h = u$ are just the Bloch waves and the dispersion relation computed at the first order in u. This is natural as in the anomalous scaling we put in the grassmanian integration for each scale h the term $\sigma_h\int d\vec x\cos(2\pi\vec x)\psi^+(\vec x)\psi^+(\vec x)$.

We call $\hat u(k,\lambda,u)$ and $\hat Z(k,\lambda,u)$ respectively σ_h and Z_h for $\gamma^h\le|k|<\gamma^{h+1}$ with $h>h^*$ and σ_{h^*} and Z_{h^*} for $|k|\le\gamma^{h^*}$ and we define:

$$\eta_1(\lambda,u) = -\frac{\log(Z_{h^*})}{\log u} \qquad 1-\eta_2(\lambda,u) = \frac{\log(\sigma_{h^*})}{\log u} \tag{94}$$

which from Lemma 2.2 and eq.(85) are respectively $O(\lambda^2)$ and $O(\lambda)$, with $\text{sign}\eta_2 = \text{sign}(\lambda)$.

From eq.(89) it follows, from a standard argument (see for instance (B.G.1)), that for $1\le |x-y|\le u^{-(1-\eta_2)}$ the Schwinger function behaviour is:

$$|S(x,y)|\le\frac{1}{|x-y|^{1+\eta_3}}$$

with $\eta_3=\eta_1(1-\eta_2)^{-1}$, while for $|x-y|\ge u^{1-\eta_2}$ we have, for any N:

$$|S(x,y)|\le\frac{Z_{h^*}^{-1}\sigma_{h^*}C_N}{1+\sigma_{h^*}^N|x-y|^N} \tag{95}$$

so that the bound eq.(17)(19) in the theorem follows. Note that such bounds do not depend on the explicit expression of $g^h(x-y)$ eq.(90). Eq.(15) follows on the other hand from eq.(90) and eq.(93).

In order to found Z^{-1} note that, from the appendix 2, $\phi(\pi^+,\vec x)\phi(\pi^+,-\vec y) = \cos(\pi(\vec x+\vec y)) + \cos(\pi(\vec x-\vec y))+O(u)$ and $\phi(\pi^-,\vec x)\phi(\pi^-,-\vec y) = -\cos(\pi(\vec x+\vec y))+\cos(\pi(\vec x-\vec y))+O(u)$ so that, using eq.(66) we find:

$$\frac{1}{L}\int d\vec x d\vec y\phi(\pi^\pm,\vec x)\phi(\pi^\pm,\vec y)\sum_{h=h^*}^{0}\sum_{\vec\omega_1,\vec\omega_2}e^{i\frac{\pi}{a}(\vec\omega_1\vec x-\vec\omega_2\vec y)}g^h_{\vec\omega_1,\vec\omega_2}(x-y) =$$

$$\frac{1}{Z_{h^*}}(1+O(\text{Max}(\lambda,u,u^{1-\eta_2}))) \tag{96}$$

It is trivial now to check that the formula for the occupation number discontinuity Z^{-1} stated in the theorem holds.

Remark 1 : In order to understand the meaning of the scale h^* let us compute the Schwinger function corresponding to the $\lambda = 0, u \neq 0$ case by the techniques developed so far. It is trivial to realize that we obtain an expression for the Schwinger function analogue to eq.(89), with $Z_h = 1, \sigma_h = u$ for any h and $h^* = [\log_\gamma u]$. On the other hand we know from Appendix 2 that, if u is small and writing the momentum as $\vec{k} + \vec{\omega} p_F$ with $\vec{\omega} = \pm 1$ and $p_F = \pi$, in the region $|\vec{k}| > \gamma^{h^*}$ but still $<< 1$ we have

$$\begin{aligned} &\varepsilon(\vec{k} + \vec{\omega} p_F, u) - \mu \simeq \\ &\text{sign}(\vec{k})\sqrt{\vec{k}^2 + u^2} \simeq 2\pi\vec{k}(1 + O(\frac{u^2}{\vec{k}^2})) \end{aligned} \tag{97}$$

while in the region $|\vec{k}| < \gamma^{h^*}$ we have:

$$\varepsilon(\vec{k} + \vec{\omega} p_F, u) - \mu \simeq u(1 + O(\frac{\vec{k}^2}{u^2})) \tag{98}$$

The meaning of the scale h^* is then clear: it separates the momenta near the Fermi surface in two regions, one in which the dispersion relation is approximately linear in $\vec{k}$ and another in which is quadratic. The two points Schwinger function has of course a different behavior for small $\vec{k}$ in these two regions. In the $\lambda \neq 0$ case σ_h is not constant and this has the effect that h^* is different with respect to the $\lambda = 0$ case but its meaning remains the same *i.e.* it separates the momenta into two regions in which the two points Schwinger function has a different behaviour. If $u = 0, \lambda \neq 0$ one finds (see (B.G.P.S.)) a formula similar to eq.(89) with $h^* = -\infty$, $\sigma_{h^*} = 0$ and $Z_h \simeq \gamma^{-\eta_1 h}$ *i.e.* the periodic potential has the effect that the infrared scales are finite. It is clear that then the scale h^* separates a region in which $S(k)$ behaves like the Schwinger function of the $u = 0$ case *i.e.* as $|k|^{-(1-\eta_3)} S_0$ to a region in which the behaviour is given by eq(17).

Remark 2 : Of course the scale h^* should be chosen so that $\frac{\sigma_h}{\gamma^h} \leq K$ for any $h > h^*$ but $\frac{\sigma_{h^*}}{\gamma^{h^*}} > K$ with any constant K (we chose $K = 1$). In this case it is easy to check that $|\lambda| \leq \min[K, \frac{1}{K}]$ and $\sigma_{h^*} \simeq u^{1-\eta_2} K^{\eta_2}$, $Z_{h^*} \simeq u^{\eta_1} K^{-\eta_1}$ so that eq.(16) still holds. We have then some freedom in the choice of h^* but of course not too much. For instance if we choose $h^* = \log_\gamma u$ then $|g^{\leq h^*}(x, y)| \leq \frac{C_N \gamma^{2h^*}}{Z_{h^*} \sigma_{h^*} (1 + (|x-y| \sigma^{h^*})^N}$ *i.e.* it does not obey to the same bound of $g^h(x - y)$ for $h \geq h^*$.

2.4 Outlook on the spinning case

Let us discuss finally what happens in the spinning case. A second order computations shows that:

$$g_{1,h-1} = g_{1,h} - (\beta + O(\frac{\sigma_h}{\gamma^h}))g_{1,h}^2$$

$$
\begin{aligned}
g_{2,h-1} &= g_{2,h} + (\beta + O(\frac{\sigma_h}{\gamma^h}))g_{1,h}^2 \\
g_{3,h-1} &= g_{3,h} - 2(\beta + O(\frac{\sigma}{\gamma^h}))g_{3,h}(g_{1,h} - 2g_{2,h}) \\
g_{4,h-1} &= g_{4,h} + O(\frac{\sigma_h}{\gamma^h}) \\
g_{5,h-1} &= g_{5,h} - (\beta + O(\frac{\sigma^h}{\gamma^h}))g_{5,h}(2g_{1,h} - g_{2,h}) \\
g_{6,h-1} &= g_{6,h} - (\beta + O(\frac{\sigma^h}{\gamma^h}))g_{6,h}(2g_{1,h} - g_{2,h}) \\
\sigma_{h-1} &= \sigma_h + (\beta_3 + O(\frac{\sigma^h}{\gamma^h}))g_{2,h} \\
\tau_{h-1} &= \tau_h + \tilde{\beta} g_{2,h} g_{5,h} \\
\theta_{h-1} &= \theta_h + \tilde{\beta} g_{2,h} g_{5,h} \\
\delta_{h-1} &= \delta_h + O(\frac{\sigma_h}{\gamma^h}) \\
\frac{Z_{h-1}}{Z_h} &= 1 + \beta_1 g_{2,h}^2 + O(\frac{\sigma_h}{\gamma^h})
\end{aligned}
\tag{99}
$$

with $\beta, \beta_1, \beta_3 > 0$. One could proceed as in the spinless case using the anomalous scaling of sec.(2.2). However in the spinning case the running coupling constants do not stay necessarily bounded until the scale h^* defined as in eq.(85). In fact by eq.(99), assuming heuristically that the beta function higher orders vanish, it follows that $g_{1,h} \simeq \frac{g_{1,0}}{1-\beta g_{1,0} h}$ which is bounded for $h \geq h^*$ for any u *only if* $g_{1,0} > 0$; if $g_{1,0} < 0$ we have to require that $|h^*| < O(\frac{1}{g_{1,0}})$ *i.e.* , from eq.(85), that $u \geq k_1 e^{-|\lambda|^{-k_2}}$. With this condition it is easy to check, from eq.(99), that also the other running couplings stay bounded, remembering that $g_{3,0}, g_{5,0}, g_{6,0}, \tau_0, \theta_0 = O(\lambda u)$.

In conclusion if the fermions are spinning the second statement of the main theorem should hold in the case of repulsive interaction or if $|u| \geq k_1 e^{-|\lambda|^{-k_2}}$. In order to prove this one has to repeat the analysis of the preceding sections and to show that the flow generated by the complete beta function qualitatively coincides with the flow generated by the beta function truncated to the second order eq.(99), exploiting some cancellations which should occur in the Beta function. This seems not too difficult and we hope to discuss it in an other paper.

In the spinning case with attractive interaction $\lambda < 0$ and no periodic potential there are no rigorous result about the Schwinger function behaviour, see (B.M.); on the other hand in the filled band spinning case the above discussion says that the Schwinger function behaviour is given by eq.(16) only if $|u| \geq k_1 e^{-|\lambda|^{-k_2}}$. This suggests, in our opinion, that in the $u = 0$ attractive spinning case the Schwinger behaviour could be given by eq.(16) with $u \simeq O(e^{-|\lambda|^{-k_2}})$ *i.e.* the interaction generates spontaneously a gap in the energy spectrum. This phenomenon was heuristically studied in (L.E.) and should be analogue to the B.C.S. superconductivity in solid state models or to the mass generation in the relativistic standard model, but as far as we known no rigorous results are known (although new insigth was given in (F.T.M.R.)).

3 Appendix 1

This Appendix is devoted to the first statement of the theorem. The *Euclidean fields* are here defined as:

$$\psi^{\varepsilon}_{x,\sigma} = \int dk \phi(\vec{k}, \vec{x}) e^{i\varepsilon k_0 x_0} \psi^{\varepsilon}_{k,\sigma}$$

with $e^{ik_0\beta} = -1$, $e^{i\vec{k}N} = 1$ and the functional integration is defined on monomials by the Wick rule eq.(10) with $g(x,y) = \int dk e^{ik_0(x-0-y_0)} \phi(\vec{k}, \vec{x}) \phi(\vec{k}, -\vec{y}) g(k)$.

The Schwinger functions are given by eq.(25) and are related to the *effective potential* V_{eff} defined by:

$$e^{-V_{eff}(\varphi)} = \lim_{\substack{\beta\to\infty \\ \Lambda\to\infty}} \frac{1}{\mathcal{N}} \int P(d\psi) e^{-\bar{V}(\psi+\varphi)}. \tag{100}$$

where $\mathcal{N} = \int P(d\psi) e^{-\bar{V}(\psi)}$ is a normalization constant so that $V_{eff}(0) = 0$ and $\bar{V}(\psi) = \lambda V$ where:

$$V = \sum_{\sigma,\sigma'} \int_{\Lambda\times\Lambda} dx_1 dx_2 v(\vec{x}_1 - \vec{x}_2) \delta(x_{0,1} - x_{0,2}) \psi^+_{x_1,\sigma} \psi^+_{x_2,\sigma'} \psi^-_{x_2,\sigma'} \psi^-_{x_1,\sigma} \tag{101}$$

with $\Lambda = (-\beta/2, \beta/2) \times (-N/2, N/2)$. As u is not small the periodic potential has to be considered as a part of the free hamiltonian.

The two points Schwinger function is given in term of the effective potential by:

$$S(x,y) = g(x,y) + \int dz dz' g(x,z) V_2(z,z') g(z',y) \tag{102}$$

with $V_2(z,z') = \frac{\partial^2 V_2(\phi)}{\partial\varphi^+_z \partial\varphi^-_{z'}}$.

It is convenient to rewrite the grassmanian integral in eq.(100) as:

$$e^{-V_{eff}(\varphi)} = \frac{1}{\mathcal{N}} \int P(d\psi_{i.r.}) e^{-V^0(\psi_{i.r.}+\varphi)} \tag{103}$$

$$e^{-V^0(\psi_{i.r.}+\varphi)} = \int P(d\psi_{u.v.}) e^{-\bar{V}(\psi_{i.r.}+\psi_{u.v.}+\varphi)} \tag{104}$$

where $\psi^{u.v.}$, $\psi^{i.r.}$, φ are anticommuting grassmanian fields, $P(d\psi^{u.v.})$, $P(d\psi^{i.r.})$ denote respectively the grassmanian integrations with vanishing cross propagator and with propagators $g_{u.v.}$ and $g_{i.r.}$ given by:

$$\begin{aligned} g_{i.r.}(x,y) &= \int dk \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 - E(\vec{k})} \phi(\vec{k}, -\vec{x}) \phi(\vec{k}, \vec{y}) (1 - h(k_0^2 + E(\vec{k})^2)) \\ g_{u.v.}(x,y) &= \int dk \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 - E(\vec{k})} \phi(\vec{k}, -\vec{x}) \phi(\vec{k}, \vec{y}) h(k_0^2 + E(\vec{k})^2) \end{aligned} \tag{105}$$

where $E(\vec{k}) = \varepsilon(\vec{k}) - \mu$ and $h(t)$ is a C^∞ function in its argument t and it is identically 1 if $t > [\varepsilon((2\pi)^+) - \mu]^2$ hence the integral in $g_{u.v}(x,y)$ involves only momenta "far" from the Fermi surface so thus justifing the u.v. name.

In (B.M.) it is shown that ultraviolet part of the propagator can be written as:

$$g_{u.v.}(x,y) = G(x-y) + R(x,y) \qquad G(x) = H(\vec{x})H(x_0)\sqrt{\frac{1}{4\pi x_0}}e^{-\frac{\vec{x}^2}{2x_0}}$$

where $H(t)$ is a smooth function of compact support such that $H(t) = e^{x_0^2\pi^2}$ if $|t| \leq 1$ and $H(t) = 0$ if $|t| \geq \gamma > 1$, and $R(x,y) \leq \frac{C_N}{1+|x-y|^N}$ for any integer N. With some minor adaptation of the study of the ultraviolet problem for the $u=0$ case in (B.G.P.S) to the $u \neq 0$ case it is possible to prove that there exists an ε such that V^0 can be written, for $|\lambda| \leq \varepsilon$, in the following way:

$$\begin{aligned}
V^0(\psi) &= \sum_{\sigma,\sigma'} \lambda \int v(x-y)\psi^+_{x,\sigma}\psi^+_{y,\sigma}\psi^-_{y,\sigma}\psi^-_{x,\sigma} + 2\lambda \sum_\sigma \int dxdy v(x-y)R(x,y)\psi^+_{x,\sigma}\psi^-_{y,\sigma} \\
&+ \sum_\sigma \int_\Lambda \psi^+_{x,\sigma}\psi^-_{x,\sigma}(\nu - 2\lambda \int dy K(x,y)) + \qquad (106) \\
&+ \sum_{\sigma_1,\dots,\sigma_n} \sum_n \int dx_1 \cdots dx_{2n} \psi^+_{x_1,\sigma_1} \dots \psi^+_{x_{n_1},\sigma_{n_1}} \psi^-_{x_1,\sigma_1} \cdots \psi^-_{x_{2n},\sigma_{2n}} W_n(x_1,\dots,x_{2n};\lambda)
\end{aligned}$$

where $K(x,y) = v(x-y)R(y,y)$, the kernels W_n are products of suitable delta functions times bounded functions analytic in λ if $|\lambda| \leq \varepsilon$ and, if $d(x_1 \dots, x_n)$ is the length of the shortest tree connecting the points ("tree distance" or "graph distance"), the following bounds hold:

$$\int dx_1 \dots dx_{2n} |W_n(x_1,\dots,x_n;z)|(1+d(x_1,\dots,x_n))^N < c(N)\Lambda |z|^{max(2,n-1)} \qquad (107)$$

In order to perform the "infrared" integration eq.(103) we can write

$$\begin{aligned}
g_{i.r.}(x,y) = \sum_{h=-\infty}^{0} g^{(h)} &= \sum_h \int d^2k \frac{f(\gamma^{-2h}(k_0^2 + (\varepsilon(\vec{k}) - \mu)^2))\phi(\vec{k},\vec{x})\phi(-\vec{k},\vec{y})e^{ik_0(x_0-y_0)}}{-ik_0 - (\varepsilon(\vec{k}) - \mu)} = \\
&= \sum_{h=-\infty}^{0} \int d^2k \phi(\vec{k},\vec{x})\phi(-\vec{k},\vec{y})e^{ik_0(x_0-y_0)} g^h(k)
\end{aligned}$$

where $f(t) = h(\gamma^{-2}t) - h(t)$ is a compact support function different from zero for $\gamma^{-2}[\varepsilon((2\pi)^+) - \mu]^2 \leq t \leq [\varepsilon((2\pi^+)) - \mu]^2$. As the chemical potential is at the center of the gap we have that $\varepsilon(\vec{k}) - \mu$ is never vanishing so that only a finite number of term $g^{(h)}(k)$ contributes to $g_{i.r.}(k)$. We call h^* the last h such that $f(\gamma^{-2h}(k_0^2 + (\varepsilon(\vec{k}) - \mu)^2))$ is not identically vanishing and does not contain π/a in its support. We call, for simplicity, $g^{h^*-1}(k)$ the sum of the $g^h(k)$ whose support contain π/a. We note that $[\varepsilon((2\pi)^+) - \mu]\gamma^{h^*-1} = O(\Delta)$ where $\Delta = [\varepsilon(\pi^+) + \varepsilon(\pi^-)]/2$ is the gap amplitude.

The following lemma holds:

Lemma 3.1 *For any integer N and for any u such that $\Delta \neq 0$ there exist continuous functions $C_N(u)$ such that:*

$$|g^h(x,y)| < \gamma^h \frac{C_N(u)}{1 + (\gamma^h |x-y|)^N} \qquad (108)$$

Proof: If $h \neq h^* - 1$ the integrand is continuous and vanishing at the integration extrema so that we can write:

$$g^{(h)}(x,y) = \int d^2k \phi(\vec{k},\bar{x})\phi(-\vec{k},\bar{y}) e^{i(k_0(x_0-y_0)+\vec{k}(n_1-n_2)} g^h(k) = \tag{109}$$
$$\frac{1}{((x_0-y_0)^2+(n_1-n_2)^2)^{\frac{N}{2}}} \int d^2k e^{i(k_0(x_0-y_0)+\vec{k}(n_1-n_2)}(\partial^2_{k_0}+\partial^2_{\vec{k}})^{\frac{N}{2}} \phi(\vec{k},\bar{x})\phi(-\vec{k},\bar{y}) g^h(k)$$

where $\vec{x} = \bar{x} + n_1$ and $\vec{y} = \bar{x} + n_1$ and $\bar{x},\bar{y} \in (0,1)$. In the Appendix it is proven that the derivatives of the functions $\varepsilon(\vec{k})$ and $\phi(\vec{k},\vec{x})$, in the support of $g^h(k)$, are bounded for any N and suitable continuous functions $C_N(u)$ by:

$$|\partial^N_{\vec{k}}[\phi(\vec{k},\vec{x})\phi(-\vec{k},\vec{y})]| \leq \frac{C_N(u)}{\gamma^{Nh}} \qquad\qquad |\partial^N_{\vec{k}} \varepsilon(\vec{k})| \leq C_N(u)\gamma^{-h(N-1)} \tag{110}$$

For the last propagator $g^{(h^*-1)}$ we write

$$\int d^2k \frac{f(\gamma^{-2(h^*-1)}(k_0^2+(\varepsilon(\vec{k})-\mu)^2))\phi(\vec{k},\vec{x})\phi(-\vec{k},\vec{y})e^{ik_0(x_0-y_0)}}{-ik_0-(\varepsilon(\vec{k})-\mu)} =$$
$$\int_{[-\pi,\pi]} d^2k \frac{f(\gamma^{-2(h^*-1)}(k_0^2+(\varepsilon(\vec{k})-\mu)^2))\phi_0(\vec{k},\vec{x})\phi_0(-\vec{k},\vec{y})e^{ik_0(x_0-y_0)}}{-ik_0-(\varepsilon_0(\vec{k})-\mu)} +$$
$$\int_{[-2\pi,-\pi]\cup[\pi,2\pi]} d^2k \frac{f(\gamma^{-2(h^*-1)}(k_0^2+(\varepsilon(\vec{k})-\mu)^2))\phi_1(\vec{k},\vec{x})\phi_1(-\vec{k},\vec{y})e^{ik_0(x_0-y_0)}}{-ik_0-(\varepsilon_1(\vec{k})-\mu)}$$

where $\varepsilon_n(\vec{k}) = \varepsilon(\vec{k})$ in the region $[-(n+1)\pi, -n\pi] \cup [n\pi, (n+1)\pi]$ with $\varepsilon_n(\vec{k})$ analytic and periodic of period 2π and an analogous definition and properties hold for $\phi_n(\vec{k},\vec{x})$ (see the appendix 2). The second integral can be rewritten as

$$\int_{[-\pi,\pi]} d^2k \frac{f(\gamma^{-2(h^*-1)}(k_0^2+(\varepsilon(\vec{k})-\mu)^2))\phi_1(\vec{k},\vec{x})\phi_1(-\vec{k},\vec{y})e^{ik_0(x_0-y_0)}}{-ik_0-(\varepsilon_1(\vec{k})-\mu)}$$

so that we have written the propagator as the sum of the integrals of two periodic function over a period and we can integrate by parts without boundary terms. Hence proceeding as for $h > h^* - 1$ using eq.(110) we have

$$g^{(h^*-1)}(x,y) \leq \frac{\gamma^{h^*-1}C_N(u)}{1+(\gamma^{h^*-1}|x-y|)^N}$$

This ends the proof.

The effective potential is given by

$$-V_{eff}(\varphi) = \log \int P(d\psi_{i.r.}) e^{-V_0(\psi_{i.r.}+\varphi)} = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{E}^T(\underbrace{V_0,\ldots,V_0}_{n}) =$$

$$\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\alpha_1,\dots\alpha_n}\sum_{\tilde X_1\in X_1,\dots\tilde X_n\in X_n}\int dX_1\cdots dX_n W_{\alpha_1}(X_1)\cdots W_{\alpha_n}(X_n)$$
$$\mathcal{E}^T(\psi_{\tilde X_1},\dots,\psi_{\tilde X_n})\varphi_{X_1\backslash\tilde X_1}\cdots\varphi_{X_n\backslash\tilde X_n}$$

where $W_{\alpha_i}(X_i)$ is one of the kernels of V^0, $X_i=(x_{i,1},\dots,x_{i,n_i})$ $dX_i=\prod_{j=1}^{n_i}dx_{i,j}$, $\psi_{X_i}=\prod_{j=1}^{n_i}\psi_{x_{i,j}}$ and $\mathcal{E}^T$ is the truncated expectation with propagator $g_{i.r.}$. Using the well known expansion (Le.)(B.G.P.S.) for truncated expectation we can write

$$\mathcal{E}^T(\psi_{X_1},\dots,\psi_{X_n})=\sum_{T_n}\prod_{l\in T_n}g(x_l,y_l)\int dP_T(S)\mathrm{det}G^T(s) \tag{111}$$

where T_n is a tree graph connecting the sets of points $X_1,\dots,X_n$, G^T is a matrix whose elements are $G^T_{i,j,i'j'}=S_{jj'}g(x_{i,j},x_{i',j'})$ with $x_{i,j}-x_{i',j'}$ not belonging to the tree, $S_{jj'}$ a suitable interpolation function belonging to $[0,1]$ and $dP_T(S)$ is a normalized measure on the S variable.

From the Grahm-Hadamard inequality (B.G.P.S.) we obtain that the last integral in eq.(111) is bounded by $C^{\sum_i n_i}$ for some C. Let us consider a *spanning tree* $\tilde T$ connecting all the points in $X_1,\dots,X_n$. By eq.(107) the integral over the coordinates belonging to $T/\tilde T_n$ and the sum over α_i can be performed trivially obtaining by eq.(111) that the second addend in eq.(102) is bounded by:

$$\sum_{n=1}^{\infty}\frac{(C\lambda)^n}{n!}\sum_{T_n}\prod_{l\in T_n}\int dx_l dy_l g_{i.r.}(x_l,y_l)g_{i.r.}(x,x_1)g_{i.r.}(y_{n-1},y) \tag{112}$$

where $(x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1})$ are the $n-1$ lines belonging to $\tilde T_n$. Writing $g_{i.r.}(x,y)=\sum_{h=0}^{h^*-1}g^h(x,y)$ and remembering from lemma 2.1 that $\sup_y\int dxg^h(x,y)\le\gamma^{-h}$ we have that eq.(112) is bounded by:

$$\sum_{n=1}^{\infty}\frac{(C\lambda)^n}{n!}\sum_{T_n}\prod_{l\in T_n}\sum_{h=1}^{h^*-1}\frac{\gamma^h C_N(u)}{1+(\gamma^h|x-y|)^N}\sum_{h_2\dots h_n=1}^{h^*-1}\gamma^{-\sum_{i=2}^n h_i}\le$$
$$\sum_{n=1}^{\infty}\left|\frac{\lambda\tilde C}{\Delta}\right|^n\sum_{h=0}^{h^*-1}\frac{\gamma^h C_N(u)}{1+(\gamma^h|x-y|)^N} \tag{113}$$

using eq.(102)(113) the first statement of the theorem easily follows.

Remark: Contrary to the small u case all the infrared scales $0,-\infty$ can be integrated togheter *i.e.* without using the multiscale decomposition eq.(31)..

4 Appendix 2

In this Appendix, based on the results of (K.), we obtain some analyticity properties of Bloch waves which we need in the estimates of sec.2. We consider the two Cauchy problems for the equation

$$-\phi"(x) + uc(x)\phi(x) = \varepsilon\phi(x) \tag{114}$$

in $0 \leq x \leq 1$ and $\phi_1(0,\varepsilon,u) = \phi_2'(0,\varepsilon,u) = 1$ or $\phi_2(0,\varepsilon,u) = \phi_1'(0,\varepsilon,u) = 0$. It is easy to see that the two power series

$$\phi_1(x,\varepsilon,u) = \mathrm{c}_\varepsilon + \sum_{n=0}^{\infty} u^n \mathrm{C}_n(x,\varepsilon) \qquad \phi_2(x,\varepsilon,u) = \sigma_\varepsilon + \sum_{n=0}^{\infty} u^n \mathrm{S}_n(x,\varepsilon) \tag{115}$$

with

$$\begin{aligned}
\mathrm{C}_n(x,\varepsilon) &= \int_{0\leq s_1\leq\ldots\leq s_{n+1}=x} \mathrm{c}_\varepsilon(s_1)\prod_{i=1}^{n}[\sigma_\varepsilon(s_{i+1}-s_i)c(s_i)]ds_1\cdots ds_n \\
\mathrm{S}_n(t,\varepsilon) &= \int_{0\leq s_1\leq\ldots\leq \sigma_{n+1}=x} s_\varepsilon(s_1)\prod_{i=1}^{n}[\sigma_\varepsilon(s_{i+1}-s_i)c(s_i)]ds_1\cdots ds_n
\end{aligned} \tag{116}$$

and $\mathrm{c}_\varepsilon(x) = \cos\sqrt{\varepsilon}x$, $\sigma_e(x) = \frac{\sin(\sqrt{\varepsilon}x)}{\sqrt{\varepsilon}}$ provide formal solution for eq(114).It holds (T.P.):

Theorem 4.1 *The formal power series eq(115) converge uniformly on the bounded set of* $[0,1]\times C\times C$ *to the unique solution of eq(114). They are a fundamental set of solutions and for every* $x\in[0,1]$ $\phi_i(x,\varepsilon,u),\phi_i'(x,\varepsilon,u)$ *are entire function on* $C\times C$ *and are real on* $R\times R$.

We now look for solutions for eq(114) of the form

$$\phi(1,\varepsilon,u) = \lambda\hat{\phi}(0,\varepsilon,u) \qquad \phi'(1,\varepsilon,u) = \lambda\phi'(0,\varepsilon,u) \tag{117}$$

This solution can be extended to all R setting $\phi(x+n,\varepsilon,u) = \lambda^n\phi(x,\varepsilon,u)$. We write $\phi(x,\varepsilon,u) = \alpha\phi_1(x,\varepsilon,u)+\beta\phi_2(x,\varepsilon,u)$ and substituting we obtain

$$\lambda^2 - 2\lambda\mu(\varepsilon,u) + 1 = 0 \qquad \frac{\alpha}{\beta} = \frac{2\phi_2(1,\varepsilon,u)}{\lambda(\varepsilon,u)-\frac{1}{\lambda(\varepsilon,u)}} \tag{118}$$

where $\mu(\varepsilon,u) = \phi_1(1,\varepsilon,u)$. Setting $\lambda = e^{ik}$ the first of eq(118) become

$$\cos k = \mu(\varepsilon,u) = \phi_1(1,\varepsilon,u) \tag{119}$$

which implicitely defines the function $\varepsilon(k,u)$. In (K.) it is proved that the zeros of $\frac{\partial\mu(\varepsilon,u)}{\partial\varepsilon}$ are simple and in correspondence of real values of ε which we call $\varepsilon_n(u)$. Defining $\mu_n = \mu(\varepsilon_n(u)) = \cos k_n$, we have

$$k_{2m} = \pm[(2j+1)\pi + ih_{2m}(u)] \qquad k_{2m+1} = \pm[2j\pi + ih_{2m+1}(u)] \qquad j = 0,\pm1,\pm2,\ldots$$

We will always suppose that $\mu_{2m} < -1, \mu_{2m+1} > 1$. This is not a limitation: see remark after lemma 2. The zeros of the derivatives of μ implies that $\varepsilon(\vec{k},u)$ must be a multivalued function. This can be represented on a Riemann surface with an infinite sequence of sheets S_n and $\varepsilon_n(\vec{k})$ are the values of $\varepsilon(\vec{k},u)$ on S_n. The points k_n are diramation points that connect the sheet S_n

with the sheet S_{n-1}: *i.e.* if one starts at a real value of k on S_n,turns clockwise around k_n and returns on the real axis one arrives on S_{n+1}, while if one turns around k_{n-1} he arrives on S_{n-1}. The periodicity of $\cos(\vec{k})$ implyes that for $\vec{k}$ real $\varepsilon_n(\vec{k})$ are analytic periodic function of period π. Finally if we cut the complex plane along the segment $[\pm(m+1)\pi+ih_m(u), \pm(m+1)\pi-ih_m(u)]$ we obtain a single valued function defined on all C and analytic every where except on the cut.

Lemma 4.1 *$h_n(u)$ is real, analytic in u and $O(u)$ for small u*

Proof We consider for definetiveness only $h_0(u)$. It is given by the equation

$$\cos\vec{k}_0 = \mu(\varepsilon_0(u), u) \tag{120}$$

Note that if $\mu(\varepsilon_0(u), u) \neq -1$ we can invert locally eq.(120) obtaining the solutions $k_0 = n\pi + ih_0(u)$ and $k_0 = n\pi - ih_0(u)$. If $\mu(\varepsilon_0(\bar{u}), \bar{u}) = -1$ we note that $\bar{u}$ must be a maximum for μ because we know that $\mu(\varepsilon_0(u), u) \leq -1$. This mean that the order of its first non zero derivatives must be even, say $2n$ so that near $\bar{u}$ eq(120) is equivalent to the two equations

$$\pm(\vec{k}_0-\pi)\tilde{c}(\vec{k}_0-\pi) \equiv \pm(\vec{k}_0-\pi)\sqrt{\frac{-\cos(\vec{k}_0-\pi)+1}{(\vec{k}_0-\pi)^2}} = u^n\tilde{\mu}(u) \equiv u^n\sqrt{\frac{\mu(\varepsilon_0(u),u)+1}{u^{2n}}} \tag{121}$$

It is easy to see that $\tilde{c}(\vec{k}_0-\pi) = \tilde{c}(-(\vec{k}_0-\pi))$ so that we can write equation (121) as $\pm(\vec{k}_0-\pi)\tilde{c}(\pm(\vec{k}_0-\pi)) = u^n\tilde{\mu}(u)$. Moreover it is clear that

$$\frac{d}{d\vec{k}_0}(\vec{k}_0-\pi)\tilde{c}(\vec{k}_0-\pi)|_{\vec{k}_0=\pi} \neq 0$$

so that by the inverse function theorem from the equation $(\vec{k}_0-\pi)\tilde{c}(\vec{k}_0-\pi) = u^n\tilde{\mu}(u)$ we obtain a function $\vec{k}_0(u) = \pi + ih_0(u)$ analytic near $\bar{u}$ Applying the same argument the equation with the minus sign we obtain that its solution is $\vec{k}_0(u) = \pi - ih_0(u)$. This completes the proof that $h_0(u)$ is analytic for all u. Choosing $\bar{u} = 0$ we have that $\mu(\varepsilon_0(0), 0) = -1$ and $\frac{\partial^2\mu(\varepsilon_0,u)}{(\partial u)^2}\Big|_{u=0} \neq 0$ so that $h_0(u)$ is $O(u)$ for small u.

Let us now define the new variable $q = \pm|k-k_0| = \sqrt{(k-\pi)^2+h_0(u)^2}$ where the square root, and so q, is intended as a double valued function. The function $\mathrm{cs}(q,u) = \cos(k(q,u))$ is analytic and single valued as $\cos(\vec{k})$ is an even and periodic function and $k = \pi + \sqrt{q^2-h_0(u)^2}$.

Lemma 4.2 *: the function $\tilde{\varepsilon}(q,u)$ defined by*

$$\mathrm{cs}(q,u) = \mu(\varepsilon, u) \tag{122}$$

with the condition $\tilde{\varepsilon}(0,u) = \varepsilon_0(u)$ is analytic for q such that $0 < |\mathrm{Re}k(q)| < 2\pi$.
Moreover $\left.\frac{\partial\varepsilon(q,u)}{\partial q}\right|_{q=0} = \frac{1}{2}\sqrt{\frac{\sin h_0}{h_0}}\left(\frac{\partial^2\mu(\varepsilon,u)}{\partial^2\varepsilon}\Big|_{\varepsilon=\varepsilon_0(u)}\right)^{-1}$

Proof: near $\varepsilon_0(u)$ we can write eq(122) as $\mathrm{cs}(q) - \mu(\varepsilon_0(u), u) = \mu(\varepsilon, u) - \mu(\varepsilon_0(u), u)$. Observe that

$$\left.\frac{d\mathrm{cs}(q)}{dq}\right|_{q=0} = 0 \qquad \left.\frac{\partial\mu(\varepsilon,u)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_0(u)} = 0$$
$$\left.\frac{d^2\mathrm{cs}(q)}{dq^2}\right|_{q=0} \neq 0 \qquad \left.\frac{\partial^2\mu(\varepsilon,u)}{\partial^2\varepsilon}\right|_{\varepsilon=\varepsilon_0(u)} \neq 0 \tag{123}$$

where the relation on μ follows from (K.) and those on $\mathrm{cs}(q)$ are trivial. Proceeding as above we have that eq(122) is equivalent to the two equation

$$\pm\, q\sqrt{\frac{\mathrm{cs}(q,u) - \mu(\varepsilon_0(u),u)}{q^2}} \equiv \pm q\bar{\mathrm{cs}}(q,u) = \varepsilon\bar{\mu}(\varepsilon,u) \equiv \varepsilon\sqrt{\frac{\mu(\varepsilon,u)-1}{\varepsilon^2}} \tag{124}$$

Noting that $\mathrm{cs}(q)$ is even in q and proceeding like lemma 1 we get, by the inverse function theorem, an analytic function $\tilde{\varepsilon}(q,u)$ such that the solution of equation (124) can be written as $\tilde{\varepsilon}^{\pm}(q,u) = \tilde{\varepsilon}(\pm q, u)$. The two functions $\tilde{\varepsilon}^{\pm}(\sqrt{(k-\pi)^2 + h_0(u)^2}, u)$ are identical as bivalued function of $\vec{k}$ as q is bivalued. We choose for definiteness $\varepsilon(q,u)$ so that $\tilde{\varepsilon}(h_0(u),u) > \tilde{\varepsilon}(-h_0(u),u)$. Eq.(124) can be inverted in a connected region in which $\frac{\partial\mu(\varepsilon)}{\partial\varepsilon} \neq 0$. This end the proof.

Remark 1 if $\mu_0 = 1$ we have that $h_0 = 0$ and so $q = \pi \pm k$. The above lemma implyes that $\varepsilon(\vec{k})$ can be considered as a couple of analytic functions $\tilde{\varepsilon}(\pi \pm k)$.

Remark 2 as $\Delta(u) = \tilde{\varepsilon}(h_0(u)) - \tilde{\varepsilon}(-h_0(u))$, from lemma 2 $\Delta(u)/h_0(u)$ is a continuous function of u.

To prove the second of the bounds eq.(110) we note that

$$\frac{d^n q}{dk^n} = \sum_{n_1=0}^{n} K_{n_1}(\vec{k}-\pi)^{n_1}((\vec{k}-\pi)^2 + h_0(u)^2)^{\frac{1-n_1-n}{2}} =$$
$$= \sum_{n_1=0}^{n} K_{n_1}\left(\frac{(\vec{k}-\pi)^2}{(\vec{k}-\pi)^2 + h_0(u)^2}\right)^{n_1} \frac{1}{((\vec{k}-\pi)^2 + h_0(u)^2)^{\frac{n-1}{2}}}$$

for suitable combinatorial coefficient K_n. From lemma 2 it follows that:

$$\left|\frac{\partial^N\varepsilon(k)}{(\partial k)^N}\right| \leq \frac{C_N(u)}{((k-\pi)^2 + \Delta^2)^{\frac{N-1}{2}}}$$

where $C_N(u)$ is a continuous function of u by remark 2.

From eq.(118) the equation for Bloch wave can be explicitly written:

$$\phi(x,\varepsilon) = \frac{\phi_2(1,\varepsilon)\phi_1(x,\varepsilon) + (\lambda(\varepsilon) - \lambda(\varepsilon)^{-1})\phi_2(x,\varepsilon)}{\sqrt{\phi_2(1,\varepsilon)\dot{\phi}_1(1,\varepsilon)}} \tag{125}$$

where $\dot{\phi}_1(1,\varepsilon) = \frac{\partial \phi_1(1,\varepsilon)}{\partial \varepsilon}$. It is easy to see that $\phi_1(1,\varepsilon) = \phi_2'(1,\varepsilon)$. The functions $\phi_1(x,\varepsilon)$ and $\phi_2(x,\varepsilon)$ are analytic in ε so that we can define $\phi_1(x,q) = \phi_1(x,\tilde{\varepsilon}(q))$ and $\phi_2(x,q) = \phi_2(x,\tilde{\varepsilon}(q))$ analytic in q. To control the analytic properties of $\phi(x,q)$ we must know where the denominator is vanishing.

We have that

$$\phi_1(x,\varepsilon)\phi_2'(x,\varepsilon) - \phi_1'(x,\varepsilon)\phi_2(x,\varepsilon) \equiv 1$$

because it is the wronskian. Moreover we have $\phi_1(1,\tilde{\varepsilon}(\pm h_0(u)) = \phi_2'(1,\tilde{\varepsilon}(\pm h_0(u)) = -1$. Substituting in the last equation we have $\phi_2(1,\tilde{\varepsilon}(\pm h_0(u))\phi_1'(1,\tilde{\varepsilon}(\pm h_0(u)) = 0$. From this it follows that at $q = h_0(u)$ one of the two functions ϕ_2 and ϕ_1' must be zero. Note that around any points u^* such that $h_0(u^*) = 0$ it is impossible that $\phi_2(1,\tilde{\varepsilon}(+h_0(u))) = \phi_2(1,\varepsilon(-h_0(u)) = 0$. In fact if this would be true $\phi_2(x,\tilde{\varepsilon}(+h_0(u))$ and $\phi_2(x,\tilde{\varepsilon}(-h_0(u)))$ would solve eq.(57) with conditions $\phi_2(0,\varepsilon) = \phi_2(1,\varepsilon) = 0$ and so should be orthogonal; but around u^* their scalar product must be different from zero because it is strictly positive in u^* and it is continuous. A similar argument holds for ϕ_1' so that we can conclude that $\phi_2(1,\varepsilon)$ is zero in one and only one of the two points $\tilde{\varepsilon}(\pm h_0(u))$.

Coming back to equation (125) we can write $\dot{\phi}_1(1,q) = \dot{\mu}(\varepsilon) = q\hat{\phi}_1(1,q)$ and $\phi_2(1,q) = (q - h_0(u))\hat{\phi}_2(1,q)$ where $\tilde{\phi}_2(1,q)$ and $\tilde{\phi}_1(1,q)$ are analytic and different from 0. This implies that the denominator can be written as $\sqrt{q(q - h_0(u))}N(q)$ with N analytic and not vanishing. Moreover we have that $\lambda(\varepsilon) - \lambda(\varepsilon)^{-1} = i\sin(ka) = i(k-\pi)\frac{\sin ka}{(k-\pi)}$. We observe that $\frac{\sin k}{(k-\pi)}$ is an even function of k and so can be written as a single defined function of q *i.e.* we have $\sin ka = \sqrt{q^2 - h_0(u)^2}\mathrm{sn}(q)$ where $sn(q)$ is defined by this equation. Substituting all that in equation (125) we obtain

$$\phi(x,q) = \sqrt{1 - \frac{h_0(u)}{q}}\tilde{\phi}_1(x,\tilde{\varepsilon}(q)) + \sqrt{1 + \frac{h_0(u)}{q}}\tilde{\phi}_2(x,\tilde{\varepsilon}(q)) \tag{126}$$

with $\tilde{\phi}_1(x,\varepsilon(q)), \tilde{\phi}_1(x,\varepsilon(q))$ analytic as function of q and u.

An important feature of the function $\phi(x,q)$ is that for $q = h_0(u)$ we have $\phi(x,q) = \tilde{\phi}_1(x,\varepsilon(q))$ $\xrightarrow[u\to 0]{} \cos(\pi\vec{x})$ and for $q = -h_0(u)$ we have
$\phi(x,q) = i\tilde{\phi}_1(x,\varepsilon(q))\xrightarrow[u\to 0]{} i\sin(\pi\vec{x})$. This representation clearly holds in any small neighbourhood of the points u^* such that $h_0(u^*) = 0$. Note moreover that eq.(93) easily follows. In fact the Bloch waves extended to all R are given by $e^{i\vec{k}n}\phi(t,q(k))$ where $t \in (0,a)$ and $x = t + n$; then eq.93) easily follows noting that, for $||k| - \pi| = \pi/2$, $\varepsilon(\vec{k}) - \vec{k}^2 = O(u)$.

To obtain the estimate eq.(110) in the case u near one of u^* we first consider the case $|\vec{k} - \pi| < O(\Delta)$ Observe that $\sqrt{q(\vec{k}) \pm h_0(u)}$ is a four-valued function of $\vec{k}$ that near $\vec{k} = \pi$ can be written as:

$$\sqrt{q(\vec{k}) \pm h_0(u)} =$$

$$h_0(u)\sqrt{\pm 1+\sqrt{\frac{(\vec{k}-\pi)^2}{h_0(u)^2}+1}}=h_0(u)f^{\pm}((\vec{k}-\pi)/h_0(u))$$

Let us study $f^+((\vec{k}-\pi)/h_0(u))=\sqrt{1+\sqrt{\frac{(\vec{k}-\pi)^2}{h_0(u)^2}+1}}$. For $\vec{k}=\pi$ we have $\sqrt{\frac{(\vec{k}-\pi)^2}{h_0(u)^2}+1}=\pm 1$ so that the two values of that function are well defined and analytic near $\vec{k}=\pi$. Moreover for both values we have that $1+\sqrt{\frac{(\vec{k}-\pi)^2}{h_0(u)^2}+1}$ is an even function of $\vec{k}$ and near $\vec{k}=\pi$ is $\pm 1+O((\vec{k}-\pi)^2)$ so that its square root is well defined and analytic. The same holds for $f^-(\vec{k})$. It follows that we can write

$$\phi(x,\vec{k})=\sqrt{\frac{h_0(u)}{q}}(f^+(\vec{k}/h_0(u))\tilde{\phi}_1(x,\vec{k})+f^-(\vec{k}/h_0(u))\tilde{\phi}_2(x,\vec{k})) \tag{127}$$

with $f^{\pm}$ four valued analytic functions. When we make the $\tilde{n}$-th derivative of this expression we have that the n-th derivative of the functions $\tilde{\phi}_2(x,\vec{k})$ and $\tilde{\phi}_1(x,\vec{k})$ is bounded by $\frac{1}{h_0(u)^{n-1}}$ because they are analytic in q. The n-th derivative of $f^{\pm}$ are bounded by $C^n h_0(u)^{-n}$. The only term to control is $\sqrt{h_0(u)/q}$. We have that the n-th derivative of this function is given by:

$$\sum_m \sum_{n_1+n_2+\cdots+n_m=n} K^m_{n_1,\ldots,n_m}\prod_{i=1}^m \frac{d^{n_i}q}{d\vec{k}^{n_i}}\frac{d^m}{dq^m}\sqrt{\frac{h_0(u)}{q}}$$

with $K^m_{n_1,\ldots,n_m}$ combinatorial coefficients. using the fact that $d^nq/dk^n\le C_1^n h_0(u)^{-(n-1)}$ we get that $d^n\left(\sqrt{h_0(u)/q}\right)/d\vec{k}^n\le K_n h_0(u)^{-n}$. All together this implies $d^n\phi(\vec{k},\vec{x})/d\vec{k}^n\le K'_n h_0(u)^{-n}$.

If $\vec{k}-\pi>O(\Delta)$ it is enough to observe that the derivatives of $\sqrt{q+h_0(u)}/\sqrt{q}$ are given by:

$$\sum_{m,m_1+m_2=m}\sum_{n_1+\cdots+n_m=n} K^{m,m_1,m_2}_{n_1,\ldots,n_m}\prod_{i=1}^m\frac{d^{n_i}q}{d\vec{k}^{n_i}}q^{-(m_1+1/2)}(q-h_0(u))^{-(m_2-1/2)}$$

where $K^{m,m_1,m_2}_{n_1,\ldots,n_m}$ is a suitable combinatorial factor. The estimates follows remembering that $d^nq/d\vec{k}^n\le C(|k|-\pi)^{-(n-1)}$.

If u is not near to one of the u^* it is enough to observe that $\phi(\vec{k},\vec{x})$ is an analytic function of $\vec{k}$ in a domain of radius $\eta_0(u)$ around $k=\pi$. By a dimensional estimate we get:

$$|\frac{\partial^n\phi(\vec{k},\vec{x})}{\partial\vec{k}^n}|\le\frac{\sup_{|\vec{k}-\pi|\le r}\phi(\vec{k},\vec{x})}{r^n}$$

Chosing $r=\sqrt{(\vec{k}-\pi)^2+h_0(u)^2/2}$ and noting that $\sup_{|\vec{k}-\pi|\le r}\phi(\vec{k},\vec{x})$ is continuous the second of the bounds eq.(110) follows.

Clearly above analysis can be repeated in correspondence of each of the point ε_n.

5 Appendix 3

In this appendix we prove lemmas 2.1 and 2.2.

Proof of lemma 2.1: The bound on $g^h_{\vec{\omega},-\vec{\omega}}(x-y)$ follows from eq.(66) by integrating by parts. We can write $g^h_{\vec{\omega},\vec{\omega}}(x-y)$ as a term independent on σ_h plus a remainder:

$$g^h_{\vec{\omega},\vec{\omega}}(x-y) = g^h_{\vec{\omega};\sigma_h=0}(k) + \int_0^1 dt \frac{\partial}{\partial t} g^h_{\vec{\omega}}(x-y)|_{t\sigma_h} = g^h_{\vec{\omega};\sigma_h=0}(k) + C_2(x-y) \tag{128}$$

where

$$C_2(x-y) = \int dtdk Z_h(k)^{-1}\sigma_h(k)^2 C_h^{-1}(k) e^{ik(x-y)} \frac{-ik_0+\vec{k}^2-2\omega\pi\vec{k}}{\left(\left(-ik_0+\vec{k}^2\right)^2-\left(\pi\vec{k}\right)^2-t\sigma_h(k)\right)^2}. \tag{129}$$

The bound on C_2 follows by a standard integration by parts.

Proof of lemma 2.2: We proceed inductively by assuming that eq.(79),(83),(82) hold for $0 \geq k \geq h$ and proving that their validity for $h-1$. From eq.(75) and the analyticity of the beta function we have that:

$$Z_k(1+2c_3\beta_1\lambda^2) < Z_{k-1} < (1+c_4\beta_1\lambda^2)Z_k$$

where c_1, c_2 are constants.

Assume inductively that:

$$\gamma^{-c_3\beta_1\lambda^2 k} < Z_k < \gamma^{-c_4\beta_1\lambda^2 k}$$

for $k \geq h$; then the same equation holds for Z_{h-1} as, from eq.(75):

$$Z_{h-1} \leq Z_h(1+\beta_1 c_4\lambda^2) \leq \gamma^{-\beta_1 c_4\lambda^2(h-1)}\frac{1+\beta_1 c_4\lambda^2}{e^{\beta_1 c_4\lambda^2}} \leq \gamma^{-\beta_1 c_4\lambda^2(h-1)}$$

$$Z_{h-1} \geq Z_h(1+2\beta_1 c_3\lambda^2) \geq \gamma^{-\beta_1 c_3\lambda^2(h-1)}\frac{1+2\beta_1 c_3\lambda^2}{e^{\beta_1 c_3\lambda^2}} \geq \gamma^{-\beta_1 c_3\lambda^2(h-1)}$$

A similar argument can be used to treat the bound on σ_h so that eq.(83) is verified.

Eq.(79) follows assuming that eq.(79)(83)(82) hold for $k \geq h$ so that there are two constants c_1, c_2 such that from eq.(75):

$$-c_1\lambda^2[\frac{\sigma_h}{\gamma^h}+\gamma^h] < \lambda_{h-1}-\lambda_h < c_2[\frac{\sigma_h}{\gamma^h}+\gamma^h]\lambda^2$$

and proceeding in the same way for δ_{h-1}.

Finally in the same way the equations from τ_h, θ_h are obtained from eq.(75).

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