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Whitney regularity for solutions to the coboundary equation on Cantor sets

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Abstract

We prove Whitney regularity results for the solutions of the coboundary equation over dynamically defined Cantor sets satisfying a natural geometric regularity condition, in particular hyperbolic basic sets in dimension two. To do this we prove an analogue of Journé's well-known result in the context of Cantor sets satisfying geometric regularity conditions.

1 Introduction

This paper is concerned with the regularity of the solution ϕ to the coboundary equation

$$(\phi \circ T - \phi)(x) = g(x), \quad x \in \Lambda \tag{1.1}$$

where M is a manifold, $\Lambda \subset M$ is a closed set, $T : \Lambda \to \Lambda$ is a hyperbolic transformation that is the restriction of a smooth map, and $g : M \to \mathbb{R}$ is smooth. We are especially

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interested in the situation when Λ might have no interior. Whenever Λ is not open, smoothness is meant in the sense of Whitney.

Results in this direction were first obtained by Livšic, for real valued cocycles over Anosov mappings. He showed [6] that if the cocycle g is C^1, then the trivialization ϕ (assumed apriori only measurable) is C^1 too. For some linear actions on a torus he also showed [7] that if the cocycle is C^∞, respectively C^ω, then so is the solution; this was obtained by studying the decay of the Fourier coefficients.

The C^∞ (smooth) case was investigated by de la Llave, Marco and Moriyón [10]. One of the technical results they proved was that if a function is smooth along two transverse foliations which are absolutely continuous and whose Jacobians have some regularity properties, then it is smooth globally. This was proved using properties of elliptic operators. Relying on Taylor expansions, careful error estimates and Morrey-Campanato theory, Journé ([4]; see Theorem 1.1 below) proved a similar result without requiring the absolutely continuity of the foliations. Another approach is presented in Hurder and Katok [3], based on an unpublished idea of C. Toll. Here the decay of the Fourier coefficients is used to characterize smoothness. Using the approach in [3], de la Llave proved analogous results in the analytic case [9]. More references are given in [12].

Journé's key result is the following:

Theorem 1.1 *[Journé [4]] Let F_s and F_u be two continuous transverse foliations with uniformly smooth leaves of some manifold. If f is uniformly smooth along the leaves of F_s and F_u then f is smooth.*

Various modifications have been proven since. We mention in particular work of de la Llave [8, Theorem 5.7], where Whitney regularity was proved, and Vano [19, Lemma 3.2.6]. Our main result is in some sense a technical improvement over de la Llave's Theorem 5.7: it applies to sets that are less regular (as opposed to de la Llave's condition (iv)), and that could have measure zero. We state our result in terms of laminations whose leaves satisfy a certain geometric regularity, but we have in mind applications to a class of hyperbolic basic sets.

Recall that if $\Lambda \subset M$ is an invariant hyperbolic set for a C^1 map T, then through each point $x \in \Lambda$ there exists a local stable manifold $W^s_\varepsilon(x)$ and a local unstable manifold $W^u_\varepsilon(x)$. We call a hyperbolic invariant set *basic* if it is locally maximal, that is, $\Lambda = \cap_{n=-\infty}^{\infty} T^n(V)$ for an open set V. A hyperbolic set Λ is basic if and only if it has local product structure: if $x, y \in \Lambda$ are close enough, then the unique point $z = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ belongs to Λ [5, Section 18.4].

Let $n \geq 1$, $\alpha \in (0, 1)$. For an open set $U \subset \mathbb{R}^d$, $C^{n,\alpha}(U)$ consists of functions that are differentiable of order n, have all derivatives bounded, and the n-th partial derivatives are α-Hölder (see [18, Chapter VI]).

Definition 1.2 A $C^{n,\alpha}$-lamination of a set $\Lambda \subset M$ is a disjoint collection of $C^{n,\alpha}$ submanifolds of a given same dimension, which vary continuously in the $C^{n,\alpha}$-topology, and whose union contains the set Λ.

Remark 1.3 Examples of $C^{n,\alpha}$-laminations are the stable and unstable foliations of a hyperbolic set for a C^{n+1} diffeomorphism.

We describe here the notion of geometric regularity that we are using.

Definition 1.4 Fix constants $0 < \gamma < 1$ and $\nu > 0$.

(a) A set A contained in a C^1-curve Γ is (γ, ν)-*homogeneous* if for each $x \in A$, there is a sequence of points $x_k \in A$ converging to x such that $\mathrm{dist}_\Gamma(x, x_1) \geq \nu$, and

$$\frac{\mathrm{dist}_\Gamma(x, x_{k+1})}{\mathrm{dist}_\Gamma(x, x_k)} \geq \gamma \text{ for } \ k \geq 1 \tag{1.2}$$

(here dist_Γ denotes the distance induced on the curve Γ).

(b) A set $\Lambda \subset \mathbb{R}^2$ is (γ, ν)-*homogeneous* with respect to two transverse C^1-laminations W^s and W^u if for each $p \in \Lambda$, the sets $W^s(p) \cap \Lambda$ and $W^u(p) \cap \Lambda$ are (γ, ν)-homogeneous in the corresponding leaves.

Our main result is an extension of Journé's Theorem 1.1.

Theorem 1.5 *Let $\Lambda \subset \mathbb{R}^2$ be a closed set, and W^s, W^u two transverse uniformly $C^{n,\alpha}$-laminations of Λ.*

Suppose that $\phi : \Lambda \to \mathbb{R}$ is uniformly Whitney-$C^{n,\alpha}$ when restricted to $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ for each $x \in \Lambda$. If

- *Λ is* (γ, ν)-homogeneous *with respect to W^s and W^u for some $\gamma, \nu > 0$,*

  and

- *Λ has a local product structure: for $x, y \in \Lambda$ close enough, the (unique) point in the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ belongs to Λ,*

then ϕ is Whitney-$C^{n,\alpha}$ on Λ.

The term "uniformly" above means that the given property holds with uniform constants over the whole set Λ.

Remark 1.6 Examples of (γ, ν)-homogeneous sets include hyperbolic basic sets in dimension two (see Proposition 2.2), Anosov systems, and the direct product of one dimensional uniformly homogeneous sets in higher dimensions (see Section 5).

The paper is organized as follows. In Section 2 we describe two-dimensional examples to which our results apply. In Section 3 we describe applications of Theorem 1.5 to cohomological equations over dynamical systems. The proof of Theorem 1.5 is given in Section 4.

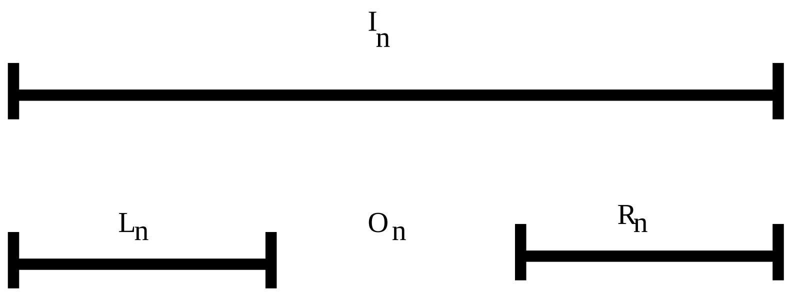

Figure 1: Construction of a Cantor set

2 Geometrically regular hyperbolic sets

We describe a class of dynamically defined two-dimensional Cantor sets to which Theorem 1.5 pertains. First we describe the notion of 'thickness' of Cantor sets. This was introduced by Newhouse [11]. A Cantor set K in $\mathbb{R}$ is formed by starting with a closed interval and successively removing open intervals of decreasing length. Suppose each open interval O_n that is removed from a closed interval I_n leaves behind two closed intervals L_n and R_n (see Figure 1). Let

$$\tau_n = \frac{\min\{|L_n|, |R_n|\}}{|O_n|}$$

and

$$\sigma_n = \frac{|O_n|}{\max\{|L_n|, |R_n|\}}$$

The quantity

$$\tau(K) = \inf_n \tau_n$$

is called the thickness of the Cantor set K. K is called a *thick* Cantor set if $\tau(K) > 0$. We define

$$\sigma(K) = \inf_n \sigma_n$$

and define a *distortion-free thick* Cantor set to be a thick Cantor set K with $\sigma(K) > 0$.

We say a two dimensional set $\Lambda \subset \mathbb{R}^2$ with a lamination is a distortion-free thick Cantor set if for each $p \in \Lambda$, $W^s(p) \cap \Lambda$ and $W^u(p) \cap \Lambda$ are distortion-free thick Cantor sets with uniform σ, τ. For more information on thick Cantor sets see [15, Chapter 4, Section 2] and [2].

Lemma 2.1 *If Λ is a distortion-free thick Cantor set then Λ is (γ, ν)-homogeneous.*

Proof: Assume $x \in \Lambda$. To simplify notation, we parametrize $W^s_\varepsilon(x)$ by arc-length, coordinatize so that $x = 0$ and assume from now on that points in $\Lambda \cap W^s_\varepsilon(x)$ are in $\mathbb{R}$. We show that there is a sequence of numbers $x_k \neq 0, x_k \in \Lambda$ converging monotonically to zero and satisfying equation (4.1) for some γ. Furthermore γ may be taken as uniform over $x \in \Lambda$.

First note that for all n:

(a) $\sigma\tau \max\{|L_n|, |R_n|\} \leq \tau|O_n| \leq \min\{|L_n|, |R_n|\}$

(b) $\frac{\sigma\tau}{1+\sigma+\sigma\tau}|I_n| \leq \min\{|L_n|, |R_n|\}$

Suppose $x \in I_n$, choose x_1 to be the endpoint of I_n furthest from x. Without loss of generality suppose that $x \in L_n \cap I_n$ (exactly the same argument holds if $x \in R_n$). Choose x_2 to be the endpoint of L_n furthest from x. Then $|x_2| \geq \frac{1}{2}|L_n| \geq \frac{1}{2}\frac{\sigma\tau}{1+\sigma+\sigma\tau}|x_1|$ as $|x_1| \leq |I_n|$. Taking $\gamma = \frac{\sigma\tau}{2(1+\sigma+\sigma\tau)}$ we have $|x_2| \geq \gamma|x_1|$. Repeat this procedure, taking L_n to be I_n and noting that x_2 is the endpoint of L_n furthest from x. The argument for the unstable foliation is the same. ∎

Proposition 2.2 *Two-dimensional hyperbolic basic sets are (γ, ν)-homogeneous.*

Proof: This follows from [15, Chapter 4, Section 1] and Lemma 2.1. Note that both local stable and local unstable manifolds are dynamically defined Cantor sets for an expanding map (an expanding map in backwards time for stable manifolds). By the uniform bounded distortion estimates on such maps [15, Chapter 4, Section 1] σ_n, τ_n are uniformly bounded away from zero. ∎

3 Livšic regularity results

The regularity results for the Livsic equation (1.1) state that under certain conditions a measurable solution is actually Hölder, or that a continuous solution is actually (Whitney) smooth.

Suppose that Λ is a hyperbolic basic set and μ is an ergodic Gibbs measure corresponding to a Hölder continuous potential.

We first show that if we have a measurable coboundary ϕ for a Hölder real-valued cocycle g then ϕ has a Hölder version, that is, there exists a Hölder ϕ' such that $\phi' = \phi$, μ a.e. This result is not new and the main idea is due to Livsic [7] (for related results see [1, 13, 14, 16, 17]). We sketch its proof only for completeness.

Proposition 3.1 *Let $\Lambda \subset U \subset M$ be a hyperbolic set for the C^1 embedding $T : U \to M$. Suppose that Λ is equipped with an ergodic Gibbs measure μ. Assume that $g : U \to \mathbb{R}$ is η-Hölder, $\eta > 0$, and there is a μ-measurable function $\phi : \Lambda \to \mathbb{R}$ such that*

$$(\phi \circ T - \phi)(x) = g(x), \quad x \in \Lambda. \tag{3.1}$$

Then there exists an η-Hölder function $\phi' : \Lambda \to \mathbb{R}$ such that $\phi' = \phi$ μ a.e.

Proof: For $x \in \Lambda$ define $g_N(x) := g(T^{N-1}x) + \cdots g(Tx) + g(x)$. If $y \in W^s_\varepsilon(x)$ then $|g_N(x) - g_N(y)| \leq \sum_{i=0}^{N-1} |g(T^i x) - g(T^i y)| \leq C_1(\sum_{i=0}^{N-1} \lambda^{\eta i}) d(x,y)^\eta$ for some $0 < \lambda < 1$. Thus for all $N \geq 0$ and $y \in W^s_\varepsilon(x)$

$$|\phi(x) - \phi(y)| \leq C_2 d(x,y)^\eta + |\phi(T^N x) - \phi(T^N y)|$$

By Lusin's theorem there exists a set $\Lambda' \subset \Lambda$ such that $\mu(\Lambda') > \frac{1}{2}$ and ϕ restricted to Λ' is uniformly continuous. Since T is ergodic with respect to μ, for μ a.e. $x \in \Lambda$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{i \mid 0 \leq i \leq N-1, T^i(x) \in \Lambda'\} = \mu(\Lambda') > \frac{1}{2} \tag{3.2}$$

The ergodic Gibbs measure μ is a product measure, that is, for μ a.e. $x \in \Lambda$, on a neighborhood of x, the measure μ is equivalent to $\mu^s_x \times \mu^u_x$, where μ^s_x and μ^u_x are the conditional measures of μ along the stable $W^s_\varepsilon(x)$ and unstable $W^u_\varepsilon(x)$ leaves respectively. Since μ is locally a product measure, μ a.e. $x \in \Lambda$ has the property that for μ^s_x a.e. $y \in W^s_\varepsilon(x)$ equation (3.2) holds. Hence for μ a.e. $x \in \Lambda$, for μ^s_x a.e. $y \in W^s_\varepsilon(x)$ we may choose a subsequence N_i so that $|\phi(T^{N_i}x) - \phi(T^{N_i}y)| \to 0$. Thus μ a.e. $x \in \Lambda$ has the property that for μ^s_x a.e. $y \in W^s_\varepsilon(x)$, $|\phi(x) - \phi(y)| \leq C_2 d(x,y)^\eta$. Similar considerations show that μ a.e. $x \in \Lambda$ has the property that for μ^u_x a.e. $y \in W^u_\varepsilon(x)$, $|\phi(x) - \phi(y)| \leq C_3 d(x,y)^\eta$. The local product structure for μ implies that ϕ has an η-Hölder version (see [5, Proposition 19.1.1]). ∎

Next, we show that the regularity can be improved from C^0 to $C^{n,\alpha}$ for hyperbolic basic sets in dimension two:

Theorem 3.2 *Let M be a* two-dimensional *manifold, and $\Lambda \subset U \subset M$ a hyperbolic basic set for the $C^{n,\alpha}$ embedding $T : U \to M$. Assume that for $g \in C^{n,\alpha}(U)$, there is a continuous function $\phi : \Lambda \to \mathbb{R}$ such that*

$$(\phi \circ T - \phi)(x) = g(x), \quad x \in \Lambda. \tag{3.3}$$

Then ϕ is $C^{n,\alpha}$ in the Whitney sense, that is, it admits a $C^{n,\alpha}$ extension to a neighborhood of Λ.

The proof of this theorem follows the "classic" approach [7, 10, 4]: first show that ϕ is regular along the stable and unstable foliations, and then prove that such a function is regular globally. The first step is straightforward, and presented in Lemma 3.3. The second part, in view of Proposition 2.2, is Theorem 1.5.

Lemma 3.3 *Let M be a manifold, and $\Lambda \subset U \subset M$ a hyperbolic basic set for the $C^{n,\alpha}$ embedding $T : U \to M$. Assume that for $g \in C^{n,\alpha}(U)$, there is a continuous function $\phi : \Lambda \to \mathbb{R}$ such that*

$$(\phi \circ T - \phi)(x) = g(x), \quad x \in \Lambda. \tag{3.4}$$

Then there is a "natural" extension of ϕ to $W^s_\varepsilon(\Lambda) \cup W^u_\varepsilon(\Lambda)$ which is $C^{n,\alpha}$ on each local stable and unstable leaf, and varies continuously (in the $C^{n,\alpha}$-topology) with the leaf.

Proof: We extend ϕ through the cohomology equation (3.4). Using the notation $F|_a^b = F(b) - F(a)$, we obtain

$$\phi(y) - \phi(x) = -\sum_{n=0}^{N} g \circ T^n|_x^y + \phi \circ T^{N+1}|_x^y$$

Therefore, define

$$\widetilde{\phi}_s(y) := \phi(x) - \sum_{n=0}^{\infty} g \circ T^n|_x^y, \qquad y \in W_\varepsilon^s(x),$$
$$\widetilde{\phi}_u(z) := \phi(x) + \sum_{n=1}^{\infty} g \circ T^{-n}|_x^z, \qquad z \in W_\varepsilon^u(x),$$

The series converge uniformly in $C^{n,\alpha}(W_\varepsilon^s(x))$, respectively $C^{n,\alpha}(W_\varepsilon^u(x))$. Therefore, $\widetilde{\phi}_s$ and $\widetilde{\phi}_u$ are in $C^{n,\alpha}$, and vary continuously with the leaf.

Assume $x, y \in \Lambda$ and $t \in W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$. By the local product structure, $t \in \Lambda$. Therefore, by (3.4) and the continuity of ϕ,

$$\widetilde{\phi}_s(t) = \phi(x) - \sum_{n=0}^{\infty}[\phi \circ T - \phi] \circ T^n|_x^t = \phi(x) - \left[-\phi|_x^t + \lim_{n\to\infty} \phi \circ T^n|_x^t\right] = \phi(t).$$

Similarly, using T^{-1} instead of T, we obtain that $\widetilde{\phi}_u(t) = \phi(t)$. Thus, the two extensions coincide over $W_\varepsilon^s(\Lambda) \cap W_\varepsilon^u(\Lambda)$. ∎

As an immediate corollary of Proposition 3.1 and Theorem 3.2,

Corollary 3.4 *Let M be a* two-dimensional *manifold, and $\Lambda \subset U \subset M$ a hyperbolic basic set for the $C^{n,\alpha}$ embedding $T : U \to M$. Assume that for $g \in C^{n,\alpha}(U)$, there is a measurable function $\phi : \Lambda \to \mathbb{R}$ such that*

$$(\phi \circ T - \phi)(x) = g(x), \quad x \in \Lambda. \tag{3.5}$$

Then ϕ is $C^{n,\alpha}$ in the Whitney sense, that is, it admits a $C^{n,\alpha}$ extension to a neighborhood of Λ.

4 Journé revisited

We prove Theorem 1.5 in this section. The proof follows the method of Journé [4], with a few adjustments.

Journé [4] constructed approximating polynomials $\widetilde{Q}(q;p)$ indexed by $p \in U$ (U an open set) which satisfied $|\widetilde{Q}(q;p) - \phi(q)| \le C|p-q|^{n+\alpha}$ for all $q \in U$ and then invoked a theorem of Campanato to show that ϕ extends to a $C^{n+\alpha}$ function.

Our proof similarly uses approximating polynomials but we will use [18, §VI.2.3, Theorem 4] which gives sufficient conditions under which a function extends in the Whitney sense from a closed set.

Given our homogeneity assumption, we can select on each leaf a grid whose spacing is close to being a geometric progression (that is, points are at distances approximatively $\{\omega^k\}$ from the origin). The local product structure yields a two-dimensional grid, on which we interpolate the function ϕ by a sequence of polynomials. We prove that the lower-order coefficients of these polynomials converge, thus yielding a local approximation $Q(q;p)$ to $\phi(q)$ (indexed by each point p of the set Λ). These approximating polynomials 'correspond' to the Taylor polynomials of ϕ. We then show that these local approximations satisfy the hypothesis of the Whitney Extension Theorem stated in [18, §VI.2.3, Theorem 4]. The homogeneity plays an important role here as well.

Notation 4.1 *Unless stated otherwise, all constants are uniform on Λ. In particular, we will use the letter C for various constants of this type, even if their values are different. For simplicity of exposition we will refer to $W^s_\varepsilon(x)$ as local stable manifolds and to $W^u_\varepsilon(x)$ as local unstable manifolds.*

Given nearby points $x, y \in \Lambda$, we denote $[x, y] := W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$.

(a) Regular grid from homogeneity

We show that the homogeneity assumption provides a quite regular "grid".

Lemma 4.2 *Suppose $\{r_k\}$ is a sequence of numbers converging to zero such that*

$$\frac{|r_{k+1}|}{|r_k|} \geq \gamma \text{ for } \; k \geq 1 \tag{4.1}$$

for some $\gamma > 0$. Let $\omega \in (0, \gamma)$ and define $k_0 := [\log_\omega(|r_1|)]$. Define $\mathcal{R} = \{|r_k|\}$. Then the intersection $\mathcal{R} \cap [\omega^{k+1}, \omega^k]$ is nonempty for each $k \geq k_0$.

In particular, there is a decreasing subsequence in $\mathcal{R}$, which we denote also by $|r_k|$, such that $|r_k| \in [\omega^{2k+1}, \omega^{2k}]$ for $2k \geq k_0$.

Proof: Notice first that we may assume that the sequence $|r_k|$ is strictly decreasing. Indeed, if $|r_{\ell+1}| \geq |r_\ell|$, then property (4.1) is preserved after dropping the term $|r_{\ell+1}|$ from the sequence and relabeling the remaining terms.

Let $\omega \in (0, \gamma)$ and k_0 be as defined in the lemma. Then $|r_1| \in \mathcal{R} \cap [\omega^{k_0+1}, \omega^{k_0}]$. We will prove the statement of the lemma by induction starting with $k = k_0$. Assume, by contradiction, that $\mathcal{R} \cap [\omega^{k+1}, \omega^k] \neq \emptyset$ but $\mathcal{R} \cap [\omega^{k+2}, \omega^{k+1}] = \emptyset$. Then there is an ℓ such that $|r_\ell| \in [\omega^{k+1}, \omega^k]$ and $|r_{\ell+1}| < \omega^{k+2}$, hence $\frac{|r_{\ell+1}|}{|r_\ell|} < \omega < \gamma$ which contradicts (4.1). ∎

(b) Approximation by polynomials

The goal of this subsection is the following local approximation result:

Theorem 4.3 *Under the assumptions of Theorem 1.5, there are constants $\varepsilon', C' > 0$ and for each $p \in \Lambda$ a polynomial $Q(\,\cdot\,;p)$ of degree n, such that*

$$|\phi(q) - Q(q;p)| \le C'|q-p|^{n+\alpha} \quad \text{for } q \in \Lambda,\ |q-p| < \varepsilon'. \tag{4.2}$$

Most arguments are adapted from [4].

Let $p \in \Lambda$. Since the consideration is local we make a $C^{n,\alpha}$ change of variables so that the local stable and unstable manifolds passing through p become the coordinate axes through the origin. The theorem is a consequence of the following Lemma, which we prove later.

Lemma 4.4 *Given $\kappa > 0$ large enough and the cone $\mathcal{K} = \{(u,v) \in \mathbb{R}^2 \,:\, |v| \le \kappa|u|\}$, there is a polynomial $Q = Q_{\mathcal{K}}$ of degree n and constants $C_1 = C_1(\kappa), \varepsilon_1 = \varepsilon_1(\kappa) > 0$ such that*

$$|\phi(z) - Q(z)| \le C_1|z|^{n+\alpha} \quad \text{for } z \in \Lambda \cap \mathcal{K} \cap B_{\varepsilon_1},$$

where B_r denotes the ball of radius r centered at the origin.

The constants C_1 and ε_1 are uniform with respect to $p \in \Lambda$.

Assuming the above Lemma, we prove next Theorem 4.3.

Proof of Theorem 4.3

Using Lemma 4.4, we can also construct a polynomial Q' approximating ϕ on the cone $\mathcal{K}' = \{(u,v) \in \mathbb{R}^2 \,:\, |u| \le \kappa|v|\}$ centered on the unstable manifold. By choosing κ large enough, we can achieve that $V = \Lambda \cap \mathcal{K} \cap \mathcal{K}' \cap B_{\varepsilon_1}$ is not Zariski closed (indeed, the homogeneity assumption implies that the origin is an accumulation point of both $\Lambda \cap W^s_\varepsilon(0,0)$ and $\Lambda \cap W^u_\varepsilon(0,0)$, and the product structure gives a two-dimensional "grid" in V).

This implies that the n-th degree polynomials Q and Q' have to coincide, because they have a contact of order higher than n on V.

This establishes that at the point p, in the coordinates used to linearize its stable and unstable manifolds, Q is a local approximation of ϕ of order $n+\alpha$.

When we return to the original coordinates, then the polynomial Q becomes a $C^{n,\alpha}$-function $\mathcal{Q}$, but the local approximation property still holds. We denote by $Q(\,\cdot\,;p)$ the n-th order Taylor polynomial of $\mathcal{Q}$ at p. To check condition (4.2), notice that $R(q) := \mathcal{Q}(q) - Q(q;p)$ satisfies

$$|R(q)| = \left|\frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}\frac{d^n}{dt^n}\left[R(p+t(q-p))\right]\, dt\right| \le C\left\|\frac{d^n}{dt^n}\mathcal{Q}\right\|_\alpha |q-p|^{n+\alpha}$$

(here we used that $D^nR(r) = D^n\mathcal{Q}(r) - D^n\mathcal{Q}(p)$, because $Q(\,\cdot\,,p)$ is a polynomial of degree n).

Note that the relevant constants of Lemma 4.4, ε_1 and C_1, are uniform in $p \in \Lambda$, hence so will the constants ε' and C' of Theorem 4.3. ∎

The remainder of the section is devoted to the proof of Lemma 4.4. This proof is divided into a few subsections. By our reduction so far, p is the origin in $\mathbb{R}^2$ and the coordinate axes are leaves of the two laminations.

Interpolating polynomials

We will use the following interpolation result of Journé, with corresponding constants from Lemma 1 of Journé denoted by the subscript J.

Lemma 4.5 (Journé [4], Lemma 1) *Fix $n \geq 1$. For each $B \geq 1$, there are $\varepsilon = \varepsilon_J(B) > 0$ and $C = C_J(B) > 0$ with the following property: if the collections of points $\{z_{k,\ell} : 0 \leq k \leq n, 0 \leq \ell \leq n\} \subset \mathbb{R}^2$, $\{x_k : 0 \leq k \leq n\} \subset \mathbb{R}$, $\{y_\ell : 0 \leq \ell \leq n\} \subset \mathbb{R}$ satisfy*

$$\begin{aligned} & R/\eta < B \\ \textit{and} \qquad & |z_{k,\ell} - (x_k, y_\ell)| \leq \varepsilon\eta \\ \textit{where} \qquad & R = \sup_{k,\ell} |z_{k,\ell}|, \qquad \eta = \inf_{(k,\ell)\neq(k',\ell')} |z_{k,\ell} - z_{k',\ell'}|, \end{aligned}$$

then for any values $\{b_{k,\ell} : 0 \leq k \leq n, 0 \leq \ell \leq n\} \subset \mathbb{R}$, there exists a unique polynomial

$$p(x,y) = \sum_{0 \leq p,q \leq n} c_{pq} x^p y^q$$

such that $p(z_{k,\ell}) = b_{k,\ell}$. Moreover,

$$\sum_{p,q} |c_{pq}| R^{p+q} \leq C \sup_{k,\ell} |b_{k,\ell}|. \tag{4.3}$$

Since $\phi(\cdot, 0)$ and $\phi(0, \cdot)$ are $C^{n,\alpha}$ (and therefore Lemma 4.4 holds for them by approximating with the Taylor polynomial), we can replace $\phi(x,y)$ by $\phi(x,y) - \phi(x,0) - \phi(0,y) + \phi(0,0)$. Thus, we may assume that ϕ vanishes along the axes.

We begin by selecting a convenient grid on each axis. Along the x-axis (which we assume to be the stable direction) we take a sequence of points $(r'_k, 0) \in \Lambda$, $k \geq 0$, converging to $(0,0)$ and satisfying the homogeneity condition (4.1). We take a sequence $(0, s'_k) \in \Lambda$ along the unstable direction (which we assume to be the y-axis) satisfying bounds similar to (4.1).

By Lemma 4.2, passing to a subsequence, we may assume that the two sequences are similarly spaced: $|r'_k|, |s'_k| \in [\omega^{2k+1}, \omega^{2k}]$ for $k \geq k_0$, where ω and k_0 are determined using Lemma 4.2 for the two sequences involved. Moreover, by our (γ, ν)-homogeneity assumption, we can arrange that $\omega = \gamma/2$, and $|r'_1|$, $|s'_1|$ are of order ε_1, uniformly with respect to $p \in \Lambda$, where $\varepsilon_1 > 0$ will be determined later.

Since we must control ϕ on the whole set Λ, we explain next how to include arbitrary points of $\Lambda \cap \mathcal{K} \cap B_{\varepsilon_1}$ in this grid.

Notation 4.6 *Assume given sequences* $(r'_k, 0), (0, s'_k) \in \Lambda$ *as above, and let* $w \in \Lambda \cap \mathcal{K} \cap B_{\varepsilon_1}$*. Consider the points* $(x_w, 0) = [p, w] \in \Lambda$*,* $(0, y_w) = [w, p] \in \Lambda$*.*

We construct a new sequence of points $(r_k, 0) \in W^s_\varepsilon(p) \cap \Lambda$ *that includes* $(x_w, 0)$ *and still has a regular spacing. We proceed as follows: if* $|r'_{m+2}| < |r'_{m+1}| < |x_w| \leq |r'_m| < |r'_{m-1}|$*, then we drop* r'_{m+1} *and* r'_m*, and define* $r_{m+1} = x_w$*,* $r_k = r'_k$ *for* $k \geq m+2$ *and* $r_k = r'_{k-1}$ *for* $k \leq m$*. Thus, the* x*-coordinates of the new sequence in* $W^s_\varepsilon(p) \cap \Lambda$ *are* $\cdots, r_{m+3} = r'_{m+3}, r_{m+2} = r'_{m+2}, r_{m+1} = x_w, r_m = r'_{m-1}, r_{m-1} = r'_{m-2}, \cdots$*.*

We proceed similarly with the points $(0, s'_\ell) \in W^u_\varepsilon(p) \cap \Lambda$ *to incorporate* $(0, y_w)$*.*

For k, ℓ large enough, use the local product structure to define $z_{k,\ell} := [(0, s_\ell), (r_k, 0)] = W^u_\varepsilon((r_k, 0)) \cap W^s_\varepsilon((0, s_\ell)) \in \Lambda$.

The continuity in C^1 of the stable and unstable leaves implies that

$$|[(x,0),(0,y)] - (x,y)| = o(|(x,y)|).$$

Introduce the "rectangular" grid $S_{k,\ell} = \{(0,0)\} \cup \{(r_{k'}, 0) : k \leq k' < k+n\} \cup \{(0, s_{\ell'}) : \ell \leq \ell' < \ell + n\} \cup \{(z_{k,\ell}) : k \leq k' \leq k+n, \ell \leq \ell' \leq \ell + n\}$.

For each k and ℓ, denote $\eta_{k,\ell} := \inf\{|z - z'| : z, z' \in S_{k,\ell},\ z \neq z'\}$, $R_{k,\ell} := \sup\{|z| : z \in S_{k,\ell}\}$, and $T_k := \max\{|r_k|, |s_k|\}$.

Then there are constants C_0, k_1, independent of z, such that for $k, \ell \geq k_1$:

- $\frac{R_{k,\ell}}{\eta_{k,\ell}} < C_0$ as long as $|k - \ell| \leq 1$.
- $|z_{k,\ell} - (r_k, s_\ell)| \leq \frac{\varepsilon_J(C_0)}{C_0} |(r_k, s_\ell)|$ for $|k - \ell| \leq n$.
- $\frac{1}{C_0} \omega^{2k} \leq T_k \leq C_0 \omega^{2k}$.

where $\varepsilon_J(C_0)$ is the corresponding constant from Lemma 4.5.

Remark 4.7 The points z to which Lemma 4.4 applies are those corresponding to r_k, s_ℓ with $k, \ell \geq k_1$, hence $\varepsilon_1 = O(\omega^{k_1})$. During the proof we might have to decrease ε_1, which means that k_1 will increase accordingly. Note however that the three properties listed above remain valid after such a change.

In light of Lemma 4.5 we conclude that for $k \geq k_1$ there exists a unique polynomial

$$P(x,y) = \sum_{0 \leq p,q \leq n} c_{pq} x^p y^q$$

which interpolates ϕ on each grid S of form $S_{k,k}$ i.e.

$$P(z) = \phi(z)$$

for $z \in S$.

Furthermore

$$\sum_{p,q} |c_{pq}| R_S^{p+q} \leq C \sup\{|\phi(z)| : z \in S\} \tag{4.4}$$

where $C = C_J(C_0)$ is given by Lemma 4.5, the supremum is taken over the grid used for the interpolating polynomial, and R_S is $R_{k,k}$.

A similar statement is valid for $S = S_{k,k+1}$.

Consecutive interpolations

Take $k \geq k_1$ and let P and P', denote respectively the polynomials of degree n in x and y which interpolate ϕ on $S_{k,k}$, respectively $S_{k,k+1}$. We denote their coefficients by c_{pq}, respectively $c_{pq}^{'}$. Recall that $T_k := \max\{|r_k|, |s_k|\}$. The results of this section and the next will show that

$$|c'_{pq} - c_{pq}| \quad \leq \quad O(T_k^{n+\alpha-p-q}).$$

By (4.3) applied for $P - P'$, it is enough to obtain an upper bound for $|P' - P|$ on $S_{k,k+1}$. To obtain the above estimate it would suffice to prove that if $(x,y) \in S_{k,k+1}$ then

$$|P'(x,y) - P(x,y)| = O(T_k^{n+\alpha}).$$

Instead, we prove in this subsection relation (4.10), which implies the intermediate result (see relation (4.11) below)

$$\sum_{p,q} |c_{p,q}^{'} - c_{p,q}| T_k^{p+q} \leq C(T_k^{n+\alpha} + \delta \sum_{p+q>n} |c_{p,q}| T_k^{p+q} + \sum_{p+q\leq n} |c_{p,q}| T_k^{n+\alpha}).$$

First note that P, P' agree on $S_{k,k+1}$ except on the n points $z_{k',k+n}$, $k \leq k' < k+n$. But on these points, by construction, $P'(z_{k',k+n}) = \phi(z_{k',k+n})$. So in fact we need only estimate $|\phi(z_{k',k+n}) - P(z_{k',k+n})|$ for $k \leq k' \leq k+n$.

For k' as above, parametrize $W_\varepsilon^u(r_{k'}, 0)$ by the y coordinate. We denote a point on $W_\varepsilon^u(r_{k'}, 0)$ by $z_{k'}(y) = (x_{k'}(y), y)$.

Consider the interval $I_k := [-C_2 T_k, C_2 T_k]$, where the constant C_2, independent of k, is chosen so that the region of the unstable leaf parametrized by I_k contains all the points in $(W_\varepsilon^u(r_{k'}, 0) \cap S_{k,k}) \cap \mathcal{K}$ for all $k \geq k_1$ and $k \leq k' < k+n$.

By the hypothesis, there is a uniformly-$C^{n,\alpha}$ extension of ϕ to the local unstable leaves of Λ. We refer to this extension whenever we evaluate ϕ outside the set Λ. We will show that $|\phi(z_{k'}(y)) - P(z_{k'}(y))| = O(T_k^{n+\alpha})$ for $k \leq k' \leq k+n$ and $y \in I_k$.

Fix k' between k and $k+n$. To simplify the notation, denote by a tilde the functions evaluated on $W_\varepsilon^u(z_{k'})$ via its parametrization $y \mapsto z_{k'}(y)$. That is, $\widetilde{f}(y) = f(z_{k'}(y))$. If not explicitly stated, these functions have domain I_k.

We collect the necessary estimates in the following lemma:

Lemma 4.8 *There is a $C > 0$ such that if $k \geq k_1$, $k \leq k' \leq k+n$, and $y \in [-C_2 T_k, C_2 T_k]$, then:*

(i)

$$|(\widetilde{\phi} - \widetilde{P})(y)| \leq C T_k^{n+\alpha} \| \frac{d^n}{dy^n} (\widetilde{\phi} - \widetilde{P}) \|_\alpha. \tag{4.5}$$

(ii) If $p, q \leq n$ and $p + q > n$ then

$$\| \frac{d^n}{dy^n} x_{k'}^p(y) y^q \|_\alpha \leq C T_k^{p+q-n-\alpha} \; \|x_{k'}\|_{C^{n,\alpha}(I_k)}$$

(iii) If $p+q \leq n$ then

$$\|\frac{d^n}{dy^n} x_{k'}^p(y) y^q\|_\alpha \leq C$$

(iv) Therefore

$$\|\frac{d^n}{dy^n}\widetilde{P}\|_\alpha \leq C\|x_{k'}\|_{C^{n,\alpha}(I_k)} \sum_{p+q>n} |c_{p,q}| T_k^{p+q-n-\alpha} + C \sum_{p+q\leq n} |c_{p,q}| \tag{4.6}$$

Proof: To prove (i), note that $\phi - P$ is a $C^{n,\alpha}$ function along $W^u_{\mathcal{E}}(r_{k'},0)$ and has $(n+1)$ zeroes in the image of the interval I_k. Thus each derivative $(\widetilde{\phi} - \widetilde{P})^{(j)}$, $j = 0, \ldots, n$, has at least one zero, denoted by t_j, in this interval. Hence,

$$|\frac{d^n}{dy^n}(\widetilde{\phi} - \widetilde{P})(t)| = |\frac{d^n}{dy^n}(\widetilde{\phi} - \widetilde{P})(t) - \frac{d^n}{dy^n}(\widetilde{\phi} - \widetilde{P})(t_n)| \leq \|\frac{d^n}{dy^n}(\widetilde{\phi} - \widetilde{P})\|_\alpha (C_2 T_k)^\alpha$$

and similarly

$$|\frac{d^j}{dy^j}(\widetilde{\phi} - \widetilde{P})(t)| = |\int_{t_j}^t \frac{d^{j+1}}{dy^{j+1}}(\widetilde{\phi} - \widetilde{P})(u) du| \leq \|\frac{d^{j+1}}{dy^{j+1}}(\widetilde{\phi} - \widetilde{P})\|_{C^0(I_k)} C_2 T_k.$$

These imply (4.5).

For (ii), notice that $\frac{d^n}{dy^n} x_{k'}^p(y) y^q$ is the sum of terms of the form $\mathcal{D} x_{k'}^{p'} y^{q'}$, where $\mathcal{D}$ is a product of differentiated $x_{k'}$-terms (if any), and $p' + q' \geq p + q - n$. The Hölder norm of such a term can be bound by

$$\begin{aligned} \|\mathcal{D} x_{k'}^{p'} y^{q'}\|_\alpha \leq{}& \|\mathcal{D}\|_\alpha \|x_{k'}^{p'}\|_{C^0} \|y^{q'}\|_{C^0} \\ &+ \|\mathcal{D}\|_{C^0} \|x_{k'}^{p'}\|_\alpha \|y^{q'}\|_{C^0} + \|\mathcal{D}\|_{C^0} \|x_{k'}^{p'}\|_{C^0} \|y^{q'}\|_\alpha. \end{aligned} \tag{4.7}$$

We will not write explicitly the uniform bound over the set Λ of the $C^{n,\alpha}$-norm of $x_{k'}$ (this is determined by the uniform lamination W^u).[1] Note that $|x_{k'}(0)| \leq T_k$, hence $\|x_{k'}\|_{C^0(I_k)} \leq C T_k$ (because its derivative is bounded). Also, $\|f\|_\alpha \leq C \|f'\|_{C^0(I_k)} T_k^{1-\alpha}$ for a differentiable function $f : I_k \to \mathbb{R}$. Thus (trivially for $p' = 0$, $q' = 0$, or $\mathcal{D}$ a constant):

$$\begin{aligned} &\|x_{k'}^{p'}\|_\alpha \leq C \|x_{k'}'\|_{C^0(I_k)} T_k^{p'-\alpha} && \|x_{k'}^{p'}\|_{C^0} \leq C T_k^{p'} \\ &\|y^{q'}\|_\alpha \leq C T_k^{q'-\alpha} && \|y^{q'}\|_{C^0} \leq C T_k^{q'} \\ &\|\mathcal{D}\|_\alpha \leq C \|\frac{d^n}{dy^n} x_{k'}\|_\alpha + C \|x_{k'}\|_{C^n(I_k)} T_k^{1-\alpha} \end{aligned}$$

These estimates prove that the first term in the right-hand side of (4.7) is bounded by $C T_k^{p+q-n-\alpha} \; \|x_{k'}\|_{C^{n,\alpha}(I_k)}$, as desired. If $\mathcal{D}$ is not a constant then $\|\mathcal{D}\|_{C^0} \leq$

[1] Here the uniform smoothness of the leaves is used.

$C\|x_{k'}\|_{C^n(I_k)}$, and we obtain the desired bound for the other two terms in the right-hand side of (4.7). If $\mathcal{D}$ is a constant, then we must have had $q = n$, $q' = 0$, $p = p'$, hence the Hölder norm in the third term of (4.7) is zero, and the second term satisfies the desired bound (in this case the first term vanished as well).

The relation (iii) is proven similarly.

Relation (iv) is an immediate consequence of the estimates (ii) and (iii). ∎

Next, we bound the $\|x_{k'}\|_{C^{n,\alpha}(I_k)}$ term by choosing $\varepsilon_1 > 0$ sufficiently small. Since the local (un)stable manifolds are continuous in the $C^{n,\alpha}$ topology and the leaf through the origin coincides with the vertical axis, given $\delta > 0$, we may choose $\varepsilon_1 > 0$ sufficiently small so that

$$\|x_{k'}\|_{C^{n,\alpha}(I_k)} < \delta \tag{4.8}$$

whenever $|r_{k'}| < \varepsilon_1$ and $y \in I_k$.[2]

Given our choice of the sequence of points r_k and s_ℓ, we may increase k_1 so that (4.8) holds for all $k \geq k_1$ (recall that r_{k_1} is at distance approximately ω^{2k_1} from the origin).

Since, by hypothesis, $\phi \in C^{n,\alpha}(W^u_\varepsilon(r_{k'}, 0))$ uniformly[3], we have

$$\|\frac{d^n}{dy^n}\widetilde{\phi}(y)\|_\alpha \leq C. \tag{4.9}$$

From (4.5), (4.9), (4.6) we obtain that

$$|(\phi - P)(z_{k'}(y))| \leq CT_k^{n+\alpha} + C\delta \sum_{p+q>n} |c_{p,q}|T_k^{p+q} + C\sum_{p+q\leq n} |c_{p,q}|T_k^{n+\alpha} \tag{4.10}$$

for $y \in I_k$.

By evaluating the above relation at $z_{k',k+n}$ (recall that $P'(z_{k',k+n}) = \phi(z_{k',k+n})$), and using (4.3) for $P - P'$ on $S_{k,k+1}$, we obtain that

$$\sum_{p,q} |c'_{p,q} - c_{p,q}|T_k^{p+q} \leq C(T_k^{n+\alpha} + \delta \sum_{p+q>n} |c_{p,q}|T_k^{p+q} + \sum_{p+q\leq n} |c_{p,q}|T_k^{n+\alpha}). \tag{4.11}$$

Convergence of interpolating polynomials

We let P_{2k} denote the interpolation polynomial corresponding to the grid $S_{k,k}$, and P_{2k+1} denote the interpolation polynomial corresponding to the grid $S_{k,k+1}$. Equation (4.11) relates the coefficients of P_{2k+1} to P_{2k} and the same line of reasoning relates the coefficients of P_{2k+2} to P_{2k+1} except that we use the smoothness of ϕ along the other lamination. Recall (see the properties of the grid listed on page 11) that $T_k = \max\{r_k, s_k\}$ is comparable to ω^{2k}. We will be sloppy with notation and let $T_{j/2}$ denote $T_{[j/2]}$.

Denote by c^m_{pq} the coefficient of $x^p y^q$ in P_m.

[2]Here the transverse $C^{n,\alpha}$-continuity of the lamination is used.

[3]This is where the uniform smoothness of the restrictions of ϕ to the leaves is used.

We will show that, by reducing ε_1, there are K, m_0 large enough, such that

$$|c_{pq}^m| \;\leq\; K \sum_{j=m_0}^{m-1} (T_{j/2})^{n+\alpha-p-q} \text{ for } \; m \geq m_0 \tag{4.12}$$

and

$$|c_{pq}^{m+1} - c_{pq}^m| \leq K T_{m/2}^{n+\alpha-p-q} \text{ for } \; m \geq m_0. \tag{4.13}$$

We proceed by induction. We first determine when (4.12) for m implies (4.12) for $m+1$. We claim that this implication (which is a consequence of equation (4.14) below) holds for all $K \geq K^*$, $m_0 \geq m_0^*$, $\delta \leq \delta^*$, where the *-ed values depend only on the constant C that appears in equation (4.11). We saw that δ can be made as small as desired by reducing ε_1. Next, we pick $m_0 \geq \max\{m_0^*, k_1\}$ (recall that k_1 depends on ε_1). Finally, we can further increase K in order to satisfy (4.12) for $m = m_0 + 1$, the initial step of the induction.

We now justify our claim. From equation (4.11) and the bound $K \sum_{j=m_0}^{m-1} (T_{j/2})^{n+\alpha-p-q}$ for $|c_{pq}^m|$ given by the induction assumption, relation (4.12) will hold for $m+1$ provided

$$C \left(T_{m/2}^{n+\alpha} + K\delta \sum_{p+q>n} \sum_{j=m_0}^{m-1} (T_{j/2})^{n+\alpha-p-q} T_{m/2}^{p+q} \right.$$
$$\left. + K \sum_{p+q\leq n} \sum_{j=m_0}^{m-1} (T_{j/2})^{n+\alpha-p-q} T_{m/2}^{n+\alpha} \right) \leq K T_{m/2}^{n+\alpha}. \tag{4.14}$$

Note that, by (4.11), (4.14) implies (4.13) as well.

The first term in (4.14) is less than $(1/3) K T_{m/2}^{n+\alpha}$ if K is chosen large enough. Consider the second term divided by the right-hand side, $K T_{m/2}^{n+\alpha}$:

$$\begin{aligned}
&\frac{CK\delta}{(K T_{m/2}^{n+\alpha})} \sum_{p+q>n} \sum_{j=m_0}^{m-1} (T_{j/2})^{n+\alpha-p-q} T_{m/2}^{p+q} \\
&\qquad = C\delta \sum_{p+q>n} \sum_{j=m_0}^{m-1} \left(\frac{T_{m/2}}{T_{j/2}} \right)^{p+q-n-\alpha} \\
&\qquad \leq C\delta \sum_{p+q>n} \sum_{u=0}^{m-m_0} (\omega^u)^{p+q-n-\alpha}
\end{aligned}$$

By taking $\delta > 0$ small enough, we can make this quantity less then $1/3$ (note that the last sum converges) and therefore bound the second term by $(1/3) K T_{m/2}^{n+\alpha}$. The third term can also be bounded by a geometric series whose sum is $CKT^{n+\alpha}_{m/2}\sum_{p+q\leq n}T^{n+\alpha-p-q}_{m_0/2}$, therefore, taking m_0 sufficiently large, we may ensure that this term is less than $(1/3)KT^{n+\alpha}_{m/2}$ as well.

This proves our claim about the values of K, m_0 and δ for which equation (4.14) holds, and therefore completes the proof of relations (4.12) and (4.13).

End of proof of Lemma 4.4

By now the constants K, m_0, are determined, and uniform on Λ. Reduce once more ε_1 so that $k_1 \geq m_0$ (recall that $\varepsilon_1 = O(\omega^{k_1})$).

We describe the coefficients ℓ_{pq} of the polynomial Q of degree n mentioned in Lemma 4.4. Recall that $P_m(x,y) = \sum_{p,q} c^m_{pq}x^p y^q$. If $p+q > n$ then we set $\ell_{pq} = 0$ whereas if $p+q \leq n$ then we define

$$\ell_{pq} = \lim_{m\to\infty} c^m_{pq}.$$

The limit exists by (4.13).

We claim that, for any $C^{(1)} > 0$, there is $C^{(2)} > 0$, such that for all $m \geq m_0$,

$$|Q - P_m| \leq C^{(2)}T^{n+\alpha}_{m/2} \tag{4.15}$$

provided $|x|, |y| \leq C^{(1)}T_{m/2}$.

By (4.12), if $p+q > n$ then $|c^m_{pq}T^{p+q}_{m/2}| = O(T^{n+\alpha}_{m/2})$. If $p+q \leq n$, by (4.13) we obtain that

$$|c^m_{p,q} - c^{m+k}_{p,q}| \leq K\sum_{j=m}^{m+k-1} T^{n+\alpha-p-q}_{j/2}$$

and hence letting $k \to \infty$

$$\begin{aligned}|(c^m_{p,q} - \ell_{p,q})T^{p+q}_{m/2}| &\leq KT^{n+\alpha}_{m/2}\sum_{j=0}^{\infty}\left(T_{(m+j)/2}/T_{m/2}\right)^{n+\alpha-p-q}\\ &\leq CKT^{n+\alpha}_{m/2}\sum_{j=0}^{\infty}(\omega^j)^{n+\alpha-p-q}\end{aligned}$$

which is $O(T^{n+\alpha}_{m/2})$. These estimates imply the validity of equation (4.15).

In addition to the bound obtained above for $|c^m_{pq}T^{p+q}_{m/2}|$, $p+q > n$, from (4.12) one also obtains upper bounds for $|c^m_{pq}|$, $p+q \leq n$. Note that all these hold uniformly on Λ for $m \geq m_0$. Using these, relation (4.10) implies that

$$|(\phi - P_m)(z_{k'}(y))| \leq C^{(3)}T^{n+\alpha}_{m/2} \tag{4.16}$$

if $m \geq m_0$, $m/2 \leq k' \leq m/2 + n$ and $y \in I_m = [-C_2T_{m/2}, C_2T_{m/2}]$.

By the triangle inequality, (4.15) and (4.16) imply $|(Q-\phi)(z_{k'}(y))| \le C^{(4)} T_{m/2}^{n+\alpha}$ if $|y| \le C_2 T_{m/2}$. Thus, we obtain that

$$|Q(w) - \phi(w)| \le C_1 |w|^{n+\alpha} \quad \text{for} \quad w \in \mathcal{K} \cap (\cup_{k \ge m_0} W^u_\varepsilon(r_k, 0)) .$$

In order to finish the proof, notice that by the relabeling introduced in Notation 4.6, we can include any point $w \in \mathcal{K} \cap \Lambda \cap B_{\varepsilon_1}$ in the above union. For $w, w' \in \mathcal{K} \cap \Lambda \cap B_{\varepsilon_1}$, we obtain two pairs of sequences $\{r_k, s_k\}_{k \ge m_0}$ on the stable, respectively unstable, manifold of the origin. But these sequences differ only at finitely many positions. Therefore, the limit polynomial Q does not depend on the point w. ∎

(c) Whitney regularity

We prove here the following:

Theorem 4.9 *The conclusion of Theorem 4.3 and our homogeneity assumptions on the set Λ imply that ϕ is $C^{n,\alpha}(\Lambda)$ in the Whitney sense.*

We will show that functions $\phi^{(\ell)}$, derived from the approximating polynomials $Q(q;p)$ defined on the closed set Λ satisfy conditions (16) and (17) of [18, §VI.2.3] and hence by [18, §VI.2.3, Theorem 4] ϕ admits a Whitney extension to $\mathbb{R}^2$.

According to [18, §VI.2.3, Theorem 4], the conclusion of Theorem 4.9 follows from Lemma 4.10 below. We begin by introducing more notation, relying on the polynomials $Q(q;p)$ constructed in Theorem 4.3.

Define $\phi^{(\ell)} : \Lambda \to \mathbb{R}$ by

$$Q(q;p) = \sum_{|\ell| \le n} \phi^{(\ell)}(p) \frac{(q-p)^\ell}{\ell!},$$

where $\ell = (\ell_1, \ell_2, \dots, \ell_d)$ is a multi-index, $\ell! = \ell_1! \dots \ell_d!$, $|\ell| = \ell_1 + \cdots + \ell_d$ and $x^\ell = x_1^{\ell_1} \dots x_d^{\ell_d}$ for $x = (x_1, \dots, x_d)$. Here $d = 2$. Note that $\phi^0 = \phi$.

For a multi-index j with $|j| \le n$, define the polynomials

$$Q_j(q,p) := \sum_{|\ell + j| \le n} \phi^{(\ell + j)}(p) \frac{(q-p)^\ell}{\ell!}.$$

Note that $Q_0(q;p) = Q(q;p)$. Let $R_j(q;p) = \phi^{(j)}(q) - Q_j(q;p)$ for $|j| \le n$.

Lemma 4.10 *Under the assumptions of Theorem 4.9,*

$$|R_j(q;p)| \le C|q-p|^{n+\alpha-|j|} \quad \textit{for } p, q \in \Lambda,\ |q-p| \le \varepsilon'/2$$

for all multi-indexes $|j| \le n$.

Proof: Note that the case $|j| = 0$ is exactly the conclusion of Theorem 4.3.

By [18, Lemma VI.2.3.1], if $a, b \in \Lambda$, then

$$Q(x;b) - Q(x;a) = \sum_{|j| \le n} R_j(b;a) \frac{(x-b)^j}{j!}.$$

For $x \in \Lambda$ then $Q(x;b) - Q(x;a) = [\phi(x) - R_0(x;b)] - [\phi(x) - R_0(x;a)] = R_0(x;a) - R_0(x;b)$, and we obtain

$$\sum_{0<|j|\le n} R_j(b;a) \frac{(x-b)^j}{j!} = F_{a,b}(x), \tag{4.17}$$

where $F_{a,b} : \Lambda \to \mathbb{R}$, $F_{a,b}(x) := R_0(x;a) - R_0(x;b) - R_0(b;a)$. That is, the values $R_j(b;a)$ are the coefficients of a polynomial of degree n interpolating the function $F_{a,b} : \Lambda \to \mathbb{R}$.

Such a polynomial is uniquely determined if relation (4.17) holds at $(n+1)^2$ points $x \in \Lambda$ spaced as in Lemma 4.5 of Journé, and relation (4.3) of that lemma gives the desired upper bound. We describe next the details.

Fix $a, b \in \Lambda$. By our assumptions on Λ, there are points $x_i \in W^s_\varepsilon(a) \cap \Lambda$, $y_i \in W^u_\varepsilon(a) \cap \Lambda$, $0 \le i \le n$, such that

$$|a-b|/C \le \min_{i \ne j} |x_i - x_j| \qquad |a-b|/C \le \min_{i \ne j} |y_i - y_j|$$

$$\max |x_i - a| \le |a-b|/2 \qquad \max |y_i - a| \le |a-b|/2.$$

Consider the grid (centered at b) determined by $z_{i,j} = W^u_\varepsilon(x_i) \cap W^s_\varepsilon(y_j) \in \Lambda$. Then $\max\{|a - z_{i,j}|, |b - z_{i,j}|\} \le C|a-b|$, and there is a uniform (with respect to $|a-b|$) bound on the "R/η" of Journé (because both R and η are of order $|a-b|$). By Theorem 4.3, the former property also implies that $|F_{a,b}(z_{i,j})| \le C|b-a|^{n+\alpha}$. Lemma 4.5 can be applied to this interpolating grid. Inequality (4.3) becomes

$$\sum_{0<|j|\le n} |R_j(b;a)| |a-b|^{|j|} \le C|a-b|^{n+\alpha},$$

which gives the desired conclusion. ∎

5 Sets in higher dimensions

Most of our discussion has concerned dimension two, where the homogeneity condition is automatically satisfied for hyperbolic basic sets. It is possible to give geometric conditions on sets in dimensions $d \ge 3$ for the analog of our results to hold: for example a set in $\mathbb{R}^d$ which is the direct product of d one-dimensional (γ, ν) homogeneous sets.

Unfortunately we do not know if these are natural conditions for hyperbolic basic sets in higher dimensions or for hyperbolic sets arising out of, for instance, transverse homoclinic intersections in $\mathbb{R}^d$.

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