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INERTIALITY IMPLIES THE LORENTZ GROUP

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ABSTRACT. In his seminal paper of 1905, Einstein derives the Lorentz group as being the coordinate transformations of Special Relativity, under the main assumption that all inertial frames are equivalent. In that paper, Einstein also assumes the coordinate transformations are linear. Since then, other investigators have weakened and varied the linearity assumption. In the present paper, we retain only the inertiality assumption, and do not even assume that the coordinate transformations are continuous. Linearity is deduced.

Our result is described in the affine space, $\mathbb{R}^{n+1}$, with coordinates $x^0, x^1, \ldots, x^n$. Using the notation $t = x^0$ and $y = (x^1, \ldots, x^n)$, the slope of a line in $\mathbb{R}^{n+1}$ is defined to be $|\Delta y/\Delta t|$, computed from any two points on the line. The slope is non-negative and possibly infinite. A line in $\mathbb{R}^{n+1}$ is said to be *time-like* if the slope of the line is strictly less than 1. Since inertial frames agree on who is inertial, coordinate transformations must carry time-like lines to time-like lines. A bijection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ is said to be *time-like* if it maps any time-like line onto another time-like line. The bijection is not assumed to be continuous. This paper proves that a time-like bijection *is* continuous (in fact, affine linear). The bijection is said to be *strictly time-like* if both it and its inverse are time-like. It is elementary to deduce that the strictly time-like bijections form the group generated by the extended Poincaré group and the dilations.

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1. Introduction

In his seminal paper [2], Einstein derives the Lorentz group as being the coordinate transformations of special relativity, under the main assumption that all inertial frames are equivalent. In that paper, Einstein also assumes (ibid. §3, page 898) that the transformations are linear. Since then, other investigators have weakened and varied the linearity assumption. Some of these results are discussed at the end of this section. In the present paper, we retain only the inertiality assumption, and do not even assume that the coordinate transformations are continuous. Linearity is deduced.

The results of this paper are valid for space-time of any dimension, with the proofs being identical to the 4-dimensional case. Accordingly, we consider an n-dimensional Euclidean [1] space, $n \geq 1$. The $n+1$ dimensional affine space, $\mathscr{S}$, is obtained by considering time to be an additional dimension. We use the notation $x = (x^0 \ldots x^n)$ where component x^0 is time. A point in $\mathscr{S}$ is an *event*.

In an observer's frame of space-time, $\mathscr{S}$, the path of another inertial observer is a straight line, which, parametrized by time, t, has the form $x_0 + (t, tv)$, where x_0 is the event on the line at $t = 0$, and v is the velocity of the other observer.

Definition 1.1. *For a given point $x \in \mathscr{S}$, let $t = x^0$ and $y = (x^1 \ldots x^n)$. The* slope *of a line $L \subset \mathscr{S}$ is defined as $|\Delta y/\Delta t|$, computed from any two points on the line.*

The slope of a line is non-negative and possibly infinite. When the line is the world line of an object, the slope of the line is the speed of the object, and corresponds to a speed less than c; cf [8]. Also, as is customary, this upper bound, c, for speeds, is taken to be of unit value.

Definition 1.2. *A line in $\mathscr{S}$ is said to be* time-like *if the slope of the line is strictly less than 1.*

The paths of inertial observers are time-like lines. For each point $p \in \mathscr{S}$, the set of time-like lines through p is called the time-like cone at p.

All observers are measuring the same real world events, so between any two reference frames there is a mapping of events, a coordinate transformation, which is necessarily a bijection of $\mathscr{S}$ with itself.

All inertial frames see each other as inertial, so the coordinate transformation between inertial frames must convert time-like lines into time-like lines. This motivates the definition

Definition 1.3. *A bijection, f, of $\mathscr{S}$, is said to be* time-like *if for any time-like line $L \subset \mathscr{S}$, it is also the case that $f(L) \subset \mathscr{S}$ is a time-like line.*

It is convenient, at times, to use the notation, $\mathscr{S}'$, to denote the range of f, but this is purely for didactic reasons.

The definition requires that a time-like bijection takes a time-like line onto an entire time-like line, and is not just a subset of the second line.

The main technical result of this paper is

Theorem 1.1. *A time-like bijection of $\mathscr{S}$ is necessarily an affine map.*

The proof is the subject of §3.

[1]By *Euclidean*, we mean a vector space over the real numbers with the usual inner product, $x \cdot y = \sum x^i y^i$, and the associated norm, $|x| = \sqrt{x \cdot x}$.

Example 1.1. *The affine map $f(t,y) = (t, y/2)$, of 2-dimensional space-time, is time-like since*

$$t^2 - y^2 > 0 \Longrightarrow t^2 - y^2/4 > 0 .$$

However, it is clear that f^{-1} is not time-like.

The above example motivates the

Definition 1.4. *A time-like bijection of $\mathscr{S}$ is said to be* strictly time-like *if also the inverse mapping is time-like.*

The *extended inhomogeneous Lorentz group* is described in [5]. We refer to the inhomogeneous Lorentz group as the *Poincaré group* cf [7], §4.

Theorem 1.2. *The set of strictly time-like bijections is generated (as a group) by the dilations and the extended Poincaré group.*

The proof is the subject of §4.

We emphasize that, in the above results, continuity of the transformation is not hypothesized, but is a consequence of Theorem 1.1. Moreover, we do not assume that there is anything (such as light) which travels at the bounding unit speed.

In §2.17 of [8] it is noted that linearity may be derived from the same hypotheses. However, some form of continuity is assumed; for example, in [3], the transformation is assumed to be twice differentiable.

A result, similar to the present paper's, was proved by Zeeman [10]:

> *A bijection, f, of space-time is said to be* causal *if, for all points x, y in space-time, $y - x$ is time-like and forward-pointing if and only if $f(y) - f(x)$ is also time-like and forward-pointing. Zeeman shows that the causal transformations are generated by the orthochronous Lorentz group, translations and dilations.*

Zeeman's hypotheses and conclusion are similar to the present paper's Theorem 1.2; the differences are worth noting:

- Zeeman's proof is by induction on the dimension of space-time. For the present paper, the proof for 2-dimensional space-time is the main step in the exposition. Interestingly, Zeeman notes that his result is not valid for a space-time of 2 dimensions, but requires the dimension to be at least 3.
- A priori, Zeeman's hypotheses map the forward and reverse-pointing time-like cones at x respectively to the forward and reverse-pointing time-like cones at $f(x)$ for each event x, whereas the present paper's Theorem 1.2 requires that time-like lines be mapped to time-like lines. It is interesting to note that Zeeman's assumption of the preservation of forward-pointing and reverse-pointing time-like cones readily shows that a causal bijection is Euclidean-continuous, since these two sets of cones together form a sub-basis of the Euclidean topology in space-time. For the present paper, however, continuity can only be deduced at the end of the main exposition.

Section 2 contains background material for the presentation of the paper.

2. Background

The time-like cone about the origin of $\mathscr{S}$ is given by the well-known inequality

$$(x^0)^2 - \sum_{i=1}^{n} (x^i)^2 > 0 . \tag{1}$$

We use the diagonal matrix

$$Q = \begin{pmatrix} 1 & 0\dots 0 \\ 0 & \\ \vdots & -I \\ 0 & \end{pmatrix}$$

to express Equation 1 as $x^t Qx > 0$. Here, I is the $n \times n$ identity matrix.

We use short hand notation to denote $Q(x) = x^t Qx$.

A null line represents a speed of exactly 1. The set of null lines through a point $p \in \mathscr{S}$ is called the null cone at p. The null cone is the boundary of the time-like cone. The null cone about the origin has the equation $Q(x) = 0$.

2.1. **Quadrics.** A *quadric* is a zero set of the form $x^t Rx = 0$ where R is a symmetric matrix. An example is the null cone about the origin, which is described using the matrix Q.

We prove, next, a special case of a general result. The proof is computational. To arrive at a more elegant proof requires an overhead which cannot be justified by the scope of this paper.

Lemma 2.1. *If the quadric $x^t Rx = 0$ contains the null cone about the origin, then the matrix R is a scalar multiple of the matrix Q.*

Proof. Let $\alpha = R_{0,0}$. We will show that $R = \alpha Q$. Let $e_0, e_1, \dots e_n$ be the standard basis for $\mathscr{S}$ i.e e_i has 0 for every component except the i^{th}, which is 1.

Use the elements

$$e_0 + e_i \text{ and } e_0 - e_i \ , i > 0 \ ,$$

both in the null cone, and hence both also in the quadric R, to show that $R_{i,0} = R_{0,i} = 0$ and $R_{i,i} = -\alpha$. If $n = 1$, there is nothing more to do.

For $n > 1$, in a similar way, use the elements

$$\sqrt{2}e_0 + e_i + e_j \ , 0 < i < j$$

to show that all off-diagonal terms of R are 0, which proves the lemma. ☐

2.2. **Affine Transformations.** An affine subspace of a vector space is defined as being a translation of a vector subspace.

Points $x_0 \dots x_k$ in an affine space are said to be *affinely independent* if the points $x_1 - x_0, \dots x_k - x_0$ are linearly independent (in the containing vector space). In this way, many results for affine spaces are reduced to the more familiar results for vector spaces.

A mapping of an affine space into a vector space $f : A \to W$ is defined to be an *affine* map if for each $x, y \in A$ and scalar r, it is true that

$$f(rx + (1-r)y) = rf(x) + (1-r)f(y) \ . \tag{2}$$

The proof of the following lemma is essentially a restatement of the definition of an affine map.

Proposition 2.2. *Let $f : A \to W$ be* any *function from an affine space to a vector space. Suppose that for every line $L \subset A$ it is true that $f|_L: L \to W$ is an affine map. Then f itself is an affine map.*

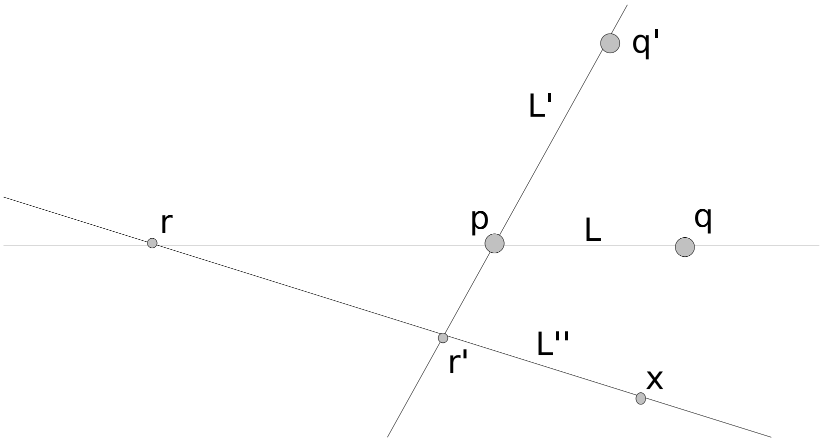

FIGURE 1. The plane P

3. TIME-LIKE BIJECTIONS

In this section, we prove our main technical result, described in §1:
Theorem 1.1 *A time-like bijection of $\mathscr{S}$ is necessarily an affine map.*

We adopt the notation that $f\colon \mathscr{S} \to \mathscr{S}'$ is a time-like bijection and begin by stating a lemma, whose proof is the subject of the next subsection, which is basically the case of $n = 1$ ($\dim(\mathscr{S}) = 2$).

Lemma 3.1. *Let $P \subset \mathscr{S}$ be any plane which contains a time-like line of $\mathscr{S}$. Then $f|_P\colon P \to \mathscr{S}'$ is an affine map.*

Assuming the validity of this lemma, the proof of Theorem 1.1 is very short:

Let $L \subset \mathscr{S}$ be any line. Let $p \in L$ be any point, and L' the line through p parallel to the time axis. Since L' represents a stationary observer, L' is time-like. Let P be a plane containing L and L' (P is unique if $L \neq L'$). By Lemma 3.1, $f|_P$ is an affine map, so that, also $f|_L$ is affine. We deduce from Proposition 2.2 that f, itself, is affine, which proves Theorem 1.1.

3.1. **Proof of Lemma 3.1.** Let L be a time-like line in P, and let p be any point on L. Rotate L in P around p by a small enough amount so that the resulting line L' is also time-like; cf Figure 1.

Choose points $q \in L$ and $q' \in L'$, distinct from p. In $\mathscr{S}'$, the 3 points $f(p)$, $f(q)$ and $f(q')$ span a plane $P' \subset \mathscr{S}'$.

We now show that $f(P) \subset P'$. First of all, $f(L)$ is a line containing $f(p)$ and $f(q)$, so $f(L) \subset P'$. Similarly, $f(L') \subset P'$. Now, let x be a point of $P \setminus (L \cup L')$. Choose a time-like line L'' through x which meets the lines L and L' at distinct points r and r' (For example, choose L'' to be almost parallel to L). It follows that both $f(r)$ and $f(r')$ are in P', so that $f(L'') \subset P'$ and $f(x) \in P'$. This shows that $f(P) \subset P'$.

For the rest of the proof of Lemma 3.1, we restrict our attention to the plane P. The 3 points p, q, q' are affinely independent (cf §2.2), so we may construct an affine map

$g : P \to P'$ by defining

$$g(p) = f(p),\ g(q) = f(q),\ g(q') = f(q').$$

Moreover, also the 3 points $f(p), f(q), f(q')$ are affinely independent, so that g is an isomorphism of affine spaces. The composition $h = g^{-1} \circ f\,|_P$ is a 1-1 map from P to P which fixes p, q and q', and which maps time-like lines onto lines.

We next show that h fixes *every* point of P. It will then follow that $f\,|_P = g$ is an affine map. We first prove a lemma which is used repeatedly through the rest of this section.

Lemma 3.2. *Let A be a time-like line contained in the plane P. Suppose that A is invariant under h i.e. $h(A) \subset A$. Let B be a time-like line in P parallel to A. Then $h(B)$ is also parallel to A.*

Proof. First note that, in fact, $h(A) = A$, since f is time-like, and g is an affine bijection of planes. If $B = A$, the result is evidently true, so assume that $B \neq A$, so that, in fact, B and A are disjoint. Since h is a 1-1 map, $h(B)$ must be disjoint from $h(A)$ i.e. $h(B)$ is parallel to $h(A) = A$. □

Since $h(p) = p$ and $h(q) = q$, it follows that $h(L) = L$ (this is not to say (yet) that h fixes each point of L, but rather that h maps the line L onto itself). It follows that there is a function $\lambda : \mathbb{R} \to \mathbb{R}$ which satisfies

$$h(p + s(q - p)) = p + \lambda(s)(q - p)\ .$$

We say that $s \in \mathbb{R}$ is the coordinate of the point $p + s(q - p) \in L$. Furthermore, since h fixes p and q, it is also true that $\lambda(0) = 0$ and $\lambda(1) = 1$.

Similarly, $h(L') = L'$, and there is a function $\mu : \mathbb{R} \to \mathbb{R}$ which satisfies

$$h(p + t(q' - p)) = p + \mu(t)(q' - p)\ , \mu(0) = 0\ , \mu(1) = 1\ .$$

As was done for L, this defines coordinates on L'. (In this section, "t" is not time, but, rather, an arbitrary parameter).

Choose coordinates on P by using grid lines parallel to the lines L and L'. A point $x \in P$ has coordinates $(s,t) \in \mathbb{R}^2$ when the line through x parallel to L' meets L at the point s, and the line through x parallel to L meets L' at the point t. In this section, "slope" refers to the s-t coordinate system, where L is horizontal with 0 slope and L' is vertical with infinite slope. We may as well have chosen q' to be the image of q after rotating L, so that the lines of non-negative slope (including horizontal & vertical lines) are time-like lines (these lines are "in between" L and L').

Let $x \in P$ have coordinates (s,t). By Lemma 3.2, the line through x parallel to L is mapped by h to another line parallel to L, in fact, the one which meets L' at $\mu(t)$.

Similarly, the line through x parallel to L' is mapped by h to the line parallel to L' meeting L at $\lambda(s)$. It follows that

$$h(s,t) = (\lambda(s), \mu(t))$$

To complete the proof of the lemma, we will show that both λ and μ are the identity functions on $\mathbb{R}$.

For $s \in \mathbb{R}$, let L_s denote the (time-like) line through $(s,0)$ with slope 1.

Since L_0 passes through $(0,0)$ and $(1,1)$, and both these points are fixed by h, then $h(L_0) = L_0$. It follows, that since $h(s,s) = (\lambda(s), \mu(s))$, that $\lambda(s) = \mu(s)\ \forall s \in \mathbb{R}$.

It remains to show that λ is the identity function on $\mathbb{R}$. The demonstration of this proceeds in several steps, which we now list; the proofs follow the list.

(1) $h(L_s) = L_{\lambda(s)}$.
(2) $\lambda(-t) = -\lambda(t)$.
(3) $\lambda(s+t) = \lambda(s) + \lambda(t)$.
(4) $\lambda(st) = \lambda(s)\lambda(t)$.

L_0 is a fixed line of h, so that $h(L_s)$ is again parallel to L_0. Also, $h(L_s)$ contains the point $(\lambda(s), 0)$, which proves 1.

L_{s-t} contains the point (s,t), which is mapped by h to $(\lambda(s), \lambda(t))$, so that $h(L_{s-t}) = L_{\lambda(s)-\lambda(t)}$. Hence, by step 1, it follows that

$$\lambda(s-t) = \lambda(s) - \lambda(t)\ . \tag{3}$$

Put $s = 0$ to prove step 2, and then replace t by $-t$ in Equation 3 to prove step 3.

The line, E, through the origin and $(1,t)$ is mapped to the line, $h(E)$, through the origin and $(1, \lambda(t))$. Since the point $(s, st) \in E$, it follows that $(\lambda(s), \lambda(st)) \in h(E)$. Comparing slopes, we have that

$$\lambda(st) = \lambda(s)\lambda(t)$$

which proves step 4.

We now see that λ is a field homomorphism of $\mathbb{R}$, and is, therefor, the identity cf [9], Corollary 2.2.

This completes the proof of Lemma 3.1. □

4. Strictly Time-like Bijections

In this section, we prove the following theorem, which is described in §1.

Theorem 1.2. *The set of strictly time-like bijections is generated (as a group) by the dilations and the extended Poincaré group.*

According to Theorem 1.1, the time-like bijection, f, is an affine map. We will consider only linear maps, as the translation component of a coordinate transformation is not relevant to our analysis. Specifically, the origins of both frames coincide at time $t = 0$, i.e. $f(0) = 0$.

We use matrix notation, and write $f(x) = Ax$, where A is an invertible $n+1$ square matrix.

Let $-v$ be the velocity at which observer O' sees O moving. Then

$$A \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} t' \\ -t'v \end{pmatrix}$$

so that $t' = \gamma t$, where $\gamma = A_{0,0}$, and we may write

$$A = \begin{pmatrix} \gamma & w^t \\ -\gamma v & B \end{pmatrix} \tag{4}$$

where w is an n-vector, and B is an n-square matrix whose well-known decompositions will not be used in this paper.

Preservation by f of the time-like cone is expressed as

$$\begin{aligned} \forall x\ x^t Q x > 0 \quad &\Rightarrow \quad (Ax)^t Q A x > 0 \\ &\Rightarrow \quad x^t A^t Q A x > 0\ . \end{aligned}$$

By continuity, and the fact that $f^{-1}$is also a time-like bijection, we deduce that f preserves the null-cone, too.

$$\forall x\ x^t Q x = 0 \Rightarrow x^t A^t Q A x = 0\ .$$

Lemma 2.1 now implies that there is some $\alpha \in \mathbb{R}$ such that

$$A^t Q A = \alpha Q \ . \tag{5}$$

Using Equation 4, and inspecting the upper left coefficient of Equation 5, we see that

$$\gamma^2(1-v^2) \quad = \quad \alpha$$

It follows that $\alpha > 0$, so that Equation 5 may be rewritten as

$$(A/\sqrt{\alpha})^t QA/\sqrt{\alpha} = Q \ ,$$

and we see that $A/\sqrt{\alpha}$ belongs to the extended Lorentz group. This completes the proof of Theorem 1.2. $\square$

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