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EVALUATING FEYNMAN PATH INTEGRALS VIA THE POLYNOMIAL PATH FAMILY

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Abstract. We present a difficult and detailed calculation of propagators for quadratic potentials using a filtration of the space of variational paths around the classical path by the dense set of polynomial paths filtered by degree. This presents evidence for a cohesive explanation of path integrals as limits of finite-dimensional integrations using a sequence of filtration measures determined for specific dense families of paths by the requirement that the limit yield the free particle propagator.

Introduction

The main purpose of this paper is to present a detailed derivation of the propagator for the harmonic oscillator potential using an interpretation of the Feynman path integral as a limit of integrations over finite-dimensional families of polynomial variations of the classical sinusoidal path. It is known [2] that the Feynman path integral cannot be interpreted as a usual Lebesgue integral since there is no measure on the space of paths connecting two space-time points that has all the properties required to produce propagators for standard potentials. Feynman's solution for calculating path integrals was to approximate arbitrary paths by piecewise linear ones, or by trigonometric paths in case of quadratic potentials. Other methods of approximation have been used, usually with orthogonal families of approximating functions [8, p. 117] where the full power of orthogonality simplifies calculations. However, in these approaches there is no general theory that identifies appropriate conditions on approximating families that would guarantee which families will work, and which won't. Several other approaches to articulating a rigorous version of the path integral have appeared in the literature since 1960. Very sophisticated finite-dimensional approximation approaches that give rigorous computations for fairly general approximating sequences, under suitable assumptions on the potential in the Schrödinger equation, appear in [5] and [1]. An informative short review of five different approaches between 1960 and 1999 appears in [17]. In comparison to these previous approaches, ours is rather transparent and direct and, though

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yet to be generalized to cover a large class of potentials, yields a beautiful, if complicated, calculation of propagators for quadratic potentials using a filtration (see the definition subsequently) of the space of variational paths by the standard polynomial path family filtered by degree.

We begin by filling out a dense subset of the space of paths by the sequence of polynomial paths of bounded degree N, $N = 1, 2 \ldots$. We want to integrate over these finite dimensional families and take a limit as N increases without bound, but the problem is that we do not quite know the measures over which to integrate. Our strategy is essentially that of Feynman's in his approximation by piecewise linear families, viz, to backward engineer the measures on the filtration sequence by calculating the known free particle propagator as the limit of integrations over these finite-dimensional subsets of paths, using 'natural' measures defined with unknown normalization constants. These constants are then determined by the requirement that the final limit must yield the known free particle propagator as well as by a certain sense of elegance and aesthetics. It turns out that there is a pleasing way to define these normalization constants so that the procedure works to yield the free particle propagator. This of course is not much. We then apply the approach to calculate propagators of particles moving in a potential field, using the backward engineered measures for our integrations. These dense families of nice paths seem to capture the full effect of the whole path family, and the method works to produce rigorous calculations of propagators, at least for quadratic potentials.

It is a surprise that the general polynomial family works as there are no orthogonality conditions invoked. The calculations can get very complicated, much more so than some of the heuristic calculations physicists use. In fact, computations using Fourier families turn out to be much easier than those using polynomial families, primarily because of the orthogonality of the Fourier families. Were the point the ease of calculations, we might as well have abandoned this project from the beginning, for the calculations others use are easier than those presented in this paper using polynomial families. But the real point of this paper is that in verifying, through extremely tough and rigorous calculations, that correct propagators crystalize from computations with polynomial families, there is the strong hint of a deep theory underlying the calculations.

We begin in Section 1 with a brief review of Feynman's approach to the propagator via the path integral. This section is for the novice and may be skipped by the initiate. In Section 2 we use polynomial path families to calculate the free particle propagator, using the known propagator to determine normalizing constants for our measures. As a quick test case, we apply this to give a rigorous calculation of the propagator for a particle in a constant field. The primary goal of our efforts is realized in Section 3, where the propagator for the harmonic oscillator is worked out. This ends up being quite an interesting and lengthy computation with beautiful surprises along the way. Next, in Section 4, we apply the procedure using finite Fourier sums instead of polynomial families, recovering Feynman's original Fourier series computations from scratch. In a final section, Section 5, we discuss how general our approach may be and give hints for further developments, as well as try to isolate our view of the significant insights this interpretation affords.

1. The Propagator via the Feynman Path Integral

Following Felsager [7, pp. 41,42], we write the laws of quantum mechanics à la Feynman.

Axiom 1: To each event E is associated a complex number $\phi(E)$, the **probability amplitude** for that event. In the case of continuous spectra, this is to be interpreted as a probability amplitude density. The probability (density) for the event is $p(E) = |\phi(E)|^2$.

Axiom 2: If E may be realized as one of several independent, alternate events E_i, the total amplitude for E may be written as

$$\phi(E) = \phi(\vee_i E_i) = \sum_i \phi(E_i).$$

Axiom 3: If E may be decomposed into several individual steps E_j, the total amplitude may be written as

$$\phi(E) = \phi(\wedge_j E_j) = \prod_j \phi(E_j).$$

Simple enough, but the real problem is in interpreting Axiom 2 when the index parameter i is uncountably infinite rather than discrete. How is the sum to be interpreted? The immediate answer is as an integral, which works rigorously in some cases, but leads to great difficulties in others. Most would and do view these axioms of quantum mechanics as intuitive and heuristic, supplying some tentative physical insights, but not to be taken too seriously. It is a tribute to the genius of Feynman that he did take these axioms seriously and produced a calculus that works even in cases where rigorous argument fails. He reproduced much of canonical quantum mechanics from this approach and even used his powerful methods to go beyond the canonical. Our interest is in Feynman's use of Axiom 2 in calculating propagators for particles moving through space-time.

1.1. Feynman's Propagator Calculus.

We consider a quantum particle in space-time. For space-time points (t_0, x_0) and (t_1, x_1), the Feynman **propagator** denoted as $K(t_1, x_1; t_0, x_0)$ is the amplitude for a particle to be at position x_1 at time t_1 given that it is at position x_0 at time t_0. Let $\Psi(t, x)$ be the amplitude for the particle to be at space-time point (t, x). Obviously, $|\Psi(t, x)|^2$ is the position probability density, so that $\Psi(t, x)$ is the Schrödinger wave function of wave mechanics. By Axiom 3, the product $K(t, x; t_0, x_0)\Psi(t_0, x_0)$ is the amplitude density for a particle to be at position x at time t and position x_0 at time t_0. Applying Axiom 2, using the natural interpretation of the sum as integral, we get

$$\Psi(t, x) = \int_{\mathbb{R}^3} K(t, x; t_0, x_0)\Psi(t_0, x_0)\, dx_0. \tag{1.1}$$

This is a case where the axioms lead to rigorous results. In the parlance of mathematicians, $K(t, x; t_0, x_0)$ is known as the Green's function for the Schrödinger operator.

One of the more important formulae for propagators that follows, at least formally, from the axioms is the **group property**:

$$K(t_1, x_1; t_0, x_0) = \int_{\mathbb{R}^3} K(t_1, x_1; t', x')K(t', x'; t_0, x_0)\, dx'. \tag{1.2}$$

Here a particle travels from position x_0 at time t_0 to position x_1 at time t_1. At a time t' intermediate between t_0 and t_1, the particle passes through point x'. The amplitude for this event, by Axiom 3, is the integrand in Equation 1.2. The equation itself then follows by applying Axiom 2.

1.2. **The Path Integral.** Taking the axioms seriously, Feynman reasoned as follows. The crux of time evolution in quantum mechanics is embodied in Equation 1.1: if the propagator $K(t_1, x_1; t_0, x_0)$ is known, then the wave function at time t_1 can be calculated from that at time t_0. Let Γ be the collection of continuous paths $x\colon [t_0, t_1] \to \mathbb{R}^3$ with $x(t_0) = x_0$ and $x(t_1) = x_1$. To get from x_0 at time t_0 to x_1 at time t_1, the particle must traverse some path x in Γ. Let $\phi(x)$ be the amplitude for the traversal of the path $x \in \Gamma$. According to Axiom 2,

$$K(t_1, x_1; t_0, x_0) = \sum_{x\in\Gamma} \phi(x). \tag{1.3}$$

The central challenge of the Feynman formulation of quantum mechanics is to determine what this sum means! This sum is ususally written as a formal integral, the **path integral**,

$$K(t_1, x_1; t_0, x_0) = \int_{\Gamma} \phi(x)\, \mathcal{D}(x(t)). \tag{1.4}$$

At this point there are two problems to overcome. The first is to determine what form the amplitude $\phi(x)$ should take, the second to determine how to perform the 'integration' given that that there is no measure on Γ that has all the properties expected of this formal integral [2]. For the former, Feynman, inspired by a comment in Dirac's classic treatise [4], defines the amplitude as

$$\phi(x) = \exp\left(\frac{i}{\hbar}S[x]\right) = \exp\left(\frac{i}{\hbar}\int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt\right), \tag{1.5}$$

where $S[x]$ is the classical action associated to the path and $L(x, \dot{x}, t)$ is the lagrangian for the particle. Of course this choice for ϕ immediately restricts the path family to piecewise differentiable paths rather than general continuous paths. Almost any text that describes the path integral gives a plausible physical explanation for this choice. We recommend Feynman and Hibbs's [6, pp. 29–31]. The latter problem is tougher to solve. Feynman indeed knows well the difficulties in providing rigor to his definition of path integral.

> There may be other cases where . . . the present definition of a sum over all paths is just too awkward to use. Such a situation arises in ordinary integration in which the Riemann definition . . . is not adequate and recourse must be had to some other definition, such as that of Lebesque.

> The necessity to redefine the method of integration does not destroy the concept of integration. So we feel that the possible awkwardness of the special definition of the sum over all paths ... may eventually require new definitions to be formulated. Nevertheless, the concept of the sum over all paths, like the concept of an ordinary integral, is independent of the special definition and valid in spite of the failure of such definitions.*

Feynman gives a general presciption, the "present definition" of the quote, for constructing the sum 1.3 by partitioning the time interval and approximating paths by piecewise linear ones. This is presented in almost any text that describes the path integral. His scheme involves the introduction of unknown normalizing constants that are determined by the requirement that the path integral approach must lead to the Schrödinger equation. Feynman's argument for this is very clever, but glosses over difficult points. He replaces the propagator between space-time points using the action for a single path, with the justification that the time variables are close, but then integrates over space variables. Moreover, he approximates the integrand by a quadratic Taylor approximation in the space variable, which he points out is good as long as the space variable varies over a small interval, but then immediately integrates over all of space. In defence of this argument, Feynman does explain the intuition that suggests that as one integrates over the space variable, it will be only a small variance of the space variable around the classical path that contributes to the integral. Though we are sympathetic to this last argument and are extremely impressed with Feynman's physical intuitions, the development at best is a formal plausibility argument that, as a whole, lacks rigor.

1.3. **Evaluating Path Integrals: Quadratic Lagrangians.** One of the most beautiful and useful results for evaluating path integrals is Feynman's observation (see [6, Section 3–5]) that when the lagrangian is quadratic in x and its derivative $\dot{x}$, the path integral reduces to a calculation for which $x_0 = x_1 = 0$. Indeed, if the lagrangian is of the form

$$L(x, \dot{x}, t) = a(t)x^2 + b(t)x\dot{x} + c(t)\dot{x}^2 + d(t)x + e(t)\dot{x} + f(t), \tag{1.6}$$

then

$$\begin{aligned} K(t_1, x_1; t_0, x_0) &= e^{\frac{i}{\hbar}S_{\mathrm{cl}}} \\ &\times \int_{\Gamma_0} \left\{ \exp\left(\frac{i}{\hbar} \int_{t_0}^{t_1} [a(t)x^2 + b(t)x\dot{x} + c(t)\dot{x}^2]\, dt \right) \right\} \mathcal{D}(x(t)), \end{aligned} \tag{1.7}$$

where S_{cl} is the action associated to the classical path the particle traverses between space-time points (t_0, x_0) and (t_1, x_1), and Γ_0 is the space of loops based at the origin. Thus, for quadratic lagrangians, the phase $e^{\frac{i}{\hbar}S_{\mathrm{cl}}}$ is expressible exactly in terms of the classical path. This is probably the single most useful result that aids in the evaluation of path integrals. The following comments of Lawrence Schulman are particularly appropriate.

*[6, p. 34]

> For all path integrals evaluated above, the result was expressible entirely in terms of the classical path. It turns out that every propagator that anyone has ever been able to evaluate exactly and in closed form is a sum over "classical paths" only. There has even appeared in the literature the claim that *all* propagators are given exactly as a sum over classical paths.[†]

1.4. **Our Approach.** In the spirit of Feynman, our approach to interpreting the path integral is to replace the idea of some sort of integration over all paths with integration over a dense subspace of paths. We begin with a **filtration** of the space Γ of all paths between (t_0, x_0) and (t_1, x_1). This is an increasing union $\cup_{N=1}^{\infty}\Gamma_N \subset \Gamma$, where $\Gamma_N \subset \Gamma_{N+1}$, each Γ_N is finite dimensional, and the union $\cup\Gamma_N$ is dense in Γ. Γ_N is parameterized by a euclidean space and, using the known free particle propagator, the euclidean measure is adjusted to a measure $d\mu_N$ so that an appropriate notion of the restriction of $d\mu_{N+1}$ to Γ_N equals the measure $d\mu_N$, and so that the free particle propagator may be written as

$$\lim_{N\to\infty}\int_{\Gamma_N}\phi(x)\,d\mu_N. \tag{1.8}$$

We then use the **filtration measures** $d\mu_N$ and the limit above to evaluate propagators of particles moving in nonzero potential fields.

2. Calculating the Path Integral with Polynomial Paths

In this section we use the family of polynomial paths to calculate path integrals for the free particle and for the particle in a constant field. We use the known free particle propagator to help define our 'measure' on the space of polynomial paths and we then use this measure to verify that we get the correct propagator for the particle in a constant field.

For a positive integer N, let Γ_N denote the collection of polynomial paths $x(t) = \sum_{n=0}^{N} a_n t^n$ between space-time points (t_0, x_0) and (t_1, x_1). Let $X = x_1 - x_0$ and $T = t_1 - t_0$, and replace $x(t)$ by $x(t + t_0)$ to assume, without loss of generality, that $t_0 = 0$ and $t_1 = T$. For the path $x(t)$ we have $x_0 = x(0) = a_0$ and $x_1 = x(T) = \sum_{n=0}^{N} a_n T^n$. The classical average velocity is

$$v = \frac{X}{T} = \sum_{n=1}^{N} a_n T^{n-1} = \sum_{n=0}^{N-1} a_{n+1} T^n$$

so that

$$\begin{aligned} v^2 T &= \sum_{i,j=0}^{N-1} a_{i+1} a_{j+1} T^{i+j+1} \\ &= a_1^2 T + 2a_1 \sum_{n=1}^{N-1} a_{n+1} T^{n+1} + \sum_{i,j=1}^{N-1} a_{i+1} a_{j+1} T^{i+j+1}. \end{aligned} \tag{2.1}$$

2.1. **The Free Particle.**

[†][15, p. 39]

2.1.1. *The Free Particle Lagrangian and Action.* The free particle lagrangian is just $L(x) = \frac{1}{2}m\dot{x}^2$ and the corresponding action is

$$S[x] = \int_0^T \tfrac{1}{2}m\dot{x}^2\,dt = \int_0^T \tfrac{1}{2}m\left(\sum_{n=0}^{N-1}(n+1)a_{n+1}t^n\right)^2 dt$$
$$= \int_0^T \tfrac{1}{2}m\sum_{i,j=0}^{N-1}(i+1)(j+1)a_{i+1}a_{j+1}t^{i+j}\,dt.$$

Integrating and substituting Equation 2.1 gives

$$\begin{aligned}
S[x] &= \tfrac{1}{2}m\sum_{i,j=0}^{N-1}\frac{(i+1)(j+1)}{i+j+1}a_{i+1}a_{j+1}T^{i+j+1}\\
&= \tfrac{1}{2}ma_1^2T + ma_1\sum_{n=1}^{N-1}a_{n+1}T^{n+1}\\
&\qquad + \tfrac{1}{2}m\sum_{i,j=1}^{N-1}\frac{(i+1)(j+1)}{i+j+1}a_{i+1}a_{j+1}T^{i+j+1}\\
&= \tfrac{1}{2}mv^2T - \tfrac{1}{2}m\sum_{i,j=1}^{N-1}a_{i+1}a_{j+1}T^{i+j+1}\\
&\qquad + \frac{1}{2}m\sum_{i,j=1}^{N-1}\frac{(i+1)(j+1)}{i+j+1}a_{i+1}a_{j+1}T^{i+j+1}\\
&= \tfrac{1}{2}mv^2T + \tfrac{1}{2}m\sum_{i,j=1}^{N-1}\left(\frac{(i+1)(j+1)}{i+j+1}-1\right)a_{i+1}a_{j+1}T^{i+j+1}\\
&= \tfrac{1}{2}mv^2T + \tfrac{1}{2}m\sum_{i,j=1}^{N-1}\frac{ij}{i+j+1}a_{i+1}a_{j+1}T^{i+j+1}.
\end{aligned}$$

Let $\mathbf{a}$ be the column vector $\mathbf{a} = (a_2\ a_3\ \cdots\ a_N)^\dagger$ and let $\mathbf{B}_{N-1}$ be the positive definite symmetric matrix

$$\mathbf{B}_{N-1} = \left[\frac{ij}{i+j+1}T^{i+j+1}\right]_{i,j=1}^{N-1}.$$

The free particle action over the path $x(t)$ may be written as

$$S[x] = \tfrac{1}{2}mv^2T + \tfrac{1}{2}m\mathbf{a}^\dagger\mathbf{B}_{N-1}\mathbf{a}. \tag{2.2}$$

2.1.2. *The Free Particle Propagator.* The free particle propagator is given by the Feynman path integral that heuristically sums 'over all possible paths' between the space-time points (t_0, x_0) and (t_1, x_1):

$$K(t_1, x_1; t_0, x_0) = \int_\Gamma \exp\left(\frac{i}{\hbar}S[x]\right)\mathcal{D}(x(t)), \tag{2.3}$$

where Γ is the space of all paths between space-time points (t_0, x_0) and (t_1, x_1). As recorded in the Section 1, the challenge of this approach is to make sense of this sum 'over all paths'. We are attempting to see how far we can get if we interpret the sum as a sum over polynomial paths. To this end, with the aid of Equation 2.2, we interpret Equation 2.3 as

$$\begin{aligned} K(t_1, x_1; t_0, x_0) &= \lim_{N\to\infty} \int_{\Gamma_N} e^{\frac{i}{\hbar}(\frac{1}{2}mv^2T + \frac{1}{2}m\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a})}\, d\mu_N \\ &= e^{\frac{imv^2T}{2\hbar}} \lim_{N\to\infty} \int_{\Gamma_N} e^{\frac{im}{2\hbar}\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a}}\, d\mu_N, \end{aligned} \tag{2.4}$$

for appropriate measures $d\mu_N$ on the $\mathbf{a}$-parameter spaces $\Gamma_N = \mathbb{R}^{N-1}$ and an appropriate limiting process, both to be determined subsequently. Notice that the integral term on the far right in Equation 2.4 depends only on the time difference T so that we may write

$$K(t_1, x_1; t_0, x_0) = F(T) \exp \frac{imv^2T}{2\hbar} = F(T) \exp \frac{im(x_1 - x_0)^2}{2\hbar(t_1 - t_0)}. \tag{2.5}$$

The term $S_{\mathrm{cl}} = \frac{mv^2T}{2} = \frac{m(x_1-x_0)^2}{2(t_1-t_0)}$ is exactly the action associated to the classical path a free particle traverses between space-time points (t_0, x_0) and (t_1, x_1). This illustrates the observation of Feynman reported in Section 1.3. In this case then the problem of the interpretation of the path integral becomes the problem of finding the dependence of the integral on the time difference $T = t_1 - t_0$. Our formula for this dependence in the setting of the free particle is

$$F(T) = \lim_{N\to\infty} \int_{\Gamma_N} e^{\frac{im}{2\hbar}\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a}}\, d\mu_N. \tag{2.6}$$

2.1.3. *Evaluating $F(T)$.* In this section we fix N, we let

$$F_N(T) = \int_{\Gamma_N} e^{\frac{im}{2\hbar}\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a}}\, d\mu_N,$$

and we assume that the measure $d\mu_N$ is of the form

$$d\mu_N = \delta_1(\delta_2 da_2 \wedge \delta_3 da_3 \wedge \cdots \wedge \delta_N da_N) = \delta(N) d\mathbf{a}, \tag{2.7}$$

where $\delta(N) = \delta_1\delta_2 \cdots \delta_N$, for constants δ_n. Since the matrix $\mathbf{B}_{N-1}$ is symmetric, there is an $(N-1) \times (N-1)$ orthogonal matrix $\mathbf{D}$ that diagonalizes $\mathbf{B}_{N-1}$:

$$\mathbf{\Lambda} = \mathbf{D}^\dagger \mathbf{B}_{N-1}\mathbf{D} = \mathrm{diag}(\lambda_2\ \lambda_3\ \cdots\ \lambda_N), \tag{2.8}$$

where the λ_i are the eigenvalues of $\mathbf{B}_{N-1}$, all positive since $\mathbf{B}_{N-1}$ is positive definite and symmetric. We perform the change of variables $\mathbf{a} = \mathbf{D}\mathbf{u}$ in the integral for $F_N(T)$ in Equation 2.6 to obtain

$$F_N(T) = \delta(N) \int_{\mathbb{R}^{N-1}} e^{\frac{im}{2\hbar}\mathbf{u}^\dagger \mathbf{\Lambda}\mathbf{u}} \det \mathbf{J}(\mathbf{D})\, d\mathbf{u}.$$

As $\mathbf{D}$ is orthogonal, its Jacobian determinant $\det \mathbf{J}(\mathbf{D})$ is unity. We then have

$$\begin{aligned} F_N(T) = \delta(N) \int_{\mathbb{R}^{N-1}} e^{\frac{im}{2\hbar}\mathbf{u}^\dagger \mathbf{\Lambda} \mathbf{u}}\, d\mathbf{u} \\ = \delta(N) \int_{\mathbb{R}^{N-1}} \exp\left\{ \frac{im}{2\hbar} \sum_{n=2}^{N} \lambda_n u_n^2 \right\} d\mathbf{u} \\ = \delta(N) \prod_{n=2}^{N} \int_{-\infty}^{\infty} e^{\frac{im\lambda_n}{2\hbar} u_n^2}\, du_n. \end{aligned}$$

Each of the integral factors is a complex gaussian integral and may be integrated using the formula

$$\int_{-\infty}^{\infty} e^{i(\alpha u^2 + \beta u + \gamma)}\, du = \sqrt{\frac{\pi i}{\alpha}} e^{-i\frac{\beta^2 - 4\alpha\gamma}{4\alpha}} \qquad \text{for } \alpha > 0, \tag{2.9}$$

where we have written $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. This gives

$$F_N(T) = \delta(N) \prod_{n=2}^{N} \sqrt{\frac{2\pi i\hbar}{m\lambda_n}} = \delta(N) \left(\frac{2\pi i\hbar}{m}\right)^{\frac{N-1}{2}} (\lambda_2\lambda_3 \cdots \lambda_N)^{-\frac{1}{2}}.$$

Of course, as $\mathbf{D}$ is orthogonal, we have $\det \mathbf{B}_{N-1} = \det \mathbf{\Lambda} = \lambda_2\lambda_3 \cdots \lambda_N$, so that

$$F_N(T) = \delta(N) \left(\frac{2\pi i\hbar}{m}\right)^{\frac{N-1}{2}} (\det \mathbf{B}_{N-1})^{-\frac{1}{2}}. \tag{2.10}$$

2.1.4. *Evaluating* $\det \mathbf{B}_{N-1}$. We shall calculate a general formula for the determinant of $\mathbf{B}_N$, for an arbitrary positive integer N. In computing this determinant, the Hilbert matrices

$$\mathbf{H}_N = \left[\frac{1}{i+j-1}\right]_{i,j=1}^{N} \quad \text{and} \quad \mathbf{C}_N = \left[\frac{1}{i+j+1}\right]_{i,j=1}^{N} \tag{2.11}$$

will play an important role. We begin by writing the determinant formula:

$$\det \mathbf{B}_N = \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn} \sigma \left(\prod_{k=1}^{N} \frac{k\sigma(k)}{k + \sigma(k) + 1} \right) T^{\sum_{n=1}^{N}(n+\sigma(n)+1)},$$

where $\mathcal{S}_N$ is the symmetric group on N letters. This simplifies to

$$\det \mathbf{B}_N = (N!)^2 T^{N(N+2)} \det \mathbf{C}_N. \tag{2.12}$$

Following Melzak [11, pp. 151–153] we may evaluate $\det \mathbf{C}_N$ as a special case of the Cauchy determinant

$$\det \left[\frac{1}{s_i + t_j}\right] = \frac{\prod_{i>j}[(s_i - s_j)(t_i - t_j)]}{\prod_{i,j}(s_i + t_j)}. \tag{2.13}$$

Taking $s_i = i - 1$ and $t_j = j$ for $\mathbf{H}_N$ and $s_i = i + 1$ and $t_j = j$ for $\mathbf{C}_N$ gives, after some manipulation,

$$\det \mathbf{H}_N = \frac{\left(\prod_{n=1}^{N-1} n!\right)^4}{\prod_{n=1}^{2N-1} n!}, \quad \det \mathbf{C}_N = \frac{\left(\prod_{n=1}^{N-1} n!\right)^4 [N!(N+1)!]^2}{\prod_{n=1}^{2N+1} n!}. \tag{2.14}$$

It then easily follows that

$$\det \mathbf{C}_N = (N+1)^2 \det \mathbf{H}_{N+1}. \tag{2.15}$$

Still following Melzak [11], we may get an idea of the size of these determinants by applying Stirling's formula to derive the approximate identity

$$\det \mathbf{H}_N \cong \frac{(2\pi)^{N-1}}{2^{2N^2}}.$$

Putting together Equations 2.12, 2.14 and 2.15, we get

$$\det \mathbf{B}_{N-1} = (N!)^2 T^{N^2-1} \det \mathbf{H}_N = (N!)^2 T^{N^2-1} \frac{\left(\prod_{n=1}^{N-1} n!\right)^4}{\prod_{n=1}^{2N-1} n!}.$$

2.1.5. *Evaluating $F(T)$, continued.* Our formula for $F_N(T)$ from Equation 2.10 becomes

$$\begin{aligned} F_N(T) &= \delta(N) \left(\frac{2\pi i \hbar}{m}\right)^{\frac{N-1}{2}} \left[(N!)^2 T^{N^2-1} \frac{\left(\prod_{n=1}^{N-1} n!\right)^4}{\prod_{n=1}^{2N-1} n!} \right]^{-\frac{1}{2}} \\ &= \frac{\delta(N)}{N!} \left(\frac{2\pi i \hbar}{m T^{N+1}}\right)^{\frac{N-1}{2}} \frac{\left(\prod_{n=1}^{2N-1} n!\right)^{\frac{1}{2}}}{\left(\prod_{n=1}^{N-1} n!\right)^2}. \end{aligned} \tag{2.16}$$

Now define

$$\delta_n = \left(\frac{m}{2\pi i \hbar T}\right)^{\frac{1}{2}} \frac{(n-1)!n!}{[(2n-2)!(2n-1)!]^{\frac{1}{2}}} T^{n - \frac{1}{2^n}}. \tag{2.17}$$

This definition is designed to cancel most of the meat of Equation 2.16 with as uniform and succinct a formula as possible for the δ_n's. From Equations 2.7 and 2.17, the measure on $\mathbb{R}^{N-1}$ is the euclidean measure $d\mathbf{a}$ scaled by

$$\delta(N) = \delta_1 \delta_2 \cdots \delta_N = \left(\frac{m}{2\pi i \hbar T}\right)^{\frac{N}{2}} \frac{\left(\prod_{n=1}^{N-1} n!\right)^2}{\left(\prod_{n=1}^{2N-1} n!\right)^{\frac{1}{2}}} N! T^{\frac{N(N+1)}{2} - 1 + \frac{1}{2^N}}. \tag{2.18}$$

A short calculation using Equation 2.18 in 2.16 yields

$$F_N(T) = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}} T^{\frac{1}{2^N}}, \tag{2.19}$$

from which we calculate the free particle propagator by taking the limit as N approaches ∞:

$$F(T) = \lim_{N\to\infty} F_N(T) = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}, \tag{2.20}$$

so that, from Equation 2.5,

$$K(t_1, x_1; t_0, x_0) = \left[\frac{m}{2\pi i\hbar(t_1 - t_0)}\right]^{\frac{1}{2}} \exp \frac{im(x_1 - x_0)^2}{2\hbar(t_1 - t_0)}. \tag{2.21}$$

We have backward engineered the measure of Equation 2.7 in our definition of δ_n in Equation 2.17 in our desire to cancel from Equation 2.16 exactly those terms necessary to leave only the known propagator for the free particle. Nonetheless, it is encouraging, if not compelling, that there is one formula dependent only on n that works for each δ_n. Having backward engineered the measure to give the correct free particle propagator, we now use this measure in calculations of the propagator for particles in nonzero potentials. One should not be too critical of the use of this backward engineering for obtaining the correct normalizing constant for our measures. Afterall, the standard approaches for obtaining the propagator from a path integral all use backward engineering to fix normalizing constants. For example, Feynman's original calculation required a proportionality constant A to take the value $\sqrt{2\pi i\hbar T/m}$ in order for his approach to yield the Schrödinger equation for a particle moving in a potential in one dimension [6, pp. 76–78]; see also [14, pp. 389–393]. Most texts use Feynman's calculation, though some, like Shankar [16], also use the known free particle propagator to fix normalizing constants.

2.2. **Particle in a Constant External Field.** The standard calculation for the propagator for the particle in a constant external field uses the results recorded in Section 1.3 and, once the free particle propagator is known, reduces to nothing more than calculating the classical action along the classical path. We shall nonetheless evaluate the propagator for the particle in a constant external field from scratch, without using the results of Section 1.3, and demonstrate the way the propagator crystalizes from our approach. This section thus represents something of a test case for the approach. It is interesting how the spacial dependence separates out early in the calculation, the full time dependence requiring all the heavy lifting.

2.2.1. *The Lagrangian and the Action.* The lagrangian for a particle in a constant external field K generated by a linear potential is

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 + Kx.$$

With the polynomial path $x(t) = \sum_{n=0}^{N} a_n t^n$, the action becomes

$$S[x] = S_{\text{free}}[x] + K \sum_{n=0}^{N} \frac{a_n}{n+1} T^{n+1},$$

where $S_{\text{free}}[x]$ is the action for the free particle from Equation 2.2. By peeling off the $n = 0$ and $n = 1$ terms of the sum and using $x(0) = x_0 = a_0$ and $x(T) = x_1$, we may write, after some manipulation,

$$S[x] = S_{\text{free}}[x] + \tfrac{1}{2}KT(x_0 + x_1) - \tfrac{1}{2}K \sum_{n=1}^{N-1} \frac{n}{n+2} a_{n+1} T^{n+2}. \tag{2.22}$$

Recalling that $\mathbf{a} = (a_2 \; a_3 \; \cdots \; a_N)^\dagger$ and setting $\mathbf{b} = (b_2 \; b_3 \; \cdots \; b_N)^\dagger$ where $b_n = \frac{n-1}{n+1} T^{n+1}$, we get from Equations 2.2 and 2.22

$$S[x] = \tfrac{1}{2}mv^2 T + \tfrac{1}{2}KT(x_0 + x_1) + \tfrac{1}{2}m\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a} - \tfrac{1}{2}K\mathbf{b}^\dagger \mathbf{a}. \tag{2.23}$$

2.2.2. *The Propagator.* The propagator becomes

$$\begin{aligned} K(t_1, x_1; t_0, x_0) &= e^{\frac{i}{\hbar}(\frac{1}{2}mv^2T + \frac{1}{2}KT(x_0+x_1))} \\ &\quad \times \lim_{N\to\infty} \int_{\Gamma_N} e^{\frac{i}{\hbar}(\frac{1}{2}m\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a} - \frac{1}{2}K\mathbf{b}^\dagger \mathbf{a})}\, d\mu_N \\ &= \mathcal{F}(T) e^{\frac{i}{\hbar}(\frac{1}{2}mv^2T + \frac{1}{2}KT(x_0+x_1))}. \end{aligned} \tag{2.24}$$

To evaluate $\mathcal{F}(T)$ we proceed exactly as in Section 2.1.3 for evaluating $F(T)$. First, diagonalize $\mathbf{B}_{N-1}$ via an orthogonal matrix $\mathbf{D}$ to obtain

$$\begin{aligned} \mathcal{F}_N(T) &= \int_{\Gamma_N} e^{\frac{i}{\hbar}(\frac{1}{2}m\mathbf{a}^\dagger \mathbf{B}_{N-1}\mathbf{a} - \frac{1}{2}K\mathbf{b}^\dagger \mathbf{a})}\, d\mu_N \\ &= \delta(N) \int_{\mathbb{R}^{N-1}} e^{\frac{im}{2\hbar}\mathbf{u}^\dagger \boldsymbol{\Lambda}\mathbf{u} - \frac{iK}{2\hbar}\mathbf{b}^\dagger \mathbf{D}\mathbf{u}}\, d\mathbf{u} \\ &= \delta(N) \prod_{n=2}^{N} \int_{-\infty}^{\infty} \exp\left\{ \frac{im\lambda_n}{2\hbar} u_n^2 - \frac{iK}{2\hbar} \left[\mathbf{b}^\dagger \mathbf{D}\right]_n u_n \right\} du_n. \end{aligned}$$

Each of the integral factors may be integrated using Formula 2.9, which results in

$$\begin{aligned} \mathcal{F}_N(T) &= F_N(T) \exp\left\{ -\frac{i}{\hbar} \left(\frac{K^2}{8m} \sum_{n=2}^{N} \frac{1}{\lambda_n} \left[\mathbf{b}^\dagger \mathbf{D}\right]_n^2 \right) \right\} \\ &= F_N(T) \exp\left\{ -\frac{i}{\hbar} \left(\frac{K^2}{8m} \mathbf{b}^\dagger \mathbf{B}_{N-1}^{-1} \mathbf{b} \right) \right\}, \end{aligned} \tag{2.25}$$

where $F_N(T)$ is the free particle time dependence of Equation 2.19. The last line of Equation 2.25 follows from Equation 2.8:

$$\sum_{n=2}^{N} \frac{1}{\lambda_n} \left[\mathbf{b}^\dagger \mathbf{D}\right]_n^2 = \mathbf{b}^\dagger \mathbf{D} \boldsymbol{\Lambda}^{-1} \mathbf{D}^\dagger \mathbf{b} = \mathbf{b}^\dagger \mathbf{B}_{N-1}^{-1} \mathbf{b}.$$

2.2.3. *Evaluating* $\mathbf{b}^\dagger \mathbf{B}_{N-1}^{-1}\mathbf{b}$. We will obtain a general formula for the N-dimensional form $\mathbf{b}^\dagger \mathbf{B}_N^{-1}\mathbf{b}$. Notice that $\mathbf{B}_N$ decomposes as the product $T\,\mathbf{T}_N\mathbf{C}_N\mathbf{T}_N$, where $\mathbf{C}_N$ is the Hilbert matrix of 2.11 and $\mathbf{T}_N = \operatorname{diag}(T\ 2T^2\ \cdots\ NT^N)$ is the diagonal matrix whose nth entry is nT^n. Writing $\mathbf{b}^\dagger = (b_1\ b_2\ \cdots\ b_N)$ where $b_n = \frac{n}{n+2}T^{n+2}$ and setting $\mathbf{c}^\dagger = (c_1\ c_2\ \cdots\ c_N)$ where $c_n = \frac{1}{n+2}$, we get

$$\begin{aligned}\mathbf{b}^\dagger \mathbf{B}_N^{-1}\mathbf{b} &= \frac{1}{T}\left(\mathbf{b}^\dagger \mathbf{T}_N^{-1}\right)\mathbf{C}_N^{-1}\left(\mathbf{T}_N^{-1}\mathbf{b}\right)\\ &= \frac{1}{T}\left(T^2\mathbf{c}^\dagger\right)\mathbf{C}_N^{-1}\left(T^2\mathbf{c}\right) = T^3\mathbf{c}^\dagger\mathbf{C}_N^{-1}\mathbf{c}. \end{aligned} \tag{2.26}$$

Instead of calculating the inverse of the Hilbert matrix $\mathbf{C}_N$, we finesse the problem by observing that $\mathbf{c}$ is exactly the first column of the matrix $\mathbf{C}_N$, which implies that $\mathbf{c} = \mathbf{C}_N\mathbf{d}$ where $\mathbf{d}^\dagger = (1\ 0\ 0\ \cdots\ 0)$ is the standard unit basis vector whose only nonzero entry is a 1 in the first position. Thus $\mathbf{C}_N^{-1}\mathbf{c} = \mathbf{d}$ and

$$\mathbf{b}^\dagger \mathbf{B}_N^{-1}\mathbf{b} = T^3\mathbf{c}^\dagger\mathbf{C}_N^{-1}\mathbf{c} = T^3\mathbf{c}^\dagger\mathbf{d} = T^3 c_1 = \tfrac{1}{3}T^3. \tag{2.27}$$

This is rather interesting in that our form $\mathbf{b}^\dagger \mathbf{B}_N^{-1}\mathbf{b}$ reduces to the constant $\frac{1}{3}T^3$ independent of N.

2.2.4. *The Propagator, continued.* From Equations 2.25 and 2.27, our formula for $\mathcal{F}_N$ becomes

$$\mathcal{F}_N(T) = F_N(T)\exp\left\{-\frac{i}{\hbar}\left(\frac{K^2T^3}{24m}\right)\right\}. \tag{2.28}$$

Taking the limit as N approaches ∞ and using Equation 2.20 gives

$$\mathcal{F}(T) = \lim_{N\to\infty}\mathcal{F}_N(T) = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}\exp\left\{-\frac{i}{\hbar}\left(\frac{K^2T^3}{24m}\right)\right\}. \tag{2.29}$$

The propagator for the particle in a constant field K becomes, from Equation 2.24,

$$\begin{aligned}K(t_1,x_1;t_0,x_0) = &\left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}\\ &\times\exp\left\{\frac{i}{\hbar}\left(\tfrac{1}{2}mv^2T + \tfrac{1}{2}KT(x_0+x_1) - \frac{K^2T^3}{24m}\right)\right\}.\end{aligned}$$

This again is of the form $F(T)e^{(i/\hbar)S_{\mathrm{cl}}}$ as Feynman and Hibbs [6] promise, for the term

$$\begin{aligned}S_{\mathrm{cl}} &= \tfrac{1}{2}mv^2T + \tfrac{1}{2}KT(x_0+x_1) - \frac{K^2T^3}{24m}\\ &= \frac{m(x_1-x_0)^2}{2T} + \tfrac{1}{2}KT(x_0+x_1) - \frac{K^2T^3}{24m}\end{aligned}$$

is precisely the action associated to the classical path a particle in a constant field K traverses between space-time points (t_0,x_0) and (t_1,x_1). The reader might notice a discrepancy between the final term $-\frac{K^2T^3}{24m}$ in our formula for the propagator and that of Feynman and Hibbs in [6, Equation (3-62)], which reports only $-\frac{KT^3}{24}$. The term in [6, Equation (3-62)] is a typo, having left off the factor $\frac{K}{m}$.

3. The Harmonic Oscillator

The calculation of the propagator for the harmonic oscillator is the primary goal of this paper and promises to be quite interesting. We are using polynomial paths, a set dense in the space of all paths, to perform our calculations. In the previous calculations for the free particle and for the constant force field, the classical path the particle traverses is in fact a polynomial path. This is why the free particle action for the classical path and two of the three terms in the constant force action for the classical path appear quickly, in the initial calculation of the action. This will not happen for the harmonic oscillator since the classical path in this case is a sinusoidal that can only be approximated by polynomial paths, and the action associated to the classical path contains sinusoidal terms. Rather than enduring the quite formidable calculation that approximates the general path family by polynomial paths, we simplify by approximating the general path family by polynomial variations around the classical path. Since the harmonic oscillator propagator is quadratic as in Equation 1.6, the results of Section 1.3 apply and an examination of Formula 1.7 shows that the general propagator for the harmonic oscillator may be written as

$$K(t_1, x_1; t_0, x_0) = K(t_1, 0; t_0, 0)e^{\frac{i}{\hbar}S_{\rm cl}}, \tag{3.1}$$

where the action $S_{\rm cl}$ over the classical path is given by

$$S_{\rm cl} = \frac{m\omega}{2\sin\omega T}\left[(x_0^2 + x_1^2)\cos\omega T - 2x_0x_1\right].$$

We are left with the still quite formidable task of calculating the propagator $K(t_1, 0; t_2, 0)$ for the oscillator particle that begins and ends at the origin, i.e., where $x_0 = x_1 = 0$. The only lapse of rigor in our derivation of the propagator of Equation 3.1 is in the use of Feynman's observations of Section 1.3, which he derives in the context of the full path integral. This easily is remedied in the context of polynomial path families by a straightforward calculation that confirms Feynman's observation in our special context.

3.1. **The Lagrangian, the Action, and the Propagator.**

The harmonic oscillator lagrangian is

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2x^2.$$

The initial conditions translate to $a_0 = 0$ and $a_1T = -\sum_{n=2}^{N} a_nT^n$ for the polynomial path $x(t) = \sum_{n=0}^{N} a_nt^n$. A lengthy calculation using these initial conditions to eliminate the a_0 and a_1 terms eventually yields the action as

$$S[x] = S_{\rm free}[x] + S_0[x] = \tfrac{1}{2}m\mathbf{a}^\dagger\mathbf{B}_{N-1}\mathbf{a} - \tfrac{1}{2}m\omega^2T^2\mathbf{a}^\dagger\mathbf{J}_{N-1}\mathbf{a}, \tag{3.2}$$

where $S_{\rm free}[x] = \frac{1}{2}m\mathbf{a}^\dagger\mathbf{B}_{N-1}\mathbf{a}$ is the free action of Equation 2.2 with average velocity $v = 0$, $\mathbf{a} = (a_2\ a_3\ \cdots\ a_N)^\dagger$ is the vector of coefficients, and $\mathbf{J}_{N-1}$ is the positive definite symmetric matrix

$$\mathbf{J}_{N-1} = \left[\frac{1}{3}\,\frac{i-1}{i+2}\,\frac{j-1}{j+2}\,\frac{i+j+4}{i+j+1}T^{i+j-1}\right]_{i,j=2}^{N}$$
$$= \left[\frac{1}{3}\,\frac{i}{i+3}\,\frac{j}{j+3}\,\frac{i+j+6}{i+j+3}T^{i+j+1}\right]_{i,j=1}^{N-1}.$$

Exactly as in Section 2.1.3 ending in Equation 2.10, since $\mathbf{B}_{N-1} - \omega^2T^2\mathbf{J}_{N-1}$ is a symmetric matrix, the propagator

$$K(t_1,0;t_0,0) = \lim_{N\to\infty}\int_{\Gamma_N} e^{\frac{im}{2\hbar}\mathbf{a}^\dagger(\mathbf{B}_{N-1}-\omega^2T^2\mathbf{J}_{N-1})\mathbf{a}}\,d\mu_N$$

may be calculated as

$$\lim_{N\to\infty}\delta(N)\left(\frac{2\pi i\hbar}{m}\right)^{\frac{N-1}{2}}\left[\det(\mathbf{B}_{N-1}-\omega^2T^2\mathbf{J}_{N-1})\right]^{-\frac{1}{2}}$$
$$= \lim_{N\to\infty}F_N(T)\left(\frac{\det\mathbf{B}_{N-1}}{\det(\mathbf{B}_{N-1}-\omega^2T^2\mathbf{J}_{N-1})}\right)^{\frac{1}{2}}, \quad (3.3)$$

where $F_N(T)$ is given by Equation 2.10. Assuming the square root term in Equation 3.3 has a limit, we get

$$K(t_1,0;t_0,0) = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left(\frac{\det\mathbf{B}_{N-1}}{\det(\mathbf{B}_{N-1}-\omega^2T^2\mathbf{J}_{N-1})}\right)^{\frac{1}{2}}. \quad (3.4)$$

3.2. **Evaluating the Limit of the Square Root Term.** We shall demonstrate in a rather lengthy calculation that the limit of the square root term in Equation 3.4 is $\left(\frac{\omega T}{\sin\omega T}\right)^{\frac{1}{2}}$, giving the propagator for the harmonic oscillator with initial conditions $x_0 = x_1 = 0$ as

$$K(t_1,0;t_0,0) = \left(\frac{m\omega}{2\pi i\hbar\sin\omega T}\right)^{\frac{1}{2}}. \quad (3.5)$$

In this section we present the overall attack on the problem and reduce the calculation to a calculation of six terms, labeled a, b, c, d, e, and f. In subsequent sections we address the calculations of each of these terms.

The reciprocal of the square root term in Equation 3.4 may be written as

$$\left(\frac{\det(\mathbf{B}_{N-1}-\omega^2T^2\mathbf{J}_{N-1})}{\det\mathbf{B}_{N-1}}\right)^{\frac{1}{2}} = \sqrt{\det(\mathbf{I}_{N-1}-\omega^2T^2\mathbf{B}_{N-1}^{-1}\mathbf{J}_{N-1})},$$

where $\mathbf{I}_{N-1}$ is the appropriate identity matrix. To evaluate the determinant of the matrix $\mathbf{I}_N - \omega^2T^2\mathbf{B}_N^{-1}\mathbf{J}_N$, exactly as in Section 2.2.3, write $\mathbf{B}_N = T\mathbf{T}_N\mathbf{C}_N\mathbf{T}_N$ and $\mathbf{J}_N = T\mathbf{T}_N\mathbf{K}_N\mathbf{T}_N$, where $\mathbf{C}_N$ is the Hilbert matrix of 2.11 and $\mathbf{K}_N$ is the matrix

$$\mathbf{K}_N = \left[\frac{1}{3}\,\frac{1}{i+3}\,\frac{1}{j+3}\,\frac{i+j+6}{i+j+3}\right]_{i.j=1}^{N}.$$

Obviously, $\det(\mathbf{I}_N - \omega^2 T^2 \mathbf{B}_N^{-1}\mathbf{J}_N) = \det(\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1}\mathbf{K}_N)$. The evaluation of the latter determinant is quite involved. The Hilbert matrix $\mathbf{C}_N$ is an example of a Cauchy matrix, one whose i,j entry is of the form $\frac{1}{s_i+t_j}$. We already have applied the general formula for the determinant of a Cauchy matrix, Formula 2.13, to find the determinant of $\mathbf{C}_N$. There is also a general formula for the inverse of a Cauchy matrix found, for example, in [10, Problem **41**, p. 38]. Applying this with $s_i = i$ and $t_j = j+1$ gives

$$\begin{aligned}\mathbf{C}_N^{-1} &= \left[(-1)^{i+j}\frac{\prod_{k=1}^{N}(i+k+1)(j+k+1)}{(i+j+1)(i-1)!(j-1)!(N-i)!(N-j)!}\right]_{i,j=1}^{N} \\ &= \left[(-1)^{i+j}\frac{ij}{i+j+1}\binom{N+i+1}{N}\binom{N}{i}\binom{N+j+1}{N}\binom{N}{j}\right]_{i,j=1}^{N}.\end{aligned}$$

It follows that the i,k entry of the matrix product $\mathbf{C}_N^{-1}\mathbf{K}_N$ is

$$\begin{aligned}\lambda_{i,k} = (-1)^i \frac{1}{3}\frac{i}{k+3}&\binom{N+i+1}{N}\binom{N}{i} \\ &\times \sum_{j=1}^{N}(-1)^j\binom{N+j+1}{N}\binom{N}{j}\frac{j}{i+j+1}\frac{j+k+6}{(j+3)(j+k+3)}.\end{aligned} \tag{3.6}$$

The key to evaluating these sums is to write explicitly the i,k entry of the matrix product $\mathbf{I}_N = \mathbf{C}_N^{-1}\mathbf{C}_N$. With $\delta_{i,k}$ as the Kronecker delta, we get

$$\begin{aligned}\delta_{i,k} = (-1)^i i&\binom{N+i+1}{N}\binom{N}{i} \\ &\times \sum_{j=1}^{N}(-1)^j\binom{N+j+1}{N}\binom{N}{j}\frac{j}{i+j+1}\frac{1}{j+k+1}.\end{aligned} \tag{3.7}$$

We expand the term $\frac{j+k+6}{(j+3)(j+k+3)}$ in Formula 3.6 as

$$\frac{j+k+6}{(j+3)(j+k+3)} = \frac{k+3}{k}\frac{1}{j+2+1} - \frac{3}{k}\frac{1}{j+(k+2)+1}. \tag{3.8}$$

Substituting this into Equation 3.6 and comparing with Equation 3.7 gives, for $1 \le i \le N$ and $1 \le k \le N-2$,

$$\lambda_{i,k} = \frac{1}{3(k+3)}\left(\frac{k+3}{k}\delta_{i,2} - \frac{3}{k}\delta_{i,k+2}\right). \tag{3.9}$$

The reader will notice that the only nonzero elements in the first $N-2$ columns of $\mathbf{C}_N^{-1}\mathbf{K}_N$ are

$$\left.\begin{aligned}\lambda_{2,k} &= \frac{1}{3k} \\ \lambda_{k+2,k} &= \frac{-1}{k(k+3)}\end{aligned}\right\} k = 1,\ldots,N-2. \tag{3.10}$$

The form of the matrix $\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1} \mathbf{K}_N$ is illustrated by the example

$$\left[\begin{array}{cccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & \mu_{1,7} & \mu_{1,8} \\
* & * & * & * & * & * & \mu_{2,7} & \mu_{2,8} \\
* & 0 & 1 & 0 & 0 & 0 & \mu_{3,7} & \mu_{3,8} \\
0 & * & 0 & 1 & 0 & 0 & \mu_{4,7} & \mu_{4,8} \\
0 & 0 & * & 0 & 1 & 0 & \mu_{5,7} & \mu_{5,8} \\
0 & 0 & 0 & * & 0 & 1 & \mu_{6,7} & \mu_{6,8} \\
0 & 0 & 0 & 0 & * & 0 & \mu_{7,7} & \mu_{7,8} \\
0 & 0 & 0 & 0 & 0 & * & \mu_{8,7} & \mu_{8,8}
\end{array}\right]$$

for $N = 8$, where each asterisk represents a nonzero entry calculated using 3.10, and where we have written the i, k entry of $\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1} \mathbf{K}_N$ as $\mu_{i,k} = \delta_{i,k} - \omega^2 T^2 \lambda_{i,k}$. The determinant will be evaluated after a sequence of column operations yielding a matrix $\mathbf{M}_N$, followed by a sequence of row operations yielding a matrix $\mathbf{A}_N$.

We begin with a sequence of column operations using the '1' entry in column k on the diagonal to eliminate the $k, k-2$ entry in column $k-2$, successively, starting with column $k = N-2$ and moving to the left one column at a time until column $k = 3$ is used to eliminate from the first column. The result is a matrix $\mathbf{M}_N$, illustrated for $N = 8$ by

$$\left[\begin{array}{cccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & \mu_{1,7} & \mu_{1,8} \\
* & p & * & * & * & * & \mu_{2,7} & \mu_{2,8} \\
0 & 0 & 1 & 0 & 0 & 0 & \mu_{3,7} & \mu_{3,8} \\
0 & 0 & 0 & 1 & 0 & 0 & \mu_{4,7} & \mu_{4,8} \\
0 & 0 & 0 & 0 & 1 & 0 & \mu_{5,7} & \mu_{5,8} \\
0 & 0 & 0 & 0 & 0 & 1 & \mu_{6,7} & \mu_{6,8} \\
* & 0 & * & 0 & * & 0 & \mu_{7,7} & \mu_{7,8} \\
0 & q & 0 & * & 0 & * & \mu_{8,7} & \mu_{8,8}
\end{array}\right], \tag{3.11}$$

where we have written p for the $2, 2$ entry and q for the $8, 2$ entry. More generally, we write p_N for the $2, 2$ entry and q_N for the $S, 2$ entry of $\mathbf{M}_N$, where $S = N$ if N is even and $N - 1$ if N is odd. Now this $2, 2$ entry is very interesting: we will show that it is the mth Taylor polynomial expanded about the origin for the function $\frac{\sin \omega T}{\omega T}$, for $m = \left\lfloor \frac{N-2}{2} \right\rfloor$. Indeed, the reader may wish to check now that

$$p_N(\omega T) = \sum_{j=0}^{\left\lfloor \frac{N-2}{2} \right\rfloor} \frac{(-1)^k}{(2j+1)!} (\omega T)^{2j}. \tag{3.12}$$

We continue to reduce the matrix $\mathbf{M}_N$ using row operations, but we shall do nothing to change the $2, 2$ and $S, 2$ entries. Apply row reduction using the diagonal

'1' to eliminate the two nonzero entires in column k, for $k = 1, 3, \ldots, N-2$. The result is a matrix $\mathbf{A}_N$, illustrated for $N = 8$ by

$$\left[\begin{array}{cccccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & \mu_{1,7} & \mu_{1,8} \\ 0 & p & 0 & 0 & 0 & 0 & e & f \\ 0 & 0 & 1 & 0 & 0 & 0 & \mu_{3,7} & \mu_{3,8} \\ 0 & 0 & 0 & 1 & 0 & 0 & \mu_{4,7} & \mu_{4,8} \\ 0 & 0 & 0 & 0 & 1 & 0 & \mu_{5,7} & \mu_{5,8} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mu_{6,7} & \mu_{6,8} \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & q & 0 & 0 & 0 & 0 & c & d \end{array}\right], \tag{3.13}$$

where a, b, c, d, e, and f are appropriate linear combinations of the $\mu_{i,7}$ and $\mu_{i,8}$'s. In the general case, the only entries in the last two columns needed for the determinant are the second row terms e_N and f_N and the lower right 2×2 block $\left(\begin{smallmatrix} a_N & b_N \\ c_N & d_N \end{smallmatrix}\right)$. We arrive at

$$\det(\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1} \mathbf{K}_N) = \begin{cases} p(ad - bc) + q(eb - fa) & \text{if } N \text{ is even;} \\ p(ad - bc) - q(ed - fc) & \text{if } N \text{ is odd,} \end{cases} \tag{3.14}$$

where we have suppressed the subscripts N. We need explicit formulae for the a, b, c, d, e, f, p, and q terms and for this we first need to identify explicit formulae for the $\mu_{i,k} = \delta_{i,k} - \omega^2 T^2 \lambda_{i,k}$, i.e., for $\lambda_{i,k}$, in the last two columns.

3.3. **Determining $\lambda_{i,N-1}$, $\lambda_{i,N}$, p_N, and q_N.** We were able to identify explicit formulae, Equations 3.9 and 3.10, for the entries in the first $N-2$ columns of $\mathbf{C}_N^{-1}\mathbf{K}_N$ by using the partial fraction decomposition of Equation 3.8. Unfortunately, no such simple trick presents itself to us for determining closed form expressions for the entries of the last two columns. Fortunately, though, our formulae for the $\lambda_{i,k}$'s, Equations 3.6, are of a special type, namely, of hypergeometric type, for which a general algorithmic theory exists for simplifying, if possible, such sums. This beautiful theory was brought to completion in the last decade and is presented fully and with great skill in the text [12], appropriately entitled **A = B**. There the details of the theory as well as descriptions of the `Maple` and `Mathematica` implementations of the algorithms are presented. We used `Maple` in our calculations.

The `Maple` simplifications of the sums of Formulae 3.6 for $\lambda_{i.k}$, for $k = N-1, N$, are

$$\lambda_{i,N-1} = (-1)^{N+i+1} \frac{N}{4^{N+1} i(i+1)} \frac{\Gamma(N+i+2)\Gamma(N-1)\sqrt{\pi}}{\Gamma(i)^2 \Gamma(N-i+2)\Gamma(\frac{3}{2}+N)}$$

and

$$\lambda_{i,N} = (-1)^{N+i} \frac{1}{4^{N+1} N(i-N-2) 2i(i+1)} \frac{\Gamma(N+i+2)\Gamma(N+3)\sqrt{\pi}}{\Gamma(i)^2 \Gamma(N-i+1)\Gamma(\frac{5}{2}+N)}.$$

Rewriting the Gamma functions in terms of factorials yields

$$\lambda_{i,N-1} = (-1)^{N+i+1} \frac{i}{2(N^2-1)(N-i+1)} \binom{N}{i} \binom{N+i+1}{N} \binom{2N+1}{N}^{-1}$$

(3.15)

and

$$\lambda_{i,N} = (-1)^{N+i+1} \frac{i}{2N(N-i+2)} \binom{N}{i} \binom{N+i+1}{N} \binom{2N+3}{N+1}^{-1}. \tag{3.16}$$

The entries of the matrix $\mathbf{M}_N$ are obtained by performing a sequence of column operations,

$$C_k \to C_k - \mu_{k+2,k} C_{k+2}, \qquad k = N-4, N-5, \ldots, 1,$$

on the matrix $\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1} \mathbf{K}_N$ in the order indicated, where C_k is the kth column of the matrix. Recall this yields a reduced matrix in the form of 3.11. Notice that the only nonzero entries in the first $N-2$ columns of $\mathbf{M}_N$ appear in the second and last two rows. A straightforward calculation using the entries of $\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1} \mathbf{K}_N$ computed using Formulae 3.10 yields the following formulae for these nonzero entries $M_{i,k}$. The second row, with $k = 1, 2, \ldots, N-2$, is

$$M_{2,k} = \delta_{2,k} + \frac{(k+1)!}{3k} \sum_{j=1}^{\lfloor \frac{N-k}{2} \rfloor} \frac{(-1)^j}{(2j+k-1)!} (\omega T)^{2j}, \tag{3.17}$$

which, for $k = 2$, yields 3.12, a Taylor polynomial for $\frac{\sin \omega T}{\omega T}$. The nonozero entries of the $(N-1)$st row are

$$M_{N-1,N-(2k+1)} = (-1)^{k+1} \frac{(N-1)(N-2k)!}{(N-2k-1)N!} \omega^{2k} T^{2k}, \tag{3.18}$$

for $k = 1, 2, \ldots, \lfloor \frac{N-2}{2} \rfloor$, and those of the Nth row are

$$M_{N,N-2k} = (-1)^{k+1} \frac{N(N-2k+1)!}{(N-2k)(N+1)!} \omega^{2k} T^{2k}, \tag{3.19}$$

for $k = 1, 2, \ldots, \lfloor \frac{N-1}{2} \rfloor$. $p_N(\omega T) = M_{2,2}$ has been identified as in Equation 3.12 and q_N is now identified as

$$q_N = M_{S,2} = (-1)^{\frac{S}{2}} \frac{3S}{(S+1)!} (\omega T)^{S-2}, \tag{3.20}$$

where $S = N-1$ if N is odd and $S = N$ if N is even.

3.4. **Determining a_N, b_N, c_N, and d_N.** We now reduce $\mathbf{M}_N$ with a sequence of row operations, first using the diagonal '1' to eliminate all the nonzero entries in the last two rows given by Formulae 3.18 and 3.19. This will give us the formulae for a_N, b_N, c_N, and d_N. For simplicity we assume $N = 2m$ is even, the case for N odd requiring no significant changes. The row operations used to obtain a_N and b_N are

$$R_{N-1} \to R_{N-1} - M_{N-1,k} R_k, \quad k = 1, 3, \ldots, N-3, \quad k \text{ odd},$$

where R_k is the kth row of the matrix. Those used to obtain c_N and d_N are

$$R_N \to R_N - M_{N,k}R_k, \quad k = 4, 6, \ldots, N-2, \quad k \text{ even}.$$

The resulting terms, simplified, are

$$\begin{aligned}
a_N &= 1 + \frac{(2m)!}{2(4m+1)!}\sum_{j=0}^{m-1}(-1)^{j+1}\frac{(4m-2j)!}{(2j+2)!(2m-2j-1)!}\omega^{2j+2}T^{2j+2} \\
b_N &= \frac{(m+1)(4m^2-1)(2m+1)!}{m(4m+3)!} \\
&\qquad \times \sum_{j=0}^{m-1}(-1)^{j+1}\frac{(4m-2j)!}{(2j+3)(2j+1)!(2m-2j-1)!}\omega^{2j+2}T^{2j+2} \\
c_N &= \frac{m(2m)!}{(4m^2-1)(4m+1)!}\sum_{j=0}^{m-2}(-1)^{j}\frac{(4m-2j+1)!}{(2j+1)!(2m-2j)!}\omega^{2j+2}T^{2j+2} \\
d_N &= 1 + \frac{(m+1)(2m+1)!}{(4m+3)!}\sum_{j=0}^{m-2}(-1)^{j}\frac{(4m-2j+1)!}{(j+1)(2j)!(2m-2j)!}\omega^{2j+2}T^{2j+2}.
\end{aligned} \tag{3.21}$$

3.5. **Determining e_N and f_N.** We continue to row reduce $\mathbf{M}_N$, this time using the diagonal '1' to eliminate the entries in the first $N-2$ columns of the second row given by Formulae 3.17, except for the diagonal term, which is left unchanged. The terms a_N, b_N, c_N, d_N, p_N, and q_N remain unaffected by these row operations and are reported in Equations 3.21, 3.12, and 3.20. The row operations we now employ are

$$R_2 \to R_2 - M_{2,k}R_k, \quad k = 1, 3, 4, \ldots, N-3.$$

The resulting matrix, $\mathbf{A}_N$, illustrated in 3.13 for $N = 8$, has determinant equal to that of $\mathbf{I}_N - \omega^2T^2\mathbf{C}_N^{-1}\mathbf{K}_N$ and given by Equation 3.14. The resulting terms e_N and f_N are very complicated. In terms of the entries of $\mathbf{M}_N$, they are

$$e_N = \mu_{2,N-1} - M_{2,1}\mu_{1,N-1} - \sum_{i=3}^{N-2} M_{2,i}\mu_{i,N-1} \tag{3.22}$$

and

$$f_N = \mu_{2,N} - M_{2,1}\mu_{1,N} - \sum_{i=3}^{N-2} M_{2,i}\mu_{i,N}, \tag{3.23}$$

where the $M_{2,k}$ are given in Formulae 3.17 and the $\mu_{i,N-1}$ and $\mu_{i,N}$ are calculated from 3.15 and 3.16. After some simplification, these formulae reduce to

$$e_N = \frac{m^2(2m-2)!(2m+3)!}{3(4m+1)!}\omega^2 T^2 + \frac{m(2m-2)!(2m)!}{3(4m+1)!}\sum_{j=2}^{m-1}\alpha_j \omega^{2j} T^{2j} \quad (3.24)$$

where $N = 2m$ and the coefficient α_j is

$$(-1)^{j+1}\frac{(2m+2)(2m+1)}{(2j-2)!} + \sum_{i=3}^{2m-2j+2}(-1)^{i+j}\frac{(2m+1+i)!}{i!(2j-3+i)!(2m+1-i)!},$$

and

$$f_N = \frac{(2m+1)!(2m+2)!(2m+3)!}{24m^2(2m-2)!(4m+3)!}\omega^2 T^2 + \frac{(2m+1)!(2m+2)!}{3(4m+3)!}\sum_{j=2}^{m-1}\beta_j \omega^{2j} T^{2j} \quad (3.25)$$

where the coefficient β_j is

$$(-1)^{j+1}\frac{(2m+2)}{(2j-2)!} + \sum_{i=3}^{2m-2j+2}\frac{(-1)^{i+j}}{2m}\frac{(2m+1+i)!}{i!(2j-3+i)!(2m-i)!(2m+2-i)}.$$

From Equation 3.14 $\det(\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1}\mathbf{K}_N) = p(ad-bc) + q(eb-fa)$, where we have suppressed the subscripts. Having calculated a, b, c, d, e, f, p, and q, the next task is to prove that the limit as N tends to ∞ of $ad-bc$ is 1 while that of $q(eb-fa)$ is 0. This will show that

$$\lim_{N\to\infty}\det(\mathbf{I}_N - \omega^2 T^2 \mathbf{C}_N^{-1}\mathbf{K}_N) = \lim_{N\to\infty} p_N = \sum_{j=0}^{\infty}\frac{(-1)^k}{(2j+1)!}(\omega T)^{2j} = \frac{\sin\omega T}{\omega T}.$$

3.6. **The limit of $ad - bc$ is 1.** Let $\Delta_N = a_N d_N - b_N c_N$. From Formulae 3.21, Δ_N is a polynomial of degree $N-1$ in $z = \omega^2 T^2$ with constant term 1. As such, we write

$$\Delta_N(z) = 1 + \sum_{k=1}^{\infty} D_{N,k} z^k,$$

where $D_{N,k} = 0$ for $k \geq N$. To verify that $\lim_{N\to\infty}\Delta_N(z) = 1$, it is enough to show: (i) for each fixed $k \geq 1$, $\lim_{N\to\infty} D_{N,k} = 0$ and (ii) there exists a sequence of positive terms ϑ_k such that the series $\sum \vartheta_k z^k$ converges and such that, for large enough k, $|D_{N,k}| < \vartheta_k$ for all N. A quick calculation yields the first two coefficients as

$$D_{N,1} = \frac{N+1}{2(2N+1)(2N+3)} \quad \text{and} \quad D_{N,2} = \frac{N}{16(2N+3)(4N^2-1)},$$

from which (i) is evident for $k = 1, 2$. For the full verification of (i) we find, after a lengthy calculation, that for $k \geq 3$,

$$D_{N,k} = D_{2m,k} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + J_1 + J_2, \tag{3.26}$$

where

$$I_1 = (-1)^{k-1} \frac{(m+1)(2m+1)!(4m-2k+3)!}{k(4m+3)!(2k-2)!(2m-2k+2)!},$$

$$I_2 = (-1)^{k-1} \frac{m(m+1)(2m+1)!((4m-2k+5)!}{2(k-1)(4m+1)(4m+3)!(2k-4)!(2m-2k+4)!},$$

$$I_3 = (-1)^{k-1} \frac{(m+1)(2m)!(4m-2k+4)!}{4(4m+3)(4m+1)!(2k-2)!(2m-2k+3)!},$$

$$I_4 = (-1)^{k} \frac{(2m)!(4m-2k+2)!}{2(4m+1)!(2k)!(2m-2k+1)!},$$

$$I_5 = (-1)^{k} \frac{m(m+1)(2m)!(4m-2k+5)!}{3(4m+1)(4m+3)(4m+1)!(2k-3)!(2m-2k+4)!},$$

$$I_6 = (-1)^{k} \frac{(m+1)(2m+1)!(4m-2k+4)!}{(2k-1)(4m+3)!(2k-3)!(2m-2k+3)!},$$

and $J_1 = J_2 = 0$ for $k = 3$, while

$$J_1 = (-1)^{k-1} \frac{(m+1)(2m)!(2m+1)!}{2(4m+1)!(4m+3)!} \times \sum_{j=2}^{k-2} \frac{(4m-2j+2)!(4m-2k+2j+3)!}{(k-j)(2j)!(2m-2j+1)!(2k-2j-2)!(2m-2k+2j+2)!},$$

$$J_2 = (-1)^{k} \frac{(m+1)(2m)!(2m+1)!}{(4m+1)!(4m+3)!} \times \sum_{j=2}^{k-2} \frac{(4m-2j+2)!(4m-2k+2j+3)!}{(2j+1)(2j-1)!(2m-2j+1)!(2k-2j-1)!(2m-2k+2j+2)!},$$

for $k \geq 4$. Assume for the moment that k is odd so that I_1, I_2, and I_3 are positive. Taking the limit of the I-terms yields

$$\begin{aligned}
\lim_{m\to\infty} \sum_{j=1}^{6} I_j = \frac{1}{2^{2k}(2k-4)!} \times \\
&\left[\frac{1}{2k(2k-2)(2k-3)} + \frac{1}{2(2k-2)} + \frac{1}{2(2k-2)(2k-3)}\right. \\
&\left. -\frac{1}{2k(2k-1)(2k-2)(2k-3)} - \frac{1}{3(2k-3)} - \frac{1}{(2k-1)(2k-3)}\right] \\
&= \frac{2k^3 - 9k^2 + 10k - 3}{3k(2k-1)!2^{2k}} = \frac{(k-1)(2k^2 - 7k + 3)}{3k(2k-1)!2^{2k}}.
\end{aligned} \tag{3.27}$$

Notice that this limit is zero when $k = 3$, verifying (i) in this case. When k is even, the limit is the negative of that above. Taking the limit of the J-terms yields

$$\begin{aligned}
\lim_{m\to\infty} J_1 + J_2 = \sum_{j=2}^{k-2} \frac{1}{2^{2k+1}(k-j)(2j)!(2k-2j-2)!} - \\
\sum_{j=2}^{k-2} \frac{1}{2^{2k}(2j+1)(2j-1)!(2k-2j-1)!}.
\end{aligned}$$

Multiplying by $(2k)!/(2k)!$ and rearranging yields

$$\lim_{m\to\infty} J_1 + J_2 = \frac{1}{2^{2k}(2k)!} \sum_{j=4}^{2k-3} (-1)^j (j-1) \binom{2k}{j},$$

which `Maple` evaluates as

$$\frac{1}{2^{2k}(2k)!} \left[\frac{2(4k+1)}{k(2k-1)} \binom{2k}{4} - \frac{(k-1)(4k^2 - 8k + 1)}{k(2k-1)} \binom{2k}{2k-2}\right].$$

This simplifies as

$$\lim_{m\to\infty} J_1 + J_2 = -\frac{(k-1)(2k^2 - 7k + 3)}{3k(2k-1)!2^{2k}}. \tag{3.28}$$

Again when k is even, the limit is the negative of that above. A comparison of the results of 3.27 and 3.28 verifies (i) for all $k \geq 4$.

We now determine a bound ϑ_k on $|D_{N,k}|$, independent of N. Notice that in Equation 3.26, there are six I-summands and a total of $2(k-3)$ summands in the J-sums. Each of these $2k$ summands is bounded by $(2^{2k-1}(k-2)!)^{-1}$ so that

$$\vartheta_k = \frac{k}{2^{2k-2}(k-2)!}$$

bounds $|D_{N,k}|$ for $k \geq 4$. Since $\sum_{k\geq 4} \vartheta_k z^k$ converges, (ii) is verified and we conclude that $\lim_{N\to\infty} \Delta_N(z) = 1$.

3.7. **The limit of $q(eb - fa)$ is 0.** Actually, each product, qeb and qfa, limits to 0 separately. We demonstrate this only for qfa as that for qeb is in no essential way different. Let $\Pi_N = q_N f_N a_N$. First we derive separate bounds on the three factors of Π_N. Immediately from Equation 3.20,

$$|q_N| = \frac{3N}{(N+1)!} z^{N-2} = \frac{6m}{(2m+1)!} z^{2m-2} \leq \frac{O(1)}{(2m)!} Z^{2m}, \tag{3.29}$$

where $Z = \max\{1, z\}$. We obtain a bound on a_N by rewriting its expression in Formulae 3.21 as

$$a_N = 1 + \sum_{j=0}^{m-1} (-1)^{j+1} \binom{2m}{2j} \binom{4m}{2j}^{-1} \frac{m-j}{(4m+1)(2j+2)!} z^{j+1}.$$

A straightforward calculation shows that, for $j = 0, \ldots, m-1$, $\binom{2m}{2j}\binom{4m}{2j}^{-1}$ is bounded by 1 while $\frac{m-j}{(4m+1)(2j+2)!}$ is bounded by $\frac{1}{8}$, so that $|a_N|$ is bounded by $1 + \frac{1}{8}\sum_{j=1}^{m} z^j$. We get the bound

$$|a_N| \leq O(m) Z^m. \tag{3.30}$$

For a bound on f_N, first note that the coefficient of $z = \omega^2 T^2$ in Equation 3.25 is bounded by $O(m^3)\binom{4m}{2m}^{-1}$. For $j = 2, \ldots, m-1$, the coefficient of z^j is bounded by $O(1)\binom{4m}{2m}^{-1}|\beta_j|$. We obtain a bound on $|\beta_j|$, independent of j, by rearranging factorials and bounding factors separately to obtain

$$\begin{aligned} |\beta_j| &\leq O(m) + \sum_{i=3}^{2m-2j+2} \frac{2m+1+i}{(2m)(2m+2-i)} \frac{i!}{(i+2j-3)!} \binom{2i}{i} \binom{2m+i}{2i} \\ &\leq O(m) + \sum_{i=0}^{2m} \binom{2i}{i} \binom{2m+i}{2i} \leq O(m) + \binom{4m}{2m} \sum_{i=0}^{2m} \binom{2m+i}{2i} \\ &\leq O(m) + \binom{4m}{2m} O(m) \binom{2m+m_0}{2m_0} \leq \binom{4m}{2m} O(m) \binom{2m+m_0}{2m_0}, \end{aligned} \tag{3.31}$$

where $0 \leq m_0 \leq 2m$ is the integer that gives a maximum for the expression $\binom{2m+i}{2i}$. To find out more about m_0, for $i = 0, \ldots, 2m-1$, note that

$$\binom{2m+i+1}{2i+2} = \binom{2m+i}{2i} \frac{(2m+i+1)(2m-i)}{(2i+2)(2i+1)} = \binom{2m+i}{2i} g(i), \tag{3.32}$$

where $g(i)$ is the fractional expression. Equation 3.32 implies that the expression $\binom{2m+i}{2i}$ increases when $g(i) > 1$ and decreases when $g(i) \leq 1$. Since $g(0) = m(2m+1) > 1$, $g(2m) = 0$, and $g'(x) < 0$ for all $0 \leq x \leq 2m$, there is an integer $0 < m_0 \leq 2m$ such that, for integer i, $g(i) > 1$ whenever $0 \leq i < m_0$ and $0 \leq g(i) \leq 1$ whenever $m_0 \leq i \leq 2m$. Of course, the maximum for the expression $\binom{2m+i}{2i}$ occurs at $i = m_0$. By setting $g(i) = 1$ and solving for i, one may find an exact formula for m_0, which is useless for us (asymptotically, $m_0 \approx 0.89m$). Instead, first note that $m_0 < m$ since $g(m-1) < 1$. We have

$$g(m_0) > g(m_0+1) > \cdots > g(m-1) > g(m) = \frac{(3m+1)m}{(2m+2)(2m+1)} > \frac{1}{2},$$

the last inequality holding for $m > 2$. Therefore,

$$\binom{3m}{2m} = \binom{2m+m_0}{2m_0} g(m_0)\cdots g(m-1) > \binom{2m+m_0}{2m_0}\left(\frac{1}{2}\right)^{m-m_0}.$$

This with the bound of 3.31 gives the bound

$$|\beta_j| \le \binom{4m}{2m} O(m) 2^{m-m_0} \binom{3m}{2m}. \tag{3.33}$$

Since the coefficient of z is bounded by $O(m^3)\binom{4m}{2m}^{-1}$ and, for $j = 2, \ldots, m-1$, the coefficient of z^j is bounded by $O(1)\binom{4m}{2m}^{-1}|\beta_j|$, it follows from the bound 3.33 that

$$|f_N| \le O(m^2) 2^{m-m_0} \binom{3m}{2m} Z^{m-1}. \tag{3.34}$$

From 3.29, 3.30, and 3.34,

$$|\Pi_N| \le 2^m O(m^3) \frac{1}{(2m)!}\binom{3m}{2m} Z^{4m-1}.$$

Since $\frac{1}{(2m)!}\binom{3m}{2m} < \frac{2^m}{(m!)^2}$, we get

$$|\Pi_N| \le \frac{2^{2m} O(m^3) Z^{4m-1}}{(m!)^2},$$

from which it follows that $\lim_{N\to\infty} \Pi_N = 0$.

This completes the proof that

$$\lim_{N\to\infty} \frac{\det \mathbf{B}_{N-1}}{\det(\mathbf{B}_{N-1} - \omega^2 T^2 \mathbf{J}_{N-1})} = \frac{\omega T}{\sin \omega T},$$

in Equation 3.4 and so verifies the propagator Formula 3.5.

4. Evaluating Path Integrals via Finite Fourier Sums

On pages 71–73 of [6], Feynman and Hibbs derive the propagator for the harmonic oscillator using Fourier series. We recover their computations in the context of Section 1.4 here. The computations are very easy and serve to illustrate the power of orthogonality. It is interesting that in the calculation of the propagator for the harmonic oscillator using polynomial families in the previous section, $\sin \omega T$ arises as a Taylor series while, in the calculation using Fourier families in this section, $\sin \omega T$ arises as its classical infinite product formula.

4.1. **An Easy Calculation.** We will derive the measures by examining the free particle propagator $K(T,0;0,0)$. The collection $\{\varphi_n(t) = \sin(\frac{n\pi t}{T})\}_{n=1}^{\infty}$ is orthogonal in the usual inner product

$$\langle f, g\rangle = \int_0^T f(t)g(t)\,dt$$

defined for real-valued functions on the interval $[0,T]$, and is complete in the sup-norm metric with respect to real-valued continuous functions that vanish at 0 and T; ie, each such function is the uniform limit of a sequence of real linear combinations of the φ_n. With $x(t) = \sum_{n=1}^{N} a_n\varphi_n(t)$, the action for the free particle is

$$S[x] = \int_0^T \tfrac{1}{2}m\dot{x}^2\,dt = \frac{m\pi^2}{4T}\sum_{n=1}^{N} n^2a_n^2.$$

Notice that the orthogonality of the **derivatives** of our approximating sequence, ie, $\{\dot{\varphi}_n(t) = \frac{n\pi}{T}\cos(\frac{n\pi t}{T})\}_{n=1}^{\infty}$, yields a particularly simple expression for $S[x]$ as all the cross-terms $\langle\dot{\varphi}_i, \dot{\varphi}_j\rangle$ vanish when $i \neq j$. Compare this with the expression 2.2 for the free action when the polynomial family is used. The propagator by 1.8 is

$$\begin{aligned}
K(T,0;0,0) &= \lim_{N\to\infty}\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\frac{m\pi^2}{4T}\sum_{n=1}^{N} n^2a_n^2}\,d\mu_N \\
&= \lim_{N\to\infty}\delta(N)\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\frac{m\pi^2}{4T}\sum_{n=1}^{N} n^2a_n^2}\,da_1\wedge\cdots\wedge da_N \\
&= \lim_{N\to\infty}\delta(N)\prod_{n=1}^{N}\int_{-\infty}^{\infty} e^{\frac{im\pi^2}{4\hbar T}n^2a_n^2}\,da_n \\
&= \lim_{N\to\infty}\delta(N)\prod_{n=1}^{N}\left(\frac{4i\hbar T}{m\pi n^2}\right)^{\frac{1}{2}} = \lim_{N\to\infty}\frac{\delta(N)}{N!}\left(\frac{4i\hbar T}{m\pi}\right)^{\frac{N}{2}}.
\end{aligned}$$

Setting

$$\delta(N) = N!\left(\frac{m\pi}{4i\hbar T}\right)^{\frac{N}{2}}\left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}$$

yields the free particle propagator.

The action for the harmonic oscillator is

$$S[x] = \frac{m\pi^2}{4T}\sum_{n=1}^{N}(n^2 - z^2)a_n^2,$$

where $z = \frac{\omega T}{\pi}$. Again the orthogonality of the derivative functions $\dot{\varphi}_n$ is exploited, but the orthogonality of the original approximating sequence $\{\varphi_n\}$ this time also comes into play to yield this particularly simple expression. Compare this with the expression 3.2 for the action when the polynomial family is used. The propagator for the oscillator is

$$\begin{aligned}
K(T,0;0,0) &= \lim_{N\to\infty} \delta(N) \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\frac{m\pi^2}{4T}\sum_{n=1}^N (n^2-z^2)a_n^2}\, da_1\wedge\cdots\wedge da_N \\
&= \lim_{N\to\infty} \delta(N) \prod_{n=1}^N \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\frac{m\pi^2}{4T}\sum_{n=1}^N (n^2-z^2)a_n^2}\, da_n \\
&= \lim_{N\to\infty} \delta(N) \prod_{n=1}^N \left(\frac{4i\hbar T}{m\pi(n^2-z^2)}\right)^{\frac12} \\
&= \lim_{N\to\infty} N! \left(\frac{m}{2\pi i\hbar T}\right)^{\frac12} \prod_{n=1}^N \left(\frac{1}{n^2-z^2}\right)^{\frac12} \\
&= \left(\frac{m}{2\pi i\hbar T}\right)^{\frac12} \prod_{n=1}^\infty \left(\frac{n^2}{n^2-z^2}\right)^{\frac12} \\
&= \left(\frac{m}{2\pi i\hbar T}\right)^{\frac12} \left(\frac{\pi z}{\sin\pi z}\right)^{\frac12} = \left(\frac{m\omega}{2\pi i\hbar\sin\omega T}\right)^{\frac12}.
\end{aligned}$$

5. Epilogue

How general are the calculations of the two previous sections? What dense sets of paths provide a filtration of the space Γ of all paths between spacetime points (t_0, x_0) and (t_1, x_1) whose filtration measures $d\mu_N$ yield correct propagators in the limit 1.8 when applied to, say, quadratic potentials? In Sections 2 and 3 we calculated the propagator for the quadratic potential using the set of polynomial paths dense in the set of all paths. The lack of orthogonality resulted in difficult calculations, but nonetheless succeeded in obtaining the correct propagators. In Section 4 the orthogonal family of trigonometric polynomials, with their orthogonal derivatives, reproduced the propagator for quadratic potentials quite easily. Would any dense filtration of Γ work to produce correct propagators?

In this last section we will explore this question very briefly, and for simplicity only for filtrations arising from orthonormal sets of differentiable functions complete in the set of paths with $t_0 = 0, t_1 = T, x_0 = x_1 = 0$ in the sup-norm metric on $C([0,T])$. Specifically, let $\{f_n\}$ be an orthonormal set of differentiable functions in the path space, so that $\langle f_i, f_j\rangle = \delta_{i,j}$. Define a filtration by $\Gamma_N = \mathrm{span}\{f_1,\dots,f_N\}$, the set of real linear combinations of $\{f_1,\dots,f_N\}$, and let $\Gamma_\infty = \cup_{n=1}^\infty \Gamma_N$. We assume that the closure in $C([0,T])$ of Γ_∞ equals Γ, the the set of continuous real-valued functions $x(t)$ with $x(0) = 0 = x(T)$. Note that this implies that each f_n vanishes at both 0 and T, and each path $x \in \Gamma$ is the uniform limit of a sequence from Γ_∞. We now derive the filtration measure as usual. The free particle action for the path $x(t) = \sum_{i=1}^N a_i f_i(t)$ in Γ_N is

$$S_{\text{free}}[x] = \int_0^T \tfrac12 m\dot{x}^2\,dt = \int_0^T \tfrac12 m(a_1\dot{f}_1 + \cdots + a_N\dot{f}_N)^2\,dt,$$

or in matrix notation,

$$S_{\text{free}}[x] = \tfrac12 m \sum_{i,j=1}^N a_i a_j C_{i,j} = \tfrac12 m\mathbf{a}^\dagger \mathbf{C}_N \mathbf{a},$$

where

$$C_{i,j} = \langle \dot{f}_i, \dot{f}_j \rangle = \int_0^T \dot{f}_i \dot{f}_j \, dt. \tag{5.1}$$

As in Section 2.1.3, the propagator is found by integrating against a measure $d\mu_N = \delta(N) d\mathbf{a}$ and yields

$$K(T, 0; 0, 0) = \lim_{N\to\infty} \int_{\Gamma_N = \mathbb{R}^N} e^{\frac{im}{2\hbar}\mathbf{a}^\dagger \mathbf{C}_N \mathbf{a}} \, d\mu_N = \lim_{N\to\infty} F_N(T), \tag{5.2}$$

where, as in Equation 2.10,

$$F_N(T) = \delta(N) \left(\frac{2\pi i\hbar}{m}\right)^{\frac{N}{2}} (\det \mathbf{C}_N)^{-\frac{1}{2}}.$$

The most straightforward and naive way to force the propagator of Equation 5.2 to take the correct value of $\left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}$ is to define the measure $d\mu_N = \delta(N) d\mathbf{a}$ by

$$\delta(N) = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}} \left(\frac{m}{2\pi i\hbar}\right)^{\frac{N}{2}} (\det \mathbf{C}_N)^{\frac{1}{2}}. \tag{5.3}$$

The validation of the definition of the measure $d\mu_N$ using $\delta(N)$ from Equation 5.3 would be accomplished with calculations of other propagators for nonzero potentials using this measure. We derive the validation condition for the harmonic oscillator potential. The action is

$$\begin{aligned} S[x] &= S_{\text{free}}[x] - \int_0^T \tfrac{1}{2} m\omega^2 x^2 \, dt \\ &= S_{\text{free}}[x] - \tfrac{1}{2} m\omega^2 \int_0^T (a_1 f_1 + \cdots + a_N f_N)^2 \, dt \\ &= \tfrac{1}{2} m \mathbf{a}^\dagger \mathbf{C}_N \mathbf{a} - \tfrac{1}{2} m\omega^2 \mathbf{a}^\dagger \mathbf{I}_N \mathbf{a}, \end{aligned}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix. Note the use of orthonormality in yielding the simple expression $\mathbf{a}^\dagger \mathbf{I}_N \mathbf{a}$. The propagator is

$$\begin{aligned} K(T, 0; 0, 0) &= \lim_{N\to\infty} \int_{\Gamma_N} e^{\frac{i}{\hbar}\left[\frac{1}{2} m \mathbf{a}^\dagger \mathbf{C}_N \mathbf{a} - \frac{1}{2} m\omega^2 \mathbf{a}^\dagger \mathbf{I}_N \mathbf{a}\right]} \, d\mu_N \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^N} e^{\frac{mi}{2\hbar}\mathbf{a}^\dagger\left[\mathbf{C}_N - \omega^2 \mathbf{I}_N\right]\mathbf{a}} \delta(N) \, d\mathbf{a} \\ &= \lim_{N\to\infty} \delta(N) \left(\frac{2\pi i\hbar}{m}\right)^{\frac{N}{2}} (\det(\mathbf{C}_N - \omega^2 \mathbf{I}_N))^{-\frac{1}{2}} \\ &= \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}} \lim_{N\to\infty} \left(\frac{\det(\mathbf{C}_N - \omega^2 \mathbf{I}_N)}{\det \mathbf{C}_N}\right)^{-\frac{1}{2}}. \end{aligned}$$

The condition then that would validate the definition of our filtration measure using Equation 5.3 is

$$\lim_{N\to\infty} \frac{\det(\mathbf{C}_N - \omega^2 \mathbf{I}_N)}{\det \mathbf{C}_N} = \lim_{N\to\infty} \det(\mathbf{I}_N - \omega^2 \mathbf{C}_N^{-1}) = \frac{\sin \omega T}{\omega T}, \tag{5.4}$$

where, of course, the entries $C_{i,j}$ are given by Equation 5.1. The confirmation of Condition 5.4 for the Fourier orthogonal family accomplished in the previous section is particularly easy, since not only is the Fourier family $f_n = \varphi_n = \sin \frac{n\pi t}{T}$ orthogonal, but its family of derivatives $\dot{f}_n$ is also othogonal, making the matrix $\mathbf{C}_N$ diagonal. In what generality Condition 5.4 holds is an open question.

If orthogonality of the family $\{f_n\}$ is not assumed, the condition for validation is

$$\lim_{N\to\infty} \det(\mathbf{I}_N - \omega^2 \mathbf{C}_N^{-1} \mathbf{K}_N) = \frac{\sin \omega T}{\omega T}, \tag{5.5}$$

for appropriate matrices $\mathbf{K}_N$, as in Section 3. The fact that Condition 5.5 holds for the polynomial family seems to us quite surprising and very suggestive that the validation condition holds quite generally.

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