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Non-trapping condition for semiclassical Schrödinger operators with matrix-valued potentials.

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Abstract

We consider semiclassical Schrödinger operators with matrix-valued, long-range, smooth potential, for which different eigenvalues may cross on a codimension one submanifold. We denote by h the semiclassical parameter and we consider energies above the bottom of the essential spectrum. Under a certain special condition on the matricial structure of the potential at the eigenvalues crossing and a certain structure condition at infinity, we prove that the boundary values of the resolvent at energy λ, as bounded operators on suitable weighted spaces, are $O(h^{-1})$ *if and only if* λ is a non-trapping energy for all the Hamilton flows generated by the eigenvalues of the operator's symbol.
Keywords: Non-trapping condition, eigenvalues crossing, Schrödinger matrix operators, Mourre theory, semiclassical resolvent estimates, coherent states, Egorov's theorem, semiclassical measure.

1 Introduction and main results.

In the scattering theory of Schrödinger operators, several semiclassical results, with respect to the "Planck constant", are based on semiclassical resolvent estimates (as a typical example, see [RT]). If one seeks for similar results in the semiclassical framework of the Born-Oppenheimer approximation for molecular Schrödinger operators, it is interesting to generalize these semiclassical resolvent estimates to Schrödinger operators with operator-valued potential (see [J2, J3]). On this way, it is worth to treat the case of matrix-valued potentials (see [KMW1, J1]). Among the molecular Schrödinger operators, the diatomic ones are simpler since they have a two-body nature with a spherical symmetry. If one focus on them then it is relevant to study two-body semiclassical Schrödinger matrix operators with radial potential.
In order to present this project in detail, we need some notation. Taking $m \in \mathbb{N}^*$, let $\mathcal{M}_m(\mathbb{C})$ be the algebra of $m \times m$ matrices with complex coefficients, endowed with the operator norm denoted by $\| \cdot \|_m$. We denote by I_m the corresponding identity matrix. Let $d \in \mathbb{N}^*$ and let $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$ be the space of $\mathbb{C}^m$-valued L^2 functions on $\mathbb{R}^d$, equipped with its usual norm $\| \cdot \|$ and scalar product $\langle \cdot, \cdot \rangle$. We denote by Δ_x the Laplacian in $\mathbb{R}^d$. The semiclassical parameter is $h \in]0; h_0]$, for some $h_0 > 0$. The semiclassical Schrödinger operator we consider is the unbounded operator

$$\hat{P}(h) \ := \ -h^2 \Delta_x \mathrm{I}_m + M(x) \, , \tag{1.1}$$

acting in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$, where $M(x)$ is the multiplication operator by a self-adjoint matrix $M(x) \in \mathcal{M}_m(\mathbb{C})$. We require that M is smooth on $\mathbb{R}^d$ and tends, as $|x| \to \infty$, to some matrix M_∞, which is of course self-adjoint. Furthermore we demand that the potential $M(x) - M_\infty$ is long-range (see (2.1)). It is well known that, under this assumption on M, the operator $\hat{P}(h)$ is self-adjoint on the domain of $\Delta_x \mathrm{I}_m$ (see [RS2] for instance). Its resolvent will be denoted by $R(z; h) := (\hat{P}(h) - z)^{-1}$, for z in the resolvent set of $\hat{P}(h)$. Setting $\langle x \rangle := (1 + |x|^2)^{1/2}$, we denote by $\mathrm{L}^{2,s}(\mathbb{R}^d; \mathbb{C}^m)$, for $s \in \mathbb{R}$, the weighted L^2 space of mesurable, $\mathbb{C}^m$-valued functions f on $\mathbb{R}^d$ such that $x \mapsto \langle x \rangle^s f(x)$ belongs to $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. It follows from Mourre theory ([Mo, CFKS]), with the dilation generator as scalar conjugate operator, that the resolvent has boundary values $R(\lambda \pm i0)$, as bounded operators from $\mathrm{L}^{2,s}(\mathbb{R}^d; \mathbb{C}^m)$ to $\mathrm{L}^{2,-s}(\mathbb{R}^d; \mathbb{C}^m)$ for any $s > 1/2$, provided that λ is outside the pure point spectrum of $\hat{P}(h)$ and above the operator norm $\|M_\infty\|_m$ of the matrix M_∞.
A careful inspection of the paper by Froese and Herbst (cf. [FH, CFKS]) shows that its result, namely the absence of eigenvalues above the bottom of the essential spectrum, extends to the present situation. This means that the boundary values $R(\lambda \pm i0; h)$ are well defined for any $\lambda > \|M_\infty\|_m$. It turns out that we do not need this fact here (see

Remark 3.2). We choose to ignore it in the formulation of our results. This could be useful for a possible generalization of our results to other operators, for which eigenvalues may be embedded in the essential spectrum.
The operator $\hat{P}(h)$ in (1.1) can be viewed as a h-pseudodifferential operator obtained by Weyl h-quantization (see (3.2)) of the symbol P defined by

$$\forall\, \mathrm{x}^* := (x, \xi) \in T^*\mathbb{R}^d\,, \quad P(\mathrm{x}^*) \ := \ |\xi|^2 \mathrm{I}_m \ + \ M(x)\,, \tag{1.2}$$

with self-adjoint values in $\mathcal{M}_m(\mathbb{C})$. Notice that this symbol does not depend on the kind of h-quantization (cf. [Ba, R]).
Let $\lambda > \|M_\infty\|_m$. Our aim in this paper is to characterize, in terms of the energy λ and of the symbol P, the following property: for all $s > 1/2$, there is an open interval I about λ and $h_0 > 0$ such that, for all $\mu \in I$, for all $h \in]0; h_0]$,

$$\text{there exists} \quad \lim_{\epsilon \to 0^+} \langle x\rangle^{-s} R(\mu + i\epsilon; h) \langle x\rangle^{-s} \ =: \ \langle x\rangle^{-s} R(\mu + i0; h) \langle x\rangle^{-s} \tag{1.3}$$

$$\text{and } \exists C_{s,I} > 0\,;\ \forall \mu \in I\,;\ \forall h \in]0; h_0]\,, \quad \left\| \langle x\rangle^{-s} R(\mu + i0; h) \langle x\rangle^{-s} \right\| \ \leq \ C_{s,I} \cdot h^{-1}\,. \tag{1.4}$$

Here $\|\cdot\|$ denotes the operator norm of the bounded operators on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. For short, we call this property "the property ((1.3) and (1.4))", which covers the semiclassical resolvent estimates mentionned above.

Remark 1.1. *Taking the adjoint of $R(\mu+i\epsilon; h)$ in* (1.3) *and* (1.4)*, we see that the property* ((1.3) *and* (1.4)) *implies the same property with $R(\mu + i\epsilon; h)$ replaced by $R(\mu - i\epsilon; h)$.*

In view of the goal described at the beginning, we just need a sufficient condition to have the property ((1.3) and (1.4)). Why are we interested in a characterization? First, having a characterization is a good way to avoid a sufficient condition which might be too strong (and perhaps unreasonable). Secondly, the property ((1.3) and (1.4)) means that there is no resonance (if it can be defined) with imaginary part $o(h)$ and real part in I (cf. [Ma]), therefore a characterization of this property essentially gives one of the semiclassically non-resonant scattering.
In the scalar case ($m = 1$), a characterization is well-known (see [RT, GM, W, VZ, B, J5]). Let us describe it. Let $q \in C^\infty(T^*\mathbb{R}^d; \mathbb{R})$ be a Hamilton function and denote by ϕ^t its Hamilton flow. An energy μ is non-trapping for q (or for the flow ϕ^t) if

$$\forall \mathrm{x}^* \in q^{-1}(\mu), \quad \lim_{t \to -\infty} |\phi^t(\mathrm{x}^*)| \ = \ +\infty \quad \text{and} \quad \lim_{t \to +\infty} |\phi^t(\mathrm{x}^*)| \ = \ +\infty\,. \tag{1.5}$$

For $m = 1$, the property ((1.3) and (1.4)) at energy λ holds true if and only if λ is a non-trapping energy for the symbol P.

In the matricial case, the usual definition of the Hamilton flow for a matrix-valued Hamilton function does not make sense. But we hope that there is a characterization on the symbol P. A priori, it is not clear that such a characterization exists and the paper [J4] does not give an optimistic impression (see below), thus it is already interesting to find one in some special (non trivial) case.
In [J1], we proved that, if the multiplicity of the eigenvalues of P is constant, then the non-trapping condition at energy λ for each eigenvalue (see (1.5)) is sufficient to get the property ((1.3) and (1.4)) at energy λ. The proof is based on a semiclassical version of the previously mentioned Mourre theory (see [RT, GM] for the scalar case). The converse actually holds true by Theorem 1.8. In this case, we thus have a satisfactory answer but requiring no eigenvalues crossing is too restrictive for physical applications (cf. [H]). The general case is, however, much more complicated. Indeed, one can actually find a smooth matrix-valued potential M such that this non-trapping condition does not make sense, since the eigenvalues are not regular enough (cf. [K]). In such a case, we showed in [J4] that the usual semiclassical version of Mourre theory does not work if a certain geometrical condition at the crossing is not satisfied.
Here, we choose to look at the case where a simple eigenvalues crossing occurs and where the eigenvalues and the corresponding eigenprojections are smooth. This situation already contains physically interesting potentials (see Remark 1.6). In [J4], we considered such a case but under a much stronger restriction (essentially, we needed $m = 2$). Assuming that the variation of the eigenprojections of M is "small enough", we showed there that the previous non-trapping condition implies the resolvent estimates. Since it is rather limited and unsatisfactory (see also Section 5), we think it is already worth to better understand this case before treating the general one.
Let us now present our main results. They concern potentials M for which eigenvalues may cross on a codimension 1 submanifold of $\mathbb{R}^d$ and satisfying a structure condition near the crossing (à la Hagedorn in [H]). We call such a potential a "model of codimension 1 crossing". The eigenvalues of P are smooth Hamilton functions for which the non-trapping condition (1.5) makes sense. Two other assumptions play an important role in the results. The first one says that the eigenprojections of M are conormal to the eigenvalues crossing and is called the "special condition at the crossing". The second one roughly means that, if eigenvalues cross "at infinity", the structure condition of the model of codimension 1 crossing is satisfied "at infinity", and is called the "structure condition at infinity". Precised definitions and details are provided in Definitions 2.6, 2.7, 2.9, 2.12, and 2.13, in Subsection 2.2.

Theorem 1.2. *Consider a model of codimension* 1 *crossing that satisfies the special condition at the crossing and the structure condition at infinity. Let* $\lambda > \|M_\infty\|_m$. *The* *property* ((1.3) *and* (1.4)) *holds true if and only if* λ *is non-trapping for all eigenvalues of the symbol* P *of* $\hat{P}(h)$.

The "if" (resp. "only if") part of Theorem 1.2 follows from the following Theorem 1.3 (resp. Theorem 1.4), which will be proved in Subsection 3.3 (resp. Subsection 4.2).

Theorem 1.3. *Consider a model of codimension* 1 *crossing that satisfies the special condition at the crossing. Let* $\lambda > \|M_\infty\|_m$. *If* λ *is non-trapping for all eigenvalues of the symbol* P *of* $\hat{P}(h)$, *then the property* ((1.3) *and* (1.4)) *holds true.*

Theorem 1.4. *Consider a model of codimension* 1 *crossing that satisfies the structure condition at infinity. Let* $\lambda > \|M_\infty\|_m$. *If the property* ((1.3) *and* (1.4)) *holds true then* λ *is non-trapping for all eigenvalues of the symbol* P *of* $\hat{P}(h)$.

Remark 1.5. *If* $d = 1$ *(i.e. if the variable* x *lies in* $\mathbb{R}$*), we do not need the special condition at the crossing to prove Theorem 1.3, as shown in Remark 3.8.*
In view of [H] and for $d > 1$, *it is natural to doubt that a condition on the eigenvalues of* P *only (namely the non-trapping condition) implies the resolvent estimates. The corresponding difficulty is removed by the special condition at the crossing. We shall explain this in detail in Section 5.*
Theorem 1.3 applies for a model of codimension 1 *crossing with a radial potential* M, *that is when* M *is a function of* $|x|$ (*cf. Remark 2.11*). *If there is "no crossing at infinity", then the structure condition at infinity is automatically satisfied and Theorem 1.4 applies with a simpler proof. We point out that we actually need the previous assumptions only "below" the considered energy* λ (*cf. Remark 2.15*).
In Example 2.17, we give concrete potentials M *satisfying the assumptions of Theorem 1.2.*

Remark 1.6. *Let us comment on our main physical motivation, namely the Born-Oppenheimer approximation for molecules (cf. [KMW1, KMW2, J1]). For the present semiclassical problem, one can see in [J1], that under certain conditions, the replacement of the true Schrödinger operator by a matricial Schrödinger operator is relevant. Since we want to focus on the role of the eigenvalues crossing, we choose to remove the Coulomb singularities appearing in the molecular operator. Notice that repulsive Coulomb singularities are included in [KMW2]. Finally, for diatomic molecules, the operator-valued potential only depends on* $|x|$ *(*x *being essentially the relative position of the two nuclei). Therefore, it is physically motivated to consider radial potentials* M, *as we did in Remark 1.5.*

Although we decided to focus on codimension 1 crossing, we derive some partial results concerning the general case in Subsections 2.1, 3.1, 3.2, 4.1, and in Section 5. Among them, we point out Proposition 1.7. ¿From Theorem 1.2, we see that the property ((1.3) and (1.4)) is in fact independent of $s > 1/2$, if the non-trapping condition is satisfied. This actually holds true in a more general framework, as proved in Proposition 1.7 below. In general, it follows from Mourre theory that, if the limiting absorption principle (1.3) holds true for some $s > 1/2$, it holds true for any $s > 1/2$. For our Schrödinger operators, the boundary values (1.3) even exist for any $\mu > 0$ and any $s > 1/2$, as already remarked above. The independence with respect to s of the estimate (1.4) is shown in

Proposition 1.7. *Consider a general model and let $\lambda > \|M_\infty\|_m$. If, for* some $s > 1/2$, *the resolvent estimate* (1.4) *holds true near λ and for h_0 small enough, then this is true for* any $s > 1/2$.

The proof appears at the end of Subsection 3.1. As announced above, we can complete the result in [J1] as explained in Theorem 1.8. The relevant eigenvalues at energy λ, appearing in Theorem 1.8, are functions $x \mapsto \alpha(x)$ such that $\alpha(x)$ is an eigenvalue of $M(x)$ and such that $\alpha^{-1}(]-\infty;\lambda]) \neq \emptyset$ (cf. Definitions 2.2 and 2.4).

Theorem 1.8. *Consider a general model, let $\lambda > \|M_\infty\|_m$, and assume that the relevant eigenvalues crossing at energy λ is empty. Then the property* ((1.3) *and* (1.4)) *holds true if and only if λ is non-trapping for all relevant eigenvalues of the symbol P at energy λ.*

In [J1], the "if" part was proved and we shall see another proof in Section 5 (cf. Proposition 5.2). The "only if" part follows from Proposition 4.1.

Concerning the proof of our results, we want to add some comments. For the proof of Theorem 1.3 (see Section 3), we do not follow the semiclassical version of Mourre method, as in [J1, J4]. We prefer to use Burq's strategy (see [B]), that we adapted to the scalar, long-range scattering in [J5]. Most of the results obtained in [J5] are easily extended to the present situation, in the general case, and we also get Proposition 1.7. Following [J5] further, we meet a matricial difficulty (in Subsection 3.2), based on the fact that matrices do not commute in general. For the model of codimension 1 crossing satisfying the special condition at the crossing, we can remove it and conclude as in [J5]. In Section 5, we comment on the nature of this condition. We also compare the present result with those in [J1, J4] and show how the later can be proved along the present lines.

To prove Theorem 1.4 (see Section 4), we follow the strategy in [W], which works in the scalar case. If the eigenvalues crossing is empty, it is easy to adapt it and get the "only if" part of Theorem 1.8. Since this strategy relies on a semiclassical Egorov's theorem, which is not available for general matrix operators (even for the model of codimension 1 crossing), we need an important modification of Wang's lines. Its main ingredient is a

Gronwall type argument (see Subsection 4.2).
To end this introduction, let us connect our results to other questions treated in the literature. If one considers empty eigenvalues crossings, we refer to [S] for scattering considerations and to [BG, BN] for the semiclassical Egorov's theorem. When eigenvalues cross, it is interesting to look at the case where resonances appear, as in [Né], since the resolvent estimates might be false in this case. Finally, it is worth to compare our results with those around the propagation of coherent states and the Landau-Zener formula. Among others, we refer to [CLP, FG, H]. Since the time is kept bounded and the evolution is measured strongly (not in norm), these results cannot imply ours, but they do give an idea of what might happen in our framework. For instance, they do not reveal any capture phenomenon at the crossing, giving some hope to find a simple non-trapping condition in some cases (cf. Section 5). Actually, this strongly motivated the present work. We believe that we can learn more from these papers and also from [Ka], where a certain norm control on finite time evolution is given.

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2 Detailled assumptions on the models.

In this section, we describe precisely the considered models. We first introduce the general case. Then we present a certain class of codimension 1 eigenvalues crossings à la Hagedorn in [H]. Finally, we explain the special condition at the crossing and the structure condition at infinity, that appear in Theorem 1.2.

2.1 General case.

In our general framework, we assume that the potential M of the semiclassical Schrödinger operator $\hat{P}(h)$, given in (1.1), is a self-adjoint matrix valued, smooth, long-range function on $\mathbb{R}^d$. This means that the values of M belong to $\mathcal{M}_m(\mathbb{C})$ and are self-adjoint, that M is C^∞ on $\mathbb{R}^d$, and that there exist some $\rho > 0$ and some self-adjoint matrix M_∞ such that

$$\forall \gamma \in \mathbb{N}^d\,,\ \forall x \in \mathbb{R}^d\,,\quad \Big\| \partial_x^\gamma \Big(M(x) - M_\infty \Big) \Big\|_m \;=\; O_\gamma\Big(\langle x \rangle^{-\rho - |\gamma|} \Big) \tag{2.1}$$

where $\|\cdot\|_m$ denotes the operator norm on $\mathcal{M}_m(\mathbb{C})$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$.
As already mentioned in Section 1, the operator $\hat{P}(h)$ is, if M satisfies (2.1), self-adjoint on the domain of $\Delta_x \mathrm{I}_m$, which is isomorphic to the mth power of the domain of the Δ_x in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C})$. Furthermore, $\hat{P}(h)$ may be viewed as the Weyl h-quantization (see (3.2)) of the symbol P given by (1.2). Since we want to consider the resolvent of $\hat{P}(h)$ near some energy $\lambda \in \mathbb{R}$, we introduce smooth localization functions near λ. Precisely, for $\epsilon > 0$, let $\theta \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that $\theta(\lambda) \neq 0$ and $\mathrm{supp}\,\theta \subset]\lambda - \epsilon; \lambda + \epsilon[$. The operator $\theta(\hat{P}(h))$ localizes in energy for $\hat{P}(h)$ near λ. Furthermore, according to [Ba], it is a h-pseudodifferential operator satisfying

$$\theta\Big(\hat{P}(h)\Big) \;=\; \Big(\theta(P)\Big)_h^w \,+\, h\, D(h)$$

for some uniformly bounded operator $D(h)$ (cf. (3.3)). Here the function $\theta(P) : T^*\mathbb{R}^d \longrightarrow \mathcal{M}_m(\mathbb{C})$ maps any $\mathrm{x}^* \in T^*\mathbb{R}^d$ to the matrix $\theta(P(\mathrm{x}^*))$ given by the functional calculus of the self-ajoint matrix $P(\mathrm{x}^*)$.
Up to some error $O(h)$, the corresponding phase space localization of $\theta(\hat{P}(h))$ is thus given by the support of its principal symbol $\theta(P)$:

$$\mathrm{supp}\ \theta(P) \;=\; \Big\{ \mathrm{x}^* \in T^*\mathbb{R}^d;\ \exists \mu \in \mathrm{supp}\,\theta;\ \det\Big(P(\mathrm{x}^*) - \mu \mathrm{I}_m\Big) = 0 \Big\}\,.$$

It is thus natural to consider the open set

$$E^*(\lambda, \epsilon_0) \;:=\; \bigcup_{\mu \in]\lambda - \epsilon_0; \lambda + \epsilon_0[} \Big\{ \mathrm{x}^* \in T^*\mathbb{R}^d;\ \det\Big(P(\mathrm{x}^*) - \mu \mathrm{I}_m\Big) = 0 \Big\}\,, \tag{2.2}$$

for some $\epsilon_0 > 0$, as an energy localization near λ. The energy shell of P at energy λ is

$$E^*(\lambda) \;:=\; \Big\{ \mathrm{x}^* \in T^*\mathbb{R}^d;\ \det\Big(P(\mathrm{x}^*) - \lambda \mathrm{I}_m\Big) = 0 \Big\}\,. \tag{2.3}$$

Now, let us explain what we mean by eigenvalues crossing. Here, we shall repeatedly make us of arguments from [K]. Let $x \in \mathbb{R}^d$. Using the min-max principle (cf. [RS4]), we find m eigenvalues $\alpha_1(x) \leq \cdots \leq \alpha_m(x)$ of $M(x)$. We can extract from them $k(x)$

distinct eigenvalues of $M(x)$, with $k(x) \in \{1; \cdots ; m\}$, namely $\lambda_1(x) < \cdots < \lambda_{k(x)}(x)$, and we denote by $\Pi_1(x), \cdots , \Pi_{k(x)}(x)$ the corresponding orthogonal eigenprojections. For $j \in \{1; \cdots ; k(x)\}$, the number $m'_j(x)$ of indices $\ell \in \{1; \cdots ; m\}$ such that $\alpha_\ell(x) = \lambda_j(x)$ is the multiplicity of the eigenvalue $\lambda_j(x)$. Alternatively, we see that, in $\alpha_1(x) \leq \cdots \leq \alpha_m(x)$, $m'_j(x)$ is the number of equalities between the $(j-1)$th inequality and the jth one. For $\ell \in \{1; \cdots ; m\}$ such that $\alpha_\ell(x) = \lambda_j(x)$, we set $m_\ell(x) = m'_j(x)$. This defines functions m_ℓ for all $\ell \in \{1; \cdots ; m\}$. Notice that all α_ℓ, all λ_j are continuous, and that (2.3) may be written as

$$E^*(\lambda) \;=\; \bigcup_{1\leq \ell\leq m} \Big\{\mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d;\ |\xi|^2 + \alpha_\ell(x) \;=\; \lambda\Big\}\,. \tag{2.4}$$

The set of discontinuities of k is also the union over ℓ of the set of discontinuities of m_ℓ. This leads to

Definition 2.1. *The set of discontinuities of k is called the eigenvalues crossing and is denoted by $\mathcal{C}$.*

Notice that, outside $\mathcal{C}$, the distinct eigenvalues and the corresponding orthogonal projections are smooth (cf. [K]). For the resolvent estimates at energy λ, the relevant part of the crossing is below the energy λ (cf. [J1]). This leads to the

Definition 2.2. *Let $\mu \in \mathbb{R}$. By definition, the relevant eigenvalues crossing at energy μ, denoted by $\mathcal{C}_r(\mu)$, is given by*

$$\begin{aligned} \mathcal{C}_r(\mu) \;&:=\; \Big\{x \in \mathbb{R}^d\,;\, \exists j \in \{1, \cdots , m\}\,;\, \alpha_j(x) \leq \mu \\ &\qquad\qquad \textit{and } m_j \textit{ is discontinuous at } x\Big\}\,. \end{aligned} \tag{2.5}$$

The influence of the relevant eigenvalues crossing at energy μ takes place in the following subset $\mathcal{C}^*(\mu)$ of $E^*(\mu)$, called the crossing region at energy μ,

$$\begin{aligned} \mathcal{C}^*(\mu) \;&:=\; \Big\{\mathrm{x}^* = (x, \xi) \in \mathcal{C}_r(\mu) \times \mathbb{R}^d\,;\, \exists j \in \{1, \cdots , m\}; \\ &\qquad |\xi|^2 + \alpha_j(x) = \mu \textit{ and } m_j \textit{ is discontinuous at } x\Big\}\,. \end{aligned} \tag{2.6}$$

In Proposition 2.3 below, we give a useful characterization of the crossing region at some energy. To this end we introduce

$$\chi \in C_0^\infty(\mathbb{R}; \mathbb{R}^+) \text{ such that } \chi^{-1}(]0; +\infty[) =]-1; 1[\text{ and } \chi(t; \mu, \epsilon) := \chi\Big((t-\mu)/\epsilon\Big) \tag{2.7}$$

for $\mu \in \mathbb{R}$, $\epsilon > 0$. Recall that $\chi(P(\mathrm{x}^*); \mu, \epsilon)$ is well-defined by the functional calculus of the self-adjoint matrix $P(\mathrm{x}^*)$.

Proposition 2.3. *Let $\mu \in \mathbb{R}$, $\epsilon > 0$. Let χ and $\chi(\cdot;\cdot,\cdot)$ be as in (2.7). As a subset of $T^*\mathbb{R}^d$, let $S^*(\mu,\epsilon)$ be the support of the multiplication operator defined by*

$$C_0^\infty\Big(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C})\Big) \ni A \;\mapsto\; \Big(\mathrm{x}^* \;\mapsto\; \chi\Big(P(\mathrm{x}^*);\mu,\epsilon\Big)\Big[P(\mathrm{x}^*),A(\mathrm{x}^*)\Big]\chi\Big(P(\mathrm{x}^*);\mu,\epsilon\Big)\Big)\,.$$

Then $S^*(\mu,\epsilon) \subset E^*(\mu,\epsilon)$, $\epsilon \mapsto S^*(\mu,\epsilon)$ is non decreasing, and

$$\mathcal{C}^*(\mu) \;=\; \bigcap_{\epsilon>0} S^*(\mu,\epsilon)\,. \tag{2.8}$$

If $\mathcal{C}^*(\mu)$ is empty and if there exist $\epsilon_0, R > 0$ such that

$$|x| \;\geq\; R \;\Longrightarrow\; \forall j,k,\ \Big(\lambda_j(x) = \lambda_k(x)\ \text{or}\ |\lambda_j(x)-\lambda_k(x)| \;\geq\; \epsilon_0\Big)\,,$$

then there exists $\epsilon > 0$ for which the multiplication operator is zero.

Proof: Let $\epsilon < \epsilon'$. Since $\chi(\cdot;\mu,\epsilon)/\chi(\cdot;\mu,\epsilon')$ is smooth, we see that the complement of $S^*(\mu,\epsilon)$ contains the complement of $S^*(\mu,\epsilon')$. This shows that $S^*(\mu,\epsilon) \subset S^*(\mu,\epsilon')$. Since $E^*(\mu,\epsilon) \supset \operatorname{supp}\chi(P(\cdot);\mu,\epsilon)$, $S^*(\mu,\epsilon) \subset E^*(\mu,\epsilon)$. Denote by $\mathcal{D}^*(\mu)$ the right hand side of (2.8). From $S^*(\mu,\epsilon) \subset E^*(\mu,\epsilon)$ for all $\epsilon > 0$, we derive that $\mathcal{D}^*(\mu) \subset E^*(\mu)$. Recall that $\mathcal{C}^*(\mu) \subset E^*(\mu)$ (cf. (2.6) and (2.4)).
Let $\mathrm{x}_0^* = (x_0;\xi_0) \in E^*(\mu) \setminus \mathcal{C}^*(\mu)$ and let N be the number of distinct eigenvalues of $M(x_0)$ below μ. Near x_0, $M(x)$ has N smooth distinct eigenvalues $\lambda_1(x) < \cdots < \lambda_N(x)$. The associated orthogonal projections $\Pi_1(x),\cdots,\Pi_N(x)$ are also smooth. For any $A \in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$, which is supported close enough to x_0^*, for small enough ϵ, for all $\mathrm{x}^* \in T^*\mathbb{R}^d$,

$$\begin{aligned}
&\chi\Big(P(\mathrm{x}^*);\mu,\epsilon\Big)\Big[P(\mathrm{x}^*),A(\mathrm{x}^*)\Big]\chi\Big(P(\mathrm{x}^*);\mu,\epsilon\Big)\\
=\;& \sum_{j,\ell\in\{1,\cdots,N\}} \chi\Big(|\xi|^2+\lambda_j(x);\mu,\epsilon\Big)\chi\Big(|\xi|^2+\lambda_\ell(x);\mu,\epsilon\Big)\Pi_j(x)\Big[P(\mathrm{x}^*),A(\mathrm{x}^*)\Big]\Pi_\ell(x)\\
=\;& \sum_{j\in\{1,\cdots,N\}} \chi^2\Big(|\xi|^2+\lambda_j(x);\mu,\epsilon\Big)\Pi_j(x)\Big[|\xi|^2+\lambda_j(x),A(x,\xi)\Big]\Pi_j(x) \;=\; 0\,.
\end{aligned}$$

Thus $\mathrm{x}_0^* \notin \mathcal{D}^*(\mu)$. This also proves the last statement of Proposition 2.8.
Let $\mathrm{x}_0^* = (x_0;\xi_0) \in \mathcal{C}^*(\mu)$. Let $c > 0$ such that $\chi(t) \geq c$ if $|t| \leq 1/2$. By Definition 2.1 and (2.6), the number of equalities between the $(r-1)$th and the rth inequalities in $\alpha_1(x) \leq \cdots \leq \alpha_m(x)$, for some r, is discontinuous at x_0 and $\mu = |\xi_0|^2 + \alpha_r(x_0)$. So, we can find, for any $\epsilon > 0$, some $y(\epsilon)$ with $|y(\epsilon)-x_0| < \epsilon$ such that, among the distinct eigenvalues $\lambda_1(y(\epsilon)),\cdots,\ \lambda_{k(y(\epsilon))}(y(\epsilon))$ of $M(y(\epsilon))$, there are two, say $\lambda_{j(\epsilon)}(y(\epsilon))$ and $\lambda_{\ell(\epsilon)}(y(\epsilon))$, such

that, for $s \in \{j(\epsilon), \ell(\epsilon)\}$, $|\lambda_s(y(\epsilon)) - \alpha_r(x_0)| < \epsilon/2$. For such s, $|\,|\xi_0|^2 + \lambda_s(y(\epsilon)) - \mu\,| < \epsilon/2$. Thus $\chi(|\xi|^2 + \lambda_{j(\epsilon)}(y(\epsilon)); \mu, \epsilon) \cdot \chi(|\xi|^2 + \lambda_{\ell(\epsilon)}(y(\epsilon)); \mu, \epsilon) \geq c^2$. Now, using a diagonalization of $M(y(\epsilon))$, we can construct the matrix $E(\epsilon)$ that exchanges an eigenvector of $\lambda_{j(\epsilon)}(y(\epsilon))$ with an eigenvector of $\lambda_{\ell(\epsilon)}(y(\epsilon))$. Let ψ_ϵ be a smooth, scalar cut-off function that localizes near $(y(\epsilon), \xi_0)$. Let

$$A(x,\xi) = \psi_\epsilon(x,\xi) \left(\Pi_{j(\epsilon)}(x) + \Pi_{\ell(\epsilon)}(x)\right) \cdot E(\epsilon) \cdot \left(\Pi_{j(\epsilon)}(x) + \Pi_{\ell(\epsilon)}(x)\right).$$

Then

$$\mathrm{x}^* \;\mapsto\; \chi\Big(P(\mathrm{x}^*); \mu, \epsilon\Big) \left[P(\mathrm{x}^*), A(\mathrm{x}^*)\right] \chi\Big(P(\mathrm{x}^*); \mu, \epsilon\Big)$$

is a non-zero function since it does not vanish at $y(\epsilon)$. Therefore $(y(\epsilon), \xi_0) \in S^*(\mu, \epsilon)$. Let $\epsilon_n \to 0$. For any $\epsilon > 0$, $(y(\epsilon_n), \xi_0) \in S^*(\mu, \epsilon_n) \subset S^*(\mu, \epsilon)$, for n large enough, since $S^*(\mu, \cdot)$ is non-decreasing. Since $S^*(\mu, \epsilon)$ is closed, $(x_0, \xi_0) = \lim_n (y(\epsilon_n), \xi_0)$ belongs to $S^*(\mu, \epsilon)$. This shows that $(x_0, \xi_0) \in \mathcal{D}^*(\mu)$. □

In view of Theorem 1.8, we need to explain in the general case what are the eigenvalues of the symbol P and the corresponding Hamilton flows.
For $j \in \{1, \cdots, m\}$, let $D_j := \{x \in \mathbb{R}^d;\, j \in \{1, \cdots, k(x)\}\}$. If $D_j \neq \emptyset$, the functions $\lambda_j, \Pi_j : D_j \longrightarrow \mathbb{R}$, $x \mapsto \lambda_j(x)$, $x \mapsto \Pi_j(x)$, are well defined and are smooth outside $\mathcal{C}$.

Definition 2.4. *Let j such that $D_j \neq \emptyset$. The functions λ_j, Π_j are called respectively the j^{th} eigenvalue and eigenprojection of M. The function $p_j : D_j \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $\mathrm{x}^* = (x,\xi) \mapsto |\xi|^2 + \lambda_j(x)$ is called the j^{th} eigenvalue of P. We denote by ϕ_j^t its Hamilton flow. If $\mu \in \mathbb{R}$, the relevant eigenvalues of P at energy μ are the p_j such that $p_j^{-1}(\mu) \neq \emptyset$.*

Assume that $\mathcal{C}_r(\mu)$, the relevant eigenvalues crossing at energy μ (cf. (2.5)), is empty. Then each relevant eigenvalues p_j of P at energy μ is smooth on $p_j^{-1}(]\mu - \epsilon; \mu + \epsilon[)$, for some $\epsilon > 0$. For those p_j, the corresponding Hamilton flow can be defined as usual there. To close this subsection, we recall a well known fact on smooth projection-valued functions.

Proposition 2.5. *Let Ω be an open set of $\mathbb{R}^d$ and $\Pi : \Omega \longrightarrow \mathcal{M}_m(\mathbb{C})$ be a smooth, projection-valued function, i.e. $\Pi(x)^2 = \Pi(x)$, for all $x \in \Omega$. Then, for all $x \in \Omega$, $\Pi(x)(\nabla_x \Pi)(x)\Pi(x) = 0$. Let $N \in \mathbb{N}$ and $(\Pi_j)_{1 \leq j \leq N}$ be a family of smooth, projection-valued functions on Ω satisfying $\Pi_j(x)\Pi_k(x) = \delta_{jk}\Pi_j(x)$, for all $x \in \Omega$ and all $(j,k) \in \{1; \cdots, N\}^2$. Let $\vec{v}_j \in C^\infty(\Omega; \mathbb{R}^d)$, for $1 \leq j \leq N$. Then, $\sum_{j=1}^N \Pi_j(x)(\vec{v}_j \cdot \nabla_x \Pi_k)(x)\Pi_j(x) = 0$, for all $x \in \Omega$ and all $k \in \{1; \cdots, N\}$.*

Proof: Expanding $0 = \Pi(\nabla_x(\Pi^2 - \Pi))\Pi$, we obtain the first property. Starting from $0 = \sum_j \Pi_j(\vec{v}_j \cdot \nabla_x(\Pi_k^2 - \Pi_k))\Pi_j = 2\Pi_k(\vec{v}_k \cdot \nabla_x \Pi_k)\Pi_k - \sum_j \Pi_j(\vec{v}_j \cdot \nabla_x \Pi_k)\Pi_j$ and applying the first property to Π_k, we obtain the second property. $\square$

2.2 Codimension 1 crossing.

Now we describe the model of codimension 1 crossing, the special condition at the crossing, and the structure condition at infinity. The precise definitions of these notions are given in Definitions 2.6, 2.9, and 2.13. To motivate the choice of this model, we want to explain how it follows from a reasonable "globalization" of the local definition of a codimension 1 crossing in [H]. This explanation may be skipped and one can directly go to Definition 2.6 below.

For simplicity, we first restrict ourselves to crossings of two eigenvalues. We assume that there is a non-empty, closed submanifold $\mathcal{C}$ of $\mathbb{R}^d$ of codimension one such that, on each connected component of $\mathbb{R}^d \backslash \mathcal{C}$ the matrix $M(x)$ has a constant number of eigenvalues with constant multiplicity, so that we can label these eigenvalues with increasing order inside the connected component. We assume that the boundary of such a connected component (which is a part of $\mathcal{C}$) is the union of different crossings of exactly two eigenvalues. Let us explain this in detail. At each point $x_0 \in \mathcal{C}$, $\mathcal{C}$ is locally given by some equation $\tau = 0$, with $\tau \in C^\infty(\mathbb{R}^d; \mathbb{R})$ and $d\tau \neq 0$ on $\tau^{-1}(0)$. Near x_0 and on some connected component $\mathcal{U}_0$ of $\mathbb{R}^d \setminus \mathcal{C}$, there exist $j(1) < j(1) + 1 < j(2) < j(2) + 1 < \cdots < j(k) < j(k) + 1$ such that, for integers $p > l$, the limits on $\tau^{-1}(0)$ of the eigenvalues λ_l and λ_p coincide if and only if $(l; p) = (j(i); j(i) + 1)$, for some $i \in \{1, \cdots, k\}$. Notice that the set $\mathcal{C}$ is the eigenvalues crossing in the sense of Definition 2.1. Inspired by [H], we require further the following matricial structure of $M(x)$ near $\mathcal{C}$.

Recall that I_m is the identity matrix in $\mathcal{M}_m(\mathbb{C})$. We denote by tr_m the normalized trace on $\mathcal{M}_m(\mathbb{C})$, i.e. $\mathrm{tr}_m \mathrm{I}_m = 1$. Let $x_0 \in \mathcal{C}$ and $i \in \{1, \cdots, k\}$. Near x_0, on $\mathcal{U}_0$, the sum $\mathcal{S}_i(x)$ of the spectral subspaces associated with $\lambda_{j(i)}(x)$ and $\lambda_{j(i)+1}(x)$ has constant dimension $m(i)$, up to $\overline{\mathcal{U}_0}$. We demand that, near x_0 and on $\overline{\mathcal{U}_0}$, the restriction $M_i(x)$ of $M(x)$ to $\mathcal{S}_i(x)$ is given by $(\mathrm{tr}_{m(i)} M_i(x))\mathrm{I}_{m(i)} + \tau(x) V_{j(i)}(x)$, where $V_{j(i)}$ is a smooth, self-adjoint matrix-valued function, defined on a vicinity of x_0 and having exactly two different eigenvalues there. Outside $\tau^{-1}(0)$, these eigenvalues are of course $((\lambda_{j(i)} - \mathrm{tr}_{m(i)} M_i)/\tau)(x)$ and $((\lambda_{j(i)+1}(x) - \mathrm{tr}_{m(i)} M_i)/\tau)(x)$. Using the smooth eigenvalues of $V_{j(i)}(x)$, we can smoothly extend $\lambda_{j(i)}(x)$ and $\lambda_{j(i)+1}(x)$ through $\mathcal{C}$ near x_0 such that they still are eigenvalues of $M(x)$. We recognize the codimension 1 crossing defined in [H].

By a connexity argument, we see that there exists an integer N (the number of different eigenvalues) and N globally defined, smooth functions on $\mathbb{R}^d$, that we again denote by $\lambda_1, \cdots, \lambda_N$, such that, for all $x \in \mathbb{R}^d$ and all $j \in \{1, \cdots, N\}$, $\lambda_j(x)$ is an eigenvalue of $M(x)$, the multiplicity $m_j(x)$ of which is constant on $\mathbb{R}^d \setminus \mathcal{C}$ (see also [K], p. 108).
Outside $\mathcal{C}$, the orthogonal eigenprojection $\Pi_j(x)$ of $M(x)$ associated with $\lambda_j(x)$ is smooth (cf. [K]). Let $j, k \in \{1, \cdots, N\}$ such that $j \neq k$ and $(\lambda_j - \lambda_k)^{-1}(0) \neq \emptyset$. Near any point $x_0 \in (\lambda_j - \lambda_k)^{-1}(0)$, the sum $\Pi_j(x) + \Pi_k(x)$ is smooth and has a constant range dimension $m(j,k) := m(j) + m(k)$, since only two different eigenvalues cross. According to the previous assumptions, there exist, near x_0, a smooth real, scalar function τ_{jk} and a smooth function V_{jk}, where $V_{jk}(x)$ is a self-adjoint matrix acting on $\mathcal{S}_{jk}(x)$, the range of $\Pi_j(x) + \Pi_k(x)$, with exactly two different eigenvalues, such that, for x near x_0, the restriction $M_{jk}(x)$ of $M(x)$ to $\mathcal{S}_{jk}(x)$ has the form

$$M_{jk}(x) \; = \; \Big(\mathrm{tr}_{m(j,k)} M_{jk}(x)\Big) \mathrm{I}_{m(j,k)} \; + \; \tau_{jk}(x)\, V_{jk}(x) \, .$$

Near x_0, the set $(\lambda_j - \lambda_k)^{-1}(0)$ coincide with the zero set of τ_{jk}, on which the differential of τ_{jk} does not vanish. Actually, we do not need that the differential of $\lambda_j - \lambda_k$ does not vanish on the crossing near x_0 (cf. Remark 2.15). Using the eigenprojections of $V_{jk}(x)$, we can smoothly extend Π_j and Π_k on $(\lambda_j - \lambda_k)^{-1}(0)$, near x_0. Therefore, we have N globally defined, smooth orthogonal projections $\Pi_1, \cdots, \Pi_N$. If the crossing is empty, there still exist, of course, globally smooth functions $\lambda_1, \cdots, \lambda_N$, $\Pi_1, \cdots, \Pi_N$, as above.
A careful analysis of the previous arguments shows that we can similarly deal with crossings of more than two eigenvalues and get a similar global situation, namely the situation described in

Definition 2.6. *We say that the potential M is a model of codimension 1 crossing if there exist some $N \in \{1; \cdots; m\}$, N real-valued, smooth functions $\lambda_1, \cdots, \lambda_N$, and N projection-valued, smooth functions $\Pi_1, \cdots, \Pi_N$, and some subset $\mathcal{C}'$ of $\mathbb{R}^d$ such that the following properties are satisfied.*

1. *The subset $\mathcal{C}'$ is precisely the subset $\mathcal{C}$ introduced in Definition 2.1 and is a codimension 1 submanifold of $\mathbb{R}^d$.*

2. *For all $x \in \mathbb{R}^d \setminus \mathcal{C}$, $\lambda_1(x), \cdots, \lambda_N(x)$ are the distinct eigenvalues of $M(x)$ and $\Pi_1(x), \cdots, \Pi_N(x)$ the associated orthogonal eigenprojections.*

3. *For each point $x_0 \in \mathcal{C}$, let*

$$I \; = \; \Big\{ j \in \{1; \cdots; N\};\, \exists k \in \{1; \cdots; N\};\, k \neq j \quad \textit{and} \quad x_0 \in (\lambda_j - \lambda_k)^{-1}(0) \Big\} \, ,$$

$\Pi(x) = \sum_{j\in I} \Pi_j(x)$, and $n \in \{1; \cdots ; m\}$ be the dimension of the range of $\Pi(x)$. There is a scalar function $\tau \in C^\infty(\mathbb{R}^d; \mathbb{R})$ and a function $V \in C^\infty(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, with empty eigenvalues crossing, such that, near x_0, $\mathcal{C}$ is given by $\tau^{-1}(0)$, $d\tau \neq 0$ on $\tau^{-1}(0)$, and the restriction of $M(x)$ to the range of $\Pi(x)$ is given by $\mathrm{tr}_n(M(x)\Pi(x))\mathrm{I}_n + \tau(x)V(x)$.

Notice that, if M is a model of codimension 1 crossing,

$$\mathcal{C} \; = \bigcup_{j,k\in\{1;\cdots;N\}, j\neq k} \Big(\lambda_j - \lambda_k\Big)^{-1}(0) \, .$$

Definition 2.7. *Let M be a model of codimension 1 crossing. For $j \in \{1, \cdots , N\}$, the smooth Hamilton functions p_j defined by $T^*\mathbb{R}^d \ni (x, \xi) \mapsto p_j(x, \xi) := |\xi|^2 + \lambda_j(x)$ are called the eigenvalues of P. For $j \in \{1, \cdots , N\}$ and $t \in \mathbb{R}$, we denote by ϕ_j^t the corresponding Hamilton flow at time t.*

Example 2.8. *A simple model of codimension 1 crossing with $m = 2$, studied in [J4], is defined as follows. Let $\tau, u, v_1, v_2 \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that $v_1^2 + v_2^2 \geq \epsilon$ everywhere, for some $\epsilon > 0$, and such that the equation $\tau(x) = 0$ defines a codimension 1 submanifold of $\mathbb{R}^d$. Set $M(x) = u(x)\mathrm{I}_2 + \tau(x)V(x)$ with*

$$V(x) \; := \; \begin{pmatrix} v_1(x) & v_2(x) \\ v_2(x) & -v_1(x) \end{pmatrix} .$$

Next we introduce the special condition at the crossing.

Definition 2.9. *Let M be a model of codimension 1 crossing. It satisfies the special condition at the crossing if, at any point on $\mathcal{C}$, all tangential first derivatives of the orthogonal eigenprojections of M are zero.*

Remark 2.10. *We point out that the special condition is intrinsic (i.e. it does not depend on the choice of a local equation for the crossing). It can be reformulated in different ways. To this end, let Π be any orthogonal eigenprojection and let $x_0 \in \mathcal{C}$. The differential map $D\Pi(x_0)$ of Π at x_0 is a linear map from $\mathbb{R}^d$ to $\mathcal{M}_m(\mathbb{C})$. The special condition means that, at any point $x_0 \in \mathcal{C}$ and for any eigenprojection Π, the kernel of $D\Pi(x_0)$ contains the tangential space of $\mathcal{C}$ at x_0. If $\mathcal{C}$ is given by some equation $\tau = 0$ near x_0, this property means that, for all $\xi \in \mathbb{R}^d$ with $\xi \cdot \nabla\tau(x_0) = 0$, $\xi \cdot \nabla\Pi(x_0) = 0$.*

Remark 2.11. *A model of codimension 1 crossing, for which M is a radial function (i.e. that only depends on $|x|$), satisfies the special condition at the crossing. Indeed, the* *crossing $\mathcal{C}$ is a union of spheres centered at 0 with positive radius and near $x_0 \in \mathcal{C}$, one can choose τ and V radial. Thus the eigenprojections are also radial. Therefore their tangential derivatives are zero.*

Now, we want to describe our requirement at infinity. Although it is perhaps easier to work (and to think) in the one point compactification of $\mathbb{R}^d$, we prefer to express our requirement in an elementary way.
By [K], we know that the eigenvalues λ_j have a limit at infinity, which are eigenvalues of M_∞ (cf. (2.1)). Two cases appear. We describe them in the following

Definition 2.12. *Let M be a model of codimension 1 crossing. If there exists $c > 0$ and $R > 0$ such that, for all $j \neq k \in \{1, \cdots, N\}$ and for all $|x| > R$, $|\lambda_j(x) - \lambda_k(x)| \geq c$, we say that there is no "crossing at infinity". Otherwise, we say that there is a "crossing at infinity".*

In the second case, we require, in the spirit of Definition 2.6, a control on the matricial structure of M near the crossing at infinity in the following way. Let

$$I_\infty \;=\; \Big\{ j \in \{1; \cdots ; N\};\, \exists k \in \{1; \cdots ; N\};\, k \neq j \quad \text{and} \quad \lim_{|x| \to \infty} \lambda_j(x) - \lambda_k(x) = 0 \Big\}\,, \quad (2.9)$$

$\Pi(x) = \sum_{j \in I_\infty} \Pi_j(x)$, and $n \in \{1; \cdots ; m\}$ be the dimension of the range of $\Pi(x)$. Notice that, at infinity, $\Pi(x)$ converges to some Π_∞, an eigenprojection of M_∞. We require that there exists some $R > 0$, a smooth, scalar function τ, and a smooth function V, with values in the self-adjoint $n \times n$-matrices and with empty eigenvalues crossing, such that, on $\{y \in \mathbb{R}^d; |y| > R\}$, the restriction of $M(x)$ to the range of $\Pi(x)$ is given by $\mathrm{tr}_n(M(x)\Pi(x))\mathrm{I}_n + \tau(x)V(x)$. In view of (2.1), we require that there exists a self-adjoint $n \times n$-matrix V_∞ with empty eigenvalues crossing such that, for all $\gamma \in \mathbb{N}^d$, for all $x \in \{y \in \mathbb{R}^d; |y| > R\}$,

$$\Big| \partial_x^\gamma \tau(x) \Big| + \Big\| \partial_x^\gamma \Big(V(x) - V_\infty \Big) \Big\|_n \;=\; O_\gamma\Big(\langle x \rangle^{-\rho - |\gamma|} \Big)\,. \quad (2.10)$$

This estimate roughly means that, at infinity, the function τ tends to zero while the matricial structure of V keeps its properties up to the limit V_∞. In other words, we treat the infinity of the configuration space $\mathbb{R}_x^d$ as a part of $\mathcal{C}$, at which we require (almost) the same assumptions as at finite distance on $\mathcal{C}$. Let us point out that (2.1) and (2.10) imply that, for all $j \in \{1; \cdots , N\}$, we can find an eigenvalue $\lambda_{j,\infty}$ of M_∞ and an orthogonal projection $\Pi_{j,\infty}$, such that

$$\forall \gamma \in \mathbb{N}^d,\, \forall x \in \mathbb{R}^d,\;\; \Big| \partial_x^\gamma \Big(\lambda_j(x) - \lambda_{j,\infty} \Big) \Big| + \Big\| \partial_x^\gamma \Big(\Pi_j(x) - \Pi_{j,\infty} \Big) \Big\|_m \;=\; O_\gamma\Big(\langle x \rangle^{-\rho - |\gamma|} \Big). \quad (2.11)$$

Notice that the orthogonal eigenprojection of M_∞ corresponding to $\lambda_{j,\infty}$ is the sum of the $\Pi_{k,\infty}$ over the set of the k for which $\lambda_k \to \lambda_{j,\infty}$ at infinity. This leads to the following

Definition 2.13. *Consider a model M of codimension 1 crossing. It satisfies the structure condition at infinity if one of the following conditions is satisfied.*

1. *There is no crossing at infinity.*
2. *There is a crossing at infinity and M satisfies the requirements just above. In particular, (2.10) and (2.11) hold true.*

Remark 2.14. *Let M be a model of codimension 1 crossing such that there is no crossing at infinity. Then (2.11) is a direct consequence of (2.1) (cf. [K]).*

Remark 2.15. *In the spirit of Definitions 2.2 and 2.4, we may introduce an energy dependent version of Definitions 2.6, 2.7, 2.9, 2.12, and 2.13. Given an energy μ, the requirements of each definition are imposed at points in $\mathcal{C}^*(\mu)$ and, at infinity, only on eigenvalues below μ. This change does not affect the proofs of Theorems 1.3 and 1.4.*
In Definitions 2.6 and 2.13, we may assume that the restriction of $M(x)$ to the range of $\Pi(x)$ is given by $\mathrm{tr}_n(M(x)\Pi(x))\mathrm{I}_n + f(\tau(x))V(x)$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is C^∞ near 0 and vanishes at 0. This does not affect the proofs of Theorems 1.3 and 1.4 either.

To end this subsection, we want to give concrete examples for which the assumptions of Theorem 1.2 are satisfied. To contruct them, we first mention some simple facts.
Let $W \in C^\infty(\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ with self-adjoint values such that (2.1) is satisfied with limit $W_\infty = 0$ at infinity. The function $x \mapsto U(x) := \exp(iW(x))$ has unitary values, is smooth, and satisfies (2.1) with $U_\infty = \mathrm{I}_m$ as limit at infinity. Let V_∞ a self-adjoint matrix with simple eigenvalues $\lambda_1^\infty < \cdots < \lambda_m^\infty$. Let $(e_j^\infty)_{1\leq j\leq m}$ be a based of eigenvectors of V_∞. For all j, let $e_j(x) = U(x)e_j^\infty$ and $\Pi_j(x) = \langle e_j(x), \cdot\rangle e_j(x)$. For all j, let λ_j be a smooth, real valued function satisfying (2.1) with λ_j^∞ as limit at infinity. Assume further that, for all x, $\lambda_1(x) < \cdots < \lambda_m(x)$. Let $V = \sum_{j=1}^n \lambda_j\Pi_j$. V is a smooth, self-adjoint matrix valued function satisfying (2.1) with V_∞ as limit at infinity. Furthermore, its eigenvalues are simple everywhere and V has no eigenvalues crossing at infinity, as defined in Definition 2.12. Let $\tilde{V} := \sum_{j<m-1} \lambda_j\Pi_j + (1/2)(\lambda_{m-1} + \lambda_m)(\Pi_{m-1} + \Pi_m)$. $\tilde{V}$ is smooth, has no eigenvalues crossing and no eigenvalues crossing at infinity but its spectrum is not simple. In the same way, we can modify $\tilde{V}$. Therefore, it is easy to construct a smooth, self-adjoint valued function V satisfying (2.1) and having constant "up to infinity" but quite arbitrary eigenvalue multiplicities. In particular, V has no eigenvalues crossing and no eigenvalues crossing at infinity.

Let V be a smooth, self-adjoint valued function satisfying (2.1). Let $\mathcal{C}$ be its eigenvalues crossing. Outside $\mathcal{C}$, $V = \sum_j \lambda_j \Pi_j$. Let μ_j be smooth, compactly supported, real valued functions such that $\mu_j = 0$ near $\mathcal{C}$. Let $W = \sum_j \mu_j \Pi_j$. We can choose the μ_j small enough to ensure that the eigenvalues crossing of $V + W$ is also $\mathcal{C}$. Thus $V + W$ satisfies the same assumptions and has the same eigenvalues crossing as V. Similarly, if V has a crossing at infinity, we can construct W such that $V + W$ satisfies the same assumptions and has the same eigenvalues crossing at infinity as V.

Example 2.16. *Let $u, \tau \in C^\infty(\mathbb{R}^d; \mathbb{R})$ tending at infinity to $u_\infty, \tau_\infty \in \mathbb{R}$ respectively and satisfying (2.1). Assume further that $\tau(x) = 0$ defines a codimension 1 submanifold of $\mathbb{R}^d$. Let $V \in C^\infty(\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ with self-adjoint values, without eigenvalues crossing, and without eigenvalues crossing at infinity. Assume further that V tends at infinity to some V_∞ and that V satisfies (2.1). Then $M(x) = u(x)\mathrm{I}_m + W(x)$ with $W(x) = \tau(x)V(x)$ is a model of codimension 1 crossing satisfying the structure condition at infinity. There is a crossing at infinity if and only if $\tau_\infty = 0$.*
M is also a model of codimension 1 crossing satisfying the structure condition at infinity if W is a direct sum $\tau_1 V_1 \oplus \cdots \oplus \tau_k V_k$, where the functions τ_j and V_j satisfy the assumptions of τ and V respectively.

Example 2.17. *We consider the example given in Example 2.16 and we require further that all the involved functions are radial (i.e. only depend on $|x|$). Then M is a model of codimension 1 crossing satisfying the structure condition at infinity and the special condition at the crossing (cf. Remark 2.11).*
As explained just before Example 2.16, we can perturb the previous function M by some non radial, self-adjoint valued $\tilde{M} \in C_0^\infty(\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, without changing the eigenvalues crossing and the matricial structure there. Therefore, $M + \tilde{M}$ is still a model of codimension 1 crossing satisfying the structure condition at infinity and the special condition at the crossing.
Let M' be the previous function M or $M + \tilde{M}$. Assume that, for all $x \in \mathbb{R}^d$, $M'(x) \le \lambda \mathrm{I}_m$. Let $W \in C^\infty(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ with self-adjoint values such that (2.1) is satisfied. Assume further that there exists $\epsilon > 0$ such that, for all $x \in \mathbb{R}^d$, $W(x) \ge (\lambda + \epsilon)\mathrm{I}_n$. Then, by Remark 2.15, Theorem 1.2 applies to $M' \oplus W$. Notice that the eigenvalues crossing of W may take place on a set, which is not a submanifold of $\mathbb{R}^d$.

Example 2.18. *Let $a > 0$. Let M_0, M_1 be two functions like the function M in Example 2.17. We require further that, near a, the functions M_0 and M_1 coincide, are radial, and have no eigenvalues crossing. Let $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that $0 \le \chi, \tilde{\chi} \le 1$. Assume that $\{t \in \mathbb{R}^+; \chi(t) > 0\} = [0; a[$, $\{t \in \mathbb{R}^+; \chi(t) = 0\} = [a; +\infty[$, $\{t \in \mathbb{R}^+; \tilde{\chi}(t) = 1\} = [0; a]$, and $\{t \in \mathbb{R}^+; \tilde{\chi}(t) < 1\} =]a; +\infty[$. Then the new function M, defined by*

$M(x) = \chi(|x|)M_0(x) + (1 - \tilde{\chi}(|x|))M_1(x)$, is a model of codimension 1 crossing in the extended sense introduced in Remark 2.15. It also satisfies the structure condition at infinity and the special condition at the crossing. We can perform the same pertubations as in Example 2.17 with the same results.

3 Semiclassical trapping.

The purpose of this section is to study the situation where the property ((1.3) and (1.4)) fails, for some $s > 1/2$, preparing that way a proof by contradiction of Theorem 1.3. In fact, we easily generalize to the general matricial case most of the results of our version of Burq's strategy (cf. [B]), developped in [J5]. As a by-product, we derive Proposition 1.7. However an important feature resists to our analysis in the general case (cf. Subsection 3.2). For codimension 1 crossing, we use the special condition at the crossing (cf. Definition 2.9) to overcome the difficulty.

3.1 Generalization of scalar results.

In this subsection, we work in the general framework defined in Subsection 2.1 and generalize several results obtained in [J5]. They allow us to derive by contradition the proof of Proposition 1.7.

We shall use well-known tools of semiclassical analysis, like h-pseudodifferential operators and semiclassical measures. We refer to [DG, GL, N, R] for details. For matrix-valued symbolic calculus, we refer to [Ba, J2]. The notation and the most important facts we need are recalled below.

Recall first that $\|\cdot\|$ denotes the usual norm of $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$ and that $\langle\cdot,\cdot\rangle$ denotes the corresponding scalar product. We also denote by $\|\cdot\|$ the operator norm of the bounded operators on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$.

For $(r,t) \in \mathbb{R}^2$, we consider the class of symbols $\Sigma_{r,t}$, composed of the smooth functions $A : T^*\mathbb{R}^d \longrightarrow \mathcal{M}_m(\mathbb{C})$ such that

$$\forall \gamma = (\gamma_x, \gamma_\xi) \in \mathbb{N}^{2d}\, ,\ \exists C_\gamma > 0\, ;\ \sup_{(x,\xi)\in T^*\mathbb{R}^d} \langle x\rangle^{-r+|\gamma_x|}\, \langle \xi\rangle^{-t+|\gamma_\xi|} \Big\|(\partial^\gamma A)(x,\xi)\Big\|_m \ \leq\ C_\gamma\, . \quad (3.1)$$

For such symbol A, we can define its Weyl h-quantization, denoted by A_h^w, which acts on $u \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^m)$ as follows.

$$\Big(A_h^w u\Big)(x) \ =\ (2\pi h)^{-n} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)/h} A\Big((x+y)/2, \xi\Big) \cdot u(y)\, dy d\xi\, . \quad (3.2)$$

By Calderon-Vaillancourt theorem, it extends to a bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, uniformly w.r.t. h, if $r,t\leq 0$ (cf. [Ba, J2]). If $\theta\in C_0^\infty(\mathbb{R};\mathbb{R})$, one can show (cf. [Ba]) that

$$\theta\Big(\hat{P}(h)\Big) \;=\; \Big(\theta(P)\Big)_h^w \,+\, h\,Q_h^w \,+\, h^2\,R(h) \tag{3.3}$$

where $Q\in\Sigma_{-1,0}$ and where, for all $k\in\mathbb{R}$, $\langle x\rangle^{k+2}R(h)\langle x\rangle^{-k}$ is a bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, uniformly w.r.t h. Here the symbol $\theta(P):T^*\mathbb{R}^d\longrightarrow\mathcal{M}_m(\mathbb{C})$ maps any $\mathrm{x}^*\in T^*\mathbb{R}^d$ to the matrix $\theta(P(\mathrm{x}^*))$ given by the functional calculus of the self-ajoint matrix $P(\mathrm{x}^*)$. We point out that we can use Helffer-Sjöstrand's formula (cf. [DG]) to express $\theta(P)$ and to show that $\nabla_{x,\xi}(\theta(P))$ is supported in $\mathrm{supp}\,\theta'(P)$.
Let $(u_n)_n$ be a bounded sequence of functions in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$ and let $h_n>0$ with $h_n\to 0$. There exists an increasing function $\phi:\mathbb{N}\longrightarrow\mathbb{N}$ and a nonnegative measure η acting on $C_0^0(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$ such that, for all $A\in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$,

$$\lim_{n\to\infty}\Big\langle u_{\phi(n)}\,,\,A^w_{h_{\phi(n)}}u_{\phi(n)}\Big\rangle \;=\; \Big\langle\eta\,,\,A\Big\rangle'$$

where $\langle\cdot,\cdot\rangle'$ denotes the duality between $(C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C})))'$ and $C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$. η is a semiclassical measure of $(u_n)_n$. If ϕ is the identity then we say that $(u_n)_n$ is pure. Possibly after some extraction of subsequence, we can find such a pure sequence. Since we can identify $C_0^0(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$ and $\mathcal{M}_m(C_0^0(T^*\mathbb{R}^d;\mathbb{C}))$, we can also identify η with $\tilde{\eta}\in\mathcal{M}_m((C_0^0(T^*\mathbb{R}^d;\mathbb{C})')$ and write, for all $A\in C_0^0(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$,

$$\Big\langle\eta\,,\,A\Big\rangle' \;=\; \mathrm{tr}_m\Big(\int_{T^*\mathbb{R}^d}A\,d\tilde{\eta}\Big) \;=\; \mathrm{tr}_m\left(\left(\sum_{k=1}^m\int_{T^*\mathbb{R}^d}A_{ik}\,d\tilde{\eta}_{kj}\right)_{i,j}\right)\,. \tag{3.4}$$

This value will be denoted simply by $\eta(A)$.
Now we come back to our aim. We consider the situation where the property ((1.3) and (1.4)) is false near $\lambda>\|M_\infty\|_m$, for some $s>1/2$. This situation is interpreted as a "semiclassical trapping". Precisely, we assume the

Hypothesis 1. *There exist a sequence $(h_n)_n\in]0;h_0]^{\mathbb{N}}$ tending to zero, a sequence $(f_n)_n$ of nonzero, $\mathbb{C}^m$-valued functions of the domain of $\Delta_x\mathrm{I}_m$, and a sequence $(z_n)_n\in\mathbb{C}^{\mathbb{N}}$ with $\Re(z_n)\to\lambda>\|M_\infty\|_m\geq 0$ and $0\leq\Im(z_n)/h_n\to r_0\geq 0$, such that*

$$\|\langle x\rangle^{-s}f_n\| \;=\; 1 \quad \textit{and} \quad \Big\|\langle x\rangle^s\Big(\hat{P}(h_n)-z_n\Big)f_n\Big\| \;=\; o(h_n)\,.$$

Furthermore the L^2-bounded sequence $(\langle x\rangle^{-s}f_n)_n$ is pure. We denote by μ_s its semiclassical measure and we set $\mu:=\langle x\rangle^{2s}\mu_s$.

Remark 3.1. *As noticed by V. Bruneau, the negation of (1.4) and the fact that* $\|R(z_n; h_n)\| \leq 1/|\Im(z_n)|$ *imply that* $r_0 = 0$. *However, we need the stronger fact* $\|f_n\|^2\Im(z_n)/h_n \to 0$ *for the proof of Proposition 3.4. This is done in Proposition 3.3.*

Remark 3.2. *If, for some* $s > 1/2$, *(1.3) would be false at some energy* $\lambda_n \to \lambda$ *for some sequence* $h_n \to 0$, *then, by Mourre's theory (cf. [Mo, CFKS]),* λ_n *would be an eigenvalue of* $\hat{P}(h_n)$, *for any* n. *Choosing, for each* n, *a corresponding eigenvector with a suitable normalization as* f_n *and setting* $z_n = \lambda_n$, *Hypothesis 1 would also hold true in this case. The generalization to matricial Schrödinger operators of the result by Froese and Herbst (cf. [FH, CFKS]) tells us that this case does not occur. This does not affect the proof of Theorem 1.3 since it is based on Hypothesis 1.*

Under Hypothesis 1 and according to (3.4), we can write, for any $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$,

$$\lim_{n\to\infty} \Big\langle \langle x\rangle^{-s} f_n \, , \, A^w_{h_n} \langle x\rangle^{-s} f_n \Big\rangle \; = \; \mu_s(A) \quad \text{and} \quad \lim_{n\to\infty} \Big\langle f_n \, , \, A^w_{h_n} f_n \Big\rangle \; = \; \mu(A) \, . \tag{3.5}$$

Now we show how two results in [J5] are generalized in the present context. Recall that the energy shell $E^*(\lambda)$ is defined in (2.3).

Proposition 3.3. *Under the previous conditions, the support of* μ, *denoted by* $\operatorname{supp}\mu$, *satisfies* $\operatorname{supp}\mu \subset E^*(\lambda)$ *and* $\|f_n\|^2\Im(z_n)/h_n \to 0$. *In particular,* $r_0 = 0$.

Proof: Since the arguments in [J5] apply, we only sketch the proof. By Hypothesis 1, $\langle (\hat{P}(h_n) - z_n)f_n, f_n\rangle = o(h_n)$. Taking the imaginary part, we obtain $\|f_n\|^2\Im(z_n) = o(h_n)$. Let $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, supported away from $E^*(\lambda)$. Then the sequence $A\langle x\rangle^{-s}(P - z_n)^{-1} \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ is bounded. Therefore

$$\Big\langle A^w_{h_n} \langle x\rangle^{-s} f_n \, , \, \langle x\rangle^{-s} f_n \Big\rangle \; = \; \Big\langle \Big(A\langle x\rangle^{-s}(P - z_n)^{-1}\Big)^w_{h_n} \Big(\hat{P}(h_n) - z_n\Big) f_n \, , \, \langle x\rangle^{-s} f_n \Big\rangle \, + \, O(h_n)$$

tends to zero, by Hypothesis 1. This yields $\operatorname{supp}\mu \subset E^*(\lambda)$. $\square$

Recall that ρ describes the behaviour of M at infinity in (2.1).

Proposition 3.4. *Let* $\rho' \in]0; \min(1; \rho)[$. *There exists* $R_0 > 0$ *such that*

$$\lim_{n\to\infty} \int_{|x|>R_0} \langle x\rangle^{-(1+\rho')} \, |f_n(x)|^2 \, dx \; = \; 0 \, . \tag{3.6}$$

In particular, the measure μ *is nonzero and has a compact support satisfying* $\operatorname{supp}\mu \subset E^*(\lambda) \cap \{\mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$.

Proof: Again the arguments in [J5] apply. To see this, we use the symbolic calculus of $\Sigma_{r,t}$ and the following important fact. For any **scalar** $a \in \Sigma_{r,t}$, $ih^{-1}[\hat{P}(h), (a\mathrm{I}_m)_h^w]$ is of order 0 in h since $[P, a\mathrm{I}_m] = 0$ everywhere and its principal symbol w.r.t. h is given by $\{P, a\mathrm{I}_m\} = (2\xi \cdot \nabla_x a)\mathrm{I}_m - \nabla_\xi a \cdot \nabla_x M$ (cf. (3.9) and (3.10) below).
Given some $\rho' \in]0; \min(1; \rho)[$, we use the same **scalar** function $a_\infty \in \Sigma_{0,0}$ as in [J5] and find some $R, c > 0$ such that $\{P, a_\infty \mathrm{I}_m\} \geq 4c\langle x\rangle^{-(1+\rho')}\mathrm{I}_m$ near $E^*(\lambda) \cap \{\mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d; |x| > R\}$. Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that χ only depends on $|x|$, $0 \leq \chi \leq 1$, and $\chi = 1$ on $\{x \in \mathbb{R}^d; |x| \leq R\}$. Let $\theta \in C_0^\infty(\mathbb{R}; \mathbb{R})$ supported near λ.
Expanding the commutator and using Proposition 3.3, we can show that

$$\epsilon_n \ := \ \left\langle ih_n^{-1}\left[\hat{P}(h_n), \Big((1-\chi)^2 a_\infty \mathrm{I}_m\Big)_{h_n}^w\right]\theta\Big(\hat{P}(h_n)\Big)f_n \, , \, \theta\Big(\hat{P}(h_n)\Big)f_n\right\rangle \ \to \ 0 \, .$$

Now, by the symbolic calculus,

$$\epsilon_n \ = \ \left\langle ih_n^{-1}\Big\{P, (1-\chi)^2 a_\infty \mathrm{I}_m\Big\}_{h_n}^w \theta\Big(\hat{P}(h_n)\Big)f_n \, , \, \theta\Big(\hat{P}(h_n)\Big)f_n\right\rangle \ + \ O(h_n) \, . \tag{3.7}$$

Thanks to the special form of a_∞, $a_\infty\{P, (1-\chi)^2\mathrm{I}_m\} \geq 0$ near $E^*(\lambda)$. Using the Gårding inequality, its contribution on the r.h.s. of (3.7) is bounded below by some $O(h_n)$. Now we write $(1-\chi)^2(\{P, a_\infty \mathrm{I}_m\} - 2c\langle x\rangle^{-(1+\rho')}\mathrm{I}_m)$ as $B^2\mathrm{I}_m$, for some $B \in \Sigma_{-1/2,0}$. Using that $(B_{h_n}^w)^* B_{h_n}^w \geq 0$, we arrive at

$$\epsilon_n \ \geq \ c\left\|(1-\chi)\theta\Big(\hat{P}(h_n)\Big)\langle x\rangle^{-(1+\rho')/2}f_n\right\|^2 \ + \ O(h_n) \, .$$

As in the proof of Proposition 3.3, it follows that

$$\epsilon_n \ \geq \ c\left\|(1-\chi)\langle x\rangle^{-(1+\rho')/2}f_n\right\|^2 \ + \ O(h_n) \, .$$

yielding (3.6). Of course, (3.6) holds true for any $\rho' > 0$, in particular for $\rho' = 2s - 1$. By [GL], this implies that the total mass of μ_s is $\lim_n \|\langle x\rangle^{-s}f_n\|$. This is 1 by Hypothesis 1. Therefore μ_s is nonzero. Finally, (3.6) for $\rho' = 2s - 1$ implies that $\mu_s(A) = 0$, if A is supported in $\{\mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d; |x| > R_0\}$. This shows that μ_s is supported in $E^*(\lambda) \cap \{\mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$, thus compactly supported. □

Proof of Proposition 1.7: We assume that, near λ, the estimate (1.4) holds true for some $s' > 1/2$ and that it fails for some $s > 1/2$. Thus, we have a "semiclassical trapping" for s and we use the notation of Hypothesis 1 (at the beginning of Subsection 3.1). Let

$\chi \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi^{-1}(0) = \{x; |x| \geq 2\}$, and $\chi^{-1}(1) = \{x; |x| \leq 1\}$. For $R > R_0$, we set $\chi_R(x) = \chi(x/R)$. By Proposition 3.4 and Hypothesis 1,

$$\liminf_{n\to\infty} \left\| \langle x \rangle^{-s'} \chi_R f_n \right\|^2 \ > \ 0 \, .$$

Furthermore,

$$\langle x \rangle^{s'} \Big(\hat{P}(h_n) - z_n \Big) \chi_R f_n \ = \ \langle x \rangle^{s'} \chi_R \Big(\hat{P}(h_n) - z_n \Big) f_n \ + \ h_n \langle x \rangle^{s'} h_n^{-1} [\hat{P}(h_n), \chi_R] \, f_n \, . \tag{3.8}$$

Since χ_R is a scalar function and varies in $\{x; 2R > |x| > R_0\}$, we see, using Propositions 3.3 and 3.4, that the L^2-norm of the last term in (3.8) is $o(h_n)$. By Hypothesis 1, so is the second term, since χ_R is compactly supported. Therefore, Hypothesis 1 is satisfied for s'. This contradicts the estimate (1.4) for s'. □

3.2 A matricial difficulty.

Trying to follow the other arguments in [J5], we meet, for the general case, a serious difficulty which is produced by the matricial nature of the symbol P.
Here we work in the framework given in Subsection 2.1 under Hypothesis 1 (cf. Subsection 3.1). As in [J5], we want to exploit properties of the commutator $ih^{-1}[\hat{P}(h), A_h^w]$, for symbol $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$. To this end, we use the symbolic calculus for $\Sigma_{r,t}$ (cf. (3.1)). If one keeps the order of the symbols, the composition formula for the corresponding Weyl h-pseudodifferential operators is the same as in the scalar case (see [Ba, J2, DG, R]). In particular, one can show (see [Ba, J2]) that, for $A \in \Sigma_{0,0}$,

$$ih^{-1}\Big[\hat{P}(h)\, , \, A_h^w\Big] \ = \ h^{-1} \Big(i[P, A] \Big)_h^w \ - \ \Big(\{P, A\} \Big)_h^w \ + \ h \, R(A; h) \, , \tag{3.9}$$

where, for all $k \in \mathbb{N}$, $\langle x \rangle^{k+2} R(A; h) \langle x \rangle^{-k}$ is a bounded operator on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$, uniformly w.r.t h, and where

$$\begin{aligned} \{P, A\} \ &:= \ (1/2)\Big(\nabla_\xi P \cdot \nabla_x A - \nabla_x P \cdot \nabla_\xi A\Big) \ + \ (1/2)\Big(\nabla_x A \cdot \nabla_\xi P - \nabla_\xi A \cdot \nabla_x P\Big) \\ &= \ 2\xi \cdot \nabla_x A \ - \ (1/2)\Big(\nabla_x M \cdot \nabla_\xi A + \nabla_\xi A \cdot \nabla_x M\Big) \, . \end{aligned} \tag{3.10}$$

Here we recognize a symmetrized, matricial version of the usual Poisson bracket. For $\eta \in (C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C})))'$, we denote by $-\{P, \eta\}$ the distribution that maps any $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ to $\eta(\{P, A\})$ (cf. (3.4)).

In contrast to the scalar case, the second term in (3.9) is nonzero in general and is responsible for the difficulty we mentionned. It turns out that its effect will take place on $\mathcal{C}^*(\lambda)$ (cf. Definition 2.2). Instead of proving $\{P,\mu\}=0$ as in the scalar case (cf. [J5]), we only have the

Proposition 3.5. *Consider the general case under Hypothesis 1 . For any $A\in C_0^\infty(T^*\mathbb{R}^d;$ $\mathcal{M}_m(\mathbb{C}))$, there exists*

$$\nu(A) \ := \ \lim_{n\to\infty}\Big\langle f_n\, ,\, h_n^{-1}\Big(i[P,A]\Big)^w_{h_n} f_n\Big\rangle\, . \tag{3.11}$$

$\nu\in(C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C})))'$ and satisfies $-\{P,\mu\}=\nu$. Furthermore, $\operatorname{supp}\nu\subset\mathcal{C}^*(\lambda)\cap$ $\operatorname{supp}\mu$ *and $\nu(A)=0$ if A commutes with P near* $\operatorname{supp}\nu$.

Proof: Let $A\in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$. If we expand the commutator in the scalar product $\langle f_n, ih_n^{-1}[\hat{P}(h_n), A^w_{h_n}]f_n\rangle$, then we see that this quantity tends to 0, thanks to Hypothesis 1, since $\langle x\rangle^s A^w_{h_n}\langle x\rangle^{-s}$ and $\langle x\rangle^s A^w_{h_n}\langle x\rangle^{s}$ are bounded operators, uniformly w.r.t. n. Using (3.9) and (3.5), we deduce from this that the limit (3.11) exists and is $\mu(\{P,A\})$, since $\langle x\rangle^s R(A;h_n)\ \langle x\rangle^{-s}$ is also a bounded operator, uniformly w.r.t. n. We have proved that $-\{P,\mu\}=\nu$ and, in particular, that $\nu\in(C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C})))'$ and $\operatorname{supp}\nu\subset\operatorname{supp}\mu$.
Let $A\in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$ that commutes with P near $\mathcal{C}^*(\lambda)$. Let χ, $\chi(\cdot;\cdot,\cdot)$ be as in (2.7) such that $\chi=1$ near 0. By Proposition 2.3, there exists some $\epsilon>0$ such that

$$T^*\mathbb{R}^d\ni \mathrm{x}^* \ \mapsto\ \chi\Big(P(\mathrm{x}^*);\lambda,\epsilon\Big)\,\Big[P(\mathrm{x}^*), A(\mathrm{x}^*)\Big]\,\chi\Big(P(\mathrm{x}^*);\lambda,\epsilon\Big)$$

is identically zero. Using the energy localization of the f_n, (3.3), and (3.5),

$$\begin{aligned}
&\langle f_n\, ,\, h_n^{-1}\Big(i[P,A]\Big)^w_{h_n} f_n\rangle\\
=\ &\Big\langle f_n\, ,\, h_n^{-1}\chi\Big(\hat{P}(h_n);\lambda,\epsilon\Big)\,\Big(i[P,A]\Big)^w_{h_n}\chi\Big(\hat{P}(h_n);\lambda,\epsilon\Big)\, f_n\Big\rangle\ +\ o(1)\\
=\ &0\cdot h_n^{-1}\ +\ \mu\bigg(\chi\Big(P;\lambda,\epsilon\Big)\, i[P-\lambda,A]\,Q\ +\ Q_1\ +\ Q\, i[P-\lambda,A]\chi\Big(P;\lambda,\epsilon\Big)\bigg)\ +\ o(1)\, ,
\end{aligned}$$

where Q_1 is supported in $\operatorname{supp}\chi'(P;\lambda,\epsilon)$ and where Q is the symbol introduced in (3.3). Thanks to Proposition 3.3, we obtain $\nu(A)=0$. Now, if $A\in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$ is supported away from $\mathcal{C}^*(\lambda)$, then it commutes with P near $\mathcal{C}^*(\lambda)$. Thus $\nu(A)=0$ and $\operatorname{supp}\nu\subset\mathcal{C}^*(\lambda)$.
Let $A\in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$ that commutes with P near $\operatorname{supp}\nu$. Then we can find $\psi\in C^\infty(T^*\mathbb{R}^d;\mathbb{R})$ such that $\psi=1$ near $\operatorname{supp}\nu$ and $[P,\psi A]=0$ everywhere. Thus

$\nu(A) = \nu(\psi A) = 0$, by (3.11). $\square$

We need a microlocalized version of the equation $-\{P, \mu\} = \nu$. Given $\mathrm{x}_0^* = (x_0, \xi_0) \in \mathcal{C}^*(\lambda)$, we introduced in Definition 2.2

$$I \ := \ \Big\{ j \in \{1, \cdots, m\}; \quad |\xi_0|^2 + \alpha_j(x_0) = \lambda \text{ and } m_j \text{ is discontinuous at } x_0 \Big\} .$$

Close enough to x_0^*, the orthogonal projection onto the sum over $j \in I$ of the spectral subspaces associated with $\alpha_j(x)$, denoted by $\Pi_I(x)$, is smooth and we set $A_I = \Pi_I A \Pi_I$, for $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$.

Proposition 3.6. *Consider the general case under Hypothesis 1. Let* $\mathrm{x}_0^* = (x_0, \xi_0) \in \mathcal{C}^*(\lambda)$. *For* $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, *supported close enough to* x_0^*,

$$\nu(A_I) \ = \ \mu\Big(\{P_I, A_I\}_I \Big) . \tag{3.12}$$

Proof: Since the eigenvalues α_k are continuous, the α_j for $j \in I$ are separated from the rest of the spectrum of M in some vicinity of x_0. Therefore Π_I is smooth there.
Let $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ with support close enough to x_0^*. We follow the beginning of the proof of Proposition 3.5, applied to $\langle f_n, i h_n^{-1} [\hat{P}(h_n), (A_I)^w_{h_n}] f_n \rangle$. To get (3.12), it suffices to show that

$$\Big\langle f_n \, , \, \Big(\{P, A_I\} \Big)^w_{h_n} f_n \Big\rangle \ = \ \Big\langle f_n \, , \, \Big(\{P_I, A_I\}_I \Big)^w_{h_n} f_n \Big\rangle \ + \ o(1) . \tag{3.13}$$

Let χ, $\chi(\cdot; \cdot, \cdot)$ be as in (2.7) such that $\chi = 1$ near 0. Taking $\epsilon > 0$, we set $\chi_0 = \chi(\cdot; \lambda, \epsilon)$. Using the energy localization of the f_n and (3.3), we obtain

$$\Big\langle f_n \, , \, \Big(\{P, A_I\} \Big)^w_{h_n} f_n \Big\rangle \ = \ \Big\langle f_n \, , \, \Big(\chi_0(P) \{P, A_I\} \chi_0(P) \Big)^w_{h_n} f_n \Big\rangle \ + \ o(1) .$$

Notice that $\chi_0(P(1 - \Pi_I))$ is localized in a region where the eigenvalues of $P(1 - \Pi_I)$ are close to λ. Near x_0^*, they are seperated from the eigenvalues of $P_I = P\Pi_I$. Thus, for ϵ small enough, x_0^* does not belong to the support of $\chi_0(P(1 - \Pi_I))$ and we may assume that A vanishes on this set. Therefore,

$$\chi_0(P) \{P, A_I\} \chi_0(P) \ = \ \chi_0(P_I) \{P, A_I\} \chi_0(P_I) \ = \ \chi_0(P) \{P, A_I\}_I \chi_0(P) .$$

Thanks to Proposition 2.5, $\Pi_I \{P(1 - \Pi_I), A_I\} \Pi_I = 0$, yielding $\chi_0(P)\{P, A_I\}\chi_0(P) = \chi_0(P)\{P_I, A_I\}_I \chi_0(P)$. Using again the energy localization of the f_n and (3.3), we arrive at (3.13). $\square$

3.3 Codimension 1 crossing with a special condition.

Now we focus on codimension 1 crossings (cf. Definition 2.6) under Hypothesis 1 (see Subsection 3.1). This allows us to consider smooth symbols that commute with P near the crossing and thus to remove the influence of the distribution ν of Proposition 3.5. Then, we exploit the special condition at the crossing (see Definition 2.9) to perform the end of the proof of Theorem 1.3.
In the framework defined in Subsection 2.2, we first want to derive other properties of μ near some point $\mathrm{x}_0^* = (x_0, \xi_0) \in \mathcal{C}^*(\lambda)$ ($\mathcal{C}^*(\lambda)$ being defined in Definition 2.2). Near x_0, $\mathcal{C}$ is given by some equation $\tau = 0$ (cf. Definition 2.6). We denote by $\mathbb{1}_{\mathcal{C}^*(\lambda)}$ the characteristic function of $\mathcal{C}^*(\lambda)$ and we use the notation of Proposition 3.6.

Proposition 3.7. *Consider a model of codimension* 1 *crossing under Hypothesis 1. Let* $\mathrm{x}_0^* = (x_0, \xi_0) \in \mathcal{C}^*(\lambda)$. *For all* $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$*, supported close enough to* x_0^**,* $\mu(\mathbb{1}_{\mathcal{C}^*(\lambda)}(\xi \cdot \nabla\tau)A) = 0$. *In particular, near* x_0^**,*

$$\operatorname{supp} \mathbb{1}_{\mathcal{C}^*(\lambda)}\mu \;\subset\; \left\{\mathrm{x}^* = (x,\xi) \in \mathcal{C}^*(\lambda)\,;\, \xi \cdot \nabla\tau(x) \;=\; 0\right\}. \tag{3.14}$$

Furthermore, if the model of codimension 1 *crossing satisfies the special condition at the crossing and if* $A = \sum_{j\in I} a_j \Pi_j$*, for smooth, scalar functions* a_j*, localized near* x_0^**, then*

$$\mu\left(\sum_{j\in I} \{p_j, a_j\}\Pi_j\right) \;=\; 0\,. \tag{3.15}$$

Proof: For $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, we set $\pi_I(A) = \sum_{j\in I} \Pi_j A \Pi_j$. Let A be localized near x_0^*. Since $[\pi_I(A), P] = 0$ and $\Pi_I \pi_I(A) \Pi_I = \pi_I(A)$, Proposition 3.5 and (3.12) yield

$$\mu\Big(\{P_I, \pi_I(A)\}_I\Big) \;=\; 0\,. \tag{3.16}$$

This is still true if we replace A by $\tau\chi(\tau/\epsilon)A$, where $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ with $\chi = 1$ near 0 and $\epsilon > 0$. By the dominated convergence theorem,

$$\lim_{\epsilon\to 0^+} \mu\Big((\tau/\epsilon)\chi'(\tau/\epsilon)(2\xi \cdot \nabla\tau)\pi_I(A)\Big) \;=\; 0\,.$$

Thus $\mu(\mathbb{1}_{\tau=0}(2\xi \cdot \nabla\tau)\pi_I(A)) = 0$, where $\mathbb{1}_{\tau=0}$ is the characteristic function of $\{(x,\xi) \in T^*\mathbb{R}^d; \tau(x) = 0\}$, which coincides locally with $\mathcal{C} \times \mathbb{R}^d$, the crossing viewed in $T^*\mathbb{R}^d$. This implies that $\mu(\mathbb{1}_{\mathcal{C}^*(\lambda)}(2\xi \cdot \nabla\tau)\pi_I(A)) = 0$. Thus, near x_0^*, $\mathbb{1}_{\mathcal{C}^*(\lambda)}\pi_I(\mu)$ is supported in the r.h.s of (3.14).

But, near x_0^*, $\operatorname{supp}\mu$ is contained in $\operatorname{supp}\pi_I(\mu)$. To prove this, let $B \in C^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$, non-negative, localized near x_0^*, such that $\operatorname{supp} B \cap \operatorname{supp}\pi_I(\mu) = \emptyset$. Let $\chi \in C_0^\infty(T^*\mathbb{R};\mathbb{R}^+)$ such that $\chi = 1$ on $\operatorname{supp} B$ and $\operatorname{supp}\chi \cap \operatorname{supp}\pi_I(\mu) = \emptyset$. By energy localization of μ (cf. Proposition 3.3), $0 \le \mu(B) = \mu(B_I)$. Let $A = \chi(1 + \|B\|_m^2)^{1/2}\Pi_I$. Since $A = \pi_I(A) \ge B_I$, we obtain $0 \le \mu(B) \le \mu(\pi_I(A))$. Since $\operatorname{supp} A \cap \operatorname{supp}\pi_I(\mu) = \emptyset$, $\mu(\pi_I(A)) = 0$. This implies that $\mu(B) = 0$, yielding $\operatorname{supp}\mu \subset \operatorname{supp}\pi_I(\mu)$ near x_0^*.
Therefore (3.14) holds true. Now let $A := \sum_{j\in I} a_j\Pi_j = A_I = \pi_I(A)$ be localized near x_0^*. By Proposition 2.5,

$$\Pi_I\,\{P_I, A\}\,\Pi_I \;=\; \sum_{j\in I}\{p_j, a_j\}\,\Pi_j \;+\; \sum_{j,k\in I, j\neq k}\Pi_j\Big(Q_1 \;+\; \tau Q_2\Big)\Pi_k\,, \tag{3.17}$$

where $Q_1 = \sum_{l\in I} a_l(2\xi\cdot\nabla\Pi_l)$ and Q_2 is smooth. Notice that the factor τ in the last term of (3.17) comes from the fact that P_I acts on the range of Π_I as $(|\xi|^2 + \mathrm{tr}_n(M(x)\Pi_I(x)))\mathrm{I}_m + \tau(x)V_I(x)$, for some matrix $V_I(x)$, since we consider a model of codimension 1 (cf. Definition 2.6). Away from $\mathcal{C}^*(\lambda)$, the last term of (3.17), say Q', is given by

$$Q' \;=\; \sum_{j,k\in I, j\neq k}(p_j - p_k)^{-1}\Pi_j[P\,,\,Q_1 \;+\; \tau Q_2]\Pi_k \;=\; [P - \lambda, Q_3]\,,$$

for some smooth Q_3. It is thus annihilated by μ (cf. Proposition 3.3). Therefore,

$$\mu\left(\sum_{j,k\in I, j\neq k}\Pi_j\Big(Q_1 \;+\; \tau Q_2\Big)\Pi_k\right) \;=\; \mu\left(\mathbb{1}_{\mathcal{C}^*(\lambda)}\sum_{j,k\in I, j\neq k}\Pi_j\,Q_1\,\Pi_k\right). \tag{3.18}$$

But, by the special condition at the crossing (cf. Definition 2.9) and (3.14), Q_1 precisely vanishes on the support of $\mathbb{1}_{\mathcal{C}^*(\lambda)}\mu$. Now (3.16) and (3.17) imply (3.15). $\square$

Remark 3.8. *If $d = 1$, the above proof gives a better result. Indeed, we can rewrite (3.14) as $\operatorname{supp}\mathbb{1}_{\mathcal{C}^*(\lambda)}\mu \subset \{(x,\xi)\in\mathcal{C}^*(\lambda);\xi = 0\}$. Since Q_1 vanishes on this last set, (3.15) holds true. We did not use the special condition at the crossing.*

Proof of Theorem 1.3: We want to prove the property ((1.3) and (1.4)) by contradiction. Thus, we assume that Hypothesis 1 (at the beginning of Subsection 3.1) holds true and can apply the results above. In particular, we know that the measure μ is compactly supported in the energy shell $E^*(\lambda)$ (cf. (2.3)) and is nonzero (cf. Proposition 3.4). As in [J5], we are going to show that the non-trapping condition on the flows of the eigenvalues actually implies that $\mu = 0$, yielding the desired contradiction.

By this non-trapping condition, we can find $c > 0$ and, for all $j \in \{1, \cdots, N\}$, a smooth scalar function a_j such that $\{p_j, a_j\} \geq c$ on $p_j^{-1}(\lambda) \cap \operatorname{supp} \mu$ (cf. [GM, J5]). If

$$\mu\left(\sum_{j=1}^{N} \{p_j, a_j\}\, \Pi_j\right) \;=\; 0\,, \tag{3.19}$$

then $0 \geq c\mu(I_m)$, yielding $\mu = 0$. Therefore, it suffices to show (3.19).
Using a partition of unity, it suffices to show (3.19) for functions a_j localized away from $\mathcal{C}^*(\lambda)$, the crossing region at energy λ defined in (2.6), and for functions a_j localized near any $\mathrm{x}_0^* \in \mathcal{C}^*(\lambda)$. In the second case, the localization implies that

$$\mu\left(\sum_{j=1}^{N} \{p_j, a_j\}\, \Pi_j\right) \;=\; \mu\left(\sum_{j \in I} \{p_j, a_j\}\, \Pi_j\right)$$

which equals to zero, by Proposition 3.7. In the first case, the arguments in the proofs of Propositions 3.5 and 3.7 directly show (see also [J4]) that

$$\mu\left(\sum_{j=1}^{N} \{p_j, a_j\}\, \Pi_j\right) \;=\; \mu\left(\left\{P\,,\, \sum_{j=1}^{N} a_j \Pi_j\right\}\right) \;=\; 0\,. \qquad \square$$

4 Toward the non-trapping condition.

Now we come to the proof of Theorem 1.4. Starting from the property ((1.3) and (1.4)) at energy $\lambda > \|M_\infty\|_m$, for $R(\lambda + i\epsilon; h)$ and for $R(\lambda - i\epsilon; h)$ (cf. Remark 1.1), we want to derive the non-trapping condition. To this end, we follow the scalar strategy of [W] based on Egorov's theorem and on the use of coherent states. For empty crossing, a minor change in Wang's arguments give the non-trapping condition, since we can derive a weak form of Egorov's theorem (see Subsection 4.1). In the general case or even for our model of codimension 1 crossing (cf. Definition 2.6), we are not able to follow the same lines. However, for our codimension 1 crossing with structure condition at infinity (cf. Definitions 2.6 and 2.13), we can adapt the previous strategy and extract the non-trapping condition (see Subsection 4.2).

4.1 Wang's strategy.

First, we recall Wang's strategy in the present framework and then show that, if the relevant eigenvalues crossing at energy λ (cf. Definition 2.2) is empty, it gives the desired

non-trapping condition (cf. Proposition 4.1). This completes the proof of Theorem 1.8. The effect of the crossing will be considered in Subsection 4.2.
Consider $s > 1/2$, $\lambda \in \mathbb{R}$, $h_0 > 0$, and an interval I about λ, such that the boundary values $\langle x\rangle^{-s}R(\mu + i0; h)\langle x\rangle^{-s}$ and $\langle x\rangle^{-s}R(\mu - i0; h)\langle x\rangle^{-s}$ exist on I, for $h \in]0; h_0]$ (cf. (1.3)). We assume further that, for μ in the same interval I and $h \in]0; h_0]$, the resolvent estimate (1.4) and the same estimate for $R(\mu - i0; h)$ hold true. Recall that $\| \cdot \|$ denotes the operator norm of the bounded operators on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. Following [W], we interpret the resolvent estimates by means of Kato's notion of locally $\hat{P}(h)$-smoothness (see [RS4]). These estimates imply that $\langle x\rangle^{-s}$, for $s > 1/2$, is $\hat{P}(h)$-smooth on I, for $h \in]0; h_0]$. Then

$$\int_{\mathbb{R}} \Big\| \langle x\rangle^{-s}\, \theta\Big(\hat{P}(h)\Big)\, U_h(t) \Big\|^2 \, dt \;\leq\; C_s \, , \tag{4.1}$$

where $U_h(t) := \exp(-ih^{-1}t\hat{P}(h))$, $\theta \in C_0^\infty(I; \mathbb{R})$ satisfies $0 \leq \theta \leq 1$ and equals 1 near λ, and $C_s > 0$ only depends on the resolvent estimates, s, and I. Now, for $B = \langle x\rangle^{-2s}\mathrm{I}_m$, we want an approximation of $U_h(t)^* B_h^w U_h(t)$ of the form $(F^t(B))_h^w$ where F^t should map symbols to symbols and F^0 be the identity. By $\tilde{O}_t(h)$, we denote a bounded operator on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$, the norm of which is $O_t(h)$. We write, using (3.9),

$$U_h(t)^* B_h^w U_h(t) \;-\; \Big(F^t(B)\Big)_h^w \;=\; -\int_0^t (d/dr)\Big(U_h(t-r)^* \Big(F^r(B)\Big)_h^w U_h(t-r)\Big)\, dr$$

$$\begin{aligned} =\; & -\int_0^t U_h(t-r)^*\, ih^{-1}\Big([P, F^r(B)]\Big)_h^w U_h(t-r)\, dr \;+\; \tilde{O}_t(h) \\ & -\int_0^t U_h(t-r)^* \Big(dF^r(B)/dr \;-\; \{P, F^r(B)\}\Big)_h^w U_h(t-r)\, dr \, . \end{aligned} \tag{4.2}$$

In the scalar case (see [W]), one can choose $F^t(B)$ to be $B \circ \phi^t$, where ϕ^t is the classical Hamiltonian flow, and one has a really good approximation for $U_h(t)^* B_h^w U_h(t)$ (Egorov's theorem), since the r.h.s of (4.2) is some $\tilde{O}_t(h)$. Then, it is easy to translate (4.1) into some integrability property of $(F^t(B))_h^w$, which leads to the non-trapping condition on ϕ^t at energy λ by use of coherent states microlocalized on the energy shell of energy λ (see [W] or the arguments below).
We come to the matricial case introduced in Subsection 2.1 with empty relevant eigenvalues crossing at energy λ (cf. Definition 2.2). According to Definition 2.4, let N be the number of relevant eigenvalues at energy λ. Assume for a while that there is no relevant eigenvalues crossing at energy λ "at infinity". This means that we can find $\delta, R > 0$ such that any two different relevant eigenvalues at energy λ, λ_j and λ_k, satisfy the property $(|x| \geq R \Longrightarrow |\lambda_j(x) - \lambda_k(x)| \geq \delta)$.

It turns out that we only need a control of the l.h.s of (4.2) localized in energy, that is after multiplying on both sides by some $\theta(\hat{P}(h))$. Recall that $\theta(\hat{P}(h))$ can be viewed as a h-pseudodifferential operator satisfying

$$\theta\Big(\hat{P}(h)\Big) \;=\; \Big(\theta(P)\Big)_h^w \,+\, h\,D(h)$$

for some uniformly bounded operator $D(h)$ (cf. (3.3)). Here the function $\theta(P) : T^*\mathbb{R}^d \longrightarrow \mathcal{M}_m(\mathbb{C})$ maps any $\mathrm{x}^* \in T^*\mathbb{R}^d$ to the matrix $\theta(P(\mathrm{x}^*))$ given by the functional calculus of the self-ajoint matrix $P(\mathrm{x}^*)$. By the h-pseudodifferential calculus, the principal symbol of $\theta(\hat{P}(h))([P, F^t(B)])_h^w\theta(\hat{P}(h))$ is given by $\theta(P)[P, F^t(B)]\theta(P)$, thanks to (3.9). By Proposition 2.3, it is zero if the support of θ is close enough to λ, since the relevant crossing is empty. In particular, we get a localized, weak version of Egorov's theorem. Indeed, we define on $E^*(\lambda, \epsilon)$ with small enough $\epsilon > 0$, for

$$A \;=\; \sum_{j=1}^N a_j \Pi_j \,, \qquad F^t(A) \;:=\; \sum_{j=1}^N \Big(a_j \circ \phi_j^t\Big)\Pi_j \,. \tag{4.3}$$

Recall that the ϕ_j^t are the Hamilton flows introduced in Definition 2.4. Thanks to Proposition 2.5, $\sum_{j=1}^N \Pi_j\{P, F^t(A)\}\Pi_j = dF^t(A)/dt$. Since the crossing is empty, $\theta(p_j)\theta(p_k) = 0$ for $k \neq j$, if the support of θ is close enough to λ. In particular,

$$\theta(P)\Bigg(\sum_{j,k=1, j\neq k}^N \Pi_j\{P, F^t(A)\}\Pi_k\Bigg)\theta(P) \;=\; 0\,. \tag{4.4}$$

Making use of (3.3) and (4.2), the previous arguments show that

$$\Big\|\, \theta\Big(\hat{P}(h)\Big)\, U_h(t)^* B_h^w U_h(t)\, \theta\Big(\hat{P}(h)\Big) \,-\, \theta\Big(\hat{P}(h)\Big)\, \Big(F^t(B)\Big)_h^w \theta\Big(\hat{P}(h)\Big) \,\Big\| \;=\; O_t(h)\,. \tag{4.5}$$

Thus, for any $T > 0$, the difference

$$\int_{-T}^T \theta\Big(\hat{P}(h)\Big)\, U_h(t)^* B_h^w U_h(t)\, \theta\Big(\hat{P}(h)\Big)\, dt \;-\; \int_{-T}^T \theta\Big(\hat{P}(h)\Big)\Big(F^t(B)\Big)_h^w \theta\Big(\hat{P}(h)\Big)\, dt \tag{4.6}$$

is some $\tilde{O}_T(h)$. Now, we introduce the coherent states operator microlocalized near $\mathrm{x}_0^* = (x_0, \xi_0) \in T^*\mathbb{R}^d$. It is the unitary operator on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$ given by

$$c(\mathrm{x}_0^*) := \exp\Big(ih^{-1/2}(x \cdot x_0 - \xi_0 \cdot D_x)\Big)\, \mathrm{I}_m \,, \tag{4.7}$$

(cf. [R, J2]). For any $S \in \Sigma_{r,t}$ with $r, t \leq 0$ (cf. (3.1)), $c(\mathrm{x}_0^*)^* S_h^w c(\mathrm{x}_0^*) = S(\mathrm{x}_0^*) + \tilde{O}_S(h)$. Since $\theta(\lambda) = 1$, we obtain, for $\mathrm{x}_0^* \in E^*(\lambda)$,

$$\int_{-T}^T c(\mathrm{x}_0^*)^* \theta\Big(\hat{P}(h)\Big)\Big(F^t(B)\Big)_h^w \theta\Big(\hat{P}(h)\Big) c(\mathrm{x}_0^*)\, dt \;=\; \int_{-T}^T F^t(B)(\mathrm{x}_0^*)\, dt \,+\, \tilde{O}_T(h)\,.$$

Notice that $F^t(B)(\mathrm{x}_0^*) = \sum_{j=1}^N \langle \pi_x \phi_j^t(\mathrm{x}_0^*)\rangle^{-2s}\Pi_j(x_0)$, if $\pi_x : T^*\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined by $\pi_x(q,p) = q$. Letting h tend to zero and using (4.1) and (4.6), we conclude that, there exists some $C > 0$ such that, for all $j \in \{1, \cdots, N\}$, all $\mathrm{x}_0^* \in p_j^{-1}(\lambda)$, and all $T > 0$,

$$\int_{-T}^{T} \Big\langle \pi_x \phi_j^t(\mathrm{x}_0^*)\Big\rangle^{-s} dt \ \leq \ C\,. \tag{4.8}$$

As in [W], this implies that λ is non-trapping for each Hamiltonian flow ϕ_j^t. Now, let us remove the assumption at infinity we made above. Let $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $0 \leq \chi \leq 1$ and $\chi(0) = 1$. For $R > 0$, set $\chi_R(x) = \chi(x/R)$. We can follow the previous lines with B replaced by $\chi_R B$. This leads to

$$\int_{-T}^{T} \chi_R\Big(\pi_x \phi_j^t(\mathrm{x}_0^*)\Big)\, \Big\langle \pi_x \phi_j^t(\mathrm{x}_0^*)\Big\rangle^{-s} dt \ \leq \ C\,.$$

Taking the supremum over $R \geq 1$, we derive (4.8), since C is independent of R. Thus, we have proved the

Proposition 4.1. *Consider the general case and take $\lambda \in \mathbb{R}$. Assume that the relevant crossing at energy λ is empty. If, for some $s > 1/2$, for h small enough, and near λ, the resolvent estimate* (1.4) *holds true, then, for all relevant eigenvalues p_j at energy λ, λ is a non-trapping energy for p_j.*

Remark 4.2. *In the proof of Proposition 4.1, we can derive* (*4.5*) *from the matricial Egorov theorems of [BG, BN], if there is no crossing at infinity* (*cf. Definition 2.12*).

4.2 A Gronwall type argument.

In this subsection, we adapt the strategy considered in Subsection 4.1 to prove Theorem 1.4. Because of the eigenvalues crossing, the first term on the r.h.s of (4.2) does not vanish at all by energy localization. Therefore, we need to modify the strategy. It is reasonable to solve the differential equation $dF^t(B)/dt = \{P, F^t(B)\}$ but it is not clear that the solution commutes with P near the considered energy shell. So we do not know how to generalize the approximation (4.5), if it is possible. We choose to avoid this question and to require that the first term on the r.h.s of (4.2) vanishes. Since we consider a model of codimension 1 crossing (cf. Definition 2.6), the functions in (4.3) are nice symbols satisfying the requirement. But we do not expect anymore to solve $dF^t(B)/dt = \{P, F^t(B)\}$ near the energy shell (see (4.4)). Therefore, up to some $O_t(h)$, there is a term left on the r.h.s of (4.2). Using the structure condition at infinity (cf. Definition 2.13), we roughly

control it by $\|(F^t(B))_h^w\|$ and get, by a Gronwall type argument, the expected time integrability of $(F^t(B))_h^w$ (cf. Proposition 4.3). Then, we conclude as in Subsection 4.1.
We point out that, if there is a crossing at infinity (cf. Definition 2.12), we do need the structure condition at infinity (cf. Definition 2.13) and we do use Proposition 1.7 to perform the Gronwall type argument. Otherwise, a simpler proof works.
Recall that ρ describes the behaviour of M at infinity in (2.1).

Proposition 4.3. *Consider the model of codimension* 1 *crossing that satisfies the structure condition at infinity. Let* $\lambda > \|M_\infty\|_m$. *Assume that* (4.1) *holds true for some* $s \in]1/2; (1+\rho)/2]$, *if there is a crossing at infinity, and for some* $s > 1/2$, *otherwise. Let* $B = \langle x\rangle^{-2s}\mathrm{I}_m$ *and define* $F^t(B)$ *as in (4.3). Let* χ, $\chi(\cdot;\cdot,\cdot)$ *be as in (2.7) with* $\chi = 1$ *on* $[-1/2; 1/2]$ *and* $0 \leq \chi \leq 1$. *Then, we can find* $C > 0$ *such that, for all* $T > 0$, *there exists* $\epsilon > 0$, *such that, for* h *small enough,*

$$\int_{-T}^{T} \left\|\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\left(F^t(B)\right)_h^w \chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\right\| dt \;\leq\; C\,.$$

Proof: First, we write a localized version of (4.2), using (3.3) and $[P, F^r(B)] = 0$.

$$\begin{aligned}
&\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\Big(U_h(t)^* B_h^w U_h(t) \;-\; \left(F^t(B)\right)_h^w\Big)\chi\Big(\hat{P}(h);\lambda,\epsilon\Big) \\
=&\;-\int_0^t U_h(t-r)^* \left(\chi(P;\lambda,\epsilon)\right)_h^w \left(dF^r(B)/dr \;-\; \Big\{P, F^r(B)\Big\}\right)_h^w \cdot \\
&\qquad \left(\chi(P;\lambda,\epsilon)\right)_h^w U_h(t-r)\,dr \;+\; \tilde{O}_t(h) \\
=&\;-\int_0^t U_h(t-r)^* \left(\chi(P;\lambda,\epsilon)\right)_h^w \left(\sum_{j,k=1, j\neq k}^{N} \Pi_j\{P, F^r(B)\}\Pi_k\right)_h^w \cdot \\
&\qquad \left(\chi(P;\lambda,\epsilon)\right)_h^w U_h(t-r)\,dr \;+\; \tilde{O}_t(h)\,,
\end{aligned} \tag{4.9}$$

since $\sum_{j=1}^N \Pi_j\{P, F^t(B)\}\Pi_j = dF^t(B)/dt$. Recall that $\tilde{O}_t(h)$ denotes a bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, the norm of which is $O_t(h)$. In view of Definitions 2.6 and 2.13, we introduce a partition of unity localized on $E^*(\lambda,\epsilon_0)$ (cf. (2.2)), for some $\epsilon_0 > 0$. Let $\psi, \psi_0, \psi_1, \cdots, \psi_p \in C^\infty(T^*\mathbb{R}^d;\mathbb{R})$ with $\psi + \sum_{q=0}^p \psi_q = 1$ on $E^*(\lambda,\epsilon_0)$ and satisfying the following conditions. ψ_0 is supported in $\{(x,\xi) \in T^*\mathbb{R}^d; |x| > R\}$, for some $R > 0$, where M has the structure mentioned in Definition 2.13. In particular, (2.10) and (2.11) hold true. The other functions are compactly supported and ψ is supported away from $\mathcal{C}^*(\lambda)$ (see (2.6)). For each $k \in \{1, \cdots, p\}$, M has the matricial structure described in Definition 2.6 on the support of ψ_k. If there is "no crossing at infinity" (cf. Definition 2.12),

we remove the function ψ_0 and take ψ supported away from $\mathcal{C}^*(\lambda)$ and at "infinity".
We insert $1=\psi_h^w+\sum_{q=0}^p(\psi_q)_h^w$ into (4.9), between the two energy localizations $(\chi(P;\lambda,\epsilon))_h^w$. The contribution of ψ is $\tilde{O}_t(h)$, since (4.4) holds true on the support of ψ. Let $q\in\{1,\cdots,p\}$ and consider the contribution of ψ_q. As in the proof of Proposition 3.6, we use the fact that $\chi(P(1-\Pi_I);\lambda,\epsilon)$ and ψ_q have disjoint supports, to see that the contribution L_q of ψ_q in (4.9) is given by

$$\begin{aligned} L_q \;&=\; -\int_0^t U_h(t-r)^*\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\Bigg(\sum_{j,k\in I, j\neq k}\Pi_j\Big\{P_I,\Big(F^r(B)\Big)_I\Big\}\Pi_k\,\psi_q\Bigg)_h^w \\ &\quad\cdot\Big(\chi(P;\lambda,\epsilon)\Big)_h^w U_h(t-r)\,dr\;+\;\tilde{O}_t(h)\,. \end{aligned} \tag{4.10}$$

As in the proof of Proposition 3.7 (see (3.17)), writing $F^r(B)=\sum_{l=1}^N a_l^r\Pi_l$ and using Proposition 2.5,

$$\sum_{j,k\in I,j\neq k}\Pi_j\Big\{P_I,\Big(F^r(B)\Big)_I\Big\}\Pi_k\,\psi_q\;=\;\sum_{j,k\in I,j\neq k}\Pi_j\Big(Q_1(r)\;+\;\tau Q_2(r)\Big)\Pi_k\,\psi_q\,,$$

where $\|(Q_2(r)\psi_q)_h^w\|=O_r(1)$ and $Q_1(r)=\sum_{l\in I}a_l^r(2\xi\cdot\nabla\Pi_l)$ on the support of ψ_q. But

$$\begin{aligned} &\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\Bigg(\sum_{j,k\in I,j\neq k}\Pi_j\,\tau\,Q_2(r)\,\Pi_k\,\psi_q\Bigg)_h^w\Big(\chi(P;\lambda,\epsilon)\Big)_h^w \\ =\;&\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\tau\chi(\tau;0,\epsilon)\Bigg(\sum_{j,k\in I,j\neq k}\Pi_j\,Q_2(r)\,\Pi_k\,\psi_q\Bigg)_h^w\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\;+\;\tilde{O}_r(h)\,, \end{aligned}$$

thus the contribution of Q_2 in (4.10) is $\tilde{O}_t(\epsilon)+\tilde{O}_t(h)$. Since ψ_q is compactly supported and $F^r(B)\Pi_l=a_l\Pi_l$, the contribution L_q' of Q_1 in (4.10) is

$$\begin{aligned} L_q' \;&=\; -\int_0^t U_h(t-r)^*\,\chi\Big(\hat{P}(h);\lambda,2\epsilon\Big)\,\langle x\rangle^{-s}\,\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\Big(F^r(B)\Big)_h^w\chi\Big(\hat{P}(h);\lambda,\epsilon\Big) \\ &\quad\cdot\Bigg(\langle x\rangle^{2s}\sum_{\substack{j,k,l\in I\\ j\neq k}}\Pi_j\Big(2\xi\cdot\nabla\Pi_l\Big)\Pi_k\,\psi_q\Bigg)_h^w\langle x\rangle^{-s}\,\chi\Big(\hat{P}(h);\lambda,2\epsilon\Big)\,U_h(t-r)\,dr\;+\;\tilde{O}_t(h)\,, \end{aligned} \tag{4.11}$$

where the factor containing $\langle x\rangle^{2s}$ is uniformly bounded w.r.t. h. Here we used $\chi(\cdot;\lambda,\epsilon)=\chi(\cdot;\lambda,2\epsilon)\chi(\cdot;\lambda,\epsilon)$. Although the support of ψ_0 is not compact, the same computation works for $q=0$ thanks to (2.10) and (2.11) (the set I must be replaced by I_∞ defined in (2.9)). The factor containing $\langle x\rangle^{2s}$ in (4.11) for $q=0$ is also uniformly bounded if $s\in]1/2;(1+\rho)/2]$.

Let $T > 0$. Let $\hat{\chi} = \chi(\hat{P}(h); \lambda, \epsilon)$ and $\hat{\chi}_0 = \chi(\hat{P}(h); \lambda, 2\epsilon)$, for short. Putting all together, we arrive at, for some $Q \in \Sigma_{0,0}$,

$$\begin{aligned}\int_0^T \hat{\chi}\Big(F^r(B)\Big)_h^w \hat{\chi}\, dt \;&=\; \int_0^T \hat{\chi} U_h(t)^* B_h^w U_h(t) \hat{\chi}\, dt \,+\, \tilde{O}_T(\epsilon) \,+\, \tilde{O}_T(h) \\ &\quad + \int_0^T U_h(r)^* \hat{\chi}_0 \langle x\rangle^{-s} \left(\int_0^t \hat{\chi}\Big(F^{t'}(B)\Big)_h^w \hat{\chi}\, dt' \right) Q_h^w \,\langle x\rangle^{-s}\, \hat{\chi}_0 U_h(r)\, dr\,.\end{aligned} \tag{4.12}$$

Let $q > 0$ such that $\|Q_h^w\| \leq q$, for small enough h. For ϵ and h small enough, the sum of the three first terms on the r.h.s of (4.12) is in norm smaller than $2C_s$, by (4.1). Using again (4.1), we can apply Gronwall's lemma (see [DG], for instance) to the function $T \mapsto \int_0^T \|\hat{\chi}(F^r(B))_h^w \hat{\chi}\|\, dt$. Thus, this function is bounded by $2C_s \exp(qC_s)$. Similarly, we can control the integral over $[-T; 0]$. $\square$

Proof of Theorem 1.4: By assumption, the property ((1.3) and (1.4)) at energy $\lambda > \|M_\infty\|_m$ holds true for some $s > 1/2$. If there is a "crossing at infinity" (cf. Definition 2.12), we can assume, by Proposition 1.7, that $s \in]1/2; (1+\rho)/2]$, where $\rho > 0$ measures the decay of M at infinity (cf. (2.1)). For this s, the same property applied to $R(\lambda - i\epsilon; h)$ holds true (cf. Remark 1.1). Denoting by $U_h(t) := \exp(-ih^{-1}t\hat{P}(h))$ the propagator of $\hat{P}(h)$, this allows us to use Kato's local smoothness to derive (4.1). Now, we use Proposition 4.3. Thus we can find some $C > 0$ such that, for all $T > 0$, there is some $\theta \in C_0^\infty(\mathbb{R}; \mathbb{R})$, with $\theta = 1$ near λ, such that, for h small enough,

$$\int_{-T}^T \left\| \theta\big(\hat{P}(h)\big)\, U_h(t)^* \left(\sum_{j=1}^N \Big\langle \pi_x \phi_j^t(\cdot) \Big\rangle^{-2s} \Pi_j \right)_h^w U_h(t)\, \theta\big(\hat{P}(h)\big) \right\| dt \;\leq\; C\,.$$

Here, $\pi_x \phi_j^t$ denotes the space component of the flow ϕ_j^t. Using the coherent states operators (4.7), we translate this integrability into (4.8), as in Subsection 4.1, yielding the non-trapping condition, as in [W]. $\square$

5 Previous results revisited.

In this last section, we first want to compare the present results with those of [J1, J4]. In some sense, we complete and generalize the later ones. Furthermore, we show that the method developped in Section 3 can be used to derive a new proof of them. Secondly, we

connect Theorem 1.3 with a certain result in [H] and explain in which sense the special condition at the crossing is natural.
Let us first explain what kind of improvement of results we have. In [J1], the relevant eigenvalues crossing at considered energy is empty and the property ((1.3) and (1.4)) is deduced from a non-trapping condition on the flows of the eigenvalues of the symbol. In [J4], we consider a codimension 1 crossing and we essentially require that a global eigenvalue only crosses another one (see Example 2.8). We assume further that the variation of eigenspaces of $M(x)$ is small enough. Then we prove the property ((1.3) and (1.4)) under the previous non-trapping condition.
These results are completed here in the sense that we show that this non-trapping condition is necessary to have the semiclassical resolvent estimates (see Theorems 1.4 and 1.8). In the case treated in [J1], we even have a simpler proof (see the proof of Proposition 4.1). Since we assume the special condition on M in Theorem 1.3, we cannot deduce from it the results in [J4]. But, we think that this special condition (cf. Definition 2.9) is more satisfactory than the previous, vague requirement on the variation of the eigenspaces. Furthermore, we are able here to consider much more general eigenvalues crossings (compare Examples 2.8 and 2.16).
Another interesting feature is that we do not loose anything if we adopt the method developped in Section 3. Let us show how it gives a new proof of the previously mentioned results in [J1, J4]. It turns out that these results can be derived from Theorem 5.1 below, that we proved in [J4] with the semiclassical Mourre theory. It concerns the general model of Subsection 2.1. Recall that the energy shell $E^*(\lambda)$ is given in (2.3) and that the generalized Poisson bracket $\{\cdot, \cdot\}$ is defined in (3.10).

Theorem 5.1. [J4] *Consider the general model and let $\lambda > \|M_\infty\|_m$. Assume that we can find some $c > 0$ and some function $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ such that $A(x, \xi) = (x \cdot \xi)\mathrm{I}_m$ for $|x|$ large enough and such that, near $E^*(\lambda)$, $[P, A]$ vanishes and $\{P, A\} \geq c\mathrm{I}_m$. Then, for all $s > 1/2$, the property* ((1.3) *and* (1.4)) *holds true.*

Here we shall show the following stronger result.

Proposition 5.2. *Consider the general model and let $\lambda > \|M_\infty\|_m$. Assume that, for any $R > 0$, we can find some $c > 0$ and some $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ such that $[P, A]$ vanishes near $\mathcal{C}^*(\lambda) \cap \{(x, \xi) \in T^*\mathbb{R}^d; |x| \leq R\}$ and such that $\{P, A\} \geq c\mathrm{I}_m$ near $E^*(\lambda) \cap \{(x, \xi) \in T^*\mathbb{R}^d; |x| \leq R\}$. Then, for all $s > 1/2$, the property* ((1.3) *and* (1.4)) *holds true.*

Proof: As in Section 3, we assume that the property ((1.3) and (1.4)) fails at energy λ for some $s > 1/2$. Thus Hypothesis 1 holds true. By Proposition 3.4, μ is nonzero and there exists some $R_0 > 0$ such that μ is supported in $E^*(\lambda) \cap \{(x,\xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$. Let $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ and $c > 0$ given by the assumption for $R = R_0$. By Proposition 3.5, ν is supported in $\mathcal{C}^*(\lambda) \cap \{(x,\xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$ and $0 = \nu(A) = \mu(\{P, A\}) \geq c\mu(\mathrm{I}_m)$. This contradicts $\mu \neq 0$. □

Now we focus on the meaning and the role of the special condition at the crossing. To simplify the explanations, we consider the potential M given in Example 2.8. The eigenvalues are $u \pm \tau v$ with $v := (v_1^2 + v_2^2)^{1/2}$ and the eigenprojections are $\Pi_\pm := (\mathrm{I}_2 + V)/2$. Let $p_\pm(x,\xi) = |\xi|^2 + u(x) \pm \tau(x)v(x)$, the eigenvalues of $P(x,\xi)$, and denote by $\phi_\pm^t$ the flows they generate. The eigenvalues crossing $\mathcal{C}$ is exactly $\tau^{-1}(0)$.
Let $\mathrm{x}_0^* = (x_0, \xi_0) \in T^*\mathbb{R}^d$ with $x_0 \in \mathcal{C}$ and $\xi_0 \cdot \nabla\tau(x_0) \neq 0$. Let $\delta > 0$ and $g_h \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^2)$ be microlocalized near $\phi_+^{-\delta}(\mathrm{x}_0^*)$ such that, near this point, $\Pi_+(x)g_h = g_h$. In [H], it is shown in this situation how the wave packets g_h evolve under $U(t;h) := \exp(-ih^{-1}t\hat{P}(h))$. In particular, they cross $\mathcal{C}$ and the localization of $U(t;h)g_h$ is driven by the flows $\phi_\pm^t$. However, if $\xi_0 \cdot \nabla\tau(x_0) = 0$, the arguments in [H] do not work anymore and, to our knowledge, no paper in the literature treats this case. In particular, it is not excluded that the eigenvalues crossing traps, at least for a long time, a part of $U(t;h)g_h$.
In view of Theorem 1.3, it is natural to ask: do we need the special condition at the crossing? Recall that the resolvent estimates imply (4.1). Applied to some $g \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^2)$, it essentially means that $U(t;h)g$ must go to infinity sufficiently fast (compared to $1/h$). According to the previous remark on [H], it would not be a surprise to have a difficulty to derive the resolvent estimates from the non-trapping condition only. It is thus natural to require some reasonable condition (independent of the non-trapping condition) that cancels the influence of the previous unknown evolutions on the result.
In the proof of Theorem 1.3, we need to control $\mu(Q_1)$ on the set introduced in (3.14) (cf. (3.17) and (3.18)). But it is precisely a part, say $T^*\mathcal{C}(\lambda)$, of the set of previous points x_0^*, for which we do not control the evolution of wave packets. Furthermore, the functions f_n introduced in Hypothesis 1 can be written as

$$R(z_n; h_n)\langle x\rangle^{-s} g_n \;=\; i\int_0^{+\infty} e^{-t\Im z_n + it\Re z_n}\, U(t;h_n)\, \langle x\rangle^{-s} g_n\, dt$$

for some functions $g_n \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^2)$. A priori, g_n could be like g_h in the previous remark on [H]. Therefore we do not know a priori the behaviour of the f_n, thus of μ, near $T^*\mathcal{C}(\lambda)$. It is thus natural to require that Q_1 "kills" the restriction of μ to $T^*\mathcal{C}(\lambda)$. Notice that the eigenprojections of P are in general independent of the eigenvalues of P. Therefore the special condition at the crossing, which requires that Q_1 vanishes on $T^*\mathcal{C}(\lambda)$, is independent of the non-trapping condition in general and is natural in the sense explained above.

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