M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 1, 1995

Paper 3
Received: September 8, 1995, Revised: October 16, 1995, Accepted: October 23, 1995
Editor: G. Gallavotti

A Single Scale Infinite Volume Expansion for Three Dimensional Many Fermion Green's Functions

Jacques Magnen,

Vincent Rivasseau

Centre de Physique Théorique, CNRS UPR 14

Ecole Polytechnique, F-91128 Palaiseau Cedex, FRANCE

Abstract In [FMRT1] we introduced a cluster expansion for many Fermion systems in two space dimensions based on a so-called sector decomposition. In this paper a completely different expansion is introduced to treat the more difficult case of three (or more) space dimensions: it is based on an auxiliary scale decomposition and the use of the Hadamard inequality. We prove that the perturbative expansion for a *single scale* model has a convergence radius *independent of the scale*. This is a typical result, proved in the two dimensional case in [FMRT1], which we cannot obtain in three dimensions by naive extrapolation of the sector method. Although we do not treat in this paper the full (multiscale) system, we hope this new method to be a significant step towards the rigorous construction of the BCS theory of superconductivity in three space dimensions.

§I Introduction

In this paper we consider many Fermion systems formally characterized by the effective potential

$$\mathcal{G}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z} \int e^{-\mathcal{V}(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)} d\mu_C(\psi, \bar{\psi}) \tag{I.0}$$

for the external fields $\psi^e, \bar{\psi}^e$. Here, $d\mu_C(\psi, \bar{\psi})$ is the Fermionic Gaussian measure in the Grassmann variables $\{\psi(\xi), \bar{\psi}(\xi) \mid \xi \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}\}$ with propagator

$$C(\xi, \bar{\xi}) = \delta_{\sigma,\bar{\sigma}} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i<p,\xi-\bar{\xi}>_-}}{ip_0 - e(\mathbf{p})} \tag{I.1}$$

where

$$< p, \xi >_- = \mathbf{p} \cdot \mathbf{x} - p_0 \tau \tag{I.2}$$

and

$$e(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \mu. \tag{I.3}$$

The variable $\xi = (\tau, \mathbf{x}, \sigma)$ consists of time, space and spin components and the $(d+1)$-momentum $p = (p_0, \mathbf{p})$. The interaction is given by

$$\mathcal{V} = \frac{\lambda}{2} \int \prod_{i=1}^{4} d\xi_i \quad V(\xi_1, \xi_2, \xi_3, \xi_4)\, \bar{\psi}(\xi_1)\bar{\psi}(\xi_2)\psi(\xi_4)\psi(\xi_3) \tag{I.4}$$

where λ is the coupling constant, the kernel $V(\xi_1, \xi_2, \xi_3, \xi_4)$ is translation invariant with $V(0, \xi_2, \xi_3, \xi_4)$ integrable and

$$\int d\xi = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^d} d\mathbf{x}. \tag{I.5}$$

The partition function is

$$Z = \int e^{-\mathcal{V}(\psi,\bar{\psi})} d\mu_C(\psi, \bar{\psi}) \tag{I.6}$$

so that $\mathcal{G}(0, 0) = 0$.

The fact that $\mathcal{V}$ is non-local (although short range) is the source of a few inessential complications. Therefore in this paper we restrict ourselves to the case of a δ function to focus the reader's attention on essential aspects. Therefore from now on, we will consider

$$\mathcal{V}_\Lambda = \frac{\lambda}{2} \int_\Lambda d\mathbf{x} d\tau \;\bar{\psi}_\uparrow(\mathbf{x}, \tau)\bar{\psi}_\downarrow(\mathbf{x}, \tau)\psi_\uparrow(\mathbf{x}, \tau)\psi_\downarrow(\mathbf{x}, \tau) \tag{$I.4.b$}$$

The Euclidean Green's functions with $2p$ points

$$G_p(\xi_1,\bar\xi_1,\dots,\xi_p,\bar\xi_p) = \prod_{i=1}^{p} \frac{\delta^2}{\delta\psi^e(\xi_i)\delta\bar\psi^e(\bar\xi_i)}\mathcal{G} \tag{I.7}$$

generated by the effective potential are the connected Green's functions amputated by the free propagator. By definition, $\mathcal{G}$ exists when the norm

$$\|G_p\| = \max_j \sup_{\xi_j} \int \prod_{i\neq j} d\xi_i \; |G_p(\xi_1,\cdots,\xi_{2p})| \tag{I.8}$$

of each of its moments, $G_p,\ p\geq 1$, is finite. Intuitively, $\|G_p\|$ is the supremum in momentum space of G_p. In fact, the supremum in momentum space was used as the standard norm on vertices in [FT2].

Our goal is to give a rigorous proof that the standard model for a weakly interacting system of electrons and phonons has a superconducting ground state at sufficiently low temperature. Perturbation theory and, in particular, the renormalization of the two point function was controlled in [FT1]. A renormalization group flow for the four point function was defined and analyzed in [FT2]. Two additional ingredients are required to complete this program. First we need a method to control non-perturbatively the theory and its renormalization group flow (governed by the Fermi surface singularity) until length scales of order Δ^{-1}, where Δ is the BCS gap. This first ingredient, which we call "constructive stability", has been provided in two space dimensions (d=2) by the results of [FMRT1]. However the other physically interesting case, namely d=3, is much harder. In this paper we prove by a completely different method that one of the results of [FMRT1] (theorem 1, page 684) extends to d=3: the radius of convergence of the single scale theory is uniformly bounded below. However up to now, we have not been able to obtain in d=3 an analogue of the multiscale result (theorem 2 page 718) of [FMRT1]. Therefore "constructive stability", namely the non-perturbative control of the theory with an infrared cutoff of order Δ^{-1}, remains in d=3 an open problem.

The second ingredient is the control of the so-called reduced BCS model for Cooper pairs (for length scales of order Δ^{-1} and larger). Here the problem is to control the associated spontaneous $U(1)$ symmetry breaking, and the masslessness of the corresponding Goldstone boson. This can presumably be done by combining another expansion, of the

$1/N$ type [FMRT4], and a renormalization group analysis of the infrared singularity of the Goldstone boson, using Ward identities. This program is under way although not yet completed. Remark that this second program, in contrast to the first is *easier* in d=3 than in d=2 (since the infrared divergences of the Goldstone boson look just renormalizable in d=3 but non-renormalizable in d=2). Remark also that the Fermi surface no longer plays a role in the Goldstone boson infrared problem. For references on this part of the program and on the general strategy, see [FMRT3-6].

In the remainder of this paper we specialize to $d = 3$. Our method extends however without difficulty to any $d \geq 2$ (see Appendix I).

As in [FT1,2], the model is sliced into energy regimes by decomposing momentum space into shells around the Fermi surface. The j^{th} slice has covariance

$$C^{(j)}(\xi, \bar{\xi}) = \delta_{\sigma,\bar{\sigma}} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{i<p,\xi-\bar{\xi}>_-}}{ip_0 - e(\mathbf{p})} f_j(p), \tag{I.9}$$

where

$$f_j(p) = f\left(M^{-2j}\left(p_0^2 + e(\mathbf{p})^2\right)\right) \tag{I.10}$$

effectively forces $|ip_0 - e(\mathbf{p})| \sim M^j$. The function $f \in C_0^\infty([1, M^4])$. The parameter M is strictly bigger than one so that the scales near the Fermi surface have j near $-\infty$. The model is defined in finite volume and at fixed scale by the following lemma:

Lemma I.1

$$\mathcal{G}_\Lambda^{(j)}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z_\Lambda^{(j)}} \int e^{-\mathcal{V}_\Lambda(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)} d\mu_{C^{(j)}}(\psi, \bar{\psi}) \tag{I.11}$$

where

$$\mathcal{V}_\Lambda = \frac{\lambda}{2} \int_{\Lambda^4} \prod_{i=1}^{4} d\xi_i \quad V(\xi_1, \xi_2, \xi_3, \xi_4)\, \bar{\psi}(\xi_1)\bar{\psi}(\xi_2)\psi(\xi_4)\psi(\xi_3) \tag{I.12}$$

and

$$Z_\Lambda^{(j)} = \int e^{-\mathcal{V}_\Lambda(\psi, \bar{\psi})} d\mu_{C^{(j)}}(\psi, \bar{\psi}) \tag{I.13}$$

is analytic in λ in a neighborhood of the origin that includes at least the disk of radius const $\left(M^{2j}|\Lambda|\right)^{-1}$.

The full proof (not very difficult) is given in [FMRT1, Lemma 1]. Remark however that the radius of convergence depends on volume and scale in a patently unsatisfactory way.

Recall that $G_p^{(j,\Lambda)}$ is the $2p$-point Green's function generated by $\mathcal{G}_\Lambda^{(j)}$. By the Lemma above the Taylor series

$$G_p^{(j,\Lambda)} = \sum_{n=0}^{\infty} g_n(p,j,\Lambda)\lambda^n \tag{I.14}$$

has a strictly positive, though possibly j and Λ dependent, radius of convergence. The main result of this paper is

Theorem I

There exists a const *, independent of j and Λ, such that*

$$\|g_n(p,j,\Lambda)\| \leq \text{const}^{\,n+p} M^{(4-2p)j} \|V\|^n \tag{I.15}$$

where

$$\|g_n(p,j,\Lambda)\| = \max_k \sup_{\xi_k} \int \prod_{i\neq k} d\xi_i \; |g_n(p,j,\Lambda)(\xi_1,\cdots,\xi_{2p})|. \tag{I.16}$$

Furthermore the limits

$$g_n(p,j) = \lim_{\Lambda\to\mathbb{R}^3} g_n(p,j,\Lambda) \tag{I.17}$$

exist and the infinite volume Green's functions at scale j

$$G_p^{(j)} = \sum_{n=0}^{\infty} g_n(p,j)\lambda^n \tag{I.18}$$

are analytic in $|\lambda| < R = (\text{const}\,\|V\|)^{-1}$.

This theorem is similar to the first theorem of [FMRT1], which applied only to $d = 2$. Only the dependence in the external fields is different. In [FMRT1] the theorem was proved with an external field dependence $M^{(2-5p/2)j}$, here the factor $M^{(4-2p)j}$ is what comes out naturally, because the oscillations in position space of the sin function in (II.1-3), are not taken into account in the bound (II.5a). Remark that neither $M^{(2-5p/2)j}$ nor $M^{(4-2p)j}$ is the desired perturbative factor, which is $M^{1/2(4-2p)j}$ (see (I.23) below and [FT1]). Since in this paper we do not address the multiscale problem we do not try here to improve this external field dependence. Remark however, as shown in [FMRT1], that it is not impossible to adapt a single scale result with an external field dependence which is not the perturbative one to a multiscale analysis, provided some extra ideas are added (see condition (III.10d), page 716 in [FMRT1]).

A theorem such as Theorem I is usually proven with a standard cluster/Mayer expansion. Space-time, $\mathbb{R}^{d+1}$, is paved by cubes Δ of side M^{-j} dual to the decay rate M^j of the propagator. The decay rate is primarily determined by the thickness of the shell in momentum space. Then one expands in coupling constants that control the interaction between boxes. One essential prerequisite for the convergence of such expansions is the j independent estimate $|Z(\Delta)| \leq \text{const}$. For our models, one can see in perturbation theory that this estimate fails. For instance the vacuum graph G_2 (see Fig.I) can be computed explicitly and in $d=3$ it leads to a second order contribution in $e^{\lambda^2 M^{-j}}$ to $|Z(\Delta)|$.

This difficulty can be traced to the fact that the Pauli exclusion principle now permits about $M^{-(d-1)j}$ electrons to be located in Δ with momentum restricted to the shell. For $d=1$ the Fermi surface consists of just two points and consequently only $O(1)$ electrons are allowed in Δ and there is no difficulty. As d grows the Pauli exclusion principle becomes progressively weaker and the estimate on the partition function in Δ becomes more and more j dependent.

In [FMRT1] the solution to this difficulty was found in a decomposition of the shell into smaller units, called sectors. This idea does not seem to work in $d=3$ or more. Our proof of Theorem I relies therefore on a completely different idea : we decompose the propagator in slices according to an auxiliary scale, and the Hadamard inequality is used to control the Fermionic determinants. Before giving this proof, we would like however to include an informal discussion of the reason for which the sector method does not work in $d \geq 3$.

Since the volume of the j-th momentum shell is large, it is natural to decompose the shell into $M^{-(d-1)j}$ sectors of side M^j, to get fields which are supported on cells of unit volume in phase space. This can be done through a smooth partition of unity

$$1 = \sum_{m=1}^{M^{-(d-1)j}} \eta_m(\mathbf{p}), \qquad \eta_m(\mathbf{p}) = \eta_m\left(\frac{\mathbf{p}}{|\mathbf{p}|}k_F\right) \tag{I.19}$$

of the Fermi surface, where η_m is supported on the union of the m^{th} sector, S_m, and its neighbors, whose number is at most $3^{d-1}-1$. There is a corresponding decomposition of the covariance

$$C^{(j)} = \sum_{m=1}^{M^{-(d-1)j}} C^{(j,m)} \tag{I.20}$$

where

$$C^{(j,m)}(\xi,\bar\xi) = \delta_{\sigma,\bar\sigma} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i<p,\xi-\bar\xi>_-}}{ip_0 - e(\mathbf{p})} f_j(p)\, \eta_m(\mathbf{p}) \tag{I.21}$$

and of the fields

$$\psi^j = \sum_{m=1}^{M^{-(d-1)j}} \psi^{(j,m)}, \qquad \bar\psi^j = \sum_{m=1}^{M^{-(d-1)j}} \bar\psi^{(j,m)}. \tag{I.22}$$

The standard power counting bound for an individual graph is still easy when there are sectors. First, one selects a spanning tree for the graph. To each line not in the tree there is a corresponding momentum loop, obtained by joining its ends through a path in the tree. This construction produces a complete set of independent loops. Ignoring unimportant constants, each propagator is bounded by its supremum M^{-j}. The volume of integration for each loop is now $M^{(d+1)j}$. A priori, there is one sector sum with $M^{-(d-1)j}$ terms for each line. But, by conservation of momentum, there is only one sector sum per loop. Thus, if there are n vertices and $e = 2p$ external lines, the supremum in momentum space of the graph is bounded by

$$\begin{aligned}\prod_{\text{lines}} M^{-j} \prod_{\text{loops}} M^{(d+1)j} M^{-(d-1)j} &= M^{-j(4n-2p)/2} M^{2j[(4n-2p)/2-(n-1)]} \\ &= M^{\frac{1}{2}(4-2p)j}\end{aligned} \tag{I.23}$$

In the course of a non-perturbative construction, estimates cannot be made graph by graph because there are too many of them. Rather, the perturbation series must be blocked and the blocks estimated as units. The blocks are estimated using the exclusion principle to implement strong cancellations between the roughly $n!^2$ graphs of order n. However, once the series is blocked, momentum loops can't be defined and the argument leading to the estimate above cannot be made. Conservation of momentum has to be implemented at each vertex, rather than through loops. Even though the volume cutoff Λ breaks exact conservation of momentum, many of the $M^{-2\ell(d-1)j}$ terms in the sector sums for a general 2ℓ-legged vertex must be zero.

In [FMRT1] the following lemma is proved:

Lemma 2

Fix $m \in \mathbf{Z}^{d+1}$ *and* $\ell \geq 2$ *. Then, the number of* 2ℓ*-tuples*

$$\{S_1, \ \cdots \ S_{2\ell}\}$$

of sectors for which there exist $\mathbf{k}_i \in \mathbb{R}^d$, $i = 1, \ \cdots, \ 2\ell$ *satisfying*

$$\mathbf{k}'_i \in S_i, \quad |\mathbf{k}_i - \mathbf{k}'_i| \leq \text{const } M^j \ , \quad i = 1, \ \cdots, \ 2\ell$$

and

$$|\mathbf{k}_1 + \ \cdots \ + \mathbf{k}_{2\ell}| \leq \text{const}\,(1 + |m|)\, M^j$$

is bounded by

$$\text{const}^{\,\ell}(1 + |m|)^d M^{-(2\ell-1)(d-1)j} M^j \,\{1 + |j|\delta_{d,2}\delta_{\ell,2}\}\,.$$

In particular, for a four legged vertex, the number of 4*-tuples is at most*

$$\text{const}\,(1 + |n|)^d M^{(-3d+4)j}\,\{1 + |j|\delta_{d,2}\}\,.$$

Here, $\mathbf{k}' = \frac{\mathbf{k}}{|\mathbf{k}|}$ *denotes the projection of* $\mathbf{k}$ *onto the Fermi surface.*

This lemma although instructive, is neither for $d = 2$ nor for $d > 2$ powerful enough for the non-perturbative construction. However for $d = 2$ the number of active sector 4-tuples at a vertex is of order $|j| M^{-2j}$; ordinary power counting at a vertex can accommodate at most M^{-2j}, hence there is only a "logarithmic" power counting deficit of order $|j|$ at each vertex. For $d = 3$ the number of active sector 4-tuples at a vertex is of order M^{-5j} , and ordinary power counting at a vertex can accommodate only M^{-4j} so that the deficit at each vertex is a full power M^{-j}. This deficit is worse and worse as d increases beyond 3. In three dimension this difficulty can be geometrically thought as the twist or torsion which occurs in the figure made by four momenta of unit length adding up to 0 (this figure is no longer a planar rhombus as in two dimensions).

To circumvent the extra logarithm at $d = 2$, in [FMRT1] we divided the Fermi surface into sectors of length $M^{j/2}$ rather than M^j. Taking into account the anisotropic spatial decay associated to these new "rectangular" sectors beats the logarithmic deficit. If we apply the same idea for $d = 3$, the number of active 4-tuples sectors at a vertex becomes of order $M^{-5j/2}$ but the power counting in the manner of [FMRT1] gives M^{-2j} , so that there still remains a deficit of order $M^{-j/2}$ per vertex. Making the sectors still longer does not work because the anisotropic decay no longer holds since the Fermi surface does not look flat for lengths bigger than $M^{-j/2}$. Remark that this difficulty is presumably linked to the graph G_2, which as we recall is in $M^{-j}\lambda^2$ in a cube of size M^{-j}.

The conclusion is that even the single slice theory in $d \geq 3$ is a non-trivial theory that contains a rather non-trivial renormalization, like ϕ^4_3. It is this renormalization that can be analyzed by means of the auxiliary scales used in this paper.

Finally let us discuss briefly in this introduction the present situation for the full (multi-scale) theory.

The most straightforward strategy in the case of multi-scale Fermionic models consists in writing the effect of inductive integration of higher scales in the form of an effective action for the single scale model, and to check some inductive estimate for this effective action. This is the road followed in [FMRT1]. However when combined with the Hadamard estimate on determinants, this approach runs into a major difficulty: even with a quartic bare action the effective action contains irrelevant operators of degree 6 and more in the Fermionic fields. The Hadamard inequality when applied to such operators does not lead to a convergent answer (it develops divergent factorials).

The obvious thing to do seems to treat the theory as a bosonic one. For bosonic theories we know well that one cannot simply exponentiate all the effective action, as in Wilson's original program, but that one must keep the irrelevant effects coming from higher scales in the form of a polymer gas with hardcore constraints [B][R]. A typical stability estimate in this strategy is to prove that the "completely convergent" multi-scale polymers (i.e. those who do not contain any two or four point subcontributions) form a geometric convergent series. At the moment our best estimates with the Hadamard method indicate that the sum over such "completely convergent" polymer, extending over at most N different scales, converges provided the (bare) coupling constant λ is small, *and* the number of scales is limited to $N \leq \epsilon/\lambda$ (ϵ being a small number). This is encouraging, but tantalizing. Indeed although this last estimate is much better than what a sector analysis would provide (namely

$N \leq K|\log\lambda|$), it is still not sufficient. For the construction of the BCS3 theory, where the BCS gap cuts the renormalization group flow at a scale $\Delta_{BCS} \simeq M^{-N} \simeq e^{-O(1)/\lambda}$, we would need $N \simeq K/\lambda$ (where K is not particularly small). For the construction of non-superconducting Fermi liquids as in [FKLT], the renormalization group flows for ever and we would need N independent of λ. Therefore at the moment we consider that the question of the non-perturbative stability of these theories remains open, but we hope that the Hadamard method, clearly much better than the sector method, is a step towards the final solution.

Acknowledgements This paper is part of a larger program in collaboration with J. Feldman and E. Trubowitz. We thank particularly J. Feldman for his comments.

II The auxiliary scale decomposition

The j-th slice propagator is

$$C^j(\xi, \bar{\xi}) = \delta_{\sigma\bar{\sigma}} \int \frac{dp_0 d^3\mathbf{p}}{(2\pi)^4} \frac{f_j(p)}{ip_0 - e(\mathbf{p})} e^{i<p(\xi-\bar{\xi})>_-}$$

$$= \delta_{\sigma\bar{\sigma}} \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} dp_0 e^{ip_0(\bar{\tau}-\tau)} \int_{-1}^{+\infty} du \; f_j(p_0, u) D(p_0, u) \frac{4\pi \sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|)}{2|\mathbf{x}-\bar{\mathbf{x}}|}$$

$$= \delta_{\sigma\bar{\sigma}} \frac{1}{8\pi^3|\mathbf{x}-\bar{\mathbf{x}}|} \int_{-M^{j+2}}^{+M^{j+2}} \int_{-M^{j+2}}^{+M^{j+2}} dp_0 du \; e^{ip_0(\bar{\tau}-\tau)} \; f_j(p_0, u) D(p_0, u) \sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|) \tag{II.1}$$

by putting $u = e(\mathbf{p})$, $f_j(p_0, u) \equiv f(M^{-2j}(p_0^2 + u^2))$, and $D(p_0, u) = (ip_0 - u)^{-1}$. This formula is proved by performing the integrals over the angular components θ and ϕ of $\mathbf{p}$. In the last line we used the fact that f has its support in $[1, M^4]$, and we assumed that $j \leq -2$*.

* The first slices $j = 0$ or $j = -1$ are trivial from the point of view of this paper.

We decompose further this propagator into *auxiliary slices* according to the scale of $|\mathbf{x}-\bar{\mathbf{x}}|$ by writing

$$C^j(\xi,\bar{\xi}) = \sum_{k=j}^{0} C^{j,k}(\xi,\bar{\xi}) \tag{II.2}$$

with:

$$C^{j,k}(\xi,\bar{\xi}) = -A^{j,k}(\mathbf{x},\bar{\mathbf{x}})\delta_{\sigma\bar{\sigma}} \frac{1}{8\pi^3|\mathbf{x}-\bar{\mathbf{x}}|} \int_{-M^{j+2}}^{+M^{j+2}} \int_{-M^{j+2}}^{+M^{j+2}} dp_0 du e^{ip_0(\bar{\tau}-\tau)}$$

$$D(p_0,u) f_j(p_0,u) \sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|) \tag{II.3}$$

$$A^{j,0}(\mathbf{x},\bar{\mathbf{x}}) = e^{-|\mathbf{x}-\bar{\mathbf{x}}|^2} \tag{II.4a}$$

$$A^{j,k}(\mathbf{x},\bar{\mathbf{x}}) = e^{-M^{2k}|\mathbf{x}-\bar{\mathbf{x}}|^2} - e^{-M^{2k+2}|\mathbf{x}-\bar{\mathbf{x}}|^2} \quad \text{for} \quad -1 \geq \mathrm{k} \geq \mathrm{j}+1 \tag{II.4b}$$

$$A^{j,j}(\mathbf{x},\bar{\mathbf{x}}) = 1 - e^{-M^{2j+2}|\mathbf{x}-\bar{\mathbf{x}}|^2} \tag{II.4c}$$

These formulas are totally explicit. We choose them so that $C^{j,k}$ is a nice convolution of C^j with an explicit spatial Gaussian, since the Fourier transform of a Gaussian is Gaussian, but other formulas are possible. We have

Lemma II.1 *There exists a constant $K(p)$ such that, for $k=0,...,j$, and any integer p the functions $C^{j,k}$ obey the bounds :*

$$\left|C^{j,k}(\xi,\bar{\xi})\right| \leq K(p) M^{j+k} (1+M^k|\mathbf{x}-\bar{\mathbf{x}}|)^{-p} (1+M^j|\tau-\bar{\tau}|)^{-p} \tag{II.5a}$$

Furthermore

$$C^{j,k}(\xi,\xi) = 0 \text{ for } \quad -1 \geq k \geq j \; ; \; \left|C^{j,0}(\xi,\xi)\right| \leq M^{2j} \tag{II.5b}$$

Proof: The proof is elementary and relies on integration by parts. It is given in detail in Appendix 2. ■

For each scale k we shall perform in the next section a cluster expansion according to a lattice $\mathcal{D}_k$ of tubes of spatial side M^{-k} and of time side M^{-j}. The power counting with respect to this auxiliary scale is superrenormalizable and identical to the one of the ultraviolet limit of the ϕ^4_3. To understand this fact, we can compute the weight of a connected graph with n vertices and l internal lines at subscale (j,k) with $k > j$ and $e = 2p = 4n - 2l$ external legs. Using the same norm as in Theorem I of section 1, we fix a particular vertex and integrate over the others. The usual estimate, selecting a tree in the graph [R], gives $M^{-(j+3k)(n-1)}$ for integration of the vertices times $M^{(j+k)l}$ for the power counting

Corresponding to this division of the covariance, there is an associated orthogonal decomposition of the fields ψ^j and $\bar\psi^j$ distributed with covariance C^j as sums of fields $\psi^{j,k}$ and $\bar\psi^{j,k}$ distributed with covariance $C^{j,k}$:

$$\psi^j(\xi) = \oplus_{k=j}^{0} \psi^{j,k}(\xi)$$

$$\bar\psi^j(\bar\xi) = \oplus_{k=j}^{0} \bar\psi^{j,k}(\bar\xi) \tag{II.6}$$

For each scale k we shall perform in the next section a cluster expansion according to a lattice $\mathcal{D}_k$ of tubes of spatial side M^{-k} and of time side M^{-j}. The power counting with respect to this auxiliary scale is superrenormalizable and identical to the one of the ultraviolet limit of the ϕ^4_3. To understand this fact, we can compute the weight of a connected graph with n vertices and l internal lines at subscale (j,k) with $k > j$ and $e = 2p = 4n - 2l$ external legs. Using the same norm as in Theorem I of section 1, we fix a particular vertex and integrate over the others. The usual estimate, selecting a tree in the graph [R], gives $M^{-(j+3k)(n-1)}$ for integration of the vertices times $M^{(j+k)l}$ for the power counting of the lines. This is $M^{(4-2p)j}M^{(n-3+p)(j-k)}$. The factor $M^{(4-2p)j}$ is that of (I.15) in Theorem I. The sum over k is convergent provided $n+p-3>0$, which selects exactly the same graphs as the condition of ultraviolet convergence in the ϕ^4_3 theory. This is to our knowledge the first example of a Fermionic model with this kind of power counting.

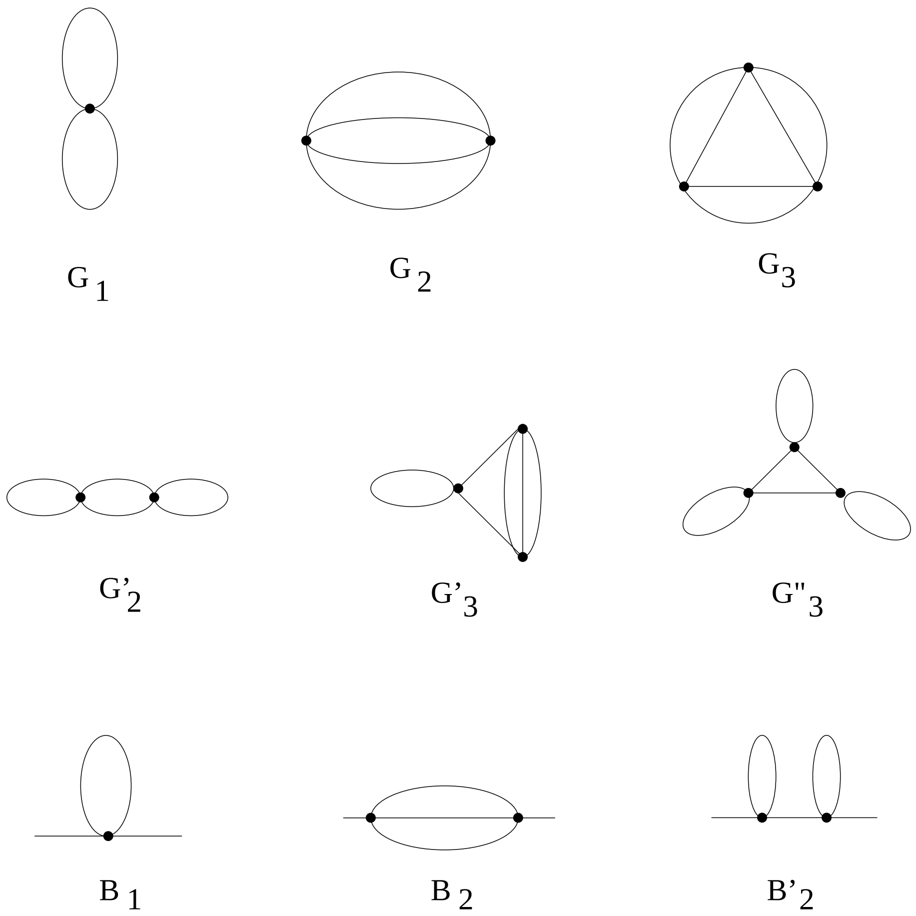

Figure 1: The divergent graphs in d=3

We conclude that as far as renormalization is concerned, the *single slice* model treated in this paper is similar in difficulty to the old ultraviolet construction of ϕ^4_3 by now well established [GJ]. Vacuum graphs are divergent up to three vertices hence the graphs G_1 G_2, G'_2 and G_3, G'_3, G'_3... of Fig. 1, are divergent and the two point subgraphs are divergent up to two vertices, hence the graphs B_1 and B_2, B'_2 of Fig. 1, are a priori divergent. But by (II.5b) the “tadpole” is convergent, hence G_1, B_1 and all the “primed” graphs in fact converge. The second order vacuum graph G_2 is linearly divergent and has to be cancelled explicitly by going to intensive quantities such as the pressure or normalized Schwinger functions. Finally the graph B_2 is apparently logarithmically divergent if we replace the propagators by the bound (II.5a). But in fact it is convergent because of the oscillations of the $\sin^3$ function, which are lost in the right hand side of (II.5a). To exploit this fact, one must renormalize this graph, hence make an explicit computation for it. Had this graph not been convergent, we would have been able only to prove a bound in $|j|^{-1/2}$ on the radius of convergence in Theorem I.

§III The expansion

Since this problem is superrenormalizable, we can treat the few divergent graphs by introducing explicit counterterms, in the manner of [GJ]. This is no problem for the vacuum graphs since they will quotient out anyway in the normalized functions. But for the two point subgraphs B_1 and B_2, we have to compensate for these counterterms by changing the propagator:

Lemma III.1 *The infinite volume connected Green's functions defined by*

$$G_p(\xi_1, \bar{\xi}_1, \dots, \xi_p, \bar{\xi}_p) = \prod_{i=1}^{p} \frac{\delta^2}{\delta\psi^e(\xi_i)\delta\bar{\psi}^e(\bar{\xi}_i)} \mathcal{G} \tag{III.1}$$

$$\mathcal{G}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z} \int e^{-\mathcal{V}(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)} d\mu_C(\psi, \bar{\psi}) \tag{III.2}$$

are identical to those of the theory

$$\bar{\mathcal{G}}(\psi^e, \bar{\psi}^e) = \log \frac{1}{\bar{Z}} \int e^{-\mathcal{I}} d\mu_{\bar{C}}(\psi, \bar{\psi}) \tag{III.3}$$

where

$$\bar{C}(p) = \sum_{n=0}^{\infty} C(p)\big(\hat{B}(\lambda, p)C(p)\big)^n \tag{III.4}$$

$$\mathcal{I} = \mathcal{V}(\psi + \psi_e, \bar{\psi} + \bar{\psi}_e) + \mathcal{M} + \mathcal{W} \tag{III.5}$$

$$\mathcal{M} = \lambda \int d\xi B_1 \bar{\psi}(\xi)\psi(\xi) + \lambda^2 \int d\xi_1 d\xi_2 \; \bar{\psi}(\xi_1) B_2(\xi_1, \xi_2)\psi(\xi_2) \tag{III.6}$$

$$\mathcal{W} = \int (a\lambda + b\lambda^2 + c\lambda^3) \tag{III.7}$$

where $\hat{B}(\lambda, p)$ is the Fourier transform of the translation invariant kernel $\lambda B_1 \delta(\xi_1, \xi_2) + \lambda^2 B_2(\xi_1, \xi_2)$.

Proof: In the perturbation expansion the counterterm $\mathcal{W}$ cancels out, and the formal power series expansion of the propagator $\bar{C}$ exactly cancels against the insertions of the mass counterterm $\mathcal{M}$ to give back the original propagator C. But by Lemma I.1, the perturbation expansion really defines the theory. This proves that the two theories are also identical at the constructive level, as defined by the same analytic germ. ■

We apply this lemma with a b and c in $\mathcal{W}$ the amplitudes of the divergent vacuum graphs with one, two and three vertices in Fig. 1, and

$$\mathcal{M} = \lambda \int d\xi B_1 \bar{\psi}(\xi)\psi(\xi) + \lambda^2 \int d\xi_1 d\xi_2 \; \bar{\psi}(\xi_1) B_2(\xi_1,\xi_2)\psi(\xi_2) \; . \tag{III.8}$$

where B_1 and $B_2(p)$ are the amplitudes for the corresponding graphs in Fig. 1. The tadpole amplitude B_1 is

$$B_1 = c_1 \int dq \frac{f_j(q)}{iq_0 - e(\mathbf{q})} \tag{III.9}$$

The Fourier transform $\hat{B}_2(p)$ of the kernel B_2 is exactly

$$\hat{B}_2(p) = c_2 \int dq_1 dq_2 \frac{f_j(p+q_1)}{i(p+q_1)_0 - e(\mathbf{p}+\mathbf{q_1})} \frac{f_j(-q_1+q_2)}{i(-q_1+q_2)_0 - e(-\mathbf{q_1}+\mathbf{q_2})} \frac{f_j(q_2)}{i(q_2)_0 - e(\mathbf{q_2})} \tag{III.10}$$

(c_1 and c_2 are some inessential numerical constant).

Lemma III.2 *For p in the support of C^j, there exists some numerical constant K such that*

$$|B_1| \leq K M^{2j} \tag{III.11}$$

$$|\hat{B}_2(p)| \leq K M^{2j} \tag{III.12}$$

Proof: (III.11) follows from (II.5b). To bound B_2 we can bound the three denominators in (III.10) by M^{-3j}, the two integrals on $dq_{1,0}$ and $dq_{2,0}$ by M^{2j}, and the two spatial integrals on $d\mathbf{q}_1$ and $d\mathbf{q}_2$ by M^{3j}. Indeed let S_a be the sphere of radius 1 centered around a. The $\mathbf{q}_1$ integral is restricted to distance M^j of $S_{-\mathbf{p}}$ by the function $e(\mathbf{p}+\mathbf{q}_1)$, hence runs on a volume of size M^j. But for fixed $\mathbf{q}_1$ generic, i.e. such that $|\mathbf{q}_1|$ is not close to 0 or $2\sqrt{2m\mu}$, the $\mathbf{q}_2$ integral is restricted to distance $\simeq M^j$ of the circle $S_0 \cap S_{-\mathbf{q}_1}$, hence runs on a volume of size $O(1)M^{2j}$. The degenerate cases $|\mathbf{q}_1| \simeq 2\sqrt{2m\mu}$ leads also to the same bound. ■

Remark also that $\hat{B}_2(p)$ is spatially rotation invariant so that it can be written solely as a function of p_0 and u:

$$\hat{B}_2(p) = \hat{\mathbf{B}}_2(p_0, u) \tag{III.13}$$

with $u = e(\mathbf{p})$.

By lemma III.2, the power series $\bar{C}^j(p) = \sum_{n=0}^{\infty} C^j(p)\Big((\lambda B_1 + \lambda^2 B_2(p))C^j(p)\Big)^n$ are convergent, uniformly in j and p, for $\lambda \leq 0(1)$*.

The propagator $\bar{C}^j$ is given by an integral representation identical to (II.1) but with $D(p_0, u) = (ip_0 - u)^{-1}$ replaced by $\bar{D}(p_0, u) = \big(ip_0 - u - f_j(p_0, u)(\lambda \bar{B}_1 + \lambda^2 \hat{\mathbf{B}}_2(p_0, u))\big)^{-1}$ We can slice the propagator $\bar{C}^j = \sum_k \bar{C}^{j,k}$ exactly as in (II.3). Lemma 1 then applies exactly in the same way, since $\bar{D}$ and its derivatives satisfy the same bounds as D on the support of f_j. This proves:

Lemma III.3 *The sliced propagator $\bar{C}^{j,k}$ satisfies the same bounds (II.5a-b) as $C^{j,k}$.*

From now on we will forget the bars, and for simplicity we write C^j, $C^{j,k}$, D instead of $\bar{C}^j$ $\bar{C}^{j,k}$, $\bar{D}$.

To obtain a better radius of convergence for our single slice model, we return to the finite volume theory. For this theory, the interaction $\mathcal{I}_\Lambda$ is equal to $\mathcal{I}$ restricted to Λ, hence is:

$$\mathcal{I}_\Lambda = \mathcal{V}_\Lambda + \mathcal{M}_\lambda + \mathcal{W}_\Lambda$$

$$\mathcal{V}_\Lambda = \frac{\lambda}{2} \int_{\Lambda^4} \prod_{i=1}^{4} d\xi_i \quad V(\xi_1, \xi_2, \xi_3, \xi_4)\, \bar{\psi}(\xi_1)\bar{\psi}(\xi_2)\psi(\xi_4)\psi(\xi_3) \tag{III.14}$$

$$\mathcal{M}_\Lambda = \lambda \int_\Lambda d\xi B_1 \bar{\psi}(\xi)\psi(\xi) + \int_{\Lambda^2} \prod_{i=1}^{2} d\xi_i \; \bar{\psi}(\xi_1) B_2(\xi_1, \xi_2)\psi(\xi_2) \tag{III.15}$$

$$\mathcal{W}_\Lambda = |\Lambda|(a\lambda + b\lambda^2 + c\lambda^3) \tag{III.16}$$

---

* By a fixed point analysis we could iterate this argument so that B_1 and B_2 themselves are replaced by similar quantities called $\bar{B}_1$and $\bar{B}_2$, computed with propagators $\bar{C}^j$ instead of C^j(this is the so-called problem of "bubbles within bubbles"). This is not really necessary for our construction, but would simplify the cancellation of the graphs B_1 and B_2 with the counterterms of $\mathcal{M}$.

where $\mathcal{W}_\Lambda$ stands for the finite volume vacuum counterterms, and $\mathcal{M}_\lambda$ for the finite volume two point function counterterms.

Remark that the finite volume Green's functions for that theory this time differ by a (small) change in the boundary condition from those of the initial theory put in a finite volume. This change in the boundary condition comes from the fact that the counterterms are limited to the volume Λ, whether the power series for the propagator is computed as if they where introduced in the whole space $\mathbb{R}^{d+1}$. But by lemma I.1 we know that we are in a single phase, at least for λ small enough, and that the thermodynamic limit is independent of boundary conditions. Therefore we can compute it in this way as well.

The theory is analyzed as usual with a sequence of cluster expansions. Let $\mathcal{D}_j$ be a covering of $\mathbb{R}^{d+1}$ by hypercubes of side size M^{-j}, and $\mathcal{D}_k$ be a covering of $\mathbb{R}^{d+1}$ by hyperrectangles which have size M^{-k} in each spatial direction and size M^{-j} in the (imaginary) time direction.

The finite volume Λ is chosen to be a finite union of $N(\Lambda)$ hypercubes in $\mathcal{D}_j$. This union is called $\mathcal{D}_j(\Lambda)$. We choose M integer so that each such hypercube is then decomposed into $M^{-3(j-k)}$ hyperrectangles of $\mathcal{D}_k$. Therefore Λ is also a union $\mathcal{D}_k(\Lambda)$ of hyperrectangles of $\mathcal{D}_k$.

For $\Delta \in \mathcal{D}_k$ we call χ_Δ the characteristic function of Δ. We decompose the external functional derivatives in $\mathcal{G}$ according to the cubes of $\mathcal{D}_j(\Lambda)$ in which they act (recall that one of them is fixed, say at the origin, and the others have to be summed in Λ). For any subset X of hyperrectangles at higher scales we can also define $\mathcal{G}_X$ as those derivatives but restricted to the support of X. We perform the first horizontal cluster expansion with respect to the $k=0$ scale. Let us denote by $<>_{\leq k}$, for $k=0,...,j$, the (normalized) Grassmann integral with respect to the Gaussian (anticommuting) measure of propagator $\sum_{l\leq k} C^{j,l}$.

We consider a given Green's function

$$\langle G_p \rangle_{\leq 0} = \frac{1}{Z_\Lambda} \int G_p e^{-\mathcal{I}_\Lambda(\psi,\bar\psi)} d\mu_{C^{(j)}}(\psi,\bar\psi) \ , \qquad \text{(III.17)}$$

where $Z_\Lambda = \int e^{-\lambda \mathcal{I}_\Lambda(\psi,\bar\psi)} d\mu_{C^{(j)}}(\psi,\bar\psi)$ and

$$G_p = \int \prod_{j=1}^{2p} d\xi_j \ G(\xi_1,\cdots,\xi_{2p}) \prod_{j=1}^{2p} \overset{(-)}{\psi}(\xi_j) \qquad \text{(III.18)}$$

is an arbitrary monomial.

The non-local counterterm B_2 is decomposed as

$$B_2(\xi,\bar\xi) = c\Big(\sum_k C^{j,k}(\xi,\bar\xi)\Big)^3 \tag{III.19}$$

A standard cluster expansion with respect to $C^{j,0}$ is now performed, including the propagators in the non-local counterterm B_2. The formula is ([B][AR]) :

$$\langle \mathcal{G}\rangle_{\le 0} = \Big\langle \sum_{\substack{X_1,\cdots,X_q \\ X_1,\cdots,X_q \text{ disjoint, non-trivial}}} \frac{1}{q\,!}\prod_{i=1}^{q}\mathcal{A}(X_i)\Big\rangle_{\le -1} \tag{III.20}$$

The polymers X_i are non-empty sets of hyperrectangles of $\mathcal{D}_0$, and the amplitude $\mathcal{A}(X_i)$ for the polymer X_i is given by a "Brydges formula"

$$\mathcal{A}(X) = \frac{1}{Z_0}\sum_T \int d^T h \int d\mu_{C^{j,0}(T,h)}\Big\{\prod_{l\in T} L_l\Big\}\mathcal{G}_X e^{-\lambda \mathcal{I}_X(\psi,\bar\psi)} \tag{III.21}$$

where Z_0 is the normalization factor of a trivial polymer made of a single hyperrectangle with no external legs. The sum is performed over oriented trees built on the polymer X. The orientation tells us when a tree line l joins X to X' if it is ξ which is located in X and $\bar\xi$ in X', in which case we put $\Delta_l = \Delta$ and $\bar\Delta_l = \Delta'$, or the contrary, in which case we put $\bar\Delta_l = \Delta$ and $\Delta_l = \Delta'$. The linking operators are

$$L_l = \int_{\Delta_l} d\xi \int_{\bar\Delta_l} d\bar\xi C^{j,0}(\xi,\bar\xi)\frac{\delta}{\delta\psi_0(\xi)}\frac{\delta}{\delta\bar\psi_0(\bar\xi)}$$

$$+3c\int_{\Delta_l} d\xi \int_{\bar\Delta_l} d\bar\xi\bar\psi(\bar\xi)\psi(\xi)C^{j,0}(\xi,\bar\xi)\Big(C^{j,0}(T,h,\xi,\bar\xi) + \sum_{k\le -1} C^{j,k}(\xi,\bar\xi)\Big)^2 \tag{III.22}$$

The second term corresponds to the non-local coupling through the B_2 counterterm.

$d^T h$ is the ordinary Lebesgue measure over interpolation parameters $h = \{h_l, h \in T\}$, each running in $[0,1]$. $d\mu_{C^{j,0}(T,h)}$ is the normalized Grassmann Gaussian measure with propagator $C^{j,0}(T,h)$, depending on the interpolation parameters h, which is equal to $C^{j,0}(\xi,\bar\xi)\cdot h^T(\xi,\bar\xi)$, where $h^T(\xi,\bar\xi)$ is 0 if ξ or $\bar\xi$ are not in the support X of the polymer, is 1 if they lie in the same hyperrectangle of X, and is otherwise the infimum over all parameters h_l on the unique path joining the hyperrectangle containing ξ to the hyperrectangle containing $\bar\xi$ [B][AR].

Then we apply a vertical expansion which tests whether a polymer X contains or not fields of lower scales (recall that external fields in $\mathcal{G}$ are considered by convention of the scale $j-1$, so lower than all other fields). This means that we introduce a parameter v_X which for each X interpolates:

$$\mathcal{V}_X = \mathcal{V}_X^0 + \mathcal{V}_{\bar{X}}^{\leq -1} + \mathcal{V}_X^{0 \text{ linked to } \leq -1} \tag{III.23}$$

$$\mathcal{V}_X^0 = \frac{1}{2}\int_X d\mathbf{x}d\tau \bar{\psi}_\uparrow^0(\mathbf{x},\tau)\bar{\psi}_\downarrow^0(\mathbf{x},\tau)\psi_\uparrow^0(\mathbf{x},\tau)\psi_\downarrow^0(\mathbf{x},\tau) \tag{III.24a}$$

$$\mathcal{V}_{\bar{X}}^{\leq -1} = \frac{1}{2}\int_X \bar{\psi}_\uparrow^{\leq -1}(\mathbf{x},\tau)\bar{\psi}_\downarrow^{\leq -1}(\mathbf{x},\tau)\psi_\uparrow^{\leq -1}(\mathbf{x},\tau)\psi_\downarrow^{\leq -1}(\mathbf{x},\tau) \tag{III.24b}$$

$$\mathcal{V}_X^{0 \text{ linked to } \leq -1} = \frac{1}{2}\int_X d\mathbf{x}d\tau \bar{\psi}_\uparrow(\mathbf{x},\tau)\bar{\psi}_\downarrow(\mathbf{x},\tau)\psi_\uparrow(\mathbf{x},\tau)\psi_\downarrow(\mathbf{x},\tau)$$

$$-\int_X d\mathbf{x}d\tau \bar{\psi}_\uparrow^0(\mathbf{x},\tau)\bar{\psi}_\downarrow^0(\mathbf{x},\tau)\psi_\uparrow^0(\mathbf{x},\tau)\psi_\downarrow^0(\mathbf{x},\tau) - \bar{\psi}_\uparrow^{\leq -1}(\mathbf{x},\tau)\bar{\psi}_\downarrow^{\leq -1}(\mathbf{x},\tau)\psi_\uparrow^{\leq -1}(\mathbf{x},\tau)\psi_\downarrow^{\leq -1}(\mathbf{x},\tau) \tag{III.24c}$$

$$\mathcal{V}_X(v_X) = \mathcal{V}_X^0 + \mathcal{V}_{\bar{X}}^{\leq -1} + v_X \mathcal{V}_X^{0 \text{ linked to } \leq -1} \ ; \ \mathcal{V}_X = \mathcal{V}_X(v_X)|_{v_X=1} \tag{III.25}$$

and we also interpolate $\mathcal{M}_X$ in a similar way. Then we perform a Taylor expansion in v_X to first order for each X. The terms with $v_X = 0$ are called vacuum polymers. For the error terms we draw a link between the polymer X and all the hyperrectangles of the next scale containing a hyperrectangle of its support; in this way we construct a polymer $\tilde{X}$ living on the hyperrectangles of the next scale.

Then we apply a contraction rule (integration by parts on the field), to compensate explicitly the vacuum polymers which contain up to three vertices with the corresponding counterterms, and the polymers with exactly one or two vertices and two low momentum fields of scale ≤ -1. This is standard [GJ]. There remains only polymers containing at least 4 vertices or at least 4 external legs (a B_2 counterterm counts for 2 vertices, etc...).

This explicit rule a la Glimm-Jaffe is slightly simpler than the modern more powerful renormalization group schemes that are needed for just renormalizable theories [B],[R]. Indeed it avoids to perform at each scale a Mayer expansion to factorize *all* the vacuum graphs, and e.g. exponentiate *all* the two point functions. The result of a Mayer expansion is a formula expressed in terms of "Mayer configurations", i.e. finite sequences of polymers. Iteration of

Mayer expansions lead to sequences of sequences, hence to formulas heavier to manipulate than ordinary polymers.

To iterate we apply the second scale cluster expansion to the set of non trivial subsets of $\mathcal{D}_{-1}$ whose elements are the polymers obtained after the first expansion, and we apply a second cluster expansion with respect to the measure $C^{j,\,-1}$ between these units, etc...

We obtain the following final formula

$$\langle \mathcal{G} \rangle_{\leq 0} = \sum_{\substack{Y_1,\cdots,Y_q \\ Y_1^k(Y),\cdots,Y_q^k(Y) \text{ disjoint}}} \frac{1}{q!} \prod_{i=1}^{q} \mathcal{A}(Y_i) \tag{III.27}$$

The polymers Y at the last scale have vertical tree structure, that is they are a collection of $1+|j|$ sets Y^k for $k=0,\cdots,j$. More precisely, $Y^k(Y)$ is a subset of $\mathcal{D}_k$, i.e. a set of hyperrectangles, made of connected components Y_l^k (taking at level k into account the connections of scales higher than or equal to k, and the set of all the Y_l^k has tree structure for the inclusion relation (see e.g. [R]).

The amplitude $\mathcal{A}(Y_i)$ for the polymer Y_i is given by applying inductively the Brydges formula, according to the collection of subsets $X^k(Y)$.

$$\mathcal{A}(Y) = \sum_{T_0,\ldots,T_j} \ldots \int d^T s \int d\mu_{C_0^j(s)} \Big\{ \prod_{l\in T} L_l' \Big\} \mathcal{G}_X e^{-\lambda \mathcal{I}_Y(\psi,\bar\psi)} \tag{III.28}$$

where the links operators L_l' include the explicit contraction rules to cancel the counterterms for the graphs B_1, B_2 and G_1,..., G_3 and the sum in (III.28) is restricted by the condition that any connected component at a given level has either more than three vertices or more than two external legs.

Theorem I then follows from the following bound

Theorem III.1 *For any $K>0$ there exists some $r(K)>0$ such that for $|\lambda|\leq r(K)$*

$$\sum_{Y|0\in X^0(Y)} |\mathcal{A}(Y)| K^{|X^0(Y)|} \leq 1 \tag{III.29}$$

Indeed performing a standard Mayer expansion we obtain analyticity of $<\mathcal{G}>$ for $|\lambda|\leq r(e)$ (where $e=2.718..$).

The next section is devoted to a proof of Theorem III.1.

§IV The bounds

The formula giving the amplitude of a polymer Y is expanded into a sum of diagrams made of explicit propagators derived by the cluster expansion, vertices, and fields. The Fermionic Gaussian integration on the fields is simply a formal device which changes the fields into products of determinants. We recall the "method of combinatorial factors" to keep track of the many sums in the expansion. This technique uses the elementary estimate

$$\kappa_i > 0 \ , \ \sum_i \kappa_i^{-1} \leq 1 \quad \Rightarrow \quad \left| \sum_i U_i \right| \leq \sup_i |\kappa_i U_i| \, . \tag{IV.1}$$

to replace each sum by a supremum. To help remember the combinatorial factor κ_i multiplying the value U_i of a given diagram, the factor will be assigned to a specific line or vertex or field of the diagram. Remember also that in the end we have a small coupling constant per vertex, hence per field, so that such constants can be forgotten in the estimates.

A vertex will be localized in the tube of its highest leg. That means that we write

$$e^{\mathcal{V}_Y} = e^{\sum_{\Delta \in Y} \int_\Delta d\xi \overset{(-)}{\psi}{}^{k(\Delta)}(\xi) \sum_{k_1,k_2,k_3 \leq k(\Delta)} \overset{(-)}{\psi}{}^{k_1}(\xi) \overset{(-)}{\psi}{}^{k_2}(\xi) \overset{(-)}{\psi}{}^{k_3}(\xi)}$$

$$= \prod_\Delta \sum_{v(\Delta)=0}^{\infty} \frac{1}{v(\Delta)!} \left(\int_\Delta d\xi \overset{(-)}{\psi}{}^{k(\Delta)}(\xi) \sum_{k_1,k_2,k_3 \leq k(\Delta)} \overset{(-)}{\psi}{}^{k_1}(\xi) \overset{(-)}{\psi}{}^{k_2}(\xi) \overset{(-)}{\psi}{}^{k_3}(\xi) \right) \tag{IV.2}$$

Furthermore by the usual "local factorial principle" we can extract, up to a constant per cube of Y, from the decay of the explicit cluster propagators a large power of the product for all tubes of the factorial of the number of the vertices produced by the explicit cluster functional derivations and integration by parts [R]. This is useful because we have no symmetry factor in $\frac{1}{v(\Delta)!}$ for these vertices. Joining this fact to (IV.2), we have a factor $\frac{1}{v(\Delta)!}$ where $v(\Delta)$ is now the total number of vertices in the polymer localized in Δ, no matter whether they are hooked to explicit cluster derivatives or not.

For any field $\overset{(-)}{\psi}{}^{k}$, we call $k(\overset{(-)}{\psi})$ its scale k and $l(\overset{(-)}{\psi})$ the scale of the localization tube of the vertex to which it is hooked. We call $\Delta_l(\overset{(-)}{\psi})$ the localization tube of the vertex to which it is hooked, not to be confused with the tube $\Delta(\overset{(-)}{\psi})$ of scale $k(\overset{(-)}{\psi})$ in which it lies. Remark that $\Delta_l(\overset{(-)}{\psi}) \subset \Delta(\overset{(-)}{\psi})$.

By a combinatoric factor $\kappa_1 = M^{(\epsilon/4)(k(\overset{(-)}{\psi})-j)}$ we can sum over all scales of all legs of any vertex. In what follows ϵ is a very small number.

By a combinatoric factor $\kappa_2 = M^{-j-3k(v)}$ per vertex of localization scale $k(v)$ we can integrate over the position of that vertex in its localization cube.

Then at each level k let us consider the piece Y_k made of the tubes of Y of scale k. Each tube contains a certain number $f(\Delta)$ of fields (or anti fields) of scale k lying in this cube.

We decompose the covariance $C^k(h,T)$ as $C^k(h,T) = \sum_{\Delta,\Delta' \in X_k} \chi_\Delta C^k(h,T)\chi_{\Delta'}$ and correspondingly we expand the fields. Let us introduce the scaled distance between Δ and Δ' as

$$d(\Delta,\Delta') = \inf_{\xi\in\Delta,\bar\xi\in\Delta'} (M^k|\mathbf{x}-\bar{\mathbf{x}}| + M^j|\tau-\bar\tau|) \tag{IV.3}$$

By a combinatoric factor $\kappa_3 = (1+d(\Delta,\Delta'))^5$ we can choose, for each field or anti field, the tube (of the same scale) to which it contracts.

Let $\overset{(-)}{f}(\Delta,\Delta')$ be the number of fields (respectively antifields) localized in Δ that contract to a field localized in Δ'. By Fermionic rules of integration, $f(\Delta,\Delta') = \bar f(\Delta',\Delta)$. We obtain therefore after Gaussian integration for each scale a product of determinants

$$\prod_{(\Delta,\Delta')\in Y_k^2} \det{}_{(\Delta,\Delta')} \tag{IV.4}$$

where $\det_{(\Delta,\Delta')}$ is a $f(\Delta,\Delta')$ by $\bar f(\Delta',\Delta) = f(\Delta,\Delta')$ determinant of propagators $C^{j,k}$. Applying the Hadamard's bound $|\det a_{ij}| \le n^{n/2}\sup|a_{ij}|^n$ we obtain by (II.5a) for these determinants a factor $f(\Delta,\Delta')^{f(\Delta,\Delta')/2}(1+d(\Delta,\Delta'))^{-pf(\Delta,\Delta')}M^{(j+k)f(\Delta,\Delta')}$. Taking p larger than 10, the factor $(1+d(\Delta,\Delta'))^{-p/2}$ per field absorbs the combinatoric factor κ_3. Joining together the lines power counting $\prod_k\prod_{(\Delta,\Delta')\in Y_k^2} M^{(j+k)f(\Delta,\Delta')}$ given by the determinant and by the explicit propagators to the integration factor κ_2 we obtain the power counting factor $M^{(j-k(v))-\frac{1}{2}\sum_{i=1,2,3}(k(v)-k_i)}$ per vertex, which is bounded by a factor $M^{(j-k(v))(1/4-\epsilon)}$ per vertex times a factor $M^{((3/4)+\epsilon)(k(\overset{(-)}{\psi})-l(\overset{(-)}{\psi}))}$ per field. Therefore combining this factor to the factor $\prod_\Delta \frac{1}{v(\Delta)!}$ we have a factor

$$M^{((3/4)+\epsilon)(k(\overset{(-)}{\psi})-l(\overset{(-)}{\psi}))}v(\Delta_l(\overset{(-)}{\psi}))^{-1/4} \tag{IV.5}$$

per field. After having absorbed the combinatoric factor κ_1 we also have still a factor $M^{(j-k(v))(1/4-2\epsilon)}$ per vertex.

Now we bound the factorials of Hadamard's inequality using in particular $f(\Delta,\Delta') = \bar{f}(\Delta',\Delta)$:

$$
\begin{aligned}
\prod_k \prod_{(\Delta,\Delta')\in Y_k^2} f(\Delta,\Delta')^{f(\Delta,\Delta')/2} &\leq \prod_k \prod_{(\Delta,\Delta')\in Y_k^2} f(\Delta,\Delta')^{f(\Delta,\Delta')/4} \bar{f}(\Delta,\Delta')^{\bar{f}(\Delta,\Delta')/4} \\
&\leq \prod_k \prod_{(\Delta,\Delta')\in Y_k^2} (f(\Delta,\Delta') + \bar{f}(\Delta,\Delta'))^{f(\Delta,\Delta')/4+\bar{f}(\Delta,\Delta')/4} \\
&\leq \prod_k \prod_{\Delta\in Y_k} f(\Delta)^{\sum_{\Delta'} f(\Delta,\Delta')/4+\bar{f}(\Delta,\Delta')/4} \\
&\leq \prod_k \prod_{\Delta\in Y_k} f(\Delta)^{f(\Delta)/4} \leq \prod_k \prod_{\Delta\in Y_k} K^{f(\Delta)} f(\Delta)!!^{1/2} \qquad \text{(IV.6)}
\end{aligned}
$$

So up to an inessential constant per field, Hadamard's bound gives a product over all tubes of the square root of the number of all Wick contractions between all fields localized in this tube, not taking into account whether they are fields or anti-fields. (Remark that $f(\Delta)$ can be odd, in which case we put by convention $f(\Delta)!! = (f(\Delta)-2)...5.3.1$, hence we do not contract one of the legs).

Then we have:

Lemma IV.1

$$
\prod_k \prod_{\Delta\in Y_k} (f(\Delta)!!)^{1/2} \leq \prod_k \prod_{\Delta\in Y_k} K^{f(\Delta)} \prod_{\psi\in\Delta} \left(M^{(3+\epsilon/4)(l(\overset{(-)}{\psi})-k(\overset{(-)}{\psi}))} v(\Delta_l(\psi)) \right)^{1/4} \qquad \text{(IV.7)}
$$

Proof: We can contract the legs starting with the ones which have largest value of $M^{(3+\epsilon/4)(l(\overset{(-)}{\psi})-k(\overset{(-)}{\psi}))} v(\Delta_l(\psi))$, and with a factor

$$
\begin{aligned}
& M^{(3+\epsilon/4)(l(\overset{(-)}{\psi}{}')-k(\overset{(-)}{\psi}{}'))} 4v(\Delta_l(\psi')) \\
& \leq \sqrt{M^{(3+\epsilon/4)(l(\overset{(-)}{\psi})-k(\overset{(-)}{\psi}))} 4v(\Delta_l(\psi)) M^{(3+\epsilon/4)(l(\overset{(-)}{\psi}{}')-k(\overset{(-)}{\psi}{}'))} 4v(\Delta_l(\psi'))} \qquad \text{(IV.8)}
\end{aligned}
$$

we can choose to which leg it contracts, by choosing first with $M^{(3+\epsilon/4)(l(\overset{(-)}{\psi}{}')-k(\overset{(-)}{\psi}{}'))}$ the localization tube of the leg to which it contracts (this is because there are $M^{3(k-k')}$ tubes of scale k in a tube of scale $k' \leq k$), then by a factor $v(\Delta_l(\psi'))$ the vertex in this localization tube, then with a factor 4 the particular leg. Collecting these factors proves the lemma. ■

Comparing the factor of (IV.7) to the one of (IV.5) we see that we get a constant per field.

It remains to sum over the positions of the tubes in Y. This is standard ordinary power counting and it can be done with the decay of the cluster propagators in the horizontal direction (at fixed k) plus the power counting of the vertices involved in the vertical connections, using the fact that no connected subgraph Y_k^l has less than two external legs and less than three vertices.

Finally the constant per field is controlled by the smallness of λ. This completes the proof of Theorem III.1. Taking into account the global fixed factor $M^{j(4-2p)}$ due to our norm, Theorem I follows.

Appendix I

In this appendix we explain the slight modifications to adapt the proof of Theorem I to all space dimensions greater than three.

The Fourier transform of the δ function of the sphere S^{d-1} in $\mathbb{R}^d$ behaves as $1/|x|^{(d-1)/2}$ at large x. Therefore reslicing the corresponding propagator C^j in auxiliary slices $C^{j,k}$ we have

$$\left|C^{j,k}(\xi,\bar{\xi})\right| \leq K(p)M^{j+k(d-1)/2}(1+M^k|\mathbf{x}-\bar{\mathbf{x}}|)^{-p}(1+M^j|\tau-\bar{\tau}|)^{-p} \qquad \text{(A1.1)}$$

The power counting of a graph with n vertices at scale k and $e = 2p$ external legs at scale j is $M^{(d+1-p(d+1)/2)j}M^{(j-k)(n+p(d-1)/2-d)}$, obtained by combining the power counting $M^{(j+k(d-1)/2)(2n-p)}$ of the legs at scale k, and the power counting $M^{-(j+dk)(n-1)}$ of the $n-1$ vertices integration. Therefore the auxiliary problem is superrenormalizable in any dimension. There is only a finite number of graphs to renormalize, those satisfying $n + p(d-1)/2 \leq d$, which are always vacuum ($p = 0$) or two point ($p = 1$) subgraphs. The main result of this paper, Theorem I extends therefore without difficulty to any dimension, with the prefactor $M^{(4-2p)j}$ in (I.15) replaced by $M^{(d+1-p(d+1)/2)j}$.

It is interesting to remark that as $d \to \infty$ more and more 2-point subgraphs diverge, and we obtain real Fermionic models (in integer dimensions) whose power counting mimic the ill-defined ϕ^4 models in fractional dimensions $d \to 4$. The "critical" dimension at which our problem becomes just renormalizable is $d = \infty$, and it would presumably be interesting to set up for it a $1/d$ expansion which would be the analogue of the Fisher-Wilson $1/\epsilon$ expansion for ϕ^4 models.

Appendix II

In this appendix we give a complete proof of Lemma II.1.

For $k \neq j$, we use the identity

$$(1 + iM^j(\tau - \bar{\tau}))^p e^{ip_0(\bar{\tau}-\tau)} = (1 + M^j \frac{d}{dp_0})^p e^{ip_0(\bar{\tau}-\tau)} \tag{A2.1}$$

and the obvious inequality $(1 + M^j|\tau - \bar{\tau}|) \leq \sqrt{2}|1 + iM^j(\tau - \bar{\tau})|$ to write

$$\begin{aligned}
(1 + M^j|\tau - \bar{\tau}|)^p |C^{j,k}| \leq (\sqrt{2})^p \Bigg| A^{j,k}(\mathbf{x}, \bar{\mathbf{x}}) \delta_{\sigma\bar{\sigma}} \frac{1}{8\pi^3 |\mathbf{x} - \bar{\mathbf{x}}|} \int_{-M^{j+2}}^{+M^{j+2}} \int_{-M^{j+2}}^{+M^{j+2}} dp_0 du \\
\left((1 + M^j \frac{d}{dp_0})^p e^{ip_0(\bar{\tau}-\tau)} \right) D(p_0, u) f_j(p_0, u) \sin(\sqrt{1+u}|\mathbf{x} - \bar{\mathbf{x}}|) \Bigg| \\
= (\sqrt{2})^p \Bigg| A^{j,k}(\mathbf{x}, \bar{\mathbf{x}}) \delta_{\sigma\bar{\sigma}} \frac{1}{8\pi^3 |\mathbf{x} - \bar{\mathbf{x}}|} \int_{-M^{j+2}}^{+M^{j+2}} \int_{-M^{j+2}}^{+M^{j+2}} dp_0 du e^{ip_0(\bar{\tau}-\tau)} \\
\left((1 - M^j \frac{d}{dp_0})^p D(p_0, u) f_j(p_0, u) \right) \sin(\sqrt{1+u}|\mathbf{x} - \bar{\mathbf{x}}|) \Bigg|
\end{aligned} \tag{A2.2}$$

The last line is obtained by integration by parts; boundary terms vanish by the support properties of f_j. Computing the action of partial derivatives $M^j \frac{d}{dp_0}$ on $D(p_0, u) f_j(p_0, u)$ completes the proof. Indeed by the scaling and support properties of f_j we obtain, up to a constant $K(p)$, the same bound as for the initial function $D(p_0, u) f_j(p_0, u)$. More precisely for $k \neq 0$ we bound $\sin(\sqrt{1+u}|\mathbf{x} - \bar{\mathbf{x}}|)$ by 1. Using then

$$\int_{-M^{j+2}}^{+M^{j+2}} \int_{-M^{j+2}}^{+M^{j+2}} dp_0 du |D(p_0, u)| \leq M^j \ . \tag{A2.3}$$

we obtain that

$$(1+M^j|\tau-\bar\tau|)^p|C^{j,k}| \le K_1(p)M^j\left|\frac{A^{j,k}(\mathbf{x},\bar{\mathbf{x}})}{\mathbf{x}-\bar{\mathbf{x}}}\right| , \tag{A2.4}$$

and we remark that for $k\neq j$, $k\neq 0$ and for any p

$$\left|\frac{A^{j,k}(\mathbf{x},\bar{\mathbf{x}})}{\mathbf{x}-\bar{\mathbf{x}}}\right| \le K_2(p)M^k(1+M^k|\mathbf{x}-\bar{\mathbf{x}}|)^{-p} . \tag{A2.5}$$

(In fact the function A^k has exponential decrease for $k\neq j$). In the case $k=0$, we use the bound

$$\frac{\sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|)}{|\mathbf{x}-\bar{\mathbf{x}}|} \le \sqrt{2} \tag{A2.6}$$

for $|u|\le 1$ (which is true on the integration domain). We combine this bound with the trivial estimate

$$\left|A^{j,0}(\mathbf{x},\bar{\mathbf{x}})\right| \le K_2(p)(1+|\mathbf{x}-\bar{\mathbf{x}}|)^{-p} \tag{A2.7}$$

to obtain (II.5a).

In the last case $k=j$, the spatial decay also has to come from integration by parts. For simplicity we assume that $p=2q$ is even (this obviously implies the general case as well). We use the identity:

$$M^{2j}|\mathbf{x}-\bar{\mathbf{x}}|^2\sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|) = (2iM^j\sqrt{1+u}\frac{d}{du})^2\sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|) \tag{A2.8}$$

Therefore we have

$$(1+M^j|\mathbf{x}-\bar{\mathbf{x}}|)^p(1+M^j|\tau-\bar\tau|)^p|C^{j,j}| \le 2^p\left|(1+M^{2j}|\mathbf{x}-\bar{\mathbf{x}}|^2)^q(1+iM^j(\tau-\bar\tau))^pC^{j,j}\right|$$

$$= 2^p\Bigg|A^{j,j}(\mathbf{x},\bar{\mathbf{x}})\delta_{\sigma\bar\sigma}\frac{1}{8\pi^3|\mathbf{x}-\bar{\mathbf{x}}|}\int_{-M^{j+2}}^{+M^{j+2}}\int_{-M^{j+2}}^{+M^{j+2}}dp_0du\bigg((1+M^j\frac{d}{dp_0})^pe^{ip_0(\bar\tau-\tau)}\bigg)$$

$$D(p_0,u)f_j(p_0,u)\bigg(\Big(1+(2iM^j\sqrt{1+u}\frac{d}{du})^2\Big)^q\sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|)\bigg)\Bigg| \tag{A2.9}$$

Then we integrate by parts, both on p_0 and u. The boundary terms vanish again by the support properties of f_j, hence partial derivatives $(1-M^j\frac{d}{dp_0})$ act as before on $D(p_0,u)f_j(p_0,u)$, and operators $2iM^j\sqrt{1+u}\frac{d}{du}$ change into transposed operators $(-2iM^j\frac{d}{du})\sqrt{1+u}\ \cdot$, also acting on $D(p_0,u)f_j(p_0,u)$. This action, by Leibniz rule, is slightly complicated to write down. But clearly, again by the support and scaling properties of f_j, we get up to a constant $K(p)$ the same estimate than for the initial p_0 and u integral, namely M^j. We bound

$|\sin(\sqrt{1+u}|\mathbf{x}-\bar{\mathbf{x}}|)|$ by one, and the last factor $M^k = M^j$ for $k = j$ comes from the easy estimate on (II.4c):

$$\left|\frac{A^{j,j}(\mathbf{x},\bar{\mathbf{x}})}{\mathbf{x}-\bar{\mathbf{x}}}\right| \leq M^j \tag{A2.10}$$

Finally (II.5b) is easy and left to the reader. (For the case $k = 0$, use the fact that $\int dp_0 du D(p_0,u) f_j(p_0,u) = 0$.) ∎

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