Memoirs on Differential Equations and Mathematical Physics

Volume 85, 2022, 133–142

Süleyman Öğrekçi, Yasemin Başcı, Adil Mısır

ON THE HYERS–ULAM STABILITY OF DELAY DIFFERENTIAL EQUATIONS

Abstract. In this paper, we consider the stability problem of delay differential equations in the sense of Hyers–Ulam and Hyers–Ulam–Rassias. By using a well known fixed point alternative on generalized complete metric spaces, we obtain some new stability criteria. Our results extend and improve the results described in literature since their proofs are based on fewer and weaker assumptions than the recent results dealing with this problem. Some illustrative examples are also given to compare these results and visualize the improvement.

2010 Mathematics Subject Classification. 24K20, 47H10.

Key words and phrases. Delay differential equations, Ulam type stability, fixed point alternative.

რეზიუმე. ნაშრომში განხილულია დაგვიანებული ტიპის დიფერენციალურ განტოლებათა მდგრადობის ამოცანა ჰაიერს-ულამ და ჰაიერს-ულამ-რასიასის აზრით. განზოგადებული სრული მეტრული სივრცეებისთვის, კარგად ცნობილი უძრავი წერტილის ალტერნატივის გამოყენებით, მიღებულია მდგრადობის ახალი კრიტერიუმები. ჩვენი შედეგები აფართოვებს და აუმჯობესებს ლიტერატურაში ადრე მიღებულ შედეგებს, რადგან დამტკიცება ეფუძნება უფრო ნაკლებ და სუსტ დაშვებებს, ვიდრე ამ ამოცანასთან დაკავშირებული ბოლოდროინდელი შედეგები. აგრეთვე მოცემულია რამდენიმე საილუსტრაციო მაგალითი ამ შედეგების შესადარებლად და გაუმჯობესების დასანახად.

1 Introduction

In 1940, Ulam [30] posed the following stability problem of functional equations: Given a group G_1 and a metric group (G_2, ρ). Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < k\varepsilon$ for all $x \in G_1$ and some $k > 0$? Roughly speaking, Ulam raised the question: suppose one has a function $f(t)$ which is close to solve an equation. Is there an exact solution (to same equation) $h(t)$ which is close to $f(t)$? If the answer is affirmative, the equation $h(xy) = h(x)h(y)$ is called *stable* in the Ulam sense. One year later, Hyers [13] gave an answer to this problem for linear functional equations on Banach spaces: Let G_1, G_2 be real Banach spaces and $\varepsilon > 0$. Then for each mapping $f : G_1 \to G_2$ satisfying $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$ for all $x, y \in G_1$, there exists a unique additive mapping $g : G_1 \to G_2$ such that $\|f(x) - h(x)\| \le \varepsilon$ holds for all $x \in G_1$. After this affirmative answer of Hyers, a new notion of stability of functional equations founded, which is today called the Hyers–Ulam stability, and is one of the central topics in mathematical analysis (see, e.g., [7, 8, 12, 22]). In 1978, Rassias [25], by considering the constant ε as a variable in Ulam's problem, made an important generalization, which is known as Hyers–Ulam–Rassias stability (see, e.g., [2, 14, 23, 24]).

The stability problem of differential equations in the Hyers–Ulam sense was initiated by the papers of Obloza [18, 19]. Later, Alsina and Ger [1] proved that assuming I is an open interval of reals, for every differentiable mapping $y : I \to \mathbb{R}$ satisfying

$$|y'(x) - y(x)| \le \varepsilon \ \text{ for all } \ x \in I \ \text{ and for a given } \varepsilon > 0,$$

there exists a solution y_0 of the differential equation

$$y'(x) = y(x)$$

such that

$$|y(x) - y_0(x)| \le 3\varepsilon \ \text{ for all } \ x \in I.$$

This result was later extended by Takahasi, Miura and Miyajima in [27] to the equation

$$y'(x) = \lambda y(x)$$

in Banach spaces, and in [16, 17] to higher order linear differential equations with constant coefficients.

Recently, Jung [15] proved the Hyers–Ulam stability as well as the Hyers–Ulam–Rassias stability of the equation

$$y' = f(t, y)$$

which extends the above-mentioned results to a nonlinear case. Jung's technique has been modified by Tunç and Biçer [29] also for functional equations in the form

$$y'(t) = F(t, y(t), y(t - \tau)), \tag{1.1}$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ is a bounded and continuous function and $\tau > 0$ is a real constant. After these pioneering works, a large number of papers devoted to this subject have been published (see, e.g., [3–6, 10, 11, 20, 21, 26, 28] and the references therein).

In this paper, we will extend and improve these result by proving the stability results for delay differential equations in the form of (1.1) with weaker assumptions.

2 Preliminaries

For some $\varepsilon \ge 0$, $\Psi \in C[t_0 - \tau, t_0]$ and $t_0, T \in \mathbb{R}$ with $T > t_0$, assume that for any continuous function $f : [t_0 - \tau, T] \to \mathbb{R}$ he following is satisfied:

$$\begin{cases} \big|f'(t) - F(t, f(t), f(t - \tau))\big| < \varepsilon, & t \in [t_0, T], \\ |f(t) - \Psi(t)| < \varepsilon, & t \in [t_0 - \tau, t_0]. \end{cases}$$

If there exists a continuous function $f_0 : [t_0 - \tau, T] \to \mathbb{R}$ satisfying

$$\begin{cases} f_0'(t) = F(t, f_0(t), f_0(t-\tau)), & t \in [t_0, T], \\ f_0(t) = \Psi(t), & t \in [t_0 - \tau, t_0] \end{cases}$$

and

$$|f(t) - f_0(t)| < K(\varepsilon), \quad t \in [t_0 - \tau, T],$$

where $K(\varepsilon)$ is an expression of ε only, we say that equation (1.1) has the Hyers–Ulam stability. If the above statement is also true when we replace ε and $K(\varepsilon)$ by φ and Φ, where $\varphi, \Phi \in C[t_0 - \tau, T]$ are the functions not depending on f and f_0 explicitly, then we say that equation (1.1) has the Hyers–Ulam–Rassias stability. These definitons may be applied to different classes of differential equations (we refer to Jung [15], Tunç and Biçer [29] and the references cited therein for more detailed definitions of Hyers–Ulam stability and Hyers–Ulam–Rassias stability).

We will use the following fixed point result on generalized complete metric spaces as the main tool in our proofs, for the proof of this result we refer to [9].

Theorem 2.1. *Let (X, d) be a generalized complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there is a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *the sequence $\{T^n x\}$ converges to a fixed point x^* of T;*

(b) *x^* is the unique fixed point of T in*

$$X^* = \{y \in X : \ d(T^k x, y) < \infty\};$$

(c) *if $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1-L}\, d(Ty, y).$$

3 Main results

Throughout this section, we define $I := [t_0 - \tau, T]$ for the given real numbers t_0, T and τ with $T > t_0$. Further, we define the set S as

$$S := \Big\{ f : I \to \mathbb{R} \mid \ f \text{ is continuous and } f(t) = \Psi(t) \text{ for } t \in [t_0 - \tau, t_0] \Big\}, \tag{3.1}$$

where $\Psi : [t_0 - \tau, t_0] \to \mathbb{R}$ is a continuous function. In our proofs, we will need a completeness of the space (S, d) which is given in the following result (see [5]).

Lemma 3.1 (see [5])**.** *Define the function $d : S \times S \to [0, \infty]$ with*

$$d(f, g) := \inf \Big\{ C \in [0, \infty] : \ |f(t) - g(t)| \mathrm{e}^{-M(t-t_0)} \le C\Phi(t), \ t \in I \Big\}, \tag{3.2}$$

where $M > 0$ is a given constant and $\Phi : I \to (0, \infty)$ is a given continuous function. Then (S, d) is a generalized complete metric space.

We are now ready to study the stability of differential equation (1.1) in the Hyers–Ulam sense.

Theorem 3.1. *Suppose that the continuous function $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition*

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \le L_1 |x_1 - x_2| + L_2 |y_1 - y_2| \tag{3.3}$$

for all $(t, x_1, y_1), (t, x_2, y_2) \in I \times \mathbb{R} \times \mathbb{R}$ and some $L_1, L_2 > 0$. Suppose also that $\Psi : [t_0 - \tau, t_0] \to \mathbb{R}$ is a continuous function. If a continuous function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} \big| y'(t) - F(t, y(t), y(t-\tau)) \big| < \varepsilon, & t \in [t_0, T], \\ |y(t) - \Psi(t)| < \varepsilon, & t \in [t_0 - \tau, t_0], \end{cases} \tag{3.4}$$

then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ *such that*

$$\begin{cases} y_0'(t) = F(t, y_0(t), y_0(t-\tau)), & t \in [t_0, T], \\ y_0(t) = \Psi(t), & t \in [t_0 - \tau, t_0] \end{cases} \tag{3.5}$$

and

$$|y(t) - y_0(t)| \le \frac{\varepsilon T}{L - (L_1 + L_2)}$$

for all $t \in I$ *and any number* L *with* $L > L_1 + L_2$.

Proof. For the set S defined by (3.1) we introduce the function $d : S \times S \to [0, \infty]$ as follows:

$$d(f, g) := \inf\Big\{C \in [0, \infty] : \ |f(t) - g(t)|\mathrm{e}^{-L(t-t_0)} \le C, \ \forall\, t \in I\Big\}.$$

Note that (S, d) is a generalized complete metric space in view of Lemma 3.1. Now, let us define the mapping $\Lambda : S \to S$ by

$$(\Lambda y)(t) := \begin{cases} \Psi(t), & \text{for } \ t \in [t_0 - \tau, t_0], \\ y(t_0) + \displaystyle\int_{t_0}^{t} F(s, y(s), y(s-\tau))\,\mathrm{d}s, & \text{for } \ t \in [t_0, T]. \end{cases} \tag{3.6}$$

For any $y \in S$, it is clear that Λy is continuous and any fixed point of Λ solves the differential equation (1.1).

Now we will show that Λ is strictly contractive on S. For any $f, g \in S$,

$$C_{f,g} \in \Big\{C \in [t_0, \infty] : \ |f(t) - g(t)|\mathrm{e}^{-L(t-t_0)} \le C, \ \forall\, t \in I\Big\}$$

and $t \in [t_0, T]$, we have

$$\begin{aligned}
\big|(\Lambda f)(t) - (\Lambda g)(t)\big| &= \Big|\int_{t_0}^{t} \big[F(s, f(s), f(s-\tau)) - F(s, g(s), g(s-\tau))\big]\,\mathrm{d}s\Big| \\
&\le \int_{t_0}^{t} \big|F(s, f(s), f(s-\tau)) - F(s, g(s), g(s-\tau))\big|\,\mathrm{d}s \\
&\le L_1 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0}^{t} |f(s-\tau) - g(s-\tau)|\,\mathrm{d}s \\
&= L_1 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0-\tau}^{t-\tau} |f(s) - g(s)|\,\mathrm{d}s \\
&= L_1 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0-\tau}^{t_0} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0}^{t-\tau} |f(s) - g(s)|\,\mathrm{d}s \\
&\le L_1 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0}^{t-\tau} |f(s) - g(s)|\,\mathrm{d}s \\
&\le L_1 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= L_1 \int_{t_0}^{t} |f(s) - g(s)| \mathrm{e}^{-L(s-t_0)} \mathrm{e}^{L(s-t_0)} \,\mathrm{d}s + L_2 \int_{t_0}^{t} |f(s) - g(s)| \mathrm{e}^{-L(s-t_0)} \mathrm{e}^{L(s-t_0)} \,\mathrm{d}s \\
&\leq L_1 C_{f,g} \int_{t_0}^{t} \mathrm{e}^{L(s-t_0)} \,\mathrm{d}s + L_2 C_{f,g} \int_{t_0}^{t} \mathrm{e}^{L(s-t_0)} \,\mathrm{d}s \\
&= \frac{L_1 + L_2}{L} C_{f,g} (\mathrm{e}^{L(t-t_0)} - 1) \leq \frac{L_1 + L_2}{L} C_{f,g} \mathrm{e}^{L(t-t_0)}
\end{aligned}
$$

and so,

$$
\big|(\Lambda f)(t) - (\Lambda g)(t)\big| \mathrm{e}^{-L(t-t_0)} \leq \frac{L_1 + L_2}{L} C_{f,g} \ \text{ for all } \ t \in [t_0, T].
$$

This inequality and

$$
\big|(\Lambda f)(t) - (\Lambda g)(t)\big| = 0 \ \text{ for all } \ t \in [t_0 - \tau, t_0]
$$

imply that

$$
d(\Lambda f, \Lambda g) \leq \frac{L_1 + L_2}{L} d(f, g),
$$

which means that the mapping Λ is strictly contractive on S.

Now, for an arbitrary $u \in S$, it is clear that $d(\Lambda u, u) < \infty$ for all $t \in I$, since $F(t, u(t), u(t - \tau))$ is bounded on I. Furthermore, we have $d(u, v) < \infty$ for any $v \in S$, since both u and v are bounded on I. That is, $\{x \in S : \ d(u, v) < \infty\} = S$. Hence, according to Theorem 2.1, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda y_0 = y_0$, $\Lambda^n u \to y_0$ in S and satisfies equation (3.5).

On the other hand, from equation (3.4), we have

$$
-\varepsilon(t - t_0) \leq \int_{t_0}^{t} y'(s) \,\mathrm{d}s - \int_{t_0}^{t} F(s, y(s), y(s - \tau)) \,\mathrm{d}s \leq \varepsilon(t - t_0)
$$

and so,

$$
|(\Lambda y)(t) - y(t)| \leq \varepsilon T \ \text{ for all } \ t \in I.
$$

Multiplying this inequality by $\mathrm{e}^{-L(t-t_0)}$, we obtain

$$
|(\Lambda y)(t) - y(t)| \mathrm{e}^{-L(t-t_0)} \leq \varepsilon T \mathrm{e}^{-L(t-t_0)}
$$

and so,

$$
d(\Lambda y, y) \leq \varepsilon T \mathrm{e}^{-L(t-t_0)} \ \text{ for all } \ t \in I.
$$

Therefore, according to Theorem 2.1, we have

$$
d(y, y_0) \leq \frac{1}{1 - (L_1 + L_2)/L} d(\Lambda y, y) \leq \frac{LT\varepsilon}{L - (L_1 + L_2)} \mathrm{e}^{-L(t-t_0)} \ \text{ for all } \ t \in I.
$$

From the definition of the metric d, this implies

$$
|y(t) - y_0(t)| \mathrm{e}^{-L(t-t_0)} \leq \frac{LT\varepsilon}{L - (L_1 + L_2)} \mathrm{e}^{-L(t-t_0)} \ \text{ for all } \ t \in I,
$$

which completes the proof. □

Theorem 3.2. *Suppose that the continuous function $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition* (3.3) *for all $(t, x_1, y_1), (t, x_2, y_2) \in I \times \mathbb{R} \times \mathbb{R}$ and some $L_1, L_2 > 0$. Suppose also that $\Psi : [t_0 - \tau, t_0] \to \mathbb{R}$ is a continuous function. Let $\varphi : I \to \mathbb{R}$ be a continuous and nondecreasing function satisfying*

$$
\left| \int_{t_0}^{t} \varphi(s) \,\mathrm{d}s \right| \leq K \varphi(t) \ \text{ for all } \ t \in I
$$

and some $K > 0$. If a continuous function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} \big|y'(t) - F(t, y(t), y(t-\tau))\big| < \varphi(t), & t \in [t_0, T], \\ |y(t) - \Psi(t)| < \varphi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{3.7}$$

then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying equation (3.5) and

$$|y(t) - y_0(t)| \le \frac{LK}{L - (L_1 + L_2)} \varphi(t)$$

for all $t \in I$ and any number L with $L > L_1 + L_2$.

Proof. Define the function $d : S \times S \to [0, \infty]$ as

$$d(f, g) := \inf\Big\{C \in [0, \infty] :\ |f(t) - g(t)|\mathrm{e}^{-L(t-t_0)} \le C\varphi(t),\ \forall\, t \in I\Big\},$$

where the set S is defined by (3.1). Then, according to Lemma 3.1, (S, d) is a complete generalized metric space. If we define the mapping $\Lambda : S \to S$ as in (3.6), every fixed point of Λ solves the differential equation (1.1). Furthermore, as in the proof of Theorem 3.1, it can be easily seen that $\{v \in S :\ d(u, v) < \infty,\ u \in S\} = S$ for arbitrary $u, v \in S$.

Now we will show that the mapping Λ is strictly contractive on I. First, note that by integration by parts and monotonicity of φ, we have

$$\int_{t_0}^{t} \varphi(s)\mathrm{e}^{L(s-t_0)}\,\mathrm{d}s \le \frac{1}{L}\,\varphi(t)\mathrm{e}^{L(t-t_0)} - \frac{1}{L}\int_{t_0}^{t} \varphi'(s)\mathrm{e}^{L(s-t_0)}\,\mathrm{d}s \le \frac{1}{L}\varphi(t)\mathrm{e}^{L(t-t_0)} \ \text{ for all }\ t \in I.$$

Now, for any pair of $f, g \in S$, let $C_{f,g}$ be any constant satisfying $d(f, g) \le C_{f,g}$. That is,

$$|f(t) - g(t)|\mathrm{e}^{L(t-t_0)} \le C_{f,g}\varphi(t) \ \text{ for all }\ t \in I.$$

Hence it follows that for any $f, g \in S$ and $t \in I$,

$$\begin{aligned} \big|(\Lambda f)(t) - (\Lambda g)(t)\big| &= \Big|\int_{t_0}^{t} [F(s, f(s), f(s-\tau) - F(s, g(s), g(s-\tau)]\,\mathrm{d}s\Big| \\ &\le \int_{t_0}^{t} \big|F(s, f(s), f(s-\tau) - F(s, g(s), g(s-\tau)\big|\,\mathrm{d}s \\ &\le L_1 \int_{t_0}^{t} |f(s) - g(s)|\,\mathrm{d}s + L_2 \int_{t_0}^{t} |f(s-\tau) - g(s-\tau)|\,\mathrm{d}s \\ &\le (L_1 + L_2) \int_{t_0}^{t} |f(s) - g(s)|\mathrm{e}^{-L(s-t_0)}\mathrm{e}^{L(s-t_0)}\,\mathrm{d}s \\ &\le C_{f,g}(L_1 + L_2) \int_{t_0}^{t} \varphi(s)\mathrm{e}^{L(s-t_0)}\,\mathrm{d}s \\ &\le \frac{L_1 + L_2}{L}\, C_{f,g}\varphi(t)\mathrm{e}^{L(t-t_0)}, \end{aligned}$$

and therefore we have

$$\big|(\Lambda f)(t) - (\Lambda g)(t)\big|\mathrm{e}^{-L(t-t_0)} \le \frac{L_1 + L_2}{L}\, C_{f,g}\varphi(t) \ \text{ for all }\ t \in I.$$

This inequality and the fact that

$$\big|(\Lambda f)(t) - (\Lambda g)(t)\big| = 0 \text{ for all } t \in [t_0 - \tau, t_0]$$

imply

$$d(\Lambda f, \Lambda g) \le \frac{L_1 + L_2}{L}\, d(f, g),$$

that is, Λ is strictly contractive on S. Therefore, all the conditions of Theorem 2.1 are satisfied and there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda y_0 = y_0$, $\Lambda^n u \to y_0$ in S and satisfies equation (3.5).

On the other hand, it follows from equation (3.7) that

$$-\varphi(t) \le y'(t) - F(t, y(t), y(t-\tau)) \le \varphi(t),$$

and so,

$$|y(t) - (\Lambda y)(t)| \le \left|\int_{t_0}^{t} \varphi(s)\,\mathrm{d}s\right| \le K\varphi(t) \text{ for all } t \in I.$$

Multiplying this inequality by $\mathrm{e}^{-L(t-t_0)}$, we obtain, for all $t \in I$,

$$|y(t) - (\Lambda y)(t)|\mathrm{e}^{-L(t-t_0)} \le K\varphi(t)\mathrm{e}^{-L(t-t_0)},$$

which means that

$$d(\Lambda y, y) \le K\varphi(t)\mathrm{e}^{-L(t-t_0)} \text{ for all } t \in I.$$

Hence, according to Theorem 2.1, we have

$$d(y_0, y) \le \frac{1}{1 - (L_1 + L_2)/L}\, d(\Lambda y, y) \le \frac{L}{L - (L_1 + L_2)}\, K\varphi(t)\mathrm{e}^{-L(t-t_0)} \text{ for all } t \in I.$$

From the definition of d, this means

$$|y(t) - y_0(t)|\mathrm{e}^{-L(t-t_0)} \le \frac{L}{L - (L_1 + L_2)}\, K\varphi(t)\mathrm{e}^{-L(t-t_0)} \text{ for all } t \in I.$$

The proof is now complete. □

4 Examples

Example 4.1. For any $\lambda_1, \lambda_2 > 0$, consider the differential equation

$$y'(t) + \lambda_1 y(t) + \lambda_2 y(t-\tau) = q(t) \tag{4.1}$$

on the interval $I := [t_0 - \tau, T]$, where t_0, τ, T are arbitrary real numbers. Since

$$F(t, y(t), y(t-\tau)) = \lambda_1 y(t) + \lambda_2 y(t-\tau) - q(t),$$

we have

$$\begin{aligned}\big|F(t, x_1, y_1) - F(t, x_2, y_2)\big| &= \big|\lambda_1 x_1 + \lambda_2 y_1 - q(t) - \lambda_1 x_2 - \lambda_2 y_2 + q(t)\big| \\ &= \big|\lambda_1(x_1 - x_2) - \lambda_2(y_1 - y_2)\big| \le \lambda_1|x_1 - x_2| + \lambda_2|y_1 - y_2| \text{ for all } t \in I.\end{aligned}$$

So, all conditions of Theorem 3.1 are satisfied and we obtain the stability of the differential equation (4.1) in the Hyers–Ulam sense.

It should be remarked that Theorem 3.1 guarantees the stability of (4.1) for any $T < \infty$, while the result of Tunç and Biçer [29] can guarantee the stability only in a small subset of I. In this example, their result works only for $T < \lambda_1 + \lambda_2$.

Example 4.2. Consider the differential equation (4.1) on the same interval I with $t_0 = 0$. We have already shown, in Example 4.1, that condition (3.3) is satisfied with $L_1 = \lambda_1$ and $L_2 = \lambda_2$. Now, if we define the function $\varphi(t) := \mathrm{e}^{\lambda t}$ $(\lambda > 0)$, we obtain

$$\Big|\int_{t_0}^{t} \varphi(t)\,\mathrm{d}s\Big| = \int_{0}^{t} \mathrm{e}^{\lambda t}\,\mathrm{d}s = \frac{1}{\lambda}\,(\mathrm{e}^{\lambda t} - 1) \le \frac{1}{\lambda}\,\mathrm{e}^{\lambda t} = \frac{1}{\lambda}\,\varphi(t) \ \text{ for all } \ t \in I.$$

Then, according to Theorem 3.2, equation (4.1) is stable in the Hyers–Ulam–Rassias sense.

Note that the result of Tunç and Biçer [29] does not work in this example if we choose $\lambda > \lambda_1 + \lambda_2$, while our result works for any $\lambda > 0$.

5 Conclusion

In this study, we consider the stability problem of a general class of non-linear differential equations with delay in the Ulam sense. We obtain some new stability criteria which extend and improve some well-known results. In Section 4, We compare our new results with some existing results. As a future research, the stability problem of more general functional differential equations (such as the equations with non-constant delays, or neutral or advanced type equations) might be considered.

References

[1] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* **2** (1998), no. 4, 373–380.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2** (1950), 64–66.

[3] Y. Başcı, A. Mısır and S. Öğrekçi, Hyers–Ulam–Rassias stability for Abel–Riccati type first-order differential equations. *GU J. Sci.* **32** (2019), no. 4, 1238–1252.

[4] Y. Başcı, A. Mısır and S. Öğrekçi, On Hyers–Ulam stability for fractional differential equations including the new Caputo–Fabrizio fractional derivative. *Mediterr. J. Math.* **16** (2019), no. 5, Paper No. 131, 14 pp.

[5] Y. Başcı, A. Mısır and S. Öğrekçi, On the stability problem of differential equations in the sense of Ulam. *Results Math.* **75** (2020), no. 1, Paper No. 6, 13 pp.

[6] E. Biçer and C. Tunç, New theorems for Hyers-Ulam stability of Lienard equation with variable time lags. I*nt. J. Math. Comput. Sci.* **13** (2018), no. 2, 231–242.

[7] J. Brzdek, D. Popa and B. Xu, The Hyers–Ulam stability of nonlinear recurrences. *J. Math. Anal. Appl.* **335** (2007), no. 1, 443–449.

[8] L. Cădariu and V. Radu, Fixed point methods for the generalized stability of functional equations in a single variable. *Fixed Point Theory Appl.* **2008**, Art. ID 749392, 15 pp.

[9] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space. *Bull. Amer. Math. Soc.* **74** (1968), 305–309.

[10] M. Eshaghi Gordji, A. Bodaghi and C. Park, A fixed point approach to the stability of double Jordan centralizers and Jordan multipliers on Banach algebras. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **73** (2011), no. 2, 65–74.

[11] M. Eshaghi Gordji, F. Sajadian, Y. J. Cho and M. Ramezani, A fixed point theorem for quasi-contraction mappings in partially order modular spaces with an application. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **76** (2014), no. 2, 135–146.

[12] G.-L. Forti, Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations. *J. Math. Anal. Appl.* **295** (2004), no. 1, 127–133.

[13] D. H. Hyers, On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.

[14] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis.* Hadronic Press, Inc., Palm Harbor, FL, 2001.

[15] S.-M. Jung, A fixed point approach to the stability of differential equations $y' = F(x, y)$. *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 1, 47–56.

[16] T. Miura, S. Miyajima and S.-E. Takahasi, A characterization of Hyers-Ulam stability of first order linear differential operators. *J. Math. Anal. Appl.* **286** (2003), no. 1, 136–146.

[17] T. Miura, S. Miyajima and S.-E. Takahasi, Hyers–Ulam stability of linear differential operator with constant coefficients. *Math. Nachr.* **258** (2003), 90–96.

[18] M. Obłoza, Hyers stability of the linear differential equation. *Rocznik Nauk.-Dydakt. Prace Mat.* No. 13 (1993), 259–270.

[19] M. Obłoza, Connections between Hyers and Lyapunov stability of the ordinary differential equations. *Rocznik Nauk.-Dydakt. Prace Mat.* No. 14 (1997), 141–146.

[20] S. Öğrekçi, Stability of delay differential equations in the sense of Ulam on unbounded intervals. *Int. J. Optim. Control. Theor. Appl. IJOCTA* **9** (2019), no. 2, 125–131.

[21] S. Öğrekçi, Y. Başcı and A. Mısır, Ulam type stability for conformable fractional differential equations. *Rend. Circ. Mat. Palermo (2)* **70** (2021), no. 2, 807–817.

[22] T. P. Petru, A. Petruşel and J.-C. Yao, Ulam–Hyers stability for operatorial equations and inclusions via nonself operators. *Taiwanese J. Math.* **15** (2011), no. 5, 2195–2212.

[23] D. Popa, Hyers–Ulam–Rassias stability of a linear recurrence. *J. Math. Anal. Appl.* **309** (2005), no. 2, 591–597.

[24] T. M. Rassias (Ed.), *Handbook of Functional Equations: Stability Theory.* Vol. 96. Springer, 2014.

[25] T. M. Rassias, On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), no. 2, 297–300.

[26] R. Saadati, C. Park and J. R. Lee, Fuzzy approximate of derivations. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **77** (2015), no. 4, 33–44.

[27] S.-E. Takahasi, T. Miura and S. Miyajima, On the Hyers–Ulam stability of the Banach space-valued differential equation $y' = \lambda y$. *Bull. Korean Math. Soc.* **39** (2002), no. 2, 309–315.

[28] C. Tunç and E. Biçer, Stability to a kind of functional differential equations of second order with multiple delays by fixed points. *Abstr. Appl. Anal.* **2014**, Art. ID 413037, 9 pp.

[29] C. Tunç and E. Biçer, Hyers–Ulam–Rassias stability for a first order functional differential equation. *J. Math. Fundam. Sci.* **47** (2015), no. 2, 143–153.

[30] S. M. Ulam, *A Collection of Mathematical Problems.* Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York–London, 1960.

(Received 05.08.2020; revised 17.07.2021; accepted 18.10.2021)

Authors' addresses:

Süleyman Öğrekçi
Department of Mathematics, Faculty of Arts and Science, Amasya University, Amasya, Turkey.
E-mail: `suleyman.ogrekci@amasya.edu.tr`

Yasemin Başcı
Department of Mathematics, Faculty of Arts and Science, Abant İzzet Baysal University, Bolu, Turkey.
E-mail: `basci_y@ibu.edu.tr`

Adil Mısır
Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey.
E-mail: `adilm@gazi.edu.tr`