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MARTINGALE SOLUTION OF STOCHASTIC HYBRID KORTEWEG–DE VRIES–BURGERS EQUATION

Abstract. In the paper, we consider a stochastic hybrid Korteweg–de Vries–Burgers type equation with multiplicative noise in the form of cylindrical Wiener process. We prove the existence of a martingale solution to the equation studied. The proof of the existence of the solution is based on two approximations of the considered problem and the compactness method. First, we introduce an auxiliary problem corresponding to the equation studied. Then, we prove the existence of a martingale solution to this problem. Finally, we show that the solution of the auxiliary problem converges, in some sense, coincides to the solution of the equation under consideration.

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რეზიუმე. ნაშრომში განხილულია სტოქასტური ჰიბრიდული კორტევეგ-დე ფრიზ-ბურგერსის ტიპის განტოლება მულტიპლიკატიური ხმაურით ცილინდრული ვინერის პროცესის ფორმით. დამტკიცებულია ამ განტოლების მარტინგალური ამონახსნის არსებობა, რომელიც დაფუძნებულია განხილული ამოცანის ორ მიახლოებასა და კომპაქტურობის მეთოდზე. თავდაპირველად შემოყვანილია დამხმარე ამოცანა, რომელიც შეესაბამება შესასწავლ განტოლებას. შემდეგ, დამტკიცებულია ამ ამოცანის მარტინგალური ამონახსნის არსებობა. და ბოლოს, ნაჩვენებია, რომ დამხმარე ამოცანის ამონახსნი გარკვეული აზრით ემთხვევა განხილული განტოლების ამოხსნას.

1 Introduction

The deterministc hybrid Korteweg–de Vries–Burgers (hKdVB) equation has been derived by Misra, Adhikary and Shuka [10] and by Elkamash and Kourakis [5] in the context of shock excitations in multicomponent plasma. The hKdVB equation derived in stretched coordinates $\xi = \epsilon^{\frac{1}{2}}(x - vt)$, $\tau = \epsilon^{\frac{3}{2}}t$ (v is the phase velocity of the wave) has the form

$$u_\tau + Au\, u_\xi + Bu_{3\xi} = Cu_{2\xi} - Du. \tag{1.1}$$

In (1.1), $u(\xi, \tau)$ represents electrostatic potential or electric field pulse in the reference frame moving with the velocity v. Indices denote partial derivatives, that is, $u_\tau = \partial u/\partial\tau$, $u_{2\xi} = \partial^2 u/\partial\xi^2$ and so on. The constants A, B, C, D are related to the parameters describing properties of plasma [5, Eq. (27)].

Although equation (1.1) was derived for dissipative dispersive waves in multicomponent plasma, it can be applied in several other physical systems, e.g., surface water waves and the motion of optical impulses in fibers. For some particular values of constants A, B, C, D, the hKdVB equation (1.1) reduces to the particular cases:

- the Korteweg–de Vries equation, when $C = D = 0$;
- the damped (dissipative) KdV equation, when $C = 0$;
- the Burgers equation, when $B = D = 0$;
- the KdV–Burgers equation, when $D = 0$;
- the damped Burgers equation, when $B = 0$.

The term with $A \neq 0$ introduces nonlinearity that with $B \neq 0$ is responsible for dispersion, $C \neq 0$ supplies diffusive term and $D \neq 0$ introduces damping. All equations of these kinds were widely studied 30-40 years ago, and most of physical ideas have been already understood (see, e.g., Lev Ostrovsky's book [11] and the references therein). On the other hand, during the last few years, one can observe renewal of interest in this field, mostly due to the extensions to higher order equations.

Studies of the full generalized hybrid KdB–Burgers equation (1.1) have appeared in the physical literature only in [5,10]. Some approximate analytic solutions and several cases of numerical solutions to (1.1) were subjects of recent studies in [6].

The paper deals with a stochastic hybrid Korteweg–de Vries–Burgers type equation. The presence of stochastic noise has deep physical grounds. In the case of waves in plasma, it can be caused by thermal fluctuations, whereas in the case of water surface waves, by air pressure fluctuations due to the wind. To the best of our knowledge, our paper is the first one which deals with the stochastic hKdVB equation.

The main result of the paper, Theorem 2.1, supplies the existence of a martingale solution to equation (2.1), which is the stochastic version of equation (1.1).

The idea of the proof of the existence of a martingale solution to (2.1) consists in the following. First, we introduce an auxiliary problem (2.6) which we can call ε-approximation of equation (2.1). Then, in Lemma 2.1, we prove that problem (2.6) has a martingale solution. Here we use the Galerkin approximation (4.1) of (2.6) and the tightness of the family of distributions of the solutions to the approximation (4.1). Next, in Lemma 2.2, we show two estimates used in the proof of Theorem 2.1. Lemma 2.3 guarantees the tightness of the family of distributions of solutions to problem (2.6) in a proper space. Finally, we prove that the solution to (2.6) converges, in some sense, to the solution of equation (2.1).

The paper is organized as follows.

In Section 2, we define the martingale solution to some kind of stochastic hybrid Korteweg–de Vries–Burgers equation (2.1) with a multiplicative Wiener noise on the interval $[0, T]$. Then we formulate and prove Theorem 2.1. In the proof, some methods introduced in [7] and extended in [4] have been adapted to the problem under consideration.

In Section 3, Lemmas 2.2 and 2.3 used in the proof of Theorem 2.1 are proved. Lemma 2.2 contains a version of estimates which are analogous to those presented in [4] and [7].

In Section 4, we give the detailed proof of Lemma 2.1. This lemma formulates the sufficient conditions for the existence of martingale solutions for m-dimensional Galerkin approximation of Korteweg–de Vries–Burgers equation with a multiplicative Wiener noise for arbitrary $m \in \mathrm{N}$.

2 Existence of martingale solution

Denote $X := [x_1, x_2] \subset \mathbb{R}$, where $-\infty < x_1 < 0 < x_2 < \infty$. We consider the following initial value problem for the hybrid Korteweg–de Vries–Burgers type equation

$$\begin{cases} du(t,x) + \big(Au(t,x)u_x(t,x) + Bu_{3x}(t,x) - Cu_{2x}(t,x) + Du(t,x)\big)\,dt = \Phi(u(t,x))\,dW(t), \\ u(0,x) = u_0(x), \quad x \in X, \quad t \geq 0. \end{cases} \tag{2.1}$$

In (2.1), $W(t)$, $t \geq 0$, is a cylindrical Wiener process on $L^2(X)$. We define W by setting $W(t) = \sum\limits_{i \in \mathbb{N}} \beta_i(t)\, e_i$, where $\{e_i\}_{i \in \mathbb{N}}$ denotes a basis on $L^2(X)$ and $\{\beta_i\}_{I \in \mathbb{N}}$ is a family of real Brownian motions, mutually independent in a fixed probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ (see [3,8]). The series defining the process W does not converge in $L^2(X)$, but it is convergent in any Hilbert space U such that the embedding $L^2(X) \subset U$ is Hilbert–Schmidt.

The initial condition $u_0 \in L^2(X)$ is a deterministic real-valued function. In (2.1), $u(\omega, \cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ for all $\omega \in \Omega$. We assume that there exists $\lambda_X > 0$ such that

$$|u(t,x)| < \lambda_X < \infty \quad \text{for all} \quad t \in \mathbb{R}_+ \quad \text{and all} \quad x \in X. \tag{2.2}$$

This assumption reflects the finiteness of the solutions to the deterministic equation (1.1) on a finite interval X (see, e.g., [6,10]).

By $H^1(X)$, $H^2(X)$, $H^s(X)$, $s < 0$, we denote the Sobolev spaces according to definitions in [1]. We assume that Φ is a continuous mapping from $H^2(X)$ to $L_2^0(L^2(X))$, the space of Hilbert–Schmidt operators from $L_2(X)$ to itself. Moreover, we assume that Φ is such that there exist the constants $\kappa_1, \kappa_2 > 0$ satisfying

$$\|\Phi(u(x))\|_{L_0^2(L^2(X))} \leq \kappa_1 \min\left\{|u(x)|^2_{L^2(X)}, |u(x)|_{L^2(X)}\right\} + \kappa_2 \quad \text{forany} \quad u \in H^2(X), \tag{2.3}$$

and that there exist the functions $a, b \in L^2(X)$ with a compact support such that the mapping

$$u \longmapsto \big(\Phi(u)a, \Phi(u)b\big)_{L^2(X)} \quad \text{is continuous in } L^2(X). \tag{2.4}$$

Definition 2.1. We say that problem (2.1) has a **martingale solution** on the interval $[0,T]$, $0 < T < \infty$, if there exists a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P}, \{W_t\}_{t \geq 0})$, where $\{W_t\}_{t \geq 0}$ is a cylindrical Wiener process, and there exists the process $\{u(t,x)\}_{t \geq 0}$ adapted to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$ with trajectories in the space

$$L^\infty(0,T; L^2(X)) \cap L^2(0,T; L^2(X)) \cap C(0,T; H^s(X)), \quad s < 0, \quad \mathbb{P}\text{-a.s.},$$

such that

$$\langle u(t,x); v(x)\rangle + \int_0^t \Big\langle Au(t,x)u_x(t,x) + Bu_{3x}(t,x) - Cu_{2x}(t,x) + Du(t,x); v(x)\Big\rangle\, ds$$
$$= \langle u_0(x); v(x)\rangle + \Big\langle \int_0^t \Phi(u(s,x))\, dW(s); v(x)\Big\rangle, \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0,T]$ and $v \in H^1(X)$.

In our consideration we assume that the coefficients of equation (2.1) satisfy the following condition:

$$B, C, D \geq 0 \ \text{ with } \ 3B \geq A+1. \tag{2.5}$$

The physical sense of the coefficients A, B, C, D and the fact that A can be positive or negative (see, e.g., [5, 6, 10]) confirm that condition (2.5) is satisfied for a wide class of physically meaningful equations which contains all particular cases listed in Section 1.

Theorem 2.1. *If conditions* (2.2)–(2.5) *hold, then for all real-valued* $u_0 \in L^2(X)$ *and* $0 < T < \infty$ *there exists a martingale solution to* (2.1).

Proof. Let $\varepsilon > 0$. Consider the following auxiliary problem:

$$\begin{cases} du^\varepsilon(t,x) + \Big[\varepsilon u^\varepsilon_{4x}(t,x) + Au^\varepsilon(t,x)u^\varepsilon_x(t,x) + Bu^\varepsilon_{3x}(t,x) - Cu^\varepsilon_{2x}(t,x) + Du^\varepsilon(t,x)\Big]\,dt \\ \qquad\qquad\qquad\qquad\qquad\qquad = \Phi(u^\varepsilon(t,x))\,dW(t), \\ u^\varepsilon_0(x) = u^\varepsilon(0,x), \ \ \varepsilon > 0. \end{cases} \tag{2.6}$$

In the proof of the theorem we use the following lemmas.

Lemma 2.1. *For any* $\varepsilon > 0$, *there exists a martingale solution to problem* (2.6) *if conditions* (2.3), (2.4) *and* (2.5) *hold.*

Lemma 2.2. *There exist* $\varepsilon_0 > 0$ *and* $\widetilde{C}_1 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon\,\mathbb{E}\left(|u^\varepsilon(t,x)|^2_{L^2(0,T;H^2(\mathbb{R}))}\right) \leq \widetilde{C}_1. \tag{2.7}$$

Moreover, there exist $\varepsilon_0 > 0$ *and* $\widetilde{C}_2(k) > 0$ *such that for all* $k \in X_k$ *and* $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{E}\left(|u^\varepsilon(t,x)|^2_{L^2(0,T;H^1(-k,k))}\right) \leq \widetilde{C}_2(k), \tag{2.8}$$

where $X_k = \{k > 0 : \ |k| \leq \min\{-x_1, x_2\}\}$.

Lemma 2.3. *Let* $\mathscr{L}(u^\varepsilon)$ *denote the family of distributions of the solutions* u^ε *to* (2.6). *Then the family* $\mathscr{L}(u^\varepsilon)$ *is tight in* $L^2(0,T;L^2(X)) \cap C(0,T;H^{-3}(X))$.

Now, in Prohorov's theorem (e.g., see Theorem 5.1 in [2]), we substitute $S := L^2(0,T;L^2(X)) \cap C(0,T;H^{-3}(X))$ and $\mathscr{K} := \{\mathscr{L}(u^\varepsilon)\}_{\varepsilon>0}$. Since $\mathscr{K}$ is tight in S, it is sequentially compact, so there exists a subsequence of $\{\mathscr{L}(u^\varepsilon)\}_{\varepsilon>0}$ converging to some measure μ in $\overline{\mathscr{K}}$. Because $\{\mathscr{L}(u^\varepsilon)\}_{\varepsilon>0}$ is convergent, in Skorohod's theorem (e.g., see Theorem 6.7 in [2]) one can substitute $\mu_\varepsilon := \{\mathscr{L}(u^\varepsilon)\}_{\varepsilon>0}$ and $\mu := \lim_{\varepsilon\to 0}\mu_\varepsilon$. Then there exist a space $(\overline{\Omega}, \overline{\mathscr{F}}, \{\overline{\mathscr{F}}_t\}_{t\geq 0}, \overline{\mathbb{P}})$ and random variables with values in $L^2(0,T;L^2(X)) \cap C(0,T;H^{-3}(X))$ such that $\overline{u}^\varepsilon \to \overline{u}$ in $L^2(0,T;L^2(X))$ and $\overline{u}^\varepsilon \to \overline{u}$ in $C(0,T;H^{-3}(X))$. Moreover, $\mathscr{L}(\overline{u}^\varepsilon) = \mathscr{L}(u^\varepsilon)$.

Then, due to Lemma 2.2, for any $p \in \mathbb{N}$ there exist the constants $\widetilde{C}_1(p)$, $\widetilde{C}_2$ such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |\overline{u}^\varepsilon(t,x)|^{2p}_{L^2(X)}\Big) \leq \widetilde{C}_1(p), \quad \mathbb{E}\left(|\overline{u}^\varepsilon(t,x)|^2_{L^2(0,T;H^2(X))}\right) \leq \widetilde{C}_2$$

and $\overline{u}^\varepsilon(t,x) \in L^2(0,T;H^1(-k,k)) \cap L^\infty(0,T;L^2(X))$, $\mathbb{P}$-a.s. Therefore, one can conclude that $\overline{u}^\varepsilon \to \overline{u}$ weakly in $L^2(\overline{\Omega}, L^2(0,T;H^1(-k,k)))$.

Let $x \in \mathbb{R}$ be fixed. Denote

$$M^\varepsilon(t) := u^\varepsilon(t,x) - u^\varepsilon_0(x) \\ + \int_0^t \Big[\varepsilon u^\varepsilon(t,x)_{4x}(t,x) + Au^\varepsilon(t,x)u^\varepsilon_x(t,x) + Bu^\varepsilon_{3x}(t,x) - Cu^\varepsilon_{2x}(t,x) + Du^\varepsilon(t,x)\Big]\,ds,$$

$$\overline{M}^\varepsilon(t) := \overline{u}^\varepsilon(t,x) - \overline{u}^\varepsilon_0(x) + \int_0^t \Big[A\,\overline{u}^\varepsilon(t,x)\overline{u}^\varepsilon_x(t,x) + B\,\overline{u}^\varepsilon_{3x}(t,x) - C\,\overline{u}^\varepsilon_{2x}(t,x) + D\,\overline{u}^\varepsilon(t,x)\Big]\,ds.$$

Note that

$$M^{\varepsilon}(t) = u_0^{\varepsilon}(x) - \int_0^t \Big[\varepsilon u^{\varepsilon}(t,x)_{4x}(t,x) + Au^{\varepsilon}(t,x)u_x^{\varepsilon}(t,x) + Bu_{3x}^{\varepsilon}(t,x) - Cu_{2x}^{\varepsilon}(t,x) + Du^{\varepsilon}(t,x)\Big]\, ds$$
$$+ \int_0^t \big(\Phi(u^{\varepsilon}(s,x))\big)\, dW(s) - u_0^{\varepsilon}(x)$$
$$+ \int_0^t \Big[Au^{\varepsilon}(t,x)u_x^{\varepsilon}(t,x) + Bu_{3x}^{\varepsilon}(t,x) - Cu_{2x}^{\varepsilon}(t,x) + Du^{\varepsilon}(t,x)\Big]\, ds$$
$$= \int_0^t \big(\Phi(u^{\varepsilon}(s,x))\big)\, dW(s).$$

So, $M^{\varepsilon}(t)$, $t \geq 0$, is a square integrable martingale with values in $L^2(X)$, adapted to the filtration $\sigma\{u^{\varepsilon}(s,x), 0 \leq s \leq t\}$ with quadratic variation, equal to

$$[M^{\varepsilon}](t) := \int_0^t \Phi(u^{\varepsilon}(s,x))\big[\Phi(u^{\varepsilon}(s,x))\big]^*\, ds.$$

In the Doob inequality (e.g., see Theorem 2.2 in [8]), substitute $M_t := M^{\varepsilon}(t)$ and $p := 2p$. Then

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} |M^{\varepsilon}(t)|^p_{L^2(X)}\Big)\Big] \leq \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\left(|M^{\varepsilon}(T)|_{L^2(X)}\right). \tag{2.9}$$

Assume that $0 \leq s \leq t \leq T$ and let φ be a bounded continuous function on $L^2(0,s;L^2(X))$ or $C(0,s;H^{-3}(X))$. Let $a, b \in H_0^3(-k,k)$, $k \in \mathbb{N}$, be arbitrary and fixed. Since $M^{\varepsilon}(t)$ is a martingale and $\mathscr{L}(\overline{u}^{\varepsilon}) = \mathscr{L}(u^{\varepsilon})$, we have (see [7, pp. 377–378])

$$\mathbb{E}\Big(\big\langle M^{\varepsilon}(t) - M^{\varepsilon}(s); a\big\rangle \varphi(u^{\varepsilon}(t,x))\Big) = 0 \text{ and } \mathbb{E}\Big(\big\langle \overline{M}^{\varepsilon}(t) - \overline{M}^{\varepsilon}(s); a\big\rangle \varphi(\overline{u}^{\varepsilon}(t,x))\Big) = 0.$$

Moreover,

$$\mathbb{E}\Bigg\{\Bigg[\langle M^{\varepsilon}(t);a\rangle\langle M^{\varepsilon}(t);b\rangle - \langle M^{\varepsilon}(s);a\rangle\langle M^{\varepsilon}(s);b\rangle$$
$$- \int_s^t \Big\langle \big[\Phi(u^{\varepsilon}(\xi,x))\big]^* a; \big[\Phi(u^{\varepsilon}(\xi,x))\big]^* b\Big\rangle\, d\xi\Bigg]\varphi(u^{\varepsilon}(t,x))\Bigg\} = 0,$$

$$\mathbb{E}\Bigg\{\Bigg[\langle \overline{M}^{\varepsilon}(t);a\rangle\langle \overline{M}^{\varepsilon}(t);b\rangle - \langle \overline{M}^{\varepsilon}(s);a\rangle\langle \overline{M}^{\varepsilon}(s);b\rangle$$
$$- \int_s^t \Big\langle \big[\Phi(\overline{u}^{\varepsilon}(\xi,x))\big]^* a; \big[\Phi(\overline{u}^{\varepsilon}(\xi,x))\big]^* b\Big\rangle\, d\xi\Bigg]\varphi(\overline{u}^{\varepsilon}(t,x))\Bigg\} = 0.$$

Denote

$$\overline{M}(t) := \overline{u}(t,x) - u_0(x) + \int_0^t \Big[A\,\overline{u}(t,x)\overline{u}_x(t,x) + B\,\overline{u}_{3x}(t,x) - C\,\overline{u}_{2x}(t,x) + D\,\overline{u}(t,x)\Big]\, ds.$$

If $\varepsilon \to 0$, then $\overline{M}^{\varepsilon}(t) \to \overline{M}(t)$ and $\overline{M}^{\varepsilon}(s) \to \overline{M}(s)$, $\overline{\mathbb{P}}$-a.s. in $H^{-3}(X)$. Moreover, since φ is continuous, we have $\varphi(\overline{u}^{\varepsilon}(s,x)) \to \varphi(\overline{u}(s,x))$, $\overline{\mathbb{P}}$-a.s. So, if $\varepsilon \to 0$, then

$$\mathbb{E}\Big(\langle \overline{M}^{\varepsilon}(t) - \overline{M}^{\varepsilon}(s); a\rangle \varphi(\overline{u}^{\varepsilon}(t,x))\Big) \to \mathbb{E}\Big(\langle \overline{M}(t) - \overline{M}(s); a\rangle \varphi(\overline{u}(t,x))\Big).$$

Moreover, since Φ is a continuous operator in the topology $L^2(X)$ and (2.9) holds, for $\varepsilon \to 0$ we have

$$\Big\langle (\Phi(\overline{u}^{\varepsilon}(s,x)))^* a; (\Phi(\overline{u}^{\varepsilon}(s,x)))^* b \Big\rangle \to \Big\langle (\Phi(\overline{u}(s,x)))^* a; (\Phi(\overline{u}(s,x)))^* b \Big\rangle$$

and

$$\begin{aligned}
&\mathbb{E}\Bigg\{\bigg[\langle \overline{M}^{\varepsilon}(t);a\rangle\langle \overline{M}^{\varepsilon}(t);b\rangle - \langle \overline{M}^{\varepsilon}(s);a\rangle\langle \overline{M}^{\varepsilon}(s);b\rangle \\
&\qquad - \int\limits_s^t \Big\langle \big[\Phi(\overline{u}^{\varepsilon}(s,\xi))\big]^* a; \big[\Phi(\overline{u}^{\varepsilon}(s,\xi))\big]^* b \Big\rangle\, d\xi\bigg]\varphi(\overline{u}^{\varepsilon}(t,x))\Bigg\} \\
&\longrightarrow \mathbb{E}\Bigg\{\bigg[\langle \overline{M}(t);a\rangle\langle \overline{M}(t);b\rangle - \langle \overline{M}(s);a\rangle\langle \overline{M}(s);b\rangle \\
&\qquad - \int\limits_s^t \Big\langle \big[\Phi(\overline{u}(s,\xi))\big]^* a; \big[\Phi(\overline{u}(s,\xi))\big]^* b \Big\rangle\, d\xi\bigg]\varphi(\overline{u}(t,x))\Bigg\}.
\end{aligned}$$

Then $\overline{M}(t)$, $t \geq 0$, is also a square integrable martingale adapted to the filtration $\sigma\{\overline{u}(s), 0 \leq s \leq t\}$ with quadratic variation $\int\limits_0^t \Phi(\overline{u}(s,x))(\Phi(\overline{u}(s,x)))^*\, ds$.

In the representation theorem (e.g., see Theorem 8.2 in [3]), substitute

$$M_t := \overline{M}(t), \quad [M_t] := \int\limits_0^t \Phi(\overline{u}(s,x))(\Phi(\overline{u}(s,x)))^*\, ds$$

and

$$\Phi(s) := \Phi(\overline{u}(s,x)).$$

Then there exists a process $\widetilde{M}(t) = \int\limits_0^t \Phi(\overline{u}(s,x))\, dW(s)$ such that $\widetilde{M}(t) = \overline{M}(t)$, $\overline{\mathbb{P}}$-a.s., and

$$\begin{aligned}
\overline{u}(t,x) - u_0(x) + \int\limits_0^t \Big[A\overline{u}(t,x)\overline{u}_x(t,x) + B\overline{u}_{3x}(t,x) - C\overline{u}_{2x}(t,x) + D\overline{u}(t,x)\Big]\, ds \\
= \int\limits_0^t \Phi(\overline{u}(s,x))\, dW(s).
\end{aligned}$$

This implies that

$$\begin{aligned}
\overline{u}(t,x) = u_0(x) - \int\limits_0^t \Big[A\overline{u}(t,x)\overline{u}_x(t,x) + B\overline{u}_{3x}(t,x) - C\overline{u}_{2x}(t,x) + D\overline{u}(t,x)\Big]\, ds \\
+ \int\limits_0^t \Phi(\overline{u}(s,x))\, dW(s).
\end{aligned}$$

Thus $\overline{u}(t,x)$ is a solution to (2.1), which completed the proof of Theorem 2.1. ☐

3 Proofs of Lemma 2.2 and Lemma 2.3

Proof of Lemma 2.2. Let $p : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the following conditions:

(i) p is increasing on X;

(ii) $p(x_1) = \delta > 0$;

(iii) $p'(x) > \alpha_X$ for all $x \in X$;

(iv) $Bp'''(x) + Cp''(x) \leq \gamma < -1$ for all $x \in X$.

Additionally, let

$$F(u^\varepsilon) := \int_X p(x)(u^\varepsilon(x))^2\,dx.$$

Application of the Itô formula for $F(u^\varepsilon)$ yields the formula

$$\begin{aligned} dF(u^\varepsilon(t,x)) = {} & \big\langle F'(u^\varepsilon(t,x)); \Phi(u^\varepsilon(t,x))\big\rangle\,dW(t) \\ & - \Big\langle F'(u^\varepsilon(t,x)); \varepsilon u^\varepsilon_{4x} + Au^\varepsilon(t,x)u^\varepsilon_x(t,x) + Bu^\varepsilon_{3x}(t,x) - Cu^\varepsilon_{2x}(t,x) + Du^\varepsilon(t,x)\Big\rangle\,dt \\ & + \frac{1}{2}\,\mathrm{Tr}\Big\{F''(u^\varepsilon(t,x))\Phi(u^\varepsilon(t,x))\big[\Phi(u^\varepsilon(t,x))\big]^*\Big\}\,dt, \end{aligned}$$

where

$$\big\langle F'(u^\varepsilon(t,x)); v(t,x)\big\rangle = 2\int_X p(x)u^\varepsilon(t,x)v(t,x)\,dx \;\text{ and }\; F'(u^\varepsilon(t,x))v(t,x) = 2p(x)v(t,x).$$

We use the following auxiliary result.

Lemma 3.1 ([4, p. 242])**.** *There exist the positive constants C_1, C_2, C_3 such that*

$$\begin{aligned} \int_X p(x)u^\varepsilon(t,x)u^\varepsilon_{4x}(t,x)\,dx \geq {} & \frac{1}{2}\int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx - C_1|u^\varepsilon(t,x)|^2_{L^2(X)} \\ & - C_2\int_X p'(x)[u_x(t,x)]^2\,dx; \end{aligned}$$

$$\int_X p(x)u^\varepsilon(t,x)u^\varepsilon_{3x}(t,x)\,dx = \frac{3}{2}\int_X p'(x)[u^\varepsilon_x(t,x)]^2\,dx - \frac{1}{2}\int_X p'''(x)[u(t,x)]^2\,dx;$$

$$\int_X p(x)[u^\varepsilon(t,x)]^2u^\varepsilon_x(t,x)\,dx \geq -C_3\big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big) - \frac{1}{2}\int_X p'(x)[u_x(t,x)]^2\,dx.$$

Similarly, as in Lemma 3.1, one has

$$\int_X p(x)u^\varepsilon(t,x)u^\varepsilon_{2x}(t,x)\,dx = \frac{1}{2}\int_X p''(x)[u^\varepsilon(t,x)]^2\,dx - \int_X p(x)[u_x(t,x)]^2\,dx.$$

These estimations imply

$$\begin{aligned} & \Big\langle F'(u^\varepsilon(t,x)); \varepsilon u^\varepsilon_{4x}(t,x) + Au^\varepsilon(t,x)u^\varepsilon_x(t,x) + Bu^\varepsilon_{3x}(t,x) - Cu^\varepsilon_{2x}(t,x) + Du^\varepsilon(t,x)\Big\rangle \\ & \quad \geq \varepsilon\int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx - 2\varepsilon C_1|u^\varepsilon(t,x)|^2_{L^2(X)} - 2\varepsilon C_2\int_X p'(x)[u_x(t,x)]^2\,dx \\ & \quad + 3B\int_X p'(x)[u^\varepsilon_x(t,x)]^2\,dx - B\int_X p'''(x)[u^\varepsilon(t,x)]^2\,dx - 2AC_3\big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big) \\ & \quad - A\int_X p'(x)[u_x(t,x)]^2\,dx - C\int_X p''(x)[u^\varepsilon(t,x)]^2\,dx \\ & \quad + 2C\int_X p(x)[u_x(t,x)]^2\,dx + 2D\int_X p(x)[u(t,x)]^2\,dx \end{aligned}$$

$$\geq \varepsilon \int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx - 2\varepsilon C_1|u^\varepsilon(t,x)|^2_{L^2(X)} + (3B - A - 2\varepsilon C_2)C_2 \int_X p'(x)[u_x(t,x)]^2\,dx$$
$$- \int_X \big[Bp'''(x) + Cp''(x)\big]\,[u^\varepsilon(t,x)]^2\,dx - 2AC_3\big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big)$$
$$\geq \varepsilon \int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx + (3B - A - 2\varepsilon C_2)\int_X p'(x)[u_x(t,x)]^2\,dx$$
$$- (\gamma + 2\varepsilon C_1)|u^\varepsilon(t,x)|^2_{L^2(X)} - 2AC_3\big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big).$$

Let $\varepsilon \leq \min\{\frac{3B-A-1}{2C_2}, -\frac{1+\gamma}{2C_1}\}$. Then

$$\Big\langle F'(u^\varepsilon(t,x)); \varepsilon u^\varepsilon_{4x}(t,x) + Au^\varepsilon(t,x)u^\varepsilon_x(t,x) + Bu^\varepsilon_{3x}(t,x) - Cu^\varepsilon_{2x}(t,x) + Du^\varepsilon(t,x)\Big\rangle$$
$$\geq \varepsilon \int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx + \int_X p'(x)[u_x(t,x)]^2\,dx + |u^\varepsilon(t,x)|^2_{L^2(X)} - 2AC_3\big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big)$$
$$\geq \varepsilon \int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx + \int_X p'(x)[u_x(t,x)]^2\,dx - 2AC_3\big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big). \quad (3.1)$$

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in $L^2(X)$. Then there exists a constant $C_4 > 0$ such that

$$\operatorname{Tr}\big(F''(u)\Phi(u)[\Phi(u)]^*\big) = 2\sum_{i\in\mathbb{N}} \int_X p(x)\big|\Phi(u^\varepsilon(t,x))e_i(x)\big|^2\,dx$$
$$\leq C_4\big|\Phi(u^\varepsilon(t,x))\big|^2_{L^2_0(L^2(X))} \leq C_4\big(\kappa_1|u^\varepsilon(t,x)|^2_{L^2(X)} + \kappa_2\big)^2. \quad (3.2)$$

From (3.1) and (3.2), we have

$$\mathbb{E}\,F(u^\varepsilon(t,x)) \leq F(u^\varepsilon_0) - \varepsilon\,\mathbb{E}\int_0^t\int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx\,dt - \mathbb{E}\int_0^t\int_X p'(x)[u_x(t,x)]^2\,dx\,dt$$
$$+ 2AC_3\,\mathbb{E}\int_0^t \big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big)\,dt + C_4\,\mathbb{E}\big(\kappa_1|u^\varepsilon(t,x)|^2_{L^2(\mathbb{R})} + \kappa_2\big)^2.$$

Thus

$$\mathbb{E}\,F(u^\varepsilon(t,x)) + \varepsilon\,\mathbb{E}\int_0^t\int_X p(x)[u^\varepsilon_{2x}(t,x)]^2\,dx\,dt + \mathbb{E}\int_0^t\int_X p'(x)[u_x(t,x)]^2\,dx\,dt$$
$$\leq F(u^\varepsilon_0) + 2AC_3\,\mathbb{E}\int_0^t \big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big)\,dt + C_4\,\mathbb{E}\big(\kappa_1|u^\varepsilon(t,x)|^2_{L^2(\mathbb{R})} + \kappa_2\big)^2$$
$$\leq F(u^\varepsilon_0) + 2AC_3\,\mathbb{E}\int_0^T \big(1 + |u^\varepsilon(t,x)|^6_{L^2(X)}\big)\,dt + C_4\,\mathbb{E}\big(\kappa_1|u^\varepsilon(t,x)|^2_{L^2(\mathbb{R})} + \kappa_2\big)^2$$
$$\leq F(u^\varepsilon_0) + 2AC_3\,\mathbb{E}\int_0^T (1 + C_5)\,dt + C_6 = F(u^\varepsilon_0) + 2AC_3T(1 + C_5) + C_6 \leq C_7.$$

Let $\varepsilon_0 > 0$ be fixed. Then for all $0 < \varepsilon < \varepsilon_0$, one has

$$\varepsilon\,\mathbb{E}\big(|u^\varepsilon(t,x)|^2_{L^2(0,T;H^2(X))}\big) = \varepsilon\,\mathbb{E}\int_0^T\int_X [u^\varepsilon(t,x)]^2\,dx\,dt + \varepsilon\,\mathbb{E}\int_0^T\int_X [u^\varepsilon_{2x}(t,x)]^2\,dx\,dt$$

$$\leq \varepsilon\, C_8 + \varepsilon\, \mathbb{E} \int_0^T \int_X [u^\varepsilon_{2x}(t,x)]^2 \; dx\, dt = \varepsilon C_8 + \varepsilon\, \mathbb{E} \int_0^T \int_X \frac{1}{p(x)}\, p(x) [u^\varepsilon_{2x}(t,x)]^2 \; dx\, dt$$
$$\leq \varepsilon\, C_8 + \varepsilon\, \mathbb{E} \int_0^T \int_X \frac{1}{\delta}\, p(x) [u^\varepsilon_{2x}(t,x)]^2 \, dx \leq \varepsilon\, C_8 + \frac{1}{\delta}\, \varepsilon\, \mathbb{E} \int_0^T \int_X p(x) [u^\varepsilon_{2x}(t,x)]^2 \, dx$$
$$\leq \varepsilon\, C_8 + \frac{1}{\delta}\, C_7 \leq C_9 (\varepsilon + \frac{1}{\delta}) \leq C_9 \Big(\varepsilon_0 + \frac{1}{\delta} \Big),$$

which proves formula (2.7). Moreover, we have

$$\mathbb{E} \left(|u^\varepsilon(t,x)|^2_{L^2(0,T;H^1(-k,k))} \right)$$
$$= \mathbb{E} \int_0^T \int_{-k}^k [u^\varepsilon(t,x)]^2 \; dx\, dt + \mathbb{E} \int_0^T \int_{-k}^k [u^\varepsilon_x(t,x)]^2 \; dx\, dt \leq C_{10} + \mathbb{E} \int_0^T \int_{-k}^k [u^\varepsilon_x(t,x)]^2 \, dx$$
$$\leq C_{10} + \mathbb{E} \int_0^T \int_X [u^\varepsilon_x(t,x)]^2 \, dx \leq C_{10} + \mathbb{E} \int_0^T \int_X \frac{1}{p'(x)}\, p'(x) [u^\varepsilon_x(t,x)]^2 \, dx.$$

Since $p'(x)$ is bounded from below on every compact set X by a positive number $\alpha_X > 0$, we have

$$\mathbb{E} \left(|u^\varepsilon(t,x)|^2_{L^2(0,T;H^1(-k,k))} \right) \leq C_{10} + \frac{1}{\alpha_X}\, \mathbb{E} \int_0^T \int_X p'(x) [u^\varepsilon_x(t,x)]^2 \, dx \leq C_{10} + \frac{1}{\alpha_X}\, C_7 \leq C_{11}.$$

This proves inequality (2.8). □

Proof of Lemma 2.3. Let $k \in X_k$ be arbitrary and fixed and let $0 < \varepsilon < \varepsilon_0 \leq 1$. Then

$$u^\varepsilon(t,x) = u^\varepsilon_0(x) - \int_0^t \Big[\varepsilon u^\varepsilon_{4x}(t,x) + A u^\varepsilon(t,x) u^\varepsilon_x(t,x) + B u^\varepsilon_{3x}(t,x) - C u^\varepsilon_{2x}(t,x) + D u^\varepsilon(t,x) \Big] \, ds$$
$$+ \int_0^t \big(\Phi(u^\varepsilon(s,x)) \big) \, dW(s). \tag{3.3}$$

Denote:

$$J_1 := u^\varepsilon_0(x); \quad J_2 := -\varepsilon \int_0^t u^\varepsilon_{4x}(t,x)\, ds; \quad J_3 := -A \int_0^t u^\varepsilon(s,x) u^\varepsilon_x(s,x)\, ds; \quad J_4 := -B \int_0^t u^\varepsilon_{3x}(t,x)\, ds;$$
$$J_5 := C \int_0^t u^\varepsilon_{2x}(t,x)\, ds; \quad J_6 := -D \int_0^t u^\varepsilon(t,x)\, ds; \quad J_7 := \int_0^t \big(\Phi(u^\varepsilon(s,x)) \big) \, dW(s).$$

Now, we start estimating the above terms.

From the assumption,
$$\mathbb{E}\, |J_1|^2_{W^{1,2}(0,T;H^{-2}(-k,k))} = C_1,$$
where $C_1 > 0$.

Next, there exists a constant $C_2 > 0$ such that

$$| - \varepsilon u^\varepsilon_{4x}(t,x)|_{H^{-2}(-k,k)} = \varepsilon |u^\varepsilon_{4x}(t,x)|_{H^{-2}(-k,k)} \leq C_2 \varepsilon |u^\varepsilon(s,x)|_{H^2(-k,k)}.$$

So, due to Lemma 2.2, there exists a constant $C_3(k) > 0$ such that

$$\mathbb{E}\,|-\varepsilon u^{\varepsilon}_{4x}(t,x)|^2_{L^2(0,T;H^{-2}(-k,k))} = \mathbb{E}\int_0^T |-\varepsilon u^{\varepsilon}_{4x}(t,x)|^2_{H^{-2}(-k,k)}\,ds \le C_2^2\varepsilon^2\,\mathbb{E}\int_0^T |u^{\varepsilon}(s,x)|^2_{H^2(-k,k)}\,ds \le C_3(k).$$

Therefore we can write

$$\mathbb{E}\,|J_2|^2_{W^{1,2}(0,T,H^{-2}(-k,k))} \le C_4(k),$$

where $C_4(k) > 0$.

Now, we use the following result from [4].

Lemma 3.2 ([4, p. 243]). *There exists a constant $C_5(k)$ such that the following inequality holds:*

$$|u^{\varepsilon}(s,x)u^{\varepsilon}_x(s,x)|_{H^{-1}(-k,k)} \le C_5(k)|u^{\varepsilon}(s,x)|^{\frac{3}{2}}_{L^2(-k,k)}|u^{\varepsilon}(s,x)|^{\frac{1}{2}}_{H^1(-k,k)}.$$

Due to Lemma 3.2, there exist the positive constants C_6, $C_7(k)$, $C_8(k)$ such that

$$\begin{aligned}\big|-Au^{\varepsilon}(s,x)u^{\varepsilon}_x(s,x)\big|_{H^{-2}(-k,k)} &= A|u^{\varepsilon}(s,x)u^{\varepsilon}_x(s,x)|_{H^{-2}(-k,k)} \le C_6A|u^{\varepsilon}(s,x)u^{\varepsilon}_x(s,x)|_{H^{-1}(-k,k)}\\ &\le AC_7(k)|u^{\varepsilon}(s,x)|^{\frac{3}{2}}_{L^2(-k,k)}|u^{\varepsilon}(s,x)|^{\frac{1}{2}}_{H^1(-k,k)}\\ &\le AC_7(k)|u^{\varepsilon}(s,x)|_{L^2(-k,k)}|u^{\varepsilon}(s,x)|^{\frac{1}{2}}_{L^2(-k,k)}|u^{\varepsilon}(s,x)|^{\frac{1}{2}}_{H^1(-k,k)}\\ &\le AC_7(k)\big[(2k\lambda_X^2)^{\frac{1}{2}}\big]\,|u^{\varepsilon}(s,x)|^{\frac{1}{2}}_{H^1(-k,k)} \le AC_8(k)\lambda_X|u^{\varepsilon}(s,x)|_{H^1(-k,k)}.\end{aligned}$$

Due to Lemma 2.2, there exists a constant $C_9(k) > 0$ such that we can write

$$\begin{aligned}\mathbb{E}\,\big|-Au^{\varepsilon}(s,x)u^{\varepsilon}_x(s,x)\big|^2_{L^2(0,T;H^{-2}(-k,k))} &= \mathbb{E}\int_0^T |-Au^{\varepsilon}(s,x)u^{\varepsilon}_x(s,x)|^2_{H^{-2}(-k,k)}\,ds \le A^2C_8^2(k)\lambda_X^2\,\mathbb{E}\int_0^T |u^{\varepsilon}(s,x)|^2_{H^1(-k,k)}\,ds\\ &= A^2C_8^2(k)\lambda_X^2\,\mathbb{E}|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^1(-k,k))} \le A^2C_9(k)\lambda_X^2.\end{aligned}$$

Therefore, we obtain

$$\mathbb{E}\,|J_3|^2_{W^{1,2}(0,T,H^{-2}(-k,k))} \le C_{10}(k),$$

where $C_{10}(k) > 0$.

Next, there exist the constants $C_{11}, C_{12} > 0$ such that

$$\begin{aligned}|-Bu^{\varepsilon}_{3x}(t,x)|_{H^{-2}(-k,k)} &= B|u^{\varepsilon}_{3x}(t,x)|_{H^{-2}(-k,k)}\\ &\le BC_{11}|u^{\varepsilon}(s,x)|_{H^1(-k,k)} \le BC_{12}|u^{\varepsilon}(s,x)|_{H^2(-k,k)}.\end{aligned}$$

Lemma 2.2 implies the existence of a constant $C_{13}(k) > 0$ such that we can write the following estimates:

$$\begin{aligned}\mathbb{E}\,|-Bu^{\varepsilon}_{3x}(t,x)|^2_{L^2(0,T;H^{-2}(-k,k))} &= \mathbb{E}\int_0^T |-Bu^{\varepsilon}_{3x}(t,x)|^2_{H^{-2}(-k,k)}\,ds \le B^2C_{12}^2\,\mathbb{E}\int_0^T |u^{\varepsilon}(s,x)|^2_{H^2(-k,k)}\,ds\\ &= B^2C_{12}^2\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^2(-k,k))} \le B^2C_{12}^2\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^2(\mathbb{R}))} \le B^2C_{13}.\end{aligned}$$

So, we obtain

$$\mathbb{E}\,|J_4|^2_{W^{1,2}(0,T,H^{-2}(-k,k))} \le C_{14},$$

where $C_{14} > 0$.

For some constant $C_{15} > 0$, we have

$$|Cu^{\varepsilon}_{2x}(t,x)|_{H^{-2}(-k,k)} = C|u^{\varepsilon}_{2x}(t,x)|_{H^{-2}(-k,k)} \leq C\, C_{15}|u^{\varepsilon}(s,x)|_{L^2(-k,k)} \leq C\, C_{16}|u^{\varepsilon}(s,x)|_{H^2(-k,k)}.$$

Lemma 2.2 implies the existence of a constant $C_{17}(k) > 0$ such that

$$\begin{aligned} &\mathbb{E}\,|Cu^{\varepsilon}_{2x}(t,x)|^2_{L^2(0,T;H^{-2}(-k,k))} \\ &\qquad = \mathbb{E}\int\limits_0^T |Cu^{\varepsilon}_{2x}(t,x)|^2_{H^{-2}(-k,k)}\,ds \leq C^2C_{16}^2\,\mathbb{E}\int\limits_0^T |u^{\varepsilon}(s,x)|^2_{H^2(-k,k)}\,ds \\ &\qquad = C^2C_{16}^2\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^2(-k,k))} \leq C^2C_{16}^2\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^2(\mathbb{R}))} \leq C^2C_{17}. \end{aligned}$$

Hence we receive

$$\mathbb{E}\,|J_5|^2_{W^{1,2}(0,T,H^{-2}(-k,k))} \leq C_{18},$$

where $C_{18} > 0$.

There exists a constant $C_{19} > 0$ such that

$$|-Du^{\varepsilon}(t,x)|_{H^{-2}(-k,k)} = D|u^{\varepsilon}(t,x)|_{H^{-2}(-k,k)} \leq DC_{19}|u^{\varepsilon}(s,x)|_{H^2(-k,k)}.$$

Due to Lemma 2.2, for some constant $C_{20}(k) > 0$, we obtain

$$\begin{aligned} \mathbb{E}\,|-Du^{\varepsilon}(t,x)|^2_{L^2(0,T;H^{-2}(-k,k))} &= \mathbb{E}\int\limits_0^T |-Du^{\varepsilon}(t,x)|^2_{H^{-2}(-k,k)}\,ds \leq D^2C_{19}^2\,\mathbb{E}\int\limits_0^T |u^{\varepsilon}(s,x)|^2_{H^2(-k,k)}\,ds \\ &= D^2C_{19}^2\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^2(-k,k))} \leq D^2C_{19}^2\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T;H^2(\mathbb{R}))} \leq D^2C_{20}. \end{aligned}$$

This implies that

$$\mathbb{E}\,|J_6|^2_{W^{1,2}(0,T,H^{-2}(-k,k))} \leq C_{21},$$

where $C_{21} > 0$.

In Lemma 2.1 from [7], insert $f(s) := \Phi(u(s,x))$, $K = H = L^2(X)$. Then

$$\mathscr{I}(f)(t) = \int\limits_0^t \Phi(u(s,x))\,dW(s)$$

and for all $p \geq 1$ and $\alpha < \frac{1}{2}$, there exists a constant $C_{22}(p,\alpha) > 0$ such that

$$\mathbb{E}\Bigg|\int\limits_0^t \Phi(u^m(s,x))\,dW(s)\Bigg|^{2p}_{W^{\alpha(p),2p}(0,T;L^2(X))} \leq C_{22}(2p,\alpha)\,\mathbb{E}\left(\int\limits_0^T |\Phi(u^m(s,x))|^{2p}_{L_2^0(L^2(X))}\,ds\right).$$

Therefore, due to condition (2.3), we can write

$$\mathbb{E}\Bigg|\int\limits_0^t \Phi(u^m(s,x))\,dW(s)\Bigg|^{2p}_{W^{\alpha,2p}(0,T;L^2(X))} \leq C_{23}(p,\alpha), \quad \text{where} \;\; C_{23} > 0.$$

Substitution of $p := 1$ in the above inequality yields

$$\mathbb{E}\,|J_7|^2_{W^{\alpha,2}(0,T;L^2(X))} = \mathbb{E}\Bigg|\int\limits_0^t \Phi(u(s,x))\,dW(s)\Bigg|^2_{W^{\alpha,2}(0,T;L^2(X))} \leq C_{23}(2,\alpha) = C_{24}(\alpha). \tag{3.4}$$

Let $\beta \in (0,\frac{1}{2})$ and $\alpha \in (\beta+\frac{1}{2},\infty)$ be arbitrarily fixed. Note that the following relations hold:

$$W^{\alpha,2}(0,T;L^2(\mathbb{R})) \subset W^{\alpha,2}(0,T;H^{-2}([-k,k)), \quad W^{1,2}(0,T,H^{-2}(-k,k)) \subset W^{\alpha,2}(0,T,H^{-2}(-k,k)).$$

Therefore, there exists a constant $C_{25}(\alpha) > 0$ such that

$$\mathbb{E}\,|u^m(s,x)|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} = \mathbb{E}\Big|\sum_{i=1}^{7} J_i\Big|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} \le \mathbb{E}\Big(\sum_{i=1}^{7}|J_i|_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}\Big)^2$$
$$= \mathbb{E}\Big[\sum_{i=1}^{7}|J_i|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} + 2\sum_{i=1}^{6}\sum_{j=i+1}^{7}|J_i|_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}|J_j|_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}\Big]$$
$$\le \mathbb{E}\Big[\sum_{i=1}^{7}|J_i|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} + 2\sum_{i=1}^{6}\sum_{j=i+1}^{7}\big(|J_i|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} + |J_j|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}\big)\Big]$$
$$= \mathbb{E}\Big[8\sum_{i=1}^{7}|J_i|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}\Big] = 8\sum_{i=1}^{7}\Big[\mathbb{E}\,|J_i|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}\Big] \le C_{25}(\alpha).$$

Moreover, one has

$$W^{\alpha,2}(0,T,H^{-2}(-k,k)) \subset C^{\beta}(0,T;H^{-3}_{loc}(-k,k), \quad W^{\alpha,2}(0,T,H^{-2}(\mathbb{R})) \subset W^{\alpha,2}(0,T,H^{-2}(-k,k)).$$

So, there exist the constants $C_{27}(k), C_{28}(k,\alpha) > 0$ such that

$$\begin{gathered}\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{C^{\beta}(0,T;H^{-3}(-k,k)} \le C_{26}\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{W^{\alpha,2}(0,T,H^{-3}(-k,k))} \le C_{27}(k,\alpha),\\ \mathbb{E}\,|u^{\varepsilon}(s,x)|_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} \le C_{28}(k,\alpha).\end{gathered} \tag{3.5}$$

Let $\eta > 0$ be arbitrarily fixed. Lemma 2.2 implies the existence of a constant $C_{30}(k) > 0$ such that

$$\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T,H^{-1}(-k,k))} \le C_{29}(k)\,\mathbb{E}\,|u^{\varepsilon}(s,x)|^2_{L^2(0,T,H^{-1}(\mathbb{R}))}\widetilde{C}_2 = C_{30}(k). \tag{3.6}$$

Substituting in Lemma 2.1 of [4] $\alpha_k := \eta^{-1}2^k(C_{30}(k) + C_{27}(k,\alpha) + C_{28}(k,\alpha))$ and using Markov's inequality [12, p. 114] for

$$X := |u^{\varepsilon}(s,x)|^2_{L^2(0,T,H^{-1}(-k,k))} + |u^{\varepsilon}(s,x)|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))} + |u^{\varepsilon}(s,x)|^2_{C^{\beta}(0,T;H^{-3}_{loc}(-k,k)},$$
$$\varepsilon := \eta^{-1}2^k(C_{30}(k) + C_{27}(k,\alpha) + C_{28}(k,\alpha)),$$

one obtains

$$\mathbb{P}\big(u^{\varepsilon} \in A(\{\alpha_k\})\big) = 1 - \mathbb{P}\Big(|u^{\varepsilon}(s,x)|^2_{L^2(0,T,H^{-1}(-k,k))} + |u^{\varepsilon}(s,x)|^2_{W^{\alpha,2}(0,T,H^{-2}(-k,k))}$$
$$+ |u^{\varepsilon}(s,x)|^2_{C^{\beta}(0,T;H^{-3}_{loc}(-k,k))} \ge \eta^{-1}2^k(C_{30}(k) + C_{27}(k,\alpha) + C_{28}(k,\alpha))\Big)$$
$$= 1 - \frac{C_{30}(k) + C_{27}(k,\alpha) + C_{28}(k,\alpha)}{\eta^{-1}2^k(C_{30}(k) + C_{27}(k,\alpha) + C_{28}(k,\alpha))} = 1 - \frac{\eta}{2^k} > 1 - \eta.$$

Let K be a mapping such that for $\eta > 0$, $K(\eta) := A(\{a_k^{(\eta)}\})$, where $\{a_k^{(\eta)}\}$ is an increasing sequence of positive numbers that may, but does not have to, depend on η. Note that due to [4, Lemma 2.1], the set $K(\eta)$ is compact for all $\eta > 0$. Moreover, $\mathbb{P}\{K(\eta)\} > 1 - \eta$, then the family $\mathscr{L}(u^{\varepsilon})$ is tight. □

4 Proof of Lemma 2.1

Proof. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in the space $L^2(X)$. Denote by P_m, for all $m \in \mathbb{N}$, the orthogonal projection on $\mathrm{Sp}(e_1, \dots, e_m)$. Consider a finite-dimensional Galerkin approximation of problem (2.6) in space $P_mL^2(X)$ in the form

$$\begin{cases} du^{m,\varepsilon}(t,x) + \Big(\varepsilon\theta\Big(\frac{|u^{m,\varepsilon}_{4x}(t,x)|^2}{m}\Big)u^{m,\varepsilon}_{4x}(t,x) + A\theta\Big(\frac{|u^{m,\varepsilon}_{x}(t,x)|^2}{m}\Big)u^{m,\varepsilon}(t,x)u^{m,\varepsilon}_x(t,x) \\ \qquad + B\theta\Big(\frac{|u^{m,\varepsilon}_{3x}(t,x)|^2}{m}\Big)u^{m,\varepsilon}_{3x}(t,x) - C\theta\Big(\frac{|u^{m,\varepsilon}_{2x}(t,x)|^2}{m}\Big)u^{m,\varepsilon}_{2x}(t,x) + Du^{m,\varepsilon}(t,x)\Big)\,dt \\ \qquad\qquad = P_m\Phi(u^{m,\varepsilon}(t,x))\,dW^m(t), \\ u_0^{m,\varepsilon}(x) = P_m u^{\varepsilon}(0,x), \end{cases} \tag{4.1}$$

where $\theta \in C^\infty(\mathbb{R})$ satisfies the conditions

$$\begin{cases} \theta(\xi) = 1, & \text{when } \ \xi \in [0,1], \\ \theta(\xi) \in [0,1], & \text{when } \ \xi \in (1,2), \\ \theta(\xi) = 0, & \text{when } \ \xi \in [2,\infty). \end{cases} \tag{4.2}$$

Let $m \in \mathbb{N}$ be arbitrarily fixed and

$$\begin{aligned} b(u(t,x)) &:= \varepsilon\theta\Big(\frac{|u_{4x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u_{4x}^{m,\varepsilon}(t,x) + A\theta\Big(\frac{|u_{x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u^{m,\varepsilon}(t,x)u_{x}^{m,\varepsilon}(t,x) \\ &\qquad + B\theta\Big(\frac{|u_{3x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u_{3x}^{m,\varepsilon}(t,x) - C\theta\Big(\frac{|u_{2x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u_{2x}^{m,\varepsilon}(t,x) + Du^{m,\varepsilon}(t,x), \\ \sigma(u(t,x)) &:= P_m\Phi(u^{m,\varepsilon}(t,x)). \end{aligned}$$

Then

$$\begin{aligned} |b(u(t,x))|_{L^2(X)} &\le \varepsilon\left|\theta\Big(\frac{|u_{4x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u^{m,\varepsilon}(t,x)u_{4x}^{m,\varepsilon}(t,x)\right|_{L^2(X)} \\ &+ A\left|\theta\Big(\frac{|u_{x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u^{m,\varepsilon}(t,x)u_{x}^{m,\varepsilon}(t,x)\right|_{L^2(X)} + B\left|\theta\Big(\frac{|u_{3x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u_{3x}^{m,\varepsilon}(t,x)\right|_{L^2(X)} \\ &+ C\left|\theta\Big(\frac{|u_{2x}^{m,\varepsilon}(t,x)|^2}{m}\Big)u_{2x}^{m,\varepsilon}(t,x)\right|_{L^2(X)} + D|u^{m,\varepsilon}(t,x)|_{L^2(X)} =: \varepsilon J_1 + AJ_2 + BJ_3 + CJ_4 + DJ_5. \end{aligned}$$

Note that

$$J_2 = \begin{cases} 0, & \text{when } \ |u_x^{m,\varepsilon}(t,x)| \ge \sqrt{2m}\,, \\ \lambda|u^{m,\varepsilon}(t,x)u_x^{m,\varepsilon}(t,x)|_{L^2(X)}, & \text{when } \ |u_x^{m,\varepsilon}(t,x)| \le \sqrt{2m}, \end{cases}$$

where $\lambda \in [0,1]$. So,

$$J_2 \le |u^{m,\varepsilon}(t,x)u_x^{m,\varepsilon}(t,x)|_{L^2(X)} \le \sqrt{2m}|u^{m,\varepsilon}(t,x)|_{L^2(X)}.$$

Similarly,

$$J_1 = \begin{cases} 0, & \text{when } \ |u_{4x}^{m,\varepsilon}(t,x)| \ge \sqrt{2m}, \\ \lambda|u_{4x}^{m,\varepsilon}(t,x)|_{L^2(X)}, & \text{when } \ |u_{4x}^{m,\varepsilon}(t,x)| \le \sqrt{2m}\,, \end{cases}$$

$$J_3 = \begin{cases} 0, & \text{when } \ |u_{3x}^{m,\varepsilon}(t,x)| \ge \sqrt{2m}, \\ \lambda|u_{3x}^{m,\varepsilon}(t,x)|_{L^2(X)}, & \text{when } \ |u_{3x}^{m,\varepsilon}(t,x)| \le \sqrt{2m}\,, \end{cases}$$

and

$$J_4 = \begin{cases} 0, & \text{when } \ |u_{2x}^{m,\varepsilon}(t,x)| \ge \sqrt{2m}\,, \\ \lambda|u_{2x}^{m,\varepsilon}(t,x)|_{L^2(X)}, & \text{when } \ |u_{2x}^{m,\varepsilon}(t,x)| \le \sqrt{2m}\,, \end{cases}$$

where $\lambda \in [0,1]$. Thus

$$J_1, J_3, J_4 \le \sqrt{2m}\,.$$

Therefore, one gets

$$\begin{aligned} \left|b(u^{m,\varepsilon}(t,x))\right|_{L^2(X)} &= \varepsilon J_1 + AJ_2 + BJ_3 + CJ_4 + DJ_5 \\ &\le \varepsilon\sqrt{2m} + A\sqrt{2m}\,|u^{m,\varepsilon}(t,x)|_{L^2(X)} + B\sqrt{2m} + C\sqrt{2m} + D|u^{m,\varepsilon}(t,x)|_{L^2(X)} \\ &= (A\sqrt{2m} + D)|u^{m,\varepsilon}(t,x)|_{L^2(X)} + \sqrt{2m}\,(\varepsilon + B + C). \end{aligned}$$

Moreover, due to condition (2.3), there exist the constants $\kappa_1, \kappa_2 > 0$ such that

$$\|\Phi(u^m(t,x))\|_{L_0^2(L^2(X))} \leq \kappa_1 |u^m(t,x)|_{L^2(X)} + \kappa_2,$$

so,

$$\begin{aligned}
&\big|b(u^{m,\varepsilon}(t,x))\big|_{L^2(X)} + \|\sigma(u^m(t,x))\|_{L_0^2(L^2(X))} \\
&\qquad \leq (A\sqrt{2m} + D)|u^{m,\varepsilon}(t,x)|_{L^2(X)} + \sqrt{2m}\,(\varepsilon + B + C) + \kappa_1 |u^m(t,x)|_{L^2(X)} + \kappa_2 \\
&\qquad = (A\sqrt{2m} + D + \kappa_1)|u^{m,\varepsilon}(t,x)|_{L^2(X)} + \sqrt{2m}\,(\varepsilon + B + C) + \kappa_2.
\end{aligned}$$

Let $\kappa := \max\{\kappa_1, \kappa_2\}$ and $\Lambda = \max\{A, \varepsilon + B + C\}$. Then

$$\begin{aligned}
&\big|b(u^{m,\varepsilon}(t,x))\big|_{L^2(X)} + \|\sigma(u^m(t,x))\|_{L_0^2(L^2(X))} \\
&\qquad \leq \big(\Lambda\sqrt{2m} + \kappa + D\big)|u^{m,\varepsilon}(t,x)|_{L^2(X)} + \Lambda\sqrt{2m} + \kappa + D \\
&\qquad = \big(\Lambda\sqrt{2m} + \kappa + D\big)\big(|u^{m,\varepsilon}(t,x)|_{L^2(X)} + 1\big).
\end{aligned}$$

Therefore, by [9, Proposition 3.6], and [9, Proposition 4.6], for all $m \in \mathbb{N}$, there exists a martingale solution to (4.1). Additionally, applying the same methods as in Section 3, one can show that for all m, there exists a constant $\widetilde{C}_1(\varepsilon) > 0$ such that

$$\mathbb{E}\left(|u^{m,\varepsilon}(t,x)|^2_{L^2(0,T;H^2(X))}\right) \leq \widetilde{C}_1(\varepsilon).$$

Moreover, for all m and all $k \in X_k$, there exists a constant $\widetilde{C}_2(k,\varepsilon) > 0$ such that

$$\mathbb{E}v\big(|u^{m,\varepsilon}(t,x)|^2_{L^2(0,T;H^1(-k,k))}\big) \leq \widetilde{C}_2(k,\varepsilon),$$

and the family of distributions $\mathscr{L}(u^{m,\varepsilon})$ is tight in $L^2(0,T;L^2(X)) \cap C(0,T;H^{-3}(X))$. Then application of the same methods as in Section 2, leads to the proof of the existence of a martingale solution to (2.6) with conditions (2.3), (2.4) and (2.5). $\square$

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