Memoirs on Differential Equations and Mathematical Physics

Volume 83, 2021, 1–11

Mohamed I. Abbas

NONLINEAR ATANGANA–BALEANU FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE MITTAG–LEFFLER INTEGRAL OPERATOR

Abstract. This paper intends to investigate the existence and uniqueness of solutions for some nonlinear Atangana–Baleanu fractional differential equations involving the Mittag–Leffler integral operator. By means of Schauder's fixed point theorem and Banach's fixed point theorem, the existence and uniqueness results are obtained. A generalized fractional order free electron laser equation is given as an application.

2010 Mathematics Subject Classification. 34A08, 33E12, 47G10.

Key words and phrases. Atangana–Baleanu fractional derivative, Schauder's fixed point theorem, Mittag–Leffler operator.

რეზიუმე. ნაშრომი ეძღვნება ამოხსნების არსებობისა და ერთადერთობის გამოკვლევას ზოგიერთი არაწრფივი ატანგანა-ბალეანუს წილადურ დიფერენციალური განტოლებებისთვის, რომლებიც შეიცავს მიტაგ-ლეფლერის ინტეგრალურ ოპერატორს. შაუდერის უძრავი წერტილის თეორემისა და ბანახის უძრავი წერტილის თეორემის საშუალებით მიღებულია არსებობისა და ერთადერთობის შედეგები. გამოყენების სახით მოცემულია თავისუფალ ელექტრონებზე მომუშავე ლაზერის წილადური რიგის განზოგადებული განტოლება.

1 Introduction

In the last decades, several significant results related to the qualitative properties of fractional differential equations have been recorded because of their ability to model real-world problems in many fields such as science, technology and engineering [11,12,19,21–23,26,29].

Recently, the interest of many researchers interested in fractional calculus has gone to a new type of fractional derivative with non-singular kernel introduced by Caputo and Fabrizio [10], this derivative is based on the exponential kernel. Later, Atangana and Baleanu [7] developed another version which used the generalized Mittag–Leffler function as non-local and non-singular kernel which appears naturally in several physical problems and the field of science and engineering [3–6,8,14,25,30,31].

On the other hand, the Mittag–Leffler function and its generalizations play a fundamental role in fractional calculus and its applications such as modelling groundwater fractal flow, viscoelasticity and probability theory [1,13].

In [24], Prabhakar studied a singular integral equation with a general Mittag–Leffler function in the kernel, namely,

$$\int_a^t (t-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t-s)^{\sigma})\phi(s)\,ds = g(t), \quad t\in[a,b],$$

where

$$\mathbb{E}^{\lambda}_{\sigma,\delta}(z) = \sum_{k=0}^{\infty}\frac{(\lambda)_k}{\Gamma(\sigma k+\delta)}\frac{z^k}{k!} \quad (\sigma,\delta,\lambda\in\mathbb{C},\ \ \mathrm{Re}(\sigma)>0).$$

The function $\mathbb{E}^{\lambda}_{\sigma,\delta}(z)$ is the three-parameter Mittag–Leffler function and $(\lambda)_k$ is the Pochhammer symbol defined as

$$(\lambda)_k = \begin{cases}(\lambda)(\lambda+1)\cdots(\lambda+k-1), & k\in\mathbb{N},\\ 1, & k=0,\ \ \lambda\neq 0.\end{cases}$$

When $\lambda=1$, $\mathbb{E}^{1}_{\sigma,\delta}(z)$ coincides with the classical two-parameter Mittag–Leffler function

$$\mathbb{E}_{\sigma,\delta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\sigma k+\delta)}\,.$$

It is useful to mention that the three-parameter Mittag–Leffler function is closely connected with the phenomenon of Havriliak–Negami relaxation [15].

In [17], Kilbas et al. investigated an integro-differential equation of the form

$$D^{\alpha}_{a^+}y(t) = \gamma\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;a^+}y(t) + f(t), \quad a<t\le b, \tag{1.1}$$

where $\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;a^+}$ is the Mittag–Leffler integral operator defined by

$$\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;a^+}y(t) = \int_a^t (t-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t-s)^{\sigma})y(s)\,ds, \tag{1.2}$$

where $\sigma,\delta,\nu,\lambda\in\mathbb{C}$, $\mathrm{Re}(\sigma)>0$, $\mathrm{Re}(\delta)>0$.

Obviously, $\mathbb{E}^{0}_{\sigma,\delta,\nu;a^+}$ is the Riemann–Liouville fractional integral operator of order δ. Therefore, operator (1.2) and its inverse can be considered as generalization of fractional integral and derivative operators involving $\mathbb{E}^{\lambda}_{\sigma,\delta}(z)$ in their kernels.

In this paper, we consider the following nonlinear Atangana–Baleanu fractional differential equation involving the Mittag–Leffler integral operator

$$\begin{cases}{}^{ABC}D^{\alpha}_{0^+}x(t) = \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x(t)), \quad \alpha\in(0,1], \quad t\in[0,1],\\ \qquad x(0)=x_0\in\mathbb{R},\end{cases} \tag{1.3}$$

where ${}^{ABC}D^{\alpha}_{0^+}$ denotes the Atangana–Baleanu fractional derivative of order α in Caputo sense, $\sigma, \delta, \nu, \lambda \in \mathbb{R}$, $\sigma, \delta > 0$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

The importance of studying such equations like (1.1) and (1.3) is that they describe the unsaturated behavior of the free electron laser [9, 27, 28], which is a kind of laser whose lasing medium consists of very-high-speed electrons moving freely through a magnetic structure.

2 Preliminaries

In [7], Atangana and Baleanu improved the Caputo–Fabrizio fractional derivative with non-singular kernel to another one with non-local and non-singular kernel. We present the basic definitions of the new fractional order derivatives.

Definition 2.1 (see [7])**.** Let $h \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then the Atangana–Baleanu fractional derivative in Caputo sense is given by

$$ {}^{ABC}D^{\alpha}_{a^+}h(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t \mathbb{E}_{\alpha}\Big[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\Big]h'(s)\,ds, \tag{2.1} $$

where $B(\alpha)$ denotes a normalization function such that $B(0) = B(1) = 1$ and $\mathbb{E}_{\alpha}$ denotes the Mittag–Leffler function defined by

$$ \mathbb{E}_{\alpha}(-t^{\alpha}) = \sum_{k=0}^{\infty}\frac{(-t)^{\alpha k}}{\Gamma(\alpha k+1)}\,. $$

However, when $\alpha = 0$, they did not recover the original function, except when at the origin the function vanishes. To avoid this issue, they proposed the following definition.

Definition 2.2 (see [7])**.** Let $h \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, and it is not necessary differentiable, then the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given by

$$ {}^{ABR}D^{\alpha}_{a^+}h(t) = \frac{B(\alpha)}{1-\alpha}\,\frac{d}{dt}\int_a^t \mathbb{E}_{\alpha}\Big[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\Big]h(s)\,ds. \tag{2.2} $$

Equations (2.1) and (2.2) have a non-local kernel. Also in equation (2.1), when the function is constant, we get zero. For more details and properties, see [7, 10].

Definition 2.3 (see [7])**.** Let $h \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then the Atangana–Baleanu fractional integral, associate to the new fractional derivative with non-local kernel is given by

$$ {}^{AB}I^{\alpha}_{a^+}h(t) = \frac{1-\alpha}{B(\alpha)}\,h(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,ds, $$

where $\Gamma(\cdot)$ denotes the well-known gamma function. The initial function is recovered when the fractional order turns to zero. Also, when the order turns to 1, we have the classical integral.

To end this section, we collect some useful lemmas.

Lemma 2.4 (see [2])**.**

$$ I^{\alpha}_{0^+}\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi) = \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(\phi), \quad \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}\mathbb{E}^{\eta}_{\sigma,\mu,\nu;0^+}(\phi) = \mathbb{E}^{\lambda+\eta}_{\sigma,\delta+\mu,\nu;0^+}(\phi), $$
$$ \|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi)\|_C \le \mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|)\|(\phi)\|_C\,. $$

Lemma 2.5 (see [2])**.** *Suppose* $z \ge 0$ *is fixed,* $\sigma, \delta, \lambda > 0$.

(i) *If* $0 \le \lambda \le 1$, *then* $\mathbb{E}^{\lambda}_{\sigma,\delta}(z) \le \mathbb{E}_{\sigma,\delta}(z)$.

(ii) *If* $\lambda \geq 1$, *then* $\mathbb{E}^{\lambda}_{\sigma,\delta}(z) \geq \mathbb{E}_{\sigma,\delta}(z)$.

Lemma 2.6 (see [18])**.** *Assume that* $\sigma, \delta, \nu, \lambda \in \mathbb{R}$, $(\sigma, \delta > 0)$, *then for a continuous function* $\phi \in C([0,1])$ *and positive integer* n, *where* $\delta > n$,

$$\frac{d^n}{dt^n}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi) = \mathbb{E}^{\lambda}_{\sigma,\delta-n,\nu;0^+}(\phi).$$

Lemma 2.7 (see [20])**.** *Suppose* $\sigma, \delta, \nu, \lambda \in \mathbb{R}$, $(\sigma, \delta > 0,\ \delta > \alpha \geq 0)$, *then for a continuous function* $\phi \in C([0,1])$,

$$D^{\alpha}_{0^+}\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi) = \mathbb{E}^{\lambda}_{\sigma,\delta-\alpha,\nu;0^+}(\phi).$$

Lemma 2.8 (Ascoli–Arzelà theorem)**.** *Let* $S = \{s(t)\}$ *be a function family of continuous mappings on a closed and bounded interval* $[a,b]$, $s : [a,b] \to \mathbb{X}$.

If S *is uniformly bounded and equicontinuous, and for any* $t^* \in [a,b]$, *the set* $\{s(t^*)\}$ *is relatively compact, then there exists a uniformly convergent function sequence* $\{s_n(t)\}$ $(n = 1, 2, \ldots,\ t \in [a,b])$ *in* S.

Lemma 2.9 (Schauder's fixed point theorem)**.** *If* U *is a closed, bounded and convex subset of a Banach space* $\mathbb{X}$ *and* $\mathcal{T} : U \to U$ *is completely continuous, then* $\mathcal{T}$ *has a fixed point in* U.

3 The Existence and Uniqueness Results

Let $C([0,1])$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|_C = \max\{|x(t)| : t \in [0,1]\}$.

Definition 3.1 ([16, Theorem 3.1])**.** A function $x \in C([0,1])$ is said to be a solution of equation (1.3) with $x(0) = x_0$ if $x(t)$ satisfies the integral equation

$$x(t) = x_0 +^{AB} I^{\alpha}_{0^+}\big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t))\big). \tag{3.1}$$

In view of Definition 2.3, together with Lemma 2.4, equation (3.1) can be reformulated as follows:

$$\begin{aligned} x(t) &= x_0 +^{AB} I^{\alpha}_{0^+}\big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t))\big) \\ &= x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t)) + \frac{\alpha}{B(\alpha)}\, I^{\alpha}_{0^+}\big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t))\big) \\ &= x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t)) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t, x(t)). \end{aligned} \tag{3.2}$$

We introduce the following assumptions:

$(A1)$ The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(A2)$ There exists a constant $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f|x - y| \ \text{ for each } \ t \in [0,1], \ \text{ and all } \ x, y \in \mathbb{R}.$$

3.1 Existence result via Schauder's fixed point theorem

Theorem 3.2. *Assume that* $(A1)$ *and* $(A2)$ *are satisfied. Then the Atangana–Baleanu fractional differential equation* (1.3) *has at least one solution on* $[0,1]$.

Proof. We define the operator $\mathcal{T} : C([0,1]) \to C([0,1])$ by

$$(\mathcal{T}x)(t) = x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t)) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t, x(t)), \quad t \in [0,1]. \tag{3.3}$$

Note that the operator $\mathcal{T}$ is well-defined on $C([0,1])$ due to $(A1)$.

Consider the set $B_r = \{x \in C([0,1]) : \ \|x\|_C \leq r\}$. Clearly, the set B_r is closed, bounded and convex. The proof is divided into several steps.

Step 1. $\mathcal{T}$ is continuous.

Let x_n be a sequence such that $x_n \to x$ in B_r . Then for each $t \in [0,1]$, we have

$$\begin{aligned}\big|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)\big| &= \Bigg|\frac{1-\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x_n(t)) - \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x(t))\Big)\\ &\quad + \frac{\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,x_n(t)) - \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,x(t))\Big)\Bigg|\\ &\leq \frac{1-\alpha}{B(\alpha)}\Big|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(f(t,x_n(t))-f(t,x(t)))\Big| + \frac{\alpha}{B(\alpha)}\Big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(f(t,x_n(t))-f(t,x(t)))\Big|\\ &\leq \Big(\frac{1-\alpha}{B(\alpha)}\|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(1)\| + \frac{\alpha}{B(\alpha)}\|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(1)\|\Big)\big\|f(\cdot\,,x_n(\cdot)) - f(\cdot\,,x(\cdot))\big\|_C\\ &\leq \Big(\frac{1-\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big)\big\|f(\cdot\,,x_n(\cdot)) - f(\cdot\,,x(\cdot))\big\|_C\,,\end{aligned}$$

which implies that

$$\|\mathcal{T}x_n - \mathcal{T}x\|_C \leq \Big(\frac{1-\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big)\big\|f(\cdot\,,x_n(\cdot)) - f(\cdot\,,x(\cdot))\big\|_C\,.$$

By $(A1)$, the continuity of the function f implies that $\mathcal{T}$ is continuous.

Step 2. $\mathcal{T}$ maps bounded sets into bounded sets in B_r.

Indeed, it is enough to show that for any $r > 0$, there exists a positive constant ℓ such that for each $x \in B_r$, one has $\|\mathcal{T}x\|_C \leq \ell$. For $t \in [0,1]$, $x \in B_r$ and in view of $(A1)$, we define $M_f = \sup_{(t,x)\in[0,1]\times B_r}\|f(t,x)\|$ and, consequently, we have

$$\begin{aligned}|(\mathcal{T}x)(t)| &= \Big|x_0 + \frac{1-\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x(t)) + \frac{\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,x(t))\Big|\\ &\leq |x_0| + \frac{(1-\alpha)M_f}{B(\alpha)}\|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(1)\| + \frac{\alpha M_f}{B(\alpha)}\|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(1)\|\\ &\leq |x_0| + \frac{(1-\alpha)M_f}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha M_f}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|) := \ell.\end{aligned}$$

Hence, $\|\mathcal{T}x\|_C \leq \ell$. This implies that $\mathcal{T}(B_r) \subset B_r$.

Step 3. $\mathcal{T}$ maps bounded sets into equicontinuous sets of B_r.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and for any $x \in B_r$, we have

$$\begin{aligned}\big|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)\big| &\leq \Bigg|\frac{1-\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t_2,x(t_2) - \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t_1,x(t_1))\Big)\Bigg|\\ &\quad + \Bigg|\frac{\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t_2,x(t_2) - \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t_1,x(t_1))\Big)\Bigg|\\ &\leq \frac{1-\alpha}{B(\alpha)}\Bigg|\int_0^{t_2}(t_2-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2-s)^{\sigma})f(s,x(s)\,ds\\ &\qquad - \int_0^{t_1}(t_1-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1-s)^{\sigma})f(s,x(s)\,ds\Bigg|\\ &\quad + \frac{\alpha}{B(\alpha)}\Bigg|\int_0^{t_2}(t_2-s)^{\delta+\alpha-1}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2-s)^{\sigma})f(s,x(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&\qquad - \int_0^{t_1} (t_1 - s)^{\delta+\alpha-1} \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1 - s)^{\sigma}) f(s, x(s)\, ds \Big| \\
&= \frac{1-\alpha}{B(\alpha)}\, I_1 + \frac{\alpha}{B(\alpha)}\, I_2,
\end{aligned}
$$

where

$$
I_1 = \Big| \int_0^{t_2} (t_2 - s)^{\delta-1} \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) f(s, x(s)\, ds - \int_0^{t_1} (t_1 - s)^{\delta-1} \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) f(s, x(s)\, ds \Big|
$$

and

$$
\begin{aligned}
I_2 = \Big| &\int_0^{t_2} (t_2 - s)^{\delta+\alpha-1} \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2 - s)^{\sigma}) f(s, x(s)\, ds \\
&\qquad - \int_0^{t_1} (t_1 - s)^{\delta+\alpha-1} \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1 - s)^{\sigma}) f(s, x(s)\, ds \Big|.
\end{aligned}
$$

For I_1, we have

$$
\begin{aligned}
I_1 \le \Big[&\int_0^{t_2} (t_2 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big| \, \|f(s, x(s))\| \, ds \\
&+ \int_0^{t_1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \|f(s, x(s))\| \, ds \\
&+ \int_{t_1}^{t_2} (t_2 - s)^{\delta-1} \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \|f(s, x(s))\| \, ds \Big] \\
\le M_f \Big[&\int_0^{1} (t_2 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big| \, ds \\
&+ \int_0^{1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \, ds \\
&+ \int_0^{1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \, ds \Big] \\
\le M_f \Big[&\Big(\int_0^{1} \big| (t_2 - s)^{\delta-1} \big|^2 \, ds \Big)^{1/2} \Big(\int_0^{1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big|^2 \, ds \Big)^{1/2} \\
&+ 2 \Big(\int_0^{1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big|^2 \, ds \Big)^{1/2} \Big(\int_0^{1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big|^2 \, ds \Big)^{1/2} \Big].
\end{aligned}
$$

Similarly, I_2 can be estimated as

$$
I_2 \le M_f \Big[\Big(\int_0^{1} \big| (t_2 - s)^{\delta+\alpha-1} \big|^2 \, ds \Big)^{1/2} \Big(\int_0^{1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1 - s)^{\sigma}) \big|^2 \, ds \Big)^{1/2}
$$

$$+ 2\Big(\int\limits_0^1 \big|(t_2-s)^{\delta+\alpha-1}-(t_1-s)^{\delta+\alpha-1}\big|^2\,ds\Big)^{1/2}\Big(\int\limits_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\Big].$$

Hence, we get

$$\begin{aligned}
\big|(\mathcal{T}x)(t_2)-(\mathcal{T}x)(t_1)\big| \le \frac{(1-\alpha)M_f}{B(\alpha)}\Big[\Big(\int\limits_0^1 \big|(t_2-s)^{\delta-1}\big|^2\,ds\Big)^{1/2} \\
\times\Big(\int\limits_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2-s)^{\sigma})-\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2} \\
+ 2\Big(\int\limits_0^1 \big|(t_2-s)^{\delta-1}-(t_1-s)^{\delta-1}\big|^2\,ds\Big)^{1/2}\Big(\int\limits_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\Big] \\
+ \frac{\alpha M_f}{B(\alpha)}\Big[\Big(\int\limits_0^1 \big|(t_2-s)^{\delta+\alpha-1}\big|^2\,ds\Big)^{1/2} \\
\times\Big(\int\limits_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2-s)^{\sigma})-\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2} \\
+ 2\Big(\int\limits_0^1 \big|(t_2-s)^{\delta+\alpha-1}-(t_1-s)^{\delta+\alpha-1}\big|^2\,ds\Big)^{1/2}\Big(\int\limits_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\Big].
\end{aligned}$$

As a result, we immediately find that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Therefore, $\mathcal{T}(B_r)$ is an equicontinuous set. It is also uniformly bounded.

Consequently, from Steps $1-3$ together with the Ascoli–Arzelà theorem (Lemma 2.8), we show that the operator $\mathcal{T}$ is completely continuous. Hence, by Schauder's fixed point theorem (Lemma 2.9), we conclude that the operator $\mathcal{T}$ has at least one fixed point which is a solution of the Atangana–Baleanu fractional differential equation (1.3) on $[0,1]$. The proof is completed. □

3.2 Uniqueness result via the Banach fixed point theorem

Theorem 3.3. *If the assumptions* $(A1)$ *and* $(A2)$ *hold, then the Atangana–Baleanu fractional differential equation* (1.3) *has a unique solution on* $[0,1]$*, provided that*

$$\Lambda := \Big(\frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|)+\frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big)L_f < 1. \tag{3.4}$$

Proof. Consider the operator $\mathcal{T}$ defined in (3.3). In what follows, we show that the operator $\mathcal{T}$ is a contraction. Repeating the same procedure as in Step 2 of the proof of Theorem 3.2, we obtain $\mathcal{T}(B_r) \subset B_r$.

Now, for $x, y \in C([0,1])$ and for each $t \in [0,1]$, by using $(A2)$, we have

$$\begin{aligned}
\big|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)\big| &= \Big|\frac{1-\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x(t))-\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,y(t))\Big) \\
&\quad + \frac{\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,x(t))-\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,y(t))\Big)\Big| \\
&\le \frac{1-\alpha}{B(\alpha)}\big|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(f(t,x(t))-f(t,y(t)))\big| + \frac{\alpha}{B(\alpha)}\big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(f(t,x(t))-f(t,y(t)))\big| \\
&\le \Big(\frac{1-\alpha}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(1)\| + \frac{\alpha}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(1)\|\Big)L_f\|x-y\|_C
\end{aligned}$$

$$\leq \Big(\frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big) L_f \|x-y\|_C\,.$$

Hence,

$$\|\mathcal{T}x - \mathcal{T}y\|_C \leq \Lambda \|x-y\|_C\,.$$

If condition (3.4) is satisfied, then, as a consequence of the Banach fixed point theorem, we conclude that the operator $\mathcal{T}$ has a unique fixed point. Thus, the Atangana–Baleanu fractional differential equation (1.3) has a unique solution. The proof is completed. □

4 An application

In this section, we consider the following generalized fractional order free electron laser equation as an application of the Atangana–Baleanu fractional differential equation (1.3).

Example 4.1.

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}_{0^+} x(t) = \mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2},2;0^+} \dfrac{|x(t)|}{50(1+e^t)(1+|x(t)|)}\,, \quad t \in [0,1], \\ \qquad x(0) = 0. \end{cases} \tag{4.1}$$

Here, t is a dimensionless time ranging from 0 to 1 and $x(t)$ is a complex-field amplitude which is assumed dimensionless and satisfies the initial condition $x(0) = 0$.

Set $\alpha = \frac{1}{2}$, $\sigma = 1$, $\delta = \frac{1}{2}$, $\nu = 2$, $\lambda = \frac{2}{5}$ and $f(t,x) = \frac{x}{50(1+e^t)(1+x)}$. Since

$$\begin{aligned} |f(t,x) - f(t,y)| &= \Big|\frac{x}{50(1+e^t)(1+x)} - \frac{y}{50(1+e^t)(1+y)}\Big| \\ &\leq \frac{|x-y|}{50(1+e^t)(1+x)(1+y)} \leq \frac{1}{50(1+e^t)}|x-y| \leq \frac{1}{100}\,\|x-y\|_C\,, \end{aligned}$$

we get the assumption $(A2)$ with $L_f = \frac{1}{100}$.

Moreover, using Lemma 2.5 and the fact that $\Gamma(k+2) \leq \Gamma(k+\frac{5}{2})$, the condition (3.4) gives

$$\begin{aligned} \Lambda &= \Big(\frac{1-\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big) L_f \\ &= \frac{1}{100}\Big(\frac{1-\frac{1}{2}}{B(\frac{1}{2})}\,\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}+1}(|2|) + \frac{\frac{1}{2}}{B(\frac{1}{2})}\,\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}+\frac{1}{2}+1}(|2|)\Big) = \frac{1}{100}\Big(\frac{1}{2}\,\mathbb{E}^{\frac{2}{5}}_{1,\frac{5}{2}}(|2|) + \frac{1}{2}\,\mathbb{E}^{\frac{2}{5}}_{1,2}(|2|)\Big) \\ &\leq \frac{1}{100}\Big(\frac{1}{2}\,\mathbb{E}_{1,\frac{5}{2}}(|2|) + \frac{1}{2}\,\mathbb{E}_{1,2}(|2|)\Big) = \frac{1}{100}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+\frac{5}{2})} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+2)}\Big) \\ &\leq \frac{1}{100}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+2)} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+2)}\Big) = \frac{1}{100}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{(k+1)!} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{(k+1)!}\Big) \\ &= \frac{1}{100}\Big(\frac{1}{2}\,\frac{e^2-1}{2} + \frac{1}{2}\,\frac{e^2-1}{2}\Big) = \frac{e^2-1}{200} = 0.03194528049 < 1. \end{aligned}$$

Therefore, all the assumptions of Theorem 3.3 are satisfied. Hence, the Atangana–Baleanu fractional differential equation (4.1) has a unique solution on $[0,1]$.

Finally, according to formula (3.2), we can obtain a unique solution $x(t)$, which is the complex-field amplitude of the generalized fractional order free electron laser equation (4.1), from the following Volterra integral equation:

$$x(t) = \frac{1}{100(1+e^t)}\Big[\int_0^t (t-s)^{-\frac{1}{2}}\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}}(2(t-s))\,\frac{x(s)}{1+x(s)}\,ds + \int_0^t \mathbb{E}^{\frac{2}{5}}_{1,1}(2(t-s))\,\frac{x(s)}{1+x(s)}\,ds\Big],$$

where
$$\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}}(2(t-s)) = \sum_{k=0}^{\infty} \frac{2^k(\frac{2}{5})_k}{\Gamma(k+\frac{1}{2})} \frac{(t-s)^k}{k!}$$
and
$$\mathbb{E}^{\frac{2}{5}}_{1,1}(2(t-s)) = \sum_{k=0}^{\infty} \frac{2^k(\frac{2}{5})_k}{\Gamma(k+1)} \frac{(t-s)^k}{k!}\,.$$

References

[1] D. P. Ahokposi, A. Atangana and D. P. Vermeulen, Modelling groundwater fractal flow with fractional differentiation via Mittag–Leffler law. *European Physical Journal Plus* **132** (2017), no. 4.

[2] H. Aktuğlu and M. A. Özarslan, Anti-periodic BVP for Volterra integro-differential equation of fractional order $1 < \alpha \leq 2$, involving Mittag–Leffler function in the kernel. *J. Nonlinear Sci. Appl.* **9** (2016), no. 2, 452–460.

[3] O. J. J. Algahtani, Comparing the Atangana-Baleanu and Caputo–Fabrizio derivative with fractional order: Allen Cahn model. *Chaos Solitons Fractals* **89** (2016), 552–559.

[4] B. S. T. Alkahtani, Chua's circuit model with Atangana–Baleanu derivative with fractional order. *Chaos Solitons Fractals* **89** (2016), 547–551.

[5] B. S. T. Alkahtani and A. Atangana, Analysis of non-homogeneous heat model with new trend of derivative with fractional order. *Chaos Solitons Fractals* **89** (2016), 566–571.

[6] R. T. Alqahtani, Atangana–Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer. *J. Nonlinear Sci. Appl.* **9** (2016), no. 6, 3647–3654.

[7] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Science* **20** (2016), no. 2, 763–769.

[8] A. Atangana and I. Koca, Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. *Chaos Solitons Fractals* **89** (2016), 447–454.

[9] L. Boyadjiev and H.-J. Dobner, Fractional free electron laser equations. *Integral Transform. Spec. Funct.* **11** (2001), no. 2, 113–136.

[10] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications* **1** (2015), no. 2, 73–85.

[11] S. Gala, Q. Liu and M. A. Ragusa, A new regularity criterion for the nematic liquid crystal flows. *Appl. Anal.* **91** (2012), no. 9, 1741–1747.

[12] S. Gala and M. A. Ragusa, Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices. *Appl. Anal.* **95** (2016), no. 6, 1271–1279.

[13] A. Giusti and I. Colombaro, Prabhakar-like fractional viscoelasticity. *Commun. Nonlinear Sci. Numer. Simul.* **56** (2018), 138–143.

[14] J. F. Gómez, L. Torres and R. F. Escobar, *Fractional Derivatives with Mittag–Leffler Kernel: Trends and Applications in Science and Engineering.* Springer, Cham, 2019.

[15] S. Havriliak and S. Negami, A complex plane representation of dielectric and mechanical relaxation processes in some polymers. *Polymer* **8** (1967), 161–210.

[16] F. Jarad, Th. Abdeljawad and Z. Hammouch, On a class of ordinary differential equations in the frame of Atangana–Baleanu fractional derivative. *Chaos Solitons Fractals* **117** (2018), 16–20.

[17] A. A. Kilbas, M. Saigo and R. K. Saxena, Solution of Volterra integrodifferential equations with generalized Mittag–Leffler function in the kernels. *J. Integral Equations Appl.* **14** (2002), no. 4, 377–396.

[18] A. A. Kilbas, M. Saigo and R. K. Saxena, Generalized Mittag–Leffler function and generalized fractional calculus operators. *Integral Transforms Spec. Funct.* **15** (2004), no. 1, 31–49.

[19] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[20] Ch. Li, D. Qian and Y. Chen, On Riemann–Liouville and Caputo derivatives. *Discrete Dyn. Nat. Soc.* **2011**, Art. ID 562494, 15 pp.

[21] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Imperial College Press, Singapore, 2010.

[22] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[23] S. Polidoro and M. A. Ragusa, Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term. *Rev. Mat. Iberoam.* **24** (2008), no. 3, 1011–1046.

[24] T. R. Prabhakar, A singular integral equation with a generalized Mittag Leffler function in the kernel. *Yokohama Math. J.* **19** (1971), 7–15.

[25] Kh. M. Saad, A. Atangana and D. Baleanu, New fractional derivatives with non-singular kernel applied to the Burgers equation. *Chaos* **28** (2018), no. 6, 063109, 6 pp.

[26] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*. Edited and with a foreword by S. M. Nikol'skiĭ. Translated from the 1987 Russian original. Gordon and Breach Science Publishers, Yverdon, 1993.

[27] R. K. Saxena and S. L. Kalla, On a fractional generalization of the free electron laser equation. *Appl. Math. Comput.* **143** (2003), no. 1, 89–97.

[28] Y. Singh, V. Gill, S. Kundu and D. Kumar, On the Elzaki transform and its applications in fractional free electron laser equation. *Acta Univ. Sapientiae Math.* **11** (2019), no. 2, 419–429.

[29] V. E. Tarasov, *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Nonlinear Physical Science. Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[30] M. Toufik and A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models. *Eur. Phys. J. Plus* **132** (2017), no. 444.

[31] S. Ullah, M. A. Khan and M. Farooq, Modeling and analysis of the fractional HBV model with Atangana–Baleanu derivative. *Eur. Phys. J. Plus* **133** (2018), no. 313.

(Received 9.06.2020)

Author's address:

Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt.

E-mail: miabbas77@gmail.com