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THIN SHELLS WITH LIPSCHITZ BOUNDARY

Abstract. In [58], we have revised an asymptotic model of a shell (Koiter, Sanchez–Palencia, Ciarlet, etc.), based on the the calculus of tangent Günter's derivatives, developed in the papers of R. Duduchava, D. Mitrea and M. Mitrea [55, 58, 64]. As a result, the 2-dimensional shell equation on a mid-surface $\mathscr{S}$ was written in terms of Günter's derivatives, unit normal vector field and the lamé constants. The principal part of the obtained equation coincides with the Lamé equation on the Hypersurface $\mathscr{S}$ investigated in [55, 58, 64].

The present investigation is inspired by the paper of G. Friesecke, R.D. James and S. Müller [77], where a hierarchy of Plate Models are derived from nonlinear elasticity by Γ-Convergence. The final goal of the present investigation is to derive and investigate 2D shell equations in terms of Günter's derivatives by Γ-Convergence.

As a first step to the final goal, by T. Buchukuri, R. Duduchava and G. Tephnadze was studied a mixed boundary value problem for the stationary heat transfer equation in a thin layer around a surface $\mathscr{C}$ with the boundary (see [16]). It was established what happens to the solution of the boundary value problem when the thickness of the layer converges to zero. In particular, there was shown that the Γ-limit of a mixed type Dirichlet–Neumann boundary value problem (BVP) for the Laplace equation in the initial thin layer is a Dirichlet BVP for the Laplace–Beltrami equation on the surface. The result was derived based on the variational reformulation of the problem using the Günter's tangent differential operators on a hypersurface and layers. The similar results were obtained for the Lamé operator. This approach allows global representation of basic differential operators and of corresponding boundary value problems in terms of the standard cartesian coordinates of the ambient Euclidean space $\mathbb{R}^n$.

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რეზიუმე. ნაშრომში [58] რ. დუდუჩავას, დ. მიტრეას, მ. მიტრეას ნაშრომებში [55,58,64] აგებული გიუნტერის მხები დიფერენციალური ოპერატორების აღრიცხვაზე დაყრდნობით ჩვენ ხელახლა გადავხედეთ გარსის ასიმპტოტურ მოდელს (კოიტერი, სანჩეს-პალენსია, სიარლე და სხვანი). შედეგად, გარსის 2-განზომილებიანი განტოლება თხელი სხეულის შუა $\mathscr{S}$ ზედაპირზე ჩაიწერა გიუნტერის მხები წამოებულების, ერთეულოვანი ნორმალის შესაბამისი ვექტორული ველისა და ლამეს მუდმივების ტერმინებში. მიღებული განტოლების მთავარი ნაწილი დაემთხვა ლამეს განტოლებას $\mathscr{S}$ ჰიპერზედაპირზე, რომელიც შესწავლილი იყო ნაშრომებში [55,58,64].

აქ წარმოდგენილი კვლევა შთაგონებული იყო გ. ფრიზეკეს, რ. დ. ჯეიმსის და ს. მ. მიულერის ნაშრომით [77], სადაც ფირფიტების იერარქიული მოდელის განტოლებები გამოყვანილ იქნა 3-განზომილებიანი დრეკადობის არაწრფივი განტოლებიდან Γ-კრებადობის გამოყენებით. მოცემული კვლევის საბოლოო მიზანია გამოვიყვანოთ და შევისწავლოთ 2-განზომილებიანი გარსის განტოლება გიუნტერის მხები წამოებულების ტერმინებში Γ-კრებადობის გამოყენებით.

ამ მიზნის მისაღწევად თ. ბუჩუკურმა, რ. დუდუჩავამ და გ. ტეფნაძემ პირველ რიგში შეისწავლეს სითბოგამტარებლობის სტაციონალური განტოლება საზღვრიანი $\mathscr{C}$ ჰიპერზედაპირის გარშემო განფენილ თხელ შრეში (იხ. [16]). დადგენილია, თუ რა მოსდის განხილული სასაზღვრო ამოცანის ამონახსნს, როდესაც შრის სისქე მიისწრაფის ნულისკენ. კერძოდ, განმარტებულია თხელ შრეში ლაპლასის განტოლებისთვის დირიხლე-ნეიმანის შერეული ტიპის სასაზღვრო ამოცანის Γ-ზღვარი, როდესაც შრის სისქე მიისწრაფის ნულისკენ, და ნაჩვენებია, რომ ასეთი Γ-ზღვარი ემთხვევა ლაპლას-ბელტრამის განტოლებისთვის დასმულ დირიხლეს ამოცანას საწყისი შრის შუა ზედაპირზე. შედეგი მიღებული იყო თავდაპირველი ამოცანის ვარიაციულ ფორმულირებაზე დაყრდნობით და გიუნტერის მხები წარმოებულების გამოყენებით შრეში და ჰიპერზედაპირზე. ანალოგიური შედეგები მიღებულია შრეში ლამეს ოპერატორისთვის. ასეთი მიდგომა საშუალებას იძლევა წარმოვადგინოთ მათემატიკური ფიზიკის ძირითადი განტოლებები და ამ განტოლებებისთვის დასმული სასაზღვრო ამოცანები გლობალურად, სტანდარტული გარემომცველი ევკლიდური $\mathbb{R}^n$ სივრცის კოორდინატთა სისტემის საშუალებით.

Introduction

Modern interest in shell theories arising from the theory of thin films caused by the widespread use of thin films in science and technology. Thin structures are encountered in engineering applications more and more often, and there emerged numerous approaches proposed for modeling linearly elastic flexural shells. Started by the Cosserats pioneering work (1909), Goldenveiser (1961), Naghdi (1963), Vekua (1965), Novozhilov (1970), Koiter (1970) and many others have introduced and developed various models of shells. The aforementioned works contributed essentially the development of the shell theory. Ellipticity of the corresponding partial differential equations was not established initially and was proved later by Roug'e (1969) for cylindrical shells, by Coutris (1973) for the shell model proposed by Naghdi, by Gordeziani (1974) for the shell model proposed by Vekua, by Shoikhet (1974) for the shell model proposed by Novozhilov, by Ciarlet and Miara (1992) for the model proposed by Koiter (cf. [22–26, 28, 29, 36] for survey and further references).

Inspired by the books and papers of Sanchez–Palencia [121, 122], Miara and Sanchez-Palencia [111], Ciarlet and Lods [26–28], Ciarlet, Lods and Miara [29] and exposed in details by Ciarlet in [23, 26, 28, 29]. in [58] we have developed the asymptotic analysis of a linearly elastic shell based on the formal calculus of tangent Günter's derivatives, developed in the papers of R. Duduchava with D. Mitrea and M. Mitrea [55, 58, 64]. The asymptotic analysis of a linearly elastic shell based on the formal calculus of tangent Günter's derivatives, was developed in the papers of R. Duduchava with D. Mitrea and M. Mitrea [55,58,64]. As a result, the 2-dimensional shell equation on a middle surface $\mathscr{S}$ is derived written in terms of Gunter's derivatives, unit normal vector field and the lamé constants. It coincides with the Lamé equation on the Hypersurface $\mathscr{S}$ investigated in [55, 58, 64].

The present investigation is inspired by the paper of G. Friesecke, R. D. James and S. Müller [77], where a hierarchy of Plate Models are derived from nonlinear elasticity by Γ-Convergence. The final goal of the investigation is to derive 2D shell equations written in terms of Günter's derivatives by Γ-Convergence.

Let us consider an example: a surface $\mathscr{S}$ be given by a local immersion

$$\Theta : \omega \to \mathscr{S}, \quad \omega \subset \mathbb{R}^{n-1}, \tag{0.0.1}$$

which means that the derivatives $\{\boldsymbol{g}_k := \partial_k\Theta\}_{k=1}^{n-1}$ are linearly independent, i.e., the Jacobi matrix $\nabla\Theta$ has the maximal rank $n-1$. Thus $\{\boldsymbol{g}_k\}_{k=1}^{n-1}$ is a **basis** (or a **covariant frame** if the basis is enriched with 0) in the space $\omega(\mathscr{S})$ of all tangent vector fields on $\mathscr{S}$. The system $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ which is biorthogonal, $\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}$, forms the **contravariant basis** (the **contravariant frame**) in the same space $\omega(\mathscr{S})$ of all tangent vector fields on $\mathscr{S}$. Let $\boldsymbol{\nu}(\mathscr{X}) = (\nu_1(\mathscr{X}), \dots, \nu_j(\mathscr{X}))^\top$ be the outer unit normal vector (the Gauß mapping) to $\mathscr{S}$ at $\mathscr{X} \in \mathscr{S}$ (see Section 1.6 for details). The Gram matrix $G_{\mathscr{S}}(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1\times n-1}$, $g_{jk} := \langle g_j, g_k\rangle$, is then positive definite, responsible for the Riemann metric on $\mathscr{S}$ and is called the **covariant metric tensor**. Moreover, it has the inverse matrix $G^{-1}_{\mathscr{S}}(\mathscr{X}) = [g^{jk}(\mathscr{X})]_{n-1\times n-1}$, $g^{jk} := \langle \boldsymbol{g}^j, \boldsymbol{g}^k\rangle$ (cf. (1.3.1), (0.0.2)), which is called the **contravariant metric tensor**.

The Gram determinant

$$\mathscr{G}\big((\partial_1\Theta(x), \dots, \partial_{n-1}\Theta)(x)\big) = \det G_{\mathscr{S}}(x), \quad x \in \omega \subset \mathbb{R}^{n-1}, \tag{0.0.2}$$

is responsible for the volume element $d\sigma$ of the surface, which is the vector product of the tangent vectors

$$d\sigma := |\partial_1\Theta \wedge \cdots \wedge \partial_{n-1}\Theta| = \sqrt{\det G_{\mathscr{S}}}\, dx, \quad dx = dx_1 \cdots dx_{n-1}. \tag{0.0.3}$$

The surface divergence and the surface gradient are defined in the intrinsic coordinates by the equalities

$$\mathbf{div}_{\mathscr{S}}\, \boldsymbol{U} := \big[\det G_{\mathscr{S}}\big]^{-1/2} \sum_{j=1}^{n} \partial_j \big\{[\det G_{\mathscr{S}}]^{1/2} U^j\big\}, \quad \nabla_{\mathscr{S}}\, f = \sum_{j,k=1}^{n-1} (g^{jk}\partial_j f)\partial_k \tag{0.0.4}$$

(see Section 1.1 and [130, Chapter 2, § 3]). Their composition is the **Laplace–Beltrami operator**

$$\boldsymbol{\Delta}_{\mathscr{S}}\, f := \mathbf{div}_{\mathscr{S}}\, \nabla_{\mathscr{S}} f = [\det G_{\mathscr{S}}]^{-1/2} \sum_{j,k=1}^{n-1} \partial_j \big\{g^{jk}[\det G_{\mathscr{S}}]^{1/2} \partial_k\, f\big\}, \quad f \in C^2(\mathscr{S}), \tag{0.0.5}$$

which is self-adjoint

$$\Delta^*_{\mathscr{S}} = (\nabla_{\mathscr{S}}\, \mathbf{div}_{\mathscr{S}})^* = (\mathbf{div}_{\mathscr{S}})^* (\nabla_{\mathscr{S}})^* = \nabla_{\mathscr{S}}\, \mathbf{div}_{\mathscr{S}} = \boldsymbol{\Delta}_{\mathscr{S}}. \tag{0.0.6}$$

The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$.

We introduce a different curvilinear system of coordinates. It differs from the covariant and contravariant metric tensors described above and used intensively by P. Ciarlet in [22, 23] for the derivation of shell equations. Moreover, the system of curvilinear coordinates introduced below is linearly dependent but, surprisingly, many partial differential equations are expressed in this system in a simple form (see [64]) including Laplace–Beltramy and shell equations on a hypersurface (see below).

Our idea is to record these operators in Cartesian coordinates. To set the conditions for precise formulations, let us consider the **natural basis**

$$\boldsymbol{e}^1 := (1, 0, \dots, 0)^\top, \dots, \boldsymbol{e}^n := (0, \dots, 0, 1)^\top \tag{0.0.7}$$

in the Euclidean space $\mathbb{R}^n$ ($\{e^j\}_{j=1}^n$ is also called the **Cartesian basis**. Each point $x = (x_1, \dots, x_n)^\top$ in the Euclidean space $\mathbb{R}^n$ is represented in the Cartesian basis $x = \sum\limits_{j=1}^{n} x_j e^j$ in a unique way.

The operator (the matrix)

$$\pi_{\mathscr{S}} : \mathbb{R}^n \to \omega(\mathscr{S}), \quad \pi_{\mathscr{S}}(t) = I - \boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(t) = \big[\delta_{jk} - \nu_j(t)\nu_k(t)\big]_{n\times n}, \ \ t \in \mathscr{S}, \tag{0.0.8}$$

represents the canonical orthogonal projection $\pi^2_{\mathscr{S}} = \pi_{\mathscr{S}}$ onto the space of tangent vector fields to $\mathscr{S}$ at the point $t \in \mathscr{S}$:

$$(\boldsymbol{\nu}, \pi_{\mathscr{S}} v) = \sum_j \nu_j v_j - \sum_{j,k} \nu_j^2 \nu_k v_k = 0 \ \text{ for all } \ v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n.$$

It turns out that the surface gradient is nothing but the collection of the weakly tangent **Günter's derivatives** (cf. [54, 86, 101])

$$\nabla_{\mathscr{S}} = \mathscr{D}_{\mathscr{S}} := (\mathscr{D}_1, \dots, \mathscr{D}_n)^\top, \quad \mathscr{D}_j := \partial_j - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}} = \partial_{\boldsymbol{d}^j}, \tag{0.0.9}$$

where $\partial_{\boldsymbol{\nu}} := \sum\limits_{j=1}^{n} \nu_j \partial_j$ denotes the normal derivative. The first-order differential operators

$$\mathscr{D}_j = \partial_{\boldsymbol{d}^j}, \ \ 1 \leqslant j \leqslant n, \tag{0.0.10}$$

are the directional derivatives along the vector fields $\boldsymbol{d}^j := \pi_{\mathscr{S}} \boldsymbol{e}^j$, $j = 1, \dots, n$.

Moreover, the surface divergence coincides with the operator

$$\mathbf{div}_{\mathscr{S}}\, \boldsymbol{U} = \sum_{j=1}^{n} \mathscr{D}_j U^0_j \ \text{ for } \ \boldsymbol{U} = \sum_{j=1}^{n} U^0_j \partial_j \in \omega(\mathscr{S}) \tag{0.0.11}$$

and the Laplace–Beltrami operator coincides with (see also [109, pp. 2ff and p. 8])

$$\boldsymbol{\Delta}_{\mathscr{S}}\,\varphi := \mathbf{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\varphi = \sum_{j=1}^{n}\mathscr{D}_j^2\varphi, \quad \varphi\in C^2(\mathscr{S}). \tag{0.0.12}$$

Relatively simple form of recorded operators enables simplified treatment of corresponding boundary value problems, which require proofs of Korn's inequalities or similar.

Calculus of Gunter's derivatives on a hypersurface, which is the main tool, together with the Γ-convergence, in the our investigation, allows representation of the most basic partial differential operators (PDO's), as well as their associated boundary value problems, on a hypersurface $\mathscr{C}$ in global form, in terms of the standard spatial coordinates in $\mathbb{R}^n$. Such BVPs arise in a variety of situations and have many practical applications. See, for example, [87, §72] for the heat conduction by surfaces, [7, §10] for the equations of surface flow, [22], [5] for the vacuum Einstein equations describing gravitational fields, [131] for the Navier-Stokes equations on spherical domains, as well as the references therein.

The Laplace–Beltrami operator (0.0.12) is the natural operator associated with the Euler-Lagrange equations for a variational integral

$$\mathscr{E}[u] = -\frac{1}{2}\int\limits_{\mathscr{S}}\|\mathscr{D}u\|^2\,dS. \tag{0.0.13}$$

A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy (Koiter's model), leads to the following form of the Lamé operator $\mathscr{L}_{\mathscr{S}}$ on $\mathscr{S}$ (cf. [64])

$$\mathscr{L}_{\mathscr{S}}\boldsymbol{U} = \mu\pi_{\mathscr{S}}\,\mathbf{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\boldsymbol{U} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\,\mathbf{div}_{\mathscr{S}}\,\boldsymbol{U} + \mu\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U} \tag{0.0.14}$$

(cf. (0.0.8) for the projection $\pi_{\mathscr{S}}$). Here $\boldsymbol{U}$ is an arbitrary (tangent) vector fields on $\mathscr{S}$, $\lambda,\mu\in\mathbb{R}$ are the Lamé moduli, whereas

$$\mathscr{H}^0_{\mathscr{S}} = -\,\mathbf{div}_{\mathscr{S}}\,\boldsymbol{\nu} := -\sum_{j=1}^{n}\mathscr{D}_j\nu_j = \operatorname{Tr}\mathscr{W}_{\mathscr{S}}, \quad \mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j\nu_k\big]_{n\times n}. \tag{0.0.15}$$

Note that $\mathscr{H}_{\mathscr{S}} := (n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}$ and $\mathscr{W}_{\mathscr{S}}$ represent, respectively, the **mean curvature** and the **Weingarten mapping** of $\mathscr{S}$. This identification ensures that the boundary-value problem

$$\begin{cases}\mathscr{L}_{\mathscr{S}}\boldsymbol{U} = 0 & \text{in } \mathscr{S},\\ \boldsymbol{U}\big|_{\Gamma} = f\in\mathbb{H}^s(\partial\mathscr{S}), \quad f\cdot\boldsymbol{\nu} = f\cdot\boldsymbol{\nu}_\Gamma = 0 & \text{on } \Gamma := \partial\mathscr{S},\end{cases} \tag{0.0.16}$$

where $\boldsymbol{U} = \sum\limits_{j=1}^{n}U^0_j\boldsymbol{d}^j \in \omega(\mathscr{S})\cap\mathbb{H}^{s+1/2}(\partial\mathscr{S})$ is the generalized displacement vector field, tangent to the elastic hypersurface $\mathscr{S}$, is well-posed, whenever $\mu>0$, $2\mu+\lambda>0$, and $0\leqslant s\leqslant 1$. Here $\mathbb{H}^s$ stands for the usual L^2-based Sobolev space, $\boldsymbol{\nu}$ is the normal vector to $\mathscr{S}$ and $\boldsymbol{\nu}_\Gamma(t)$ is the unit tangent vector to $\mathscr{S}$ at the boundary point $t\in\Gamma := \partial\mathscr{S}$ and outer normal vector to the boundary $\Gamma = \partial\mathscr{S}$.

Chapter 1

Auxiliary

In the present chapter, we have collected, for the readers convenience, some auxiliary information, mostly from [21–23, 25, 64, 77, 129].

1.1 Auxiliary from the operator theory

The results exposed in the present section will be applied to complex-valued matrices, which are identified with operators in the finite-dimensional space $\mathbb{C}^n$ of complex values n-vectors. Nevertheless, we will formulate results in general setting of operators in a Hilbert space.

Throughout this section we assume that $\mathfrak{H}$ is a Hilbert space with respect to some continuous scalar product, a bilinear form $(\,\cdot\,,\,\cdot\,) : \mathfrak{H}\times\mathfrak{H}\to\mathbb{C}$, i.e.,

$$(\lambda\, u+\mu\, w, v)=\overline{\lambda}(u,v)+\overline{\mu}(w,v),\quad (u,\lambda\, v+\mu\, z)=\lambda(u,v)+\mu(u,z),$$
$$\big|(u,v)\big|\leqslant C\|u\|_{\mathfrak{H}}\|v\|_{\mathfrak{H}},\;\; \forall\, u,w\in\mathfrak{H},\;\; \forall\, v,z\in\mathfrak{H},$$
$$(\varphi,\psi)=\overline{(\psi,\varphi)},\;\; \forall\,\varphi,\psi\in\mathfrak{H}.$$

Denote by $\mathscr{L}(\mathfrak{H},\mathfrak{H})$ or $\mathscr{L}(\mathfrak{H})$ the space of linear operators $A:\mathfrak{H}\to\mathfrak{H}$. Recall that the dual operator $(\boldsymbol{A}^*\varphi,\psi)=(\varphi,\boldsymbol{A}\psi)$ maps continuously the same space $\boldsymbol{A}^*:\mathfrak{H}\to\mathfrak{H}$ and $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is **self-adjoint** operator if

$$(\boldsymbol{A}\varphi,\psi)=(\varphi,\boldsymbol{A}\psi),\;\; \forall\,\varphi,\psi\in\mathfrak{H}. \tag{1.1.1}$$

$\boldsymbol{A}\in\mathscr{L}(\mathfrak{H},\mathfrak{H})$ is **positive definite** (or **coercive**) if the inequality

$$(\boldsymbol{A}\varphi,\varphi)\geqslant C\big\|\varphi\mid\mathfrak{H}\big\|^2 \tag{1.1.2}$$

holds for some constant $C>0$ and all $\varphi\in\mathfrak{H}$.

Lemma 1.1.1. *Let $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$. The inequality*

$$\big\|\boldsymbol{A}\varphi\mid\mathfrak{H}\big\|\geqslant C\big\|\varphi\mid\mathfrak{H}\big\| \tag{1.1.3}$$

with some constant $C>0$ holds if and only if the operator $\boldsymbol{A}$ is normally solvable (i.e., has the closed image $\mathrm{Im}A=\overline{\mathrm{Im}A}$*) and injective,* $\mathbf{Ker}\,\boldsymbol{A}=\{0\}$.

Proof. If inequality (1.1.3) holds, then $\boldsymbol{A}\varphi=0$, $\varphi\in\mathfrak{H}$, implies $\varphi=0$ and $\mathbf{Ker}\,\boldsymbol{A}=\{0\}$. Now let $\psi_j=\boldsymbol{A}\varphi_j\to\psi_0$ (convergence in the norm). The inequality (1.1.3) implies the convergence $\varphi_j\to\varphi_0$. Due to continuity of $\boldsymbol{A}$ this implies $\boldsymbol{A}\varphi_0=\psi_0\in\mathrm{Im}\boldsymbol{A}$ and the image $\mathrm{Im}\boldsymbol{A}$ is closed.

Vice versa, let $\boldsymbol{A}$ be normally solvable and $\mathbf{Ker}\,\boldsymbol{A}=\{0\}$. Then $\mathrm{Im}\boldsymbol{A}$ is a Hilbert space, subspace of $\mathfrak{H}$ and the operator $\boldsymbol{A}:\mathfrak{H}\to\mathrm{Im}\boldsymbol{A}$ is bijective. Due to the Banach Inverse mapping theorem, $\boldsymbol{A}$ is invertible: there exists $\mathbf{B}\in\mathscr{L}(\mathrm{Im}\boldsymbol{A})$ such that $\mathbf{AB}x=x$ and $\mathbf{BA}y=y$ for all $x\in\mathrm{Im}\boldsymbol{A}$ and all $y\in\mathfrak{H}$. Inserting in $\|\mathbf{B}\psi\big|\mathfrak{H}\|\leqslant C\|\psi\big|\mathrm{Im}\boldsymbol{A}\|:=\|\psi\big|\mathfrak{H}\|$ the equality $\psi=\boldsymbol{A}\varphi$, $\varphi\in\mathfrak{H}$, we get (1.1.3). □

Definition 1.1.1. For an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ the closed set

$$\Sigma(\boldsymbol{A}):=\overline{\big\{(\boldsymbol{A}\varphi,\varphi):\ \varphi\in\mathfrak{H}\big\}}\,, \tag{1.1.4}$$

where the overbar denotes closing of the set, is called the **spectral set** of $\boldsymbol{A}$.

Lemma 1.1.2. *If the spectral set $\Sigma(\boldsymbol{A})$ of an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is real-valued $\Sigma(\boldsymbol{A})\subset\mathbb{R}$, then $\boldsymbol{A}$ is self-adjoint.*

Proof. We proceed as follows:

$$\begin{aligned}(\boldsymbol{A}\varphi,\psi)&=\frac{1}{4}\Big\{(\boldsymbol{A}[\varphi+\psi],\varphi+\psi)-(\boldsymbol{A}[\varphi-\psi],\varphi-\psi)+i(\boldsymbol{A}[\varphi+i\psi],\varphi+i\psi)-i(\boldsymbol{A}[\varphi-i\psi],\varphi-i\psi)\Big\}\\&=\frac{1}{4}\Big\{\overline{(\boldsymbol{A}[\varphi+\psi],\varphi+\psi)}-\overline{(\boldsymbol{A}[\varphi-\psi],\varphi-\psi)}+i\overline{(\boldsymbol{A}[\varphi+i\psi],\varphi+i\psi)}-i\overline{(\boldsymbol{A}[\varphi-i\psi],\varphi-i\psi)}\Big\}\\&=\frac{1}{4}\Big\{(\varphi+\psi,\boldsymbol{A}[\varphi+\psi])-(\varphi-\psi,\boldsymbol{A}[\varphi-\psi])+i(\varphi+i\psi,\boldsymbol{A}[\varphi+i\psi])-i(\varphi-i\psi,\boldsymbol{A}[\varphi-i\psi])\Big\}\\&=(\varphi,\boldsymbol{A}\psi),\quad\varphi,\psi\in\mathfrak{H},\end{aligned}$$

since $(\boldsymbol{A}u,u)=\overline{(\boldsymbol{A}u,u)}$ by the condition $\Sigma(\boldsymbol{A})\subset\mathbb{R}$ and $\overline{(\boldsymbol{A}u,u)}=(u,\boldsymbol{A}u)$ by the definition. □

Corollary 1.1.1. *If an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is positive definite, it is self-adjoint and invertible.*

Proof. If $\boldsymbol{A}$ is positive definite, its spectral set is real-valued and $\boldsymbol{A}$ is self-adjoint.

From (1.1.2) we get

$$\big\|\boldsymbol{A}\varphi\mid\mathfrak{H}\big\|\,\big\|\varphi\mid\mathfrak{H}\big\|\geqslant(\boldsymbol{A}\varphi,\varphi)\geqslant C\big\|\varphi\mid\mathfrak{H}\big\|^2$$

and, further,

$$\big\|\boldsymbol{A}\varphi\mid\mathfrak{H}\big\|\,\big\|\geqslant C\big\|\varphi\mid\mathfrak{H}\big\|,\quad\varphi\in\mathfrak{H}. \tag{1.1.5}$$

Due to Lemma 1.1.1, inequality (1.1.2) implies that $\boldsymbol{A}$ is normally solvable and has a trivial kernel $\mathbf{Ker}\,\boldsymbol{A}=\{0\}$. Being self-adjoint $\boldsymbol{A}^*=\boldsymbol{A}$, the operator has the trivial cokernel $\dim\mathbf{Coker}\,\boldsymbol{A}=\dim\mathbf{Ker}\,\boldsymbol{A}=0$ (due to (1.1.2), $\boldsymbol{A}\varphi=0$ implies that $\varphi=0$). Therefore, $\boldsymbol{A}$ is invertible. □

Let $\mathbb{SO}(\mathfrak{H})$ denote the set of **orthogonal (unitary)** operators: $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$ if and only if $\mathbf{R}^*=\mathbf{R}^{-1}$. Note that $\mathbb{SO}(\mathfrak{H})$ is a group and the set $\mathbf{R}(\mathbb{SO}(\mathfrak{H}))$ coincides with $\mathbb{SO}(\mathfrak{H})$ for arbitrary $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$.

Let $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ and $\boldsymbol{A}=\mathbf{R}\mathbf{H}_{\boldsymbol{A}}$ be its left polar decomposition, where $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$ is orthogonal and $\mathbf{H}_{\boldsymbol{A}}$ is positive, self-adjoint (Hermitian) operator

$$\langle\mathbf{H}_{\boldsymbol{A}}\varphi,\varphi\rangle\geqslant C_0\|\varphi\|^2,\ \ C_0>0,\ \ \mathbf{H}^*_{\boldsymbol{A}}=\mathbf{H}_{\boldsymbol{A}},\ \ \forall\,\varphi\in\mathfrak{H}.$$

Let us check that $\mathbf{H}_{\boldsymbol{A}}=\sqrt{\boldsymbol{A}^*\boldsymbol{A}}$. Indeed, if $\boldsymbol{A}=\mathbf{R}\mathbf{H}_{\boldsymbol{A}}$, then $\boldsymbol{A}^*=\mathbf{H}^*_{\boldsymbol{A}}\mathbf{R}^*=\mathbf{H}_{\boldsymbol{A}}\mathbf{R}^{-1}$ and $\sqrt{\boldsymbol{A}^*\boldsymbol{A}}=\sqrt{\mathbf{H}_{\boldsymbol{A}}\mathbf{R}^{-1}\mathbf{R}\mathbf{H}_{\boldsymbol{A}}}=\sqrt{\mathbf{H}^2_{\boldsymbol{A}}}=\mathbf{H}_{\boldsymbol{A}}$.

Similarly, for the right polar decomposition $\boldsymbol{A}=\mathbf{H}'_{\boldsymbol{A}}\mathbf{R}'$ we get $\mathbf{H}'_{\boldsymbol{A}}=\sqrt{\boldsymbol{A}\,\boldsymbol{A}^*}$.

Note that if $\boldsymbol{A}$ is positive definite (or, at least, has a real-valued spectral set), then $\boldsymbol{A}$ is self-adjoint $\boldsymbol{A}^*=\boldsymbol{A}$ and the polar decomposition is trivial $\mathbf{H}_A=\mathbf{H}'_A=\sqrt{\boldsymbol{A}\boldsymbol{A}}=\boldsymbol{A}$, $\mathbf{R}=\mathbf{R}'=\mathbf{I}$.

The next Lemma 1.1.3 generalizes essentially the statement formulated in [77], §2.

Lemma 1.1.3. *For $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ and $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$ the norm has the following property:*

$$\|\mathbf{R}\mathbf{A}\mathbf{R}\|=\|\mathbf{R}\mathbf{A}\|=\|\mathbf{A}\mathbf{R}\|=\|\boldsymbol{A}\|. \tag{1.1.6}$$

Moreover, if $\boldsymbol{A}=\mathbf{R}\mathbf{H}_{\boldsymbol{A}}$ is the left polar decomposition of $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$, then

$$\mathbf{dist}\,\big(\boldsymbol{A},\mathbb{SO}(\mathfrak{H})\big)=\mathbf{dist}\,\big(\mathbf{H}_{\boldsymbol{A}},\mathbb{SO}(\mathfrak{H})\big), \tag{1.1.7a}$$

$$\mathbf{dist}\,\big(\boldsymbol{A},\mathbb{SO}(\mathfrak{H})\big)=\|\mathbf{H}_{\boldsymbol{A}}-\mathbf{I}\|\ \ \textit{if } \boldsymbol{A} \textit{ is positive definite}, \tag{1.1.7b}$$

$$\mathbf{dist}\,\big(\boldsymbol{A},\mathbb{SO}(\mathfrak{H})\big)\leqslant\|\mathbf{H}_{\boldsymbol{A}}-\mathbf{I}\|\ \ \textit{otherwise}. \tag{1.1.7c}$$

Proof. To prove (1.1.6) we proceed as follows:

$$\|\mathbf{RA}\| = \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{RA})^*\mathbf{RA}\varphi,\varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{(\boldsymbol{A}^*\mathbf{R}^*\mathbf{RA}\varphi,\varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{(\boldsymbol{A}^*\boldsymbol{A}\varphi,\varphi)} = \|\boldsymbol{A}\|.$$

By using the obtained equality and recalling that $\|\boldsymbol{A}\| = \|\boldsymbol{A}^*\|$ and $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$ implies $\mathbf{R}^*\in\mathbb{SO}(\mathfrak{H})$, we prove the following

$$\|\mathbf{AR}\| = \|(\mathbf{AR})^*\| = \|\mathbf{R}^*\boldsymbol{A}^*\| = \|\boldsymbol{A}\|.$$

Equalities (1.1.6) are proved and, due to them,

$$\begin{aligned}\mathbf{dist}\left(\boldsymbol{A},\mathbb{SO}(\mathfrak{H})\right) &= \inf_{\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})} \|\mathbf{RH}_{\boldsymbol{A}} - \boldsymbol{V}\| = \inf_{\boldsymbol{V}\in\mathbb{SO}(fH)} \|\mathbf{R}^*(\mathbf{RH}_{\boldsymbol{A}} - \boldsymbol{V})\| \\ &= \inf_{\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})} \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{R}^*\boldsymbol{V}\| = \mathbf{dist}\left(\mathbf{H}_{\boldsymbol{A}},\mathbb{SO}(\mathfrak{H})\right) \leqslant \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\|,\end{aligned}$$

since $\mathbf{I},\mathbf{R}^*\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})$ and the set $\{\mathbf{R}^*\boldsymbol{V} \;:\; \boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})\}$ coincides with the orthogonal group $\mathbb{SO}(\mathfrak{H})$.

Equality (1.1.7a) and inequality (1.1.7c) are proved.

To prove equality (1.1.7b) we can assume $\boldsymbol{A}$ is non-negative, i.e., also self adjoint (see Lemma 1.1.2). Then, due to (1.1.7a), we can take $\boldsymbol{A} = \mathbf{H}_{\boldsymbol{A}}$. Then the spectral set of $\boldsymbol{A}$ is non-negative

$$0 \leqslant m(\boldsymbol{A}) := \inf_{\|x\|=1} (\boldsymbol{A}x,x).$$

It follows from the spectral theorem that

$$m(\boldsymbol{A}) = \inf_{\|x\|=1} \|\boldsymbol{A}x\|. \tag{1.1.8}$$

Moreover, it is well known that for every self-adjoint operator $\boldsymbol{A}$ the spectral radius coincides with the norm:

$$\|A\| = \sup_{\|x\|=1} |(\boldsymbol{A}x,x)|.$$

It is easy to see that

$$\begin{aligned}\|\boldsymbol{A}-I\| &= \sup_{\|x\|=1} |((\boldsymbol{A}-I)x,x)| = \sup_{\|x\|=1} |(\boldsymbol{A}x,x)-1| \\ &= \max\Big\{\sup_{\|x\|=1} (\boldsymbol{A}x,x)-1,\ 1-\inf_{\|x\|=1}(\boldsymbol{A}x,x)\Big\} \\ &= \max\{\|\boldsymbol{A}\|-1,\ 1-m(\boldsymbol{A})\}.\end{aligned} \tag{1.1.9}$$

For any $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$, one has

$$\begin{aligned}\|\boldsymbol{A}-\mathbf{R}\| &= \sup_{\|x\|=1} \|(\boldsymbol{A}-\mathbf{R})x\| \geq \sup_{\|x\|=1} (\|(\boldsymbol{A}x\| - \|\mathbf{R}x\|) = \sup_{\|x\|=1} (\|\boldsymbol{A}x\|-1) = \|\boldsymbol{A}\|-1, \\ \|\boldsymbol{A}-\mathbf{R}\| &= \sup_{\|x\|=1} \|(\boldsymbol{A}-\mathbf{R})x\| \geq \sup_{\|x\|=1} (\|\mathbf{R}x\| - \|\boldsymbol{A}x\|) \\ &= \sup_{\|x\|=1} (1-\|\boldsymbol{A}x\|) = 1 - \inf_{\|x\|=1} \|\boldsymbol{A}x\| = 1-m(\boldsymbol{A})\end{aligned}$$

(see (1.1.8)) and, therefore (cf. (1.1.9)),

$$\|\boldsymbol{A}-\mathbf{R}\| \geqslant \max\{\|\boldsymbol{A}\|-1,\ 1-m(\boldsymbol{A})\} = \|\boldsymbol{A}-I\|. \tag{1.1.10}$$

Now it follows from (1.1.10) that

$$\mathbf{dist}\left(\boldsymbol{A},\mathbb{SO}(\mathfrak{H})\right) = \inf_{\mathbf{R}\in\mathbb{SO}(\mathfrak{H})} \|\boldsymbol{A}-\mathbf{R}\| \geqslant \|\boldsymbol{A}-I\|$$

and, together with (1.1.7c) proved above, this proves (1.1.7b). □

Let $\mathfrak{H}$ be a Hilbert space and consider a linear Variational problem

$$a(\varphi,\psi)=L(\psi)\quad\forall\,\psi\in\mathfrak{H}\,, \tag{1.1.11}$$

where $\varphi\in\mathfrak{H}$ is unknown and

$$a(\cdot,\cdot)\,:\,\mathfrak{H}\times\mathfrak{H}\to\mathbb{R},\quad L(\cdot)\,:\,\mathfrak{H}\to\mathbb{R} \tag{1.1.12}$$

are, respectively, a continuous bilinear form and a continuous linear form (a functional) on $\mathfrak{H}$.

Definition 1.1.2. We say Variational problem (1.1.11) is **well-posed** if, and only if, for all $\psi\in\mathfrak{H}$, it has one and only one solution $\varphi\in\mathfrak{H}^*$, with continuous dependence $\|\varphi|\mathfrak{H}^*\|\leqslant M\|\psi|\mathfrak{H}\|$ for some constant $M>0$.

Next, we expose the simple but very powerful Lax-Milgram Lemma with the elegant proof of these authors (see [102]).

Lemma 1.1.4 (Lax-Milgram)**.** *Let the continuous bilinear form* $a(\cdot,\cdot)\ :\ \mathfrak{H}\times\mathfrak{H}\to\mathbb{R}$ *in* (1.1.11) *be coercive* (*cf.* (1.1.2)).

Then Variational problem (1.1.11)*, is well posed: has a unique solution* $\varphi\in\mathfrak{B}$ *for all* $\psi\in\mathfrak{B}$*.*

This unique solution of Variational problem (1.1.11) *also is the unique solution to the following Minimization problem: find* $\psi\in\mathfrak{H}$ *such that*

$$\min_{\psi\in\mathfrak{B}}\left[\frac{1}{2}a(\psi,\psi)-L(\psi)\right]=\frac{1}{2}a(\varphi,\varphi)-L(\varphi), \tag{1.1.13}$$

i.e., which minimizes the functional

$$F(\psi):=\frac{1}{2}a(\psi,\psi)-L(\psi)\,. \tag{1.1.14}$$

Proof. From coerciveness (1.1.2) and the continuity of the bilinear form in (1.1.11) follows

$$C\|\varphi\big|\mathfrak{H}\|^2\leqslant a(\varphi,\varphi)\leqslant M\|\varphi\big|\mathfrak{H}\|^2\,;$$

hence the equality

$$\varphi\mapsto\|\varphi\big|\mathfrak{H}\|_0:=[a(\varphi,\varphi)]^{1/2}\,,\ \ \varphi\in\mathfrak{H}, \tag{1.1.15}$$

defines an equivalent norm on $\mathfrak{H}$. Moreover, $a(\varphi,\psi)$ defines an alternative scalar product on a Hilbert space $\mathfrak{H}$. According the Riesz representation theorem for a given $\psi\in\mathfrak{H}$ there exists one and only one element $\varphi\in\mathfrak{H}$ such that (1.1.11) holds. Thus, we have found the unique solution to linear equation (1.1.11) with a prescribed $\psi\in\mathfrak{H}$.

Returning to the Minimization problem: a direct verification shows that

$$F(\varphi+\psi)=F(\varphi)+\big[a(\varphi,\psi)-L(\psi)\big]+\frac{1}{2}a(\psi,\psi)\,. \tag{1.1.16}$$

The obtained equality can be interpreted as the Taylor expansion of the functional $F(\varphi+\psi)$ (note that $F'(\psi)\varphi=\big[a(\varphi,\psi)-L(\psi)\big]$ and $F''(\psi)(\varphi,\varphi)=\frac{1}{2}a(\psi,\psi)$). Then

$$a(\varphi,\psi)-L(\psi)=0\quad\text{for}\ \ \text{all}\quad\psi\in\mathfrak{H}$$

implies

$$F(\varphi+\psi)-F(\varphi)=\frac{1}{2}a(\psi,\psi)\geqslant\frac{C}{2}\|\psi\big|\mathfrak{H}\|^2\quad\forall\psi\in\mathfrak{H}$$

and, thus, $\varphi\in\mathfrak{H}$ is the minimizer of the functional F under the asserted condition $a(\varphi,\psi)-L(\psi)=0$.

Conversely: Let $\varphi\in\mathfrak{H}$ be the minimizer of F and $\psi\in\mathfrak{H}$ be arbitrary. The inequality (cf. (1.1.16))

$$0\leqslant F(\varphi+\theta\psi)-F(\varphi)=\theta\big\{a(\varphi,\psi)-L(\psi)\big\}+\frac{\theta^2}{2}a(\psi,\psi)\quad\forall\,\theta\in\mathbb{R}$$

implies that $a(\varphi,\psi)=L(\psi)$, since the first summand in the right-hand side of the equality dominates for small θ and the second is non-negative. Indeed, if $a(\varphi,\psi)\neq L(\psi)$ the difference $F(\varphi+\theta\psi)-F(\varphi)$ would become negative for certain small θ, which is a contradiction. □

In conclusion of the present section, we expose definitions an propositions about Fredholm operators. We drop the proofs since the results are well known and exposed, for example, in [83, 89, 130] and also in many other books.

Further, we assume that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are Banach spaces.

For a linear operator $A : \mathfrak{B}_1 \to \mathfrak{B}_2$ by **Ker** A is denoted the kernel, i.e., the linear space of all solutions of homogeneous equation $A\varphi = 0$, $\varphi \in \mathfrak{B}_1$.

By $\Im A := A\mathfrak{B}_1$ is denoted the image (the range) of A.

By **Coker** $A := \mathfrak{B}_2/\Im A$ (or **Coker** $A_{\mathfrak{B}_1\to\mathfrak{B}_2}$) is denoted the dimension of the quotient space $\mathfrak{B}_2/\Im A$ in the algebraic sense, i.e., regardless of a topology.

Definition 1.1.3. An operator $A \in \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ is said to be **Fredholm** (or have the **Fredholm property**, and we write $A \in \mathscr{F}(\mathfrak{B}_1, \mathfrak{B}_2)$, if A has finite-dimensional kernel and cokernel

$$\dim \mathbf{Ker}\ A < \infty, \quad \dim \mathbf{Coker}\ A < \infty\,.$$

We say A is **normally solvable** if the image $\Im A$ is a closed subspace in $\mathfrak{B}_2$.

The index

$$\mathbf{Ind}\ A = \mathbf{Ind}_{\mathfrak{B}_1\to\mathfrak{B}_2} A := \dim \mathbf{Ker}\ A - \dim \mathbf{Coker}\ A = \dim \mathbf{Ker}\ A - \dim \mathbf{Ker}\ A^* \tag{1.1.17}$$

(see below Proposition 1.1.2) maps the set of Fredholm operators into the group of integers $\mathbf{Ind} : \mathscr{F}(\mathfrak{B}_1, \mathfrak{B}_2) \to \mathbb{Z}$.

Proposition 1.1.1. *A linear operator $A \in \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ is normally solvable if and only if the equation $A\varphi = \psi$ has a solution $\varphi \in \mathfrak{B}_1$ only for those $\psi \in \mathfrak{B}_2$ for which the following orthogonality condition holds:*

$$F(\psi) = 0 \quad \textit{for all solutions} \quad A^*F = 0,$$

i.e., $\dim \mathbf{Coker}\ A = \dim \mathbf{Ker}\ A^*$.

Proposition 1.1.2. *Let $A \in \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ and* $\dim \mathbf{Coker}\ A < \infty$*. Operator A is normally solvable if and only if* $\dim \mathbf{Coker}\ A < \infty$ *and, then,* $\dim \mathbf{Coker}\ A = \dim \mathbf{Ker}\ A^*$*.*

Moreover, **Coker** A *can be identified (is isomorphic) with a linear space $\mathfrak{M}_A$ which is complementary to the image*

$$\Im A \oplus \mathfrak{M}_A = \mathfrak{B}_2\,. \tag{1.1.18}$$

Proposition 1.1.3. *If $A \in \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ is a Fredholm operator, then the adjoint operator $A^* \in \mathscr{L}(\mathfrak{B}_2^*, \mathfrak{B}_1^*)$ is Fredholm and*

$$\mathbf{Ind}\ A = \dim \mathbf{Ker}\ A - \dim \mathbf{Ker}\ A^* = -\,\mathbf{Ind}\ A^*\,. \tag{1.1.19}$$

Proposition 1.1.4. *Let $A \in \mathscr{B}(\mathfrak{B}_1, \mathfrak{B}_2)$ be a Fredholm operator between Banach spaces. There exists a small $\varepsilon > 0$ such that a perturbation $A + B + T$ by arbitrary operator $B \in \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ with a small norm $\|B\| < \varepsilon$ and by arbitrary compact operator $T \in \mathscr{C}(\mathfrak{B}_1, \mathfrak{B}_2)$ remains Fredholm $A + B + T \in \mathscr{F}(\mathfrak{B}_1, \mathfrak{B}_2)$. Moreover, such a perturbation has a stable index*

$$\mathbf{Ind}\,(A + B + T) = \mathbf{Ind}\ A\,. \tag{1.1.20}$$

Corollary 1.1.2. *For a compact operator $T \in \mathscr{C}(\mathfrak{B})$ in a Banach space $\mathfrak{B}$, the sum with the identity operator is Fredholm $I + T \in \mathscr{F}(\mathfrak{B})$ and*

$$\mathbf{Ind}\,(I + T) = 0\,. \tag{1.1.21}$$

Proposition 1.1.5. *A linear operator $A \in \mathscr{F}(\mathfrak{B}_1, \mathfrak{B}_2)$ is Fredholm if and only if there exists an operator $R \in \mathscr{F}(\mathfrak{B}_2, \mathfrak{B}_1)$, called* **regularizer***, such that*

$$RA = I - T_1, \quad AR = I - T_2, \tag{1.1.22}$$

where I is the identity operator in the corresponding space and $T_1 \in \mathscr{L}(\mathfrak{B}_1)$, $T_2 \in \mathscr{L}(\mathfrak{B}_2)$ are compact operators

The set of linear operators $\mathscr{L}(\mathfrak{B})$ is a Banach algebra with respect to the standard operator norm: if $A, B \in \mathscr{L}(\mathfrak{B})$, then the compositions AB, BA also belong to $\mathscr{L}(\mathfrak{B})$ and $\mathscr{C}(\mathfrak{B})$ is an ideal in $\mathscr{L}(\mathfrak{B})$ and $\|AB\| \leqslant \|A\|\|B\|$, $\|BA\| \leqslant \|A\|\|B\|$.

The subset of compact operators $\mathscr{C}(\mathfrak{B}) \subset \mathscr{L}(\mathfrak{B})$ is an ideal in $\mathscr{L}(\mathfrak{B})$:

$$AT, TA \in \mathscr{C}(\mathfrak{B}) \quad \text{for all} \quad A \in \mathscr{L}(\mathfrak{B}), \quad T \in \mathscr{C}(\mathfrak{B}).$$

Therefore, the quotient $\mathscr{L}(\mathfrak{B})/\mathscr{C}(\mathfrak{B})$ represents a Banach algebra and is known as the **Calkin algebra**. The norm in the Calkin algebra, the usual quotient norm

$$|||A||| := \inf_{T \in \mathscr{C}(\mathfrak{B})} \|A + T\| \tag{1.1.23}$$

is called the **essential norm** of A.

Note that definition (1.1.23) of the essential norm extends, obviously, to more general setting of all operators $A \in \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$.

Corollary 1.1.3. *$A \in \mathscr{L}(\mathfrak{B})$ is a Fredholm operator $A \in \mathscr{F}(\mathfrak{B})$ if and only if the coset (the quotient class) $[A]$ is invertible in the Calkin algebra $\mathscr{L}(\mathfrak{B})/\mathscr{C}(\mathfrak{B})$.*

Proposition 1.1.6. *Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B}_3$ be Banach spaces and $A \in \mathscr{F}(\mathfrak{B}_1, \mathfrak{B}_2)$, $B \in \mathscr{F}(\mathfrak{B}_2, \mathfrak{B}_3)$ be Fredholm operators. Then the composition $BA \in \mathscr{F}(\mathfrak{B}_1, \mathfrak{B}_3)$ is a Fredholm operator and*

$$\mathbf{Ind}\, BA = \mathbf{Ind}\, B + \mathbf{Ind}\, A. \tag{1.1.24}$$

We will expose proofs of the next assertions, because they are not well known.

Let $\mathbf{Coker}\, A_{\mathfrak{B}_k \to \mathfrak{D}_k}$ denote a direct complement to the image $\Im A_{\mathfrak{B}_k \to \mathfrak{D}_k}$, which is not unique in general.

Theorem 1.1.1. *Let*

$$A \in \mathscr{F}(\mathfrak{B}_1, \mathfrak{D}_1) \cap \mathscr{F}(\mathfrak{B}_2, \mathfrak{D}_2),$$

where $\mathfrak{B}_1$, $\mathfrak{D}_1$, $\mathfrak{B}_2$ and $\mathfrak{D}_2$ are Banach spaces and the first embedding

$$\mathfrak{B}_1 \subset \mathfrak{B}_2, \quad \mathfrak{D}_1 \subset \mathfrak{D}_2 \tag{1.1.25}$$

holds, while the second embedding is dense. If the indices of A in both pairs of spaces coincide

$$\mathbf{Ind}\, A_{\mathfrak{B}_1 \to \mathfrak{D}_1} = \mathbf{Ind}\, A_{\mathfrak{B}_2 \to \mathfrak{D}_2}, \tag{1.1.26}$$

then the corresponding kernels and the cokernels coincide as well:

$$\begin{aligned} \mathbf{Ker}\, A_{\mathfrak{B}_1 \to \mathfrak{D}_1} &= \mathbf{Ker}\, A_{\mathfrak{B}_2 \to \mathfrak{D}_2}, \\ \mathbf{Coker}\, A_{\mathfrak{B}_1 \to \mathfrak{D}_1} &= \mathbf{Coker}\, A_{\mathfrak{B}_2 \to \mathfrak{D}_2}. \end{aligned} \tag{1.1.27}$$

Proof. Due to the first embedding in (1.1.25)

$$\alpha_1 \leqslant \alpha_2, \quad \text{where} \quad \alpha_k := \dim \mathbf{Ker}\, A_{\mathfrak{B}_k \to \mathfrak{D}_k}. \tag{1.1.28}$$

Since

$$\dim \mathbf{Coker}\, A_{\mathfrak{B}_k \to \mathfrak{D}_k} = \dim \mathbf{Ker}\, A^*_{\mathfrak{D}^*_k \to \mathfrak{B}^*_k}$$

(see Proposition 1.1.2), the density of the second embedding in (1.1.25) yields $\mathfrak{D}^*_2 \subset \mathfrak{D}^*_1$. Indeed, any functional $F \in \mathfrak{D}^*_2$ is automatically included in $\mathfrak{D}^*_1$: we get $|(F, u) \leqslant \|F\|\|u|\mathfrak{D}_2\| \leqslant C\,\|F\|\|u|\mathfrak{D}_1\|$ and, therefore, $F \in \mathfrak{D}^*_1$.

On the other hand, any non-trivial functional $F \in \mathfrak{D}^*_2$, $F \neq 0$, restricted to $\mathfrak{D}_1$, does not vanish $F\big|_{\mathfrak{D}_1} \neq 0$. Otherwise, the dense embedding $\mathfrak{D}_1 \subset \mathfrak{D}_2$ implies $F = 0$. This completes the proof that the embedding $\mathfrak{D}^*_2 \subset \mathfrak{D}^*_1$ holds.

Analogously to (1.1.28) we get

$$\beta_2 \leqslant \beta_1, \quad \text{where} \quad \beta_k := \dim \mathbf{Coker}\, A_{\mathfrak{B}_k \to \mathfrak{D}_k}. \tag{1.1.29}$$

From (1.1.28), (1.1.29) and (1.1.26), which can yet be written as follows

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2,$$

we obtain

$$0 \geqslant \beta_1 - \beta_2 = \alpha_1 - \alpha_2 \leqslant 0.$$

The latter relations show that $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$. These equalities and embedding (1.1.25) entail (1.1.27). □

Lemma 1.1.5. *Let $\mathfrak{H}$ be a Hilbert space. If $\boldsymbol{A} \in \mathscr{L}(\mathfrak{H}, \mathfrak{H}^*)$ is coercive, then $\boldsymbol{A}$ is an invertible operator.*

Proof. Due to the coerciveness (see (1.1.2)), from $\boldsymbol{A}\varphi = 0$, $\varphi \in \mathfrak{H}$, it follows $\|\varphi\| = 0$. Therefore, $\mathbf{Ker}\ \boldsymbol{A} = \{0\}$.

Again, due to coerciveness (1.1.2), the convergence $\psi_n \to \psi$, where

$$\{\psi_n = \boldsymbol{A}\varphi_n\}_{n=1}^{\infty} \subset \Im\, \boldsymbol{A}, \quad \psi \in \mathfrak{H}, \quad \{\psi_n = \boldsymbol{A}\varphi_n\}_{n=1}^{\infty} \subset \mathfrak{H},$$

implies the convergence $\varphi_n \to \varphi \in \mathfrak{H}$, since

$$\|\psi_n - \psi_k\| = \|\boldsymbol{A}\varphi_n - \boldsymbol{A}\varphi_k\| \geqslant C\|\varphi_n - \varphi_k\|$$

with some fixed constant C independent of $\boldsymbol{A}$. Then, due to the continuity of $\boldsymbol{A}$, we conclude that $\psi = \lim_{n\to\infty} \boldsymbol{A}\varphi_n = \boldsymbol{A}\varphi \in \Im\, \boldsymbol{A}$ and, therefore, $\Im\, \boldsymbol{A}$ is closed (i.e., $\boldsymbol{A}$ is normally solvable).

From the coerciveness inequality (1.1.2) we also get

$$|(\varphi, \boldsymbol{A}^*\varphi)| \;=\; (\boldsymbol{A}\varphi, \varphi) \;\geqslant\; C\|\varphi\|^2$$

and, as above, conclude that $\mathbf{Ker}\ \boldsymbol{A}^* = \{0\}$.

Due to the inverse mapping theorem $\boldsymbol{A}$ is invertible. □

Corollary 1.1.4. *If $\mathfrak{H}$ is a Hilbert space, $\boldsymbol{A} \in \mathscr{L}(\mathfrak{H})$ is a linear, $\mathbf{T} \in \mathscr{C}(\mathfrak{H})$ is a compact operator and $\boldsymbol{A} + \mathbf{T} \in \mathscr{L}(\mathfrak{H})$ is coercive*

$$|\ ((\boldsymbol{A} + \mathbf{T})\varphi, \varphi)\ | \geqslant \gamma\|\varphi\|^2 \quad \forall \varphi \in \mathfrak{H},$$

then $\boldsymbol{A}$ is a Fredholm operator and $\mathbf{Ind}\ \boldsymbol{A} = 0$.

Proof. Due to Lemma 1.1.5, $\boldsymbol{A} + \mathbf{T}$ is invertible, while, due to Proposition 1.1.4, the difference $\boldsymbol{A} = (\boldsymbol{A} + \mathbf{T}) - \mathbf{T}$ is Fredholm and $\mathbf{Ind}\ \boldsymbol{A} = \mathbf{Ind}\,(\boldsymbol{A} + \mathbf{T}) = 0$. □

Lemma 1.1.6. *Let $\mathfrak{H}$ be a Hilbert space and $\boldsymbol{A} \in \mathscr{L}(\mathfrak{H})$. If the inequality*

$$|(\boldsymbol{A}\varphi, \varphi)| \;\geqslant\; C\|\varphi\|^2 - \|\mathbf{T}\varphi\|^2 \quad \forall \varphi \in \mathfrak{H} \tag{1.1.30}$$

holds for some constant $C > 0$ and a compact operator $\mathbf{T} \in \mathscr{C}(\mathfrak{B}, \mathfrak{D})$, then $\boldsymbol{A}$ is a Fredholm operator.

Proof. From (1.1.30) it follows that

$$\|\varphi\| \;\leqslant\; \frac{1}{\sqrt{C}}\,\|\mathbf{T}\varphi\| \quad \forall \varphi \in \mathbf{Ker}\ \boldsymbol{A}. \tag{1.1.31}$$

Then, due to the compactness of $\mathbf{T}$, from a bounded sequence $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbf{Ker}\ \boldsymbol{A}$ we can always select a convergent subsequence. That means $\mathbf{Ker}\ \boldsymbol{A} \subset \mathfrak{H}$ is a locally compact subspace and $\dim \mathbf{Ker}\ \boldsymbol{A} < \infty$ (only finite dimensional spaces are locally compact).

Since $\dim \mathbf{Ker}\ \boldsymbol{A} < \infty$, the linear closed set $\mathbf{Ker}\ \boldsymbol{A}$ has a complemented space $\mathfrak{H}_0 \oplus \mathbf{Ker}\, \boldsymbol{A} = \mathfrak{H}$ for some $\mathfrak{H}_0 \subset \mathfrak{H}$.

Assume that the operator $\boldsymbol{A}$ is not normally solvable. Then the operator $\boldsymbol{A} : \mathfrak{H}_0 \to \mathfrak{H}$ is not as well because they have the same ranges $\boldsymbol{A}(\mathfrak{H}_0) = \boldsymbol{A}(\mathfrak{H}) =: \Im\, \boldsymbol{A}$. Then there exists a sequence $(\varphi_j)_1^{\infty}$ in $\mathfrak{H}_0$ such that $\|\varphi_j\| = 1$, $\|\boldsymbol{A}\varphi_j\| \to 0$ as $j \to \infty$. Taking into account that $\mathbf{T}$ is compact, we can choose a subsequence which is transformed by $\mathbf{T}$ into a convergent sequence. For brevity, we assume that this subsequence is again the same sequence $\{\varphi_j\}_1^\infty$. By applying (1.1.30) we proceed as follows

$$\begin{aligned}
C\|\varphi_j-\varphi_k\|^2 &\leqslant \|\mathbf{T}(\varphi_j-\varphi_k)\|^2+|(\boldsymbol{A}(\varphi_j-\varphi_k),\varphi_j-\varphi_k)|\\
&\leqslant \|\mathbf{T}\varphi_j-\mathbf{T}\varphi_k\|^2+\|\boldsymbol{A}\varphi_j-\boldsymbol{A}\varphi_k\|\|\varphi_j-\varphi_k\|\\
&\leqslant \|\mathbf{T}\varphi_j-\mathbf{T}\varphi_k\|^2+(\|\boldsymbol{A}\varphi_j\|+\|Ax_k\|)\,(\,\|\varphi_j-\varphi_k\|). \qquad (1.1.32)
\end{aligned}$$

For sufficiently large j and k the right–hand side of (1.1.32) becomes arbitrarily small and the sequence $\{\varphi_j\}_{j=1}^\infty$ converges in $\mathfrak{H}_0$: $\lim\limits_{j\to\infty}\varphi_j=\varphi_0\in\mathfrak{H}_0$. Obviously, $\boldsymbol{A}\varphi_0=\lim\limits_{j\to\infty}\boldsymbol{A}\varphi_j=0$ and since $\varphi_0\in\mathfrak{H}_0\cap\mathbf{Ker}\,\boldsymbol{A}=\{0\}$, we get $\varphi_0=0$. This contradicts the equality $\|\varphi_0\|=\lim\limits_{j\to\infty}\|\varphi_j\|=1$ and the obtained contradiction proves that the operator $\boldsymbol{A}$ has the closed range $\Im\,\boldsymbol{A}$.

Let us rewrite inequality (1.1.30) in the form

$$|(\varphi,\boldsymbol{A}^*\varphi)|\ \geqslant C\|\varphi\|^2-\|\mathbf{T}\varphi\|^2,\quad \forall\,\varphi\in\mathfrak{H}. \qquad (1.1.33)$$

We already know that if the operator has the closed range $\Im\,A$, inequality (1.1.33) implies

$$\dim\,\mathbf{Coker}\,\boldsymbol{A}=\dim\,\mathbf{Ker}\,\boldsymbol{A}^*<\infty$$

and, thus, $\boldsymbol{A}$ is a Fredholm operator. □

1.2 Differentiation and implicit function theorem

In the present section, we expose implicit and inverse function theorems, which are applied later.

Let us recall some standard notation: $\mathbb{N}:=\{1,2,\dots\}$, $\mathbb{N}_0:=\{0,1,\dots\}$. For a natural number $n\in\mathbb{N}$ let $\mathbb{R}^n$ and C^n denote the n-dimensional spaces of vectors $x=(x_1,\dots,x_n)^\top$ with real $x_j\in\mathbb{R}$ and complex $x_j\in\mathbb{C}$ entries and standard metrics, based on the scalar product

$$\begin{aligned}
\langle x,y\rangle &:= x_1\overline{y_1}+\cdots+x_n\overline{y_n}\ \text{ for }\ x,y\in C^n,\\
\langle x,y\rangle &:= x_1y_1+\cdots+x_ny_n\ \text{ for }\ x,y\in\mathbb{R}^n.
\end{aligned}$$

$\mathbb{N}^n$ and $\mathbb{N}_0^n$ denote the sets of n-tuples multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$ with components from the corresponding sets and we use the notation

$$\partial^\alpha u(x)=\partial_x^\alpha u(x):=\frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},\ \ \partial_j:=\frac{\partial}{\partial x_j},\ \ j=1,2,\dots,n,\ \ \alpha\in\mathbb{N}_0^n,\ \ |\alpha|:=\alpha_1+\cdots+\alpha_n. \quad (1.2.1)$$

Let $\Omega\subset\mathbb{R}^n$ be an open domain. A continuous function $\Phi:\Omega\to\mathbb{R}^m$ is called **differentiable** at a point $x\in\Omega$ with **derivative** $D\Phi(x):\mathbb{R}^n\to\mathbb{R}^m$ if $D\Phi(x)$ is a linear mapping (i.e., a matrix) and

$$\Phi(x+y)=\Phi(x)+D\Phi(x)y+R(x,y),\ \ R(x,y)=\mathscr{o}(|y|)\ \text{ as }\ |y|\to 0 \qquad (1.2.2)$$

for small $y\in\mathbb{R}^n$.

With respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, the derivative $D\Phi(x)$ is the matrix of partial derivatives

$$D\Phi(x)=\big([\partial_j\Phi_k(x)]_{n\times m}\big)^\top \qquad (1.2.3)$$

and transforms a column vector $U=(u_1,\dots,u_n)^\top$ into the new column vector

$$D\Phi(x)U=\Big(\sum_{j=1}^n\partial_j\Phi_1(x)u_j,\dots,\sum_{j=1}^n\partial_j\Phi_m(x)u_j\Big)^\top.$$

The matrix $D\Phi$ in (1.2.3) is called the **Jacobi matrix**. If $n=m$, the corresponding determinant is called **Jacobi determinant** or **Jacobian**.

Φ is differentiable whenever all the partial derivatives exist.

Let $\Omega \subset \mathbb{R}^n$ be an open domain (Ω can be non-compact, e.g., $\Omega = \mathbb{R}^n$). For $r, m \in \mathbb{N}_0$, by $C^r(\Omega, \mathbb{R}^m)$ (or by $C^r(\Omega)$) it is denoted the set of r-times continuously differentiable mappings $\Phi : \Omega \to \mathbb{R}^m$ and $C^\infty(\Omega, \mathbb{R}^m) := \bigcap\limits_{r=1}^{\infty} C^r(\Omega, \mathbb{R}^m)$.

The set of complex-valued mappings will be denoted by $C^r(\Omega, C^m)$ (or by $C^r(\Omega)$).

The subspace $\mathbb{C}_0^\infty(\Omega)$ consists of infinitely differentiable functions on Ω with compact supports.

A composition of functions

$$F = \Psi \circ \Phi : \Omega \to \mathbb{R}^k, \ \ \Phi : \Omega \to \mathscr{M} \subset \mathbb{R}^m, \ \ \Phi : \mathscr{M} \to \mathbb{R}^k,$$

where Φ is differentiable at a point $x \in \Omega$ and Ψ is differentiable at a point $z = \Phi(x) \in \mathscr{M}$, is differentiable at a point x and the following **chain rule** holds:

$$D(\Psi \circ \Phi)(x) = (D\Psi)(\Phi(x)) D\Phi(x). \tag{1.2.4}$$

Let us recall that $\Omega \subset \mathbb{R}^n$ is called a **star-like domain** with respect to the point $x_0 \in \Omega$ if $y \in \Omega$ implies $x_0 + t(y - x_0) \in \Omega$ for all $0 \leqslant t \leqslant 1$.

The fundamental theorem of calculus, applied to $\varphi(t) = \Phi(x + ty)$ in a star-like domain with respect to $x \in \Omega$, gives the **Lagrange formula**

$$\Phi(x + y) = \Phi(x) + \int_0^1 D\Phi(x + ty) y \, dt = \Phi(x) + D\Phi(x + t_0 y) y \tag{1.2.5}$$

for $\Phi \in C^1(\Omega)$, all $y \in \Omega$ and some $0 \leq t_0 < 1$.

Let us consider a function

$$\Phi : \Omega \to \mathbb{R}^n, \quad \Phi \in C^k, \tag{1.2.6}$$

which maps a domain $\Omega \subset \mathbb{R}^n$ to the same Euclidean space and $\Phi(x_0) = y_0$. It is important to know the conditions ensuring the existence of the **inverse mapping**

$$\Phi^{-1} : \boldsymbol{V} \to \boldsymbol{U} \subset \Omega, \ \ \Phi(\Phi^{-1}(y)) \equiv y, \ \ y \in \boldsymbol{V}, \tag{1.2.7}$$

and its smoothness properties, at least locally, in a neighborhood of some y_0. The next inverse function theorem provides such conditions and, together with the implicit function theorem (cf. Theorem 1.2.2), represents most fundamental results of multivariable analysis.

Theorem 1.2.1 (Inverse function theorem)**.** *Let Ω be a domain in $\mathbb{R}^n$, $k \in \mathbb{N}$ and $\Phi \in C^k(\Omega, \mathbb{R}^n)$. Let the differential $D\Phi(x)$ be an invertible matrix at $x_0 \in \Omega$ and $\Phi(x_0) = y_0 \in \mathbb{R}^n$.*

*There exist neighborhoods $\boldsymbol{U} \subset \Omega$ of x_0 and $\boldsymbol{V} \subset \mathbb{R}^n$ of y_0 such that the mapping $\Phi : \boldsymbol{U} \to \boldsymbol{V}$ is one-to-one and the inverse mapping $\Phi^{-1} : \boldsymbol{V} \to \boldsymbol{U}$ is C^k-smooth (i.e., Φ^{-1} is a C^k-**diffeomorphism**).*

Proof. Let

$$\Psi(x) := (D\Phi)(x_0)]^{-1}[\Phi(x_0 + x) - y_0]. \tag{1.2.8}$$

Then, obviously,

$$\Psi(0) = 0 \ \text{ and } \ (D\Psi)(0) = I.$$

Thus, the case reduces to $\Phi(0) = 0$, $(D\Phi)(0) = I$, $0 \in \Omega$, which we suppose fulfilled. Then we have to solve the equation $\Phi(u) = v$ for small v. Due to formula (1.2.2), this can be written as an equation

$$u + R(u) = v, \ \ R(0) = 0, \ \ (DR)(0) = 0, \ \text{ where } \ R(u) = \mathscr{O}(|u|), \tag{1.2.9}$$

with the mapping $R \in C^{k-1}(\Omega, \mathbb{R}^n)$. Solving (1.2.9) is equivalent to solving

$$T_v(u) = u, \ \ T_v(u) = v - R(u). \tag{1.2.10}$$

Thus, we look for a fixed point $u = K(v) = \Phi^{-1}(v)$ and show that $(DK)(0) = I$ or, equivalently, $K(v) = v + \mathscr{o}(|v|)$. The latter implies that for all x close to the origin (small enough),

$$(DK)(x) = (D\Psi(K(x)))^{-1} \tag{1.2.11}$$

and, taking further derivatives, by induction it follows that $K \in C^k$. To implement this idea we consider a metric space

$$\mathfrak{M}_v := \{u \in \Omega : |u - v| \leqslant \mathscr{A}_v\},$$

where (cf. (1.2.2) and (1.2.9))

$$\mathscr{A}_v := \sup_{|w|\leqslant 2|v|} |R(w)| = \mathscr{o}(|w|) = \mathscr{o}(|v|). \tag{1.2.12}$$

Let us check that $\mathfrak{M}_v$ is invariant under the mapping

$$T_v : \mathfrak{M}_v \to \mathfrak{M}_v \tag{1.2.13}$$

provided that v is small enough. Indeed, since $T_v(u) - v = -R(u)$, we only need to check that $|R(u)| \leqslant \mathscr{A}_v$ for all $u \in \mathfrak{M}_v$ provided that v is small enough. Indeed, if $u \in \mathfrak{M}_v$, then, due to (1.2.12), $|u| \leqslant |v| + \mathscr{A}_v \leqslant 2|v|$ for v small enough and

$$|R(u)| \leqslant \sup_{|w|\leqslant 2|v|} |R(w)| = \mathscr{A}_v.$$

This completes the proof of the mapping property (1.2.13).

Due to the Lagrange formulae (1.2.5) and the property $(DR)(0) = 0$ (see (1.2.5), by taking v sufficiently small, mapping (1.2.13) becomes a contraction

$$|T_v(u) - T_v(w)| = |R(u) - R(w)| = |(DR)(u + t_0(w - u))(u - w)| \leqslant r|u - w|, \quad 0 < r < 1.$$

Then, by virtue of the fixed point theorem, there exists a unique fixed point $u = K(v) \in \mathfrak{M}_v$. Moreover, from $u \in \mathfrak{M}_v$ we conclude that

$$|K(v) - v| = |u - v| \leqslant \mathscr{A}_v = \mathscr{o}(|v|).$$

This completes the proof. □

Theorem 1.2.2 (Implicit function theorem)**.** *Let $\Omega \subset \mathbb{R}^m$, $\mathscr{E} \subset \mathbb{R}^n$ be domains and $k = 1, 2, \dots$. Let $\Psi(x, y) : \Omega \times \mathscr{E} \to \mathbb{R}^n$ be a C^k-mapping, $\Psi(x_0, y_0) = 0$ and the partial $n \times n$ Jacobi matrix $D_y\Psi(x, y)$ be invertible at $(x_0, y_0) \in \Omega \times \mathscr{E}$.*

*There exists a neighborhood $U_0 \subset \Omega$ of x_0 and a C^k-smooth mapping $y = \psi(x)$, $\psi : U_0 \to \mathscr{E}$ (called the **implicit function**) such that $\Psi(x, \psi(x)) \equiv 0$.*

The function $\psi(x)$ is unique: if there exists another continuous implicit function $\psi_1 : U^1 \to \mathscr{E}$, the functions coincide $\psi_1(x) = \psi(x)$ in the common neighborhood $x \in U^0 \cap U^1$ of x_0.

Proof. Consider the mapping $\Phi : \Omega \times \mathscr{E} \to \mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Phi(x, y) := \big(x, \Psi(x, y)\big). \tag{1.2.14}$$

The corresponding differential (the Jacobi matrix)

$$(D_{(x,y)}\Phi) = \begin{pmatrix} I & D_x\Psi \\ 0 & D_y\Psi \end{pmatrix} \tag{1.2.15}$$

is, obviously, invertible. Therefore, by virtue of the foregoing Theorem 1.2.1, there exists the inverse function $\Phi^{-1} : V^0 \times U_0 \to \mathbb{R}^m \times \mathbb{R}^n$ and at the point (x, y_0) acquires the form

$$\Phi^{-1}(x, y_0) = (x, \psi(x, y_0)).$$

The function $\psi(x) = \psi(x, y_0)$ is the desired implicit function.

The uniqueness of the implicit function follows since, according to Theorem 1.2.1, there exists only the unique inverse function to $\Phi(x, y) = (x, \Psi(x, y))$. □

1.3 Calculus of tangent differential operators

The content of the present section follows from [64, § 4] with a slight modification.

Throughout the present section we keep the convention similar to that in Introduction: Let $\mathscr{S}$ be a hypersurface given by a collection of charts $\{(\mathscr{S}_j, \Theta_j)\}_{j=1}^M$, where

$$\Theta_j : \omega_j \to \mathscr{S}_j, \quad \mathscr{S} = \bigcup_{j=1}^M \mathscr{S}_j, \quad \omega_j \subset \mathbb{R}^{n-1}, \quad j = 1, \dots, M. \tag{1.3.1}$$

The corresponding differentials

$$D\Theta_j(p) := \mathbf{matr}\left[\partial_1\Theta_j(p), \dots, \partial_{n-1}\Theta_j(p)\right] \tag{1.3.2}$$

have the full rank

$$\operatorname{rank} D\Theta_j(p) = n-1, \quad \forall\, p \in \Omega, \quad j = 1, \dots, M,$$

i.e., all points of Ω are regular for Θ_j.

The derivatives

$$\boldsymbol{g}_k = \partial_k \Theta_j, \quad k = 1, \dots, n-1, \tag{1.3.3}$$

are tangent vector fields on $\mathscr{S}_j$ and this system is a basis in the space of tangent vector fields $\omega(\mathscr{S})_j$. The symmetric **Gram matrix**

$$G_{\mathscr{S}}(x) := \left[\langle \boldsymbol{g}_k(\mathscr{X}), \boldsymbol{g}_m(\mathscr{X})\rangle\right]_{n-1\times n-1} = \left[\langle \partial_k\Theta_j(x), \partial_m\Theta_j(x)\rangle\right]_{n-1\times n-1}, \quad x \in \omega_j \subset \mathbb{R}^{n-1}, \tag{1.3.4}$$

defines the natural metric on the space of tangent vector fields $\omega(\mathscr{S}_j)$, which is inherited from the ambient space $\mathbb{R}^n$. Namely, for arbitrary tangent vectors

$$u_k(x) = \alpha_k^1 \partial_1\Theta_j(x) + \dots + \alpha_k^{n-1}\partial_{n-1}\Theta_j(x) \in \omega(\mathscr{S}_j), \quad \alpha_k^m \in \mathbb{R}, \quad k = 1, 2,$$

the inner product is defined by the bilinear **first fundamental form**

$$\langle u_1, u_2\rangle = \langle G_{\mathscr{S}} a_1, a_2\rangle, \quad a_k = (\alpha_k^1, \dots, \alpha_k^{n-1})^\top, \quad k = 1, 2. \tag{1.3.5}$$

The system of tangent vectors $\{\boldsymbol{g}_k\}_{k=1}^{n-1}$ to $\mathscr{S}$ (cf. (1.3.3)) is known as the **covariant basis**. There exists the unique system $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$, biorthogonal to it – the **contravariant basis**:

$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}, \quad j, k = 1, \dots, n-1.$$

The contravariant basis is defined by the formula

$$\boldsymbol{g}^k = \frac{1}{\det G_{\mathscr{S}}}\, \boldsymbol{g}_1 \wedge \dots \wedge \boldsymbol{g}_{k-1} \wedge \boldsymbol{\nu} \wedge \boldsymbol{g}_{k+1} \wedge \dots \wedge \boldsymbol{g}_{n-1}, \quad k = 1, \dots, n-1, \tag{1.3.6}$$

where $G_{\mathscr{S}}(\mathscr{X})$ is the Gram matrix (see (1.3.4)).

$\boldsymbol{\nu}_\Gamma(t)$ is the outer normal vector field to the boundary Γ, which is tangent to $\mathscr{S}$ and $\boldsymbol{\nu}(\mathscr{X})$ is the outer unit normal vector field to $\mathscr{S}$, which has the most important role in the calculus of tangent differential operators we are going to apply. The **unit normal vector field** to the surface $\mathscr{S}$, also known as the **Gauß mapping**, is defined by the vector product of the covariant basis

$$\boldsymbol{\nu}(\mathscr{X}) := \pm\frac{\boldsymbol{g}_1(\mathscr{X}) \wedge \dots \wedge \boldsymbol{g}_{n-1}(\mathscr{X})}{|\boldsymbol{g}_1(\mathscr{X}) \wedge \dots \wedge \boldsymbol{g}_{n-1}(\mathscr{X})|}, \quad \mathscr{X} \in \mathscr{S}. \tag{1.3.7}$$

The choice of sign in this formula determines the orientation of the hypersurface. In what follows, we will choose the orientation corresponding to the plus sign in (1.3.7).

Next, we expose yet another definition of a hypersurface – an **implicit** one.

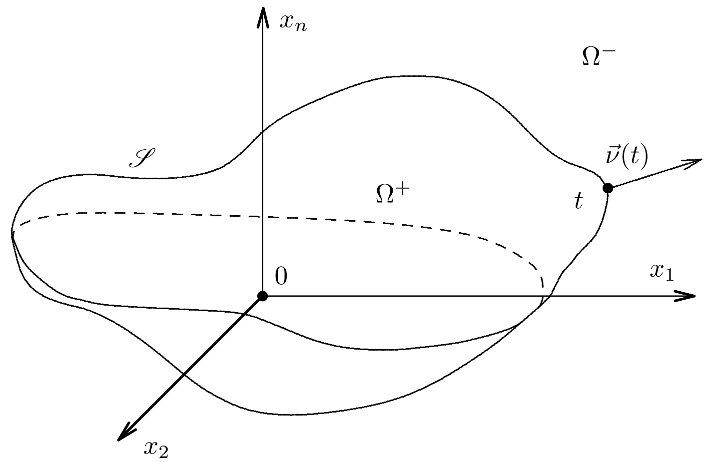

Figure 1.1.

Definition 1.3.1. Let $k \geq 1$ and $\omega \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth (an implicit Lipschitz) hypersurface in $\mathbb{R}^n$ is defined as the set

$$\mathscr{S} = \left\{ \mathscr{X} \in \omega : \Psi_{\mathscr{S}}(\mathscr{X}) = 0 \right\}, \tag{1.3.8}$$

where $\Psi_{\mathscr{S}} : \omega \to \mathbb{R}$ is a C^k-mapping (or Lipschitz mapping) which is regular: $\nabla\Psi(\mathscr{X}) \neq 0$.

Note that that the definition of a hypersurface $\mathscr{S}$ by charts in (1.3.1) and Definition 1.3.1 are equivalent and by taking a single function $\Psi_{\mathscr{S}}$ for the implicit definition of a hypersurface $\mathscr{S}$ we do not restrict the generality (see, e.g., [55]).

It is well known that using implicit surface functions gradient (see (1.3.8)) we can write an alternative definition of the unit normal vector field on the surface (see (1.3.7)):

$$\boldsymbol{\nu}(t) := \lim_{x \to t} \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|}, \quad t \in \mathscr{S}. \tag{1.3.9}$$

In applications it is necessary to extend the vector field $\boldsymbol{\nu}(t)$ in a neighborhood of $\mathscr{S}$, preserving some important features. Here is the precise definition of such extension.

Definition 1.3.2. Let $\mathscr{S}$ be a surface in $\mathbb{R}^n$ with the unit normal vector field $\boldsymbol{\nu}$. A vector field $\mathscr{N} \in C^1(\Omega_{\mathscr{S}})$ in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$ will be referred to as a **proper extension** if $\mathscr{N}\big|_{\mathscr{S}} = \boldsymbol{\nu}$, it is unitary ($|\mathscr{N}| = 1$) in $\Omega_{\mathscr{S}}$ and if $\mathscr{N}$ satisfies the condition

$$\partial_j \mathscr{N}_k(x) = \partial_k \mathscr{N}_j(x) \ \text{ for all } \ x \in \Omega_{\mathscr{S}}, \ \ j, k = 1, \ldots, n. \tag{1.3.10}$$

Such extension is needed, for example, to define correctly the normal derivative (the derivative along normal vector fields, outer or inner). It turned out that the "naive" extension (cf. (1.3.9))

$$\boldsymbol{\nu}(t) := \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|}, \quad x \in \Omega_{\mathscr{S}}, \tag{1.3.11}$$

is not proper. Indeed (see [66]), let $n = 2$ and $\mathscr{S}$ be the ellipse

$$\left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \ \ \Psi_{\mathscr{S}}(x_1, x_2) := x_1^2 + 2x_2^2 - 1 = 0 \right\}.$$

Then

$$\mathscr{N}(x) := \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|} = \frac{(x_1, 2x_2)}{\sqrt{x_1^2 + 4x_2^2}},$$

$$\partial_1 \mathscr{N}_2(x) = -\frac{2x_2 x_1}{(x_1^2 + 4x_2^2)^{3/2}}, \quad \partial_2 \mathscr{N}_1(x) = -\frac{4x_1 x_2}{(x_1^2 + 4x_2^2)^{3/2}}.$$

Hence $\partial_1\mathscr{N}_2(x)\neq\partial_2\mathscr{N}_1(x)$, unless $x_1=0$ or $x_2=0$.

For the proof of the next Proposition 1.3.1 and Corollary 1.3.1 on extension of the normal vector field we refer to [66].

Proposition 1.3.1. *Let $\mathscr{S}\subset\mathbb{R}^n$ be a hypersurface given by an implicit function*

$$\mathscr{S}=\big\{\mathscr{X}\in\mathbb{R}^n:\Psi_{\mathscr{S}}(\mathscr{X})=0\big\}.$$

Then the gradient $\nabla\Phi_{\mathscr{S}}(x)$ of the function

$$\Phi_{\mathscr{S}}(\mathscr{X}+t\boldsymbol{\nu}(\mathscr{X})):=t,\ \ \mathscr{X}+t\boldsymbol{\nu}(\mathscr{X})\in\Omega_{\mathscr{S}},\tag{1.3.12}$$

defined in the parameterized neighborhood

$$\Omega_{\mathscr{S}}:=\big\{x=\mathscr{X}+t\boldsymbol{\nu}(\mathscr{X}):\ \ \mathscr{X}\in\mathscr{S},\ \ -\varepsilon<t<\varepsilon\big\}$$

represents a unique proper extension of the unit normal vector field on the surface

$$\boldsymbol{\nu}(\mathscr{X})=\lim_{x\to\mathscr{X}}\nabla\Phi_{\mathscr{S}}(x),\ \ \mathscr{X}\in\mathscr{S}.$$

Corollary 1.3.1. *For any proper extension $\mathscr{N}(x)$, $x\in\Omega_{\mathscr{S}}\subset\mathbb{R}^n$, of the unit normal vector field $\boldsymbol{\nu}$ to the surface $\mathscr{S}\subset\Omega_{\mathscr{S}}$ the following equality holds:*

$$\partial_{\mathscr{N}}\mathscr{N}(x)=0\ \text{ for all }\ x\in\Omega_{\mathscr{S}}.\tag{1.3.13}$$

In particular, for the derivatives

$$\mathscr{D}_k=\partial_k-\mathscr{N}_k\partial_{\mathscr{N}},\ \ k=1,\dots,n,\tag{1.3.14}$$

which are extensions into the domain $\Omega_{\mathscr{S}}$ of Günter's derivatives $\mathscr{D}_k=\partial_k-\nu_k\partial_{\boldsymbol{\nu}}$ on the surface $\mathscr{S}$, the following equalities are valid:

$$\mathscr{D}_k\mathscr{N}_j=\partial_k\mathscr{N}_j-\mathscr{N}_k\partial_{\mathscr{N}}=\partial_k\mathscr{N}_j,\ \ \mathscr{D}_k\mathscr{N}_j=\mathscr{D}_j\mathscr{N}_k,\ \text{ for all }\ j,k=1,\dots,n.\tag{1.3.15}$$

In the sequel, we dwell on a proper extension and apply the properties of $\mathscr{N}$ listed above.

Lemma 1.3.1 (see [64])**.** *For an arbitrary unitary extension $\mathscr{N}(x)\in C^1(\Omega_{\mathscr{S}})$, $|\mathscr{N}(x)|\equiv1$, of $\boldsymbol{\nu}(\mathscr{X})$, in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, the following conditions are equivalent:*

(i) $\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}}=0$, *i.e., $\partial_{\mathscr{N}}\mathscr{N}_j(x)\to0$ for $x\to\mathscr{X}\in\mathscr{S}$ and $j=1,2,\dots,n$;*

(ii) $[\partial_k\mathscr{N}_j-\partial_j\mathscr{N}_k]\big|_{\mathscr{S}}=0$ *for $k,j=1,2,\dots,n$.*

The **second fundamental form** of $\mathscr{S}$ has the form

$$\begin{aligned}II(\boldsymbol{U}(\mathscr{X}),\boldsymbol{V}(\mathscr{X}))\boldsymbol{\nu}(\mathscr{X})&:=\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{X})-\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{X})\\&=(I-\pi_{\mathscr{S}})\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{X})=\langle\boldsymbol{\nu}(\mathscr{X})\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{X})\rangle\boldsymbol{\nu}(\mathscr{X}),\ \ \forall\,\mathscr{X}\in\mathscr{S},\ \ \boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S})\end{aligned}\tag{1.3.16}$$

and the **Weingarten matrix** (or the Weingarten mapping)

$$\mathscr{W}_{\mathscr{S}}:\omega(\mathscr{S})\to\omega(\mathscr{S})\tag{1.3.17}$$

is defined uniquely by the requirement that

$$\langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle=II(\boldsymbol{U},\boldsymbol{V})=\langle\boldsymbol{\nu},\partial_{\boldsymbol{U}}\boldsymbol{V}\rangle=-\langle\partial_{\boldsymbol{U}}\boldsymbol{\nu},\boldsymbol{V}\rangle=-\langle\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{\nu},\boldsymbol{V}\rangle,\ \ \forall\,\boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S}).\tag{1.3.18}$$

In the last equality in (1.3.18) we have applied the following: for a tangent vector field $\boldsymbol{V}\in\omega(\mathscr{S})$ there holds $\langle\boldsymbol{\nu}(\mathscr{X}),\boldsymbol{V}(\mathscr{X})\rangle\equiv0$, $\mathscr{X}\in\mathscr{S}$, and, by differentiating,

$$\big\langle\partial_{\boldsymbol{U}}\boldsymbol{\nu}(\mathscr{X}),\boldsymbol{V}(\mathscr{X})\big\rangle+\big\langle\boldsymbol{\nu}(\mathscr{X}),\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{X})\big\rangle\equiv0,\ \ \mathscr{X}\in\mathscr{S},\ \ j=1,\dots,n,\tag{1.3.19}$$

$$\text{for all }\ \boldsymbol{U}=\sum_{j=1}^nU_jd_j,\ \ \boldsymbol{V}=\sum_{j=1}^nV_jd_j,\ \ \boldsymbol{d}^j=\pi_{\mathscr{S}}\boldsymbol{e}^j,\ \ \partial^{\mathscr{S}}_{\boldsymbol{U}}:=\sum_{j=1}^nU_j\mathscr{D}_j.$$

We can extend the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(x)$ from the surface $\mathscr{S}$ to a neighbourhood as follows:

$$\mathscr{W}_{\mathscr{S}}(x):=-\nabla\mathscr{N}(x)=-\big[\partial_j\mathscr{N}_k(x)\big]_{n\times n},\ \ x\in\Omega_{\mathscr{S}}.\tag{1.3.20}$$

Lemma 1.3.2. *The extended Weingarten matrix* $\mathscr{W}_{\mathscr{S}}(x)$ *in* (1.3.20) *has the following properties:*

(i) $\mathscr{W}_{\mathscr{S}}(x)\mathscr{N}(x) = 0$ *for all* $x \in \Omega_{\mathscr{S}}$;

(ii) *even if extension* $\mathscr{N}(x)$ *is not proper, the restriction to the hypersurface* $\mathscr{W}_{\mathscr{S}}\big|_{\mathscr{S}}$ *coincides with the Weingarten mapping of* $\mathscr{S}$ *and only depends on* $\mathscr{S}$ *(is independent of the choice of the extension* $\mathscr{N}$*);*

(iii) *even if extension* $\mathscr{N}(x)$ *is not proper,* $\mathrm{Tr}(\mathscr{W}_{\mathscr{S}})\big|_{\mathscr{S}} = \mathscr{H}^0_{\mathscr{S}}$*, where* $\mathscr{H}^0_{\mathscr{S}}$ *is the mean curvature of* $\mathscr{S}$*;*

(iv) $\mathscr{W}_{\mathscr{S}}(x)\boldsymbol{V}(x)$*,* $x \in \mathscr{S}_C$*, is tangent to the level surface*

$$\mathscr{S}_C := \big\{y \in \mathbb{R}^n : \Psi_{\mathscr{S}}(y) = C := \Psi_{\mathscr{S}}(x)\big\} \tag{1.3.21}$$

for arbitrary vector field $\boldsymbol{V} : \mathscr{S} \to \mathbb{R}^n$.

Proof. First, $\mathscr{W}_{\mathscr{S}}\mathscr{N} = \nabla\|\mathscr{N}\|^2 = \nabla 1 = 0$ in $\Omega_{\mathscr{S}}$, justifying (i). Assertions (ii) and (iii) follow from Lemma 1.3.1.

Next, (iv) is proved as follows:

$$\langle \mathscr{N}(x), \mathscr{W}_{\mathscr{S}}\boldsymbol{V}(x)\rangle = -\sum_{j,k=1}^{n} \mathscr{N}_j(\partial_j\mathscr{N}_k)V_k = -\sum_{k=1}^{n}(\partial_{\mathscr{N}}\mathscr{N}_k)\omega = 0$$

due to (1.3.13), proved above. □

We remind that

$$G_{\mathscr{S}}(\mathscr{X}) = G(\mathscr{X}) = \big[g_{jk}(\mathscr{X})\big]_{n-1\times n-1}, \quad g_{jk} := \langle \boldsymbol{g}_j, \boldsymbol{g}_k\rangle,$$

is the positive definite Gram matrix, which is known as the **covariant Riemannian metric tensor** and defines the metric on the surface $\mathscr{S}$ (cf. Section 1.5).

Let $d\sigma = \sqrt{\det G_{\mathscr{S}}}dx$ and $d\mathfrak{s} = \sqrt{\det G_{\Gamma}}dx'$ stand for the volume elements on $\mathscr{S}$ and $\Gamma := \partial\mathscr{S}$, respectively ($x \in \mathbb{R}^{n-1}$, $x' \in \mathbb{R}^{n-2}$; cf. Section 1.5).

Let

$$P(\nabla)u = \sum_{j=1}^{n} a_j\partial_j u + bu, \quad a_j, b \in C^1(\mathbb{R}^{m\times m}), \tag{1.3.22}$$

be a first-order differential operator with real-valued (variable) matrix coefficients, acting on vector-valued functions ($u = (u_\beta)^n_{\beta=1}$) in $\mathbb{R}^n$, and its **principal symbol** is given by the matrix-valued function

$$\sigma(P;\xi) := \sum_{j=1}^{n} a_j\xi_j, \quad \xi = \{\xi_j\}^n_{j=1} \in \mathbb{R}^n. \tag{1.3.23}$$

Definition 1.3.3. We say that P is a **weakly tangent operator** to the hypersurface $\mathscr{S}$, with unit normal $\boldsymbol{\nu}$, provided that

$$\sigma(P;\boldsymbol{\nu}) = 0 \ \text{ on the hypersurface } \mathscr{S}. \tag{1.3.24}$$

Next, call P a **strongly tangent operator** to $\mathscr{S}$ provided that the symbol vanishes,

$$\sigma(P;\mathscr{N}) = 0 \ \text{ in an open neighborhood of } \mathscr{S} \text{ in } \mathbb{R}^n \tag{1.3.25}$$

on a proper extension of the unit normal vector field $\mathscr{N}$ in some neighbourhood of the surface $\mathscr{S}$ (see Definition 1.3.2.

Note that in a strongly tangent operator the coordinate derivatives ∂_j can be replaced by the Günter's derivatives $\mathscr{D}_j$:

$$P(\nabla)u = \sum_{j=1}^{n} a_j\partial_j u + bu = \sum_{j=1}^{n} a_j\mathscr{D}_j u + bu = P(\mathscr{D})u, \quad a_j, b \in C^1(\mathbb{R}^{m\times m}). \tag{1.3.26}$$

The most important tangent differential operators to the hypersurface for us are:

A. The weakly tangent Günter derivatives (see (0.0.9))

$$\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}} = \partial_j - \nu_j\sum_{k=1}^{n}\nu_k\partial_k, \quad j = 1,\dots,n.$$

B. The weakly tangent Stokes derivatives $\mathscr{M}_{jk} = \nu_j\partial_k - \nu_k\partial_j$ (for details see Section 1.6 below).

Günter and Stokes derivatives are tangent, since the corresponding vector fields are tangent

$$\begin{gathered}\mathscr{D}_j := \partial_{\boldsymbol{d}^j} = \boldsymbol{d}^j\cdot\nabla, \quad \mathscr{M}_{jk} := \partial_{\mathfrak{m}_{jk}} = \mathfrak{m}_{jk}\cdot\nabla,\\ \boldsymbol{d}^j := \pi_{\mathscr{S}}\boldsymbol{e}^j = \boldsymbol{e}^j - \nu_j\boldsymbol{\nu} = \boldsymbol{\nu}\wedge(\boldsymbol{\nu}\wedge\boldsymbol{e}^j) = \sum_{k=1}^{n}(\delta_{jk} - \nu_j\nu_k)\boldsymbol{e}^k,\\ \mathfrak{m}_{jk} := \nu_j e_k - \nu_k e_j, \quad \langle\boldsymbol{d}^j,\boldsymbol{\nu}\rangle = 0, \quad \langle\mathfrak{m}_{jk},\boldsymbol{\nu}\rangle = 0, \quad j,k = 1,\dots,n,\end{gathered} \tag{1.3.27}$$

where $\pi_{\mathscr{S}}$ is the projection on the tangent space to the surface (see (0.0.8)). Therefore, $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ can be applied to functions which are defined only on the surface $\mathscr{S}$.

The generating vector fields $\{\boldsymbol{d}^j\}_{j=1}^n$ $\{\mathfrak{m}_{jk}\}_{j,k=1}^n$ cannot constitute frames, since they are linearly dependent:

$$\sum_{j=1}^{n}\nu_j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X}) \equiv 0, \quad \mathfrak{m}_{jj} = 0, \tag{1.3.28}$$

but both systems $\{\boldsymbol{d}^j\}_{j=1}^n$ and $\{\mathfrak{m}_{jk}\}_{j,k=1}^n$ are full in the space of all tangent vector fields: any vector field $\boldsymbol{U} \in \omega(\mathscr{S})$ is represented as

$$\boldsymbol{U}(\mathscr{X}) = \sum_{j=1}^{n}U^j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X}) = \sum_{0\leqslant j<k\leqslant 1}^{n}c_{jk}(\mathscr{X})\mathfrak{m}_{jk}(\mathscr{X}). \tag{1.3.29}$$

For example, the covariant vector fields $\boldsymbol{g}_1(\mathscr{X}) := \partial_1\Theta_k(\mathscr{X}),\dots,\boldsymbol{g}_{n-1}(\mathscr{X}) := \partial_{n-1}\Theta_k(\mathscr{X})$, $\mathscr{X} \in \mathscr{S}_k$, $k = 1,\dots,N$, on $\mathscr{S}$, which generate the derivatives $\partial_j = \partial_{dx_j}$, are represented as follows:

$$\boldsymbol{g}_j(\mathscr{X}) = \sum_{m=1}^{n}g_j^m(\mathscr{X})\boldsymbol{e}^m = \sum_{m=1}^{n}g_j^m(\mathscr{X})\boldsymbol{d}^m(\mathscr{X}) \tag{1.3.30}$$

and $\{\boldsymbol{e}^m\}_{m=1}^n$ is a Cartesian frame in $\mathbb{R}^n$. Indeed, by applying the derivative to Θ_k we get

$$\boldsymbol{g}_j = \sum_{m=1}^{n}g_j^m\boldsymbol{e}^m = \sum_{m=1}^{n}g_j^m\boldsymbol{d}^m,$$

since

$$\sum_{m=1}^{n}g_j^m[\boldsymbol{e}^m - \boldsymbol{d}^m] = \sum_{m=1}^{n}g_j^m\nu_m\boldsymbol{\nu} = \langle\boldsymbol{g}_j,\boldsymbol{\nu}\rangle\boldsymbol{\nu} = 0, \quad j = 1,\dots,n-1.$$

An example of a hypersurface $\mathscr{S}$ is given in (0.0.1).

The system $\{\partial_k X\}_{k=1}^{n-1}$ of derivatives is a basis in the tangent space $\omega(\mathscr{S})$ of vectors. Consider the following **differential 1-form** ω_f

$$\omega_f(V) := \mathscr{L}_V f = \sum_{k=1}^{n-1}V^k\partial_k f \quad \text{for} \quad f \in C^1(\mathscr{S}), \quad V = \sum_{k=1}^{n-1}V^k\partial_k X \in \mathscr{T}\mathscr{S}\,. \tag{1.3.31}$$

The form is correctly defined because the differential operator $\mathscr{L}_V$ is tangential and can be applied to a function f defined on the surface $\mathscr{S}$ only.

Then, for a given f, there exists a vector field $\nabla_{\mathscr{S}}f \in \mathscr{T}\mathscr{S}$ such that

$$\omega_f(V) := (\nabla_{\mathscr{S}}f, V) \quad \text{for all} \quad V \in \mathscr{V}(X)\,, \tag{1.3.32}$$

which is, in classical differential geometry, the definition of **surface Gradient** of a function $f \in \mathbb{C}^1(\mathscr{S})$ and maps

$$\nabla_{\mathscr{S}} : \mathbb{C}^\infty(\mathscr{S}) \to \mathscr{T}\mathscr{S} . \tag{1.3.33}$$

The **surface divergence**

$$\mathbf{div}_{\mathscr{S}} : \mathscr{V}(\mathscr{S}) \to \mathbb{C}^\infty(\mathscr{S}) \tag{1.3.34}$$

of a smooth tangent vector field is, by the definition, the dual operator with the opposite sign

$$(\mathbf{div}_{\mathscr{S}} V, f) := -(V, \nabla_{\mathscr{S}} f), \quad \forall V \in \mathscr{T}\mathscr{S}, \quad \forall f \in \mathbb{C}^1(\mathscr{S}). \tag{1.3.35}$$

These operators expressing in intrinsic parameters of the surface $\mathscr{S}$ (tangent vector fields, Metric tensor etc.) are exposed in (0.0.4). In (0.0.5), it is exposed their composition-Laplace-Beltrami operator, which is self-adjoint (see (0.0.6)). The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$. Below we expose another concept – represent these operators on hypersurfaces in coordinates of the ambient Euclidean space.

Theorem 1.3.1 ([64])**.** *For any function $\varphi \in C^1(\mathscr{S})$ we have*

$$\nabla_{\mathscr{S}}\varphi = \Big\{\mathscr{D}_1\varphi, \mathscr{D}_2\varphi, \dots, \mathscr{D}_n\varphi\Big\}^\top. \tag{1.3.36}$$

Also, for a 1-smooth tangent vector field $\boldsymbol{V} = \sum\limits_{j=1}^n V^j e_j \in \omega(\mathscr{S})$,

$$\mathbf{div}_{\mathscr{S}}\, \boldsymbol{V} = -\nabla^*_{\mathscr{S}}\boldsymbol{V} := \sum_{j=1}^n \mathscr{D}_j V^j, \quad \mathbf{div}^*_{\mathscr{S}} = \nabla_{\mathscr{S}}. \tag{1.3.37}$$

The Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ on $\mathscr{S}$ takes the form

$$\begin{aligned}\boldsymbol{\Delta}_{\mathscr{S}}\,\psi &= \mathbf{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\,\psi = -\nabla^*_{\mathscr{S}}(\nabla_{\mathscr{S}}\psi) = \sum_{j=1}^n \mathscr{D}_j^2\psi \\ &= \sum_{j<k}\mathscr{M}_{jk}^2\psi = \frac{1}{2}\sum_{j,k=1}^n \mathscr{M}_{jk}^2\psi, \quad \forall\,\psi \in C^2(\mathscr{S}).\end{aligned} \tag{1.3.38}$$

Proof. According to the definition of the surface gradient (1.3.32) we have $\nabla_{\mathscr{S}}\varphi = \pi\nabla\varphi\big|_{\mathscr{S}}$, where $\pi V = \psi - (\nu, V)\nu$ denotes, for arbitrary vector field V on cS, the orthogonal projection onto the tangent vector fields from $\mathscr{T}\mathscr{S}$ (see (0.0.8). It is easy to ascertain that indeed, by the definition in (0.0.9), $\nabla_{\mathscr{S}}$ is the projection.

Now we consider the divergence operator $\mathbf{div}_{\mathscr{S}} = \nabla_{\mathscr{S}}$ (cf. (1.3.34), (1.3.35)). Let a scalar function φ and a tangent vector field $V \in \mathscr{T}\mathscr{S}$ be both smooth and $\mathscr{S}$ has the boundary $\partial\mathscr{S} \neq \emptyset$, the sent supp φ and supp V have no intersection with $\partial\mathscr{S}$. By applying duality, the proved formula (1.3.36) and formula (1.3.56) for the dual $(\mathscr{D}_j)^*_{\mathscr{S}}$, we get

$$\begin{aligned}(\mathbf{div}_{\mathscr{S}} V, \psi)_{\mathscr{S}} &= -(V, \nabla_{\mathscr{S}}\psi)_{\mathscr{S}} = \oint_{\mathscr{S}}\sum_{j=1}^n V^j(\mathscr{x})\mathscr{D}_j\varphi(\mathscr{x})\,dS(\mathscr{x}) \\ &= -\oint_{\mathscr{S}}\sum_{j=1}^n(\mathscr{D}_j)^*_{\mathscr{S}}V^j(\mathscr{x})\varphi(\mathscr{x})\,dS(\mathscr{x}) = \oint_{\mathscr{S}}\sum_{j=1}^n \mathscr{D}_jV^j(\mathscr{x})\varphi(\mathscr{x})\,dS(\mathscr{x}) \\ &\quad - (n-1)\mathscr{H}_{\mathscr{S}}\oint_{\mathscr{S}}\sum_{j=1}^n \nu_j(\mathscr{x})V^j(\mathscr{x})\varphi(\mathscr{x})\,dS(\mathscr{x}) \\ &= \sum_{j=1}^n(\mathscr{D}_jV^j, \varphi)_{\mathscr{S}}.\end{aligned}$$

We applied above that V is tangent: $\nu(\mathscr{X})\cdot V(\mathscr{X})=\sum\limits_{j=1}^{n}\nu_j(\mathscr{X})V^j(\mathscr{X})\equiv 0$. Since the function φ is arbitrary, (1.3.37) follows.

To prove (1.3.38), we apply (1.3.36), 1.3.56 and proceed as follows

$$\Delta_{\mathscr{S}}\,\varphi=\mathbf{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\,\varphi=-\sum_{j=1}^{n}(\mathscr{D}_j)^*_{\mathscr{S}}\mathscr{D}_j\varphi=\sum_{j=1}^{n}\mathscr{D}_j^2\varphi-(n-1)\mathscr{H}_{\mathscr{S}}\sum_{j=1}^{n}\nu_j\mathscr{D}_j\varphi=\sum_{j=1}^{n}\mathscr{D}_j^2\varphi\,,$$

since $\nu\cdot\mathscr{D}=\sum\limits_{j=1}^{n}\nu_j\mathscr{D}_j=0$ (cf. Lemma 1.3.3.v).

To prove the last equality (1.3.38) we insert $\mathscr{M}_{jk}=\nu_j\mathscr{D}_k-\nu_k\mathscr{D}_j$ (cf. Lemma 1.3.3.vi)) and proceed as follows:

$$\begin{aligned}\frac12\sum_{j,k=1}^{n}\mathscr{M}_{jk}^2\varphi&=\frac12\sum_{j,k=1}^{n}\left[\nu_j\mathscr{D}_k-\nu_k\mathscr{D}_j\right]^2\varphi\\&=\frac12\sum_{j,k=1}^{n}\left[\nu_j\mathscr{D}_k\nu_j\mathscr{D}_k\varphi-\nu_j\mathscr{D}_k\nu_k\mathscr{D}_j\varphi+\nu_k\mathscr{D}_j\nu_k\mathscr{D}_j\varphi-\nu_k\mathscr{D}_j\nu_j\mathscr{D}_k\varphi\right]\\&=\sum_{j,k=1}^{n}\left[\nu_j\mathscr{D}_k\nu_j\mathscr{D}_k\varphi-\nu_j\mathscr{D}_k\nu_k\mathscr{D}_j\varphi\right]\\&=\sum_{k=1}^{n}\mathscr{D}_k^2\varphi-\sum_{j,k=1}^{n}\left[\nu_j\nu_k\mathscr{D}_k\mathscr{D}_j\varphi+\big(\mathscr{D}_k\nu_k\big)\nu_j\mathscr{D}_j\varphi\right]=\sum_{k=1}^{n}\mathscr{D}_k^2\varphi=\Delta_{\mathscr{S}}\varphi\,.\end{aligned}$$

We have again applied that $\sum_{j=1}^{n}\nu_j\mathscr{D}_j=0$ and, like (1.3.64),

$$\sum_{j=1}^{n}\nu_j\mathscr{D}_k\nu_j\varphi=\nu^2\mathscr{D}_k\varphi+\sum_{j=1}^{n}\nu_j\big(\mathscr{D}_k\nu_j\big)\varphi=\mathscr{D}_k\varphi+\frac12\big(\mathscr{D}_k\nu^2\big)\varphi=\mathscr{D}_k\varphi \tag{1.3.39}$$

for $k=1,\dots,n$. □

Corollary 1.3.2 (cf. [64]). *Let $\mathscr{S}$ be a smooth closed hypersurface. The homogeneous equation*

$$\boldsymbol{\Delta}_{\mathscr{S}}\,\psi=0 \tag{1.3.40}$$

has only a constant solution in the space $\mathbb{W}^1(\mathscr{S})$.

Proof. Due to (1.3.37), (1.3.38) and (1.3.40), we get

$$0=\big(-\boldsymbol{\Delta}_{\mathscr{S}}\psi,\psi\big)=\big(-\nabla^*_{\mathscr{S}}\nabla_{\mathscr{S}}\psi,\psi\big)=\big(\nabla_{\mathscr{S}}\psi,\nabla_{\mathscr{S}}\psi\big)=\|\nabla_{\mathscr{S}}\psi\big|\mathbb{L}_2(\mathscr{S})\|,$$

which gives $\nabla_{\mathscr{S}}\psi=0$. But the trivial surface gradient means constant function $\psi=\mathrm{const}$ (this is easy to ascertain by analysing the definition of Günter's derivatives; see e.g. [56]). □

An important operator on forms is the **exterior derivative**. The **derivative of a** 0**-form**, i.e., of a scalar function

$$f:\mathscr{S}\to\mathbb{R},\ \ f\in C^1(\mathscr{S}), \tag{1.3.41}$$

is a 1-form and maps

$$df(w):\mathbb{T}_w\mathscr{S}\to\mathbb{R}. \tag{1.3.42}$$

Thus, $df(w)$ is a linear functional $df(w)\in\mathbb{T}^*_w\mathscr{S}$ over $\mathbb{T}_w\mathscr{S}$ for all $w\in\mathscr{S}$: being a vector $df(w)=Df(w)=(\partial_1f(w),\dots,\partial_{n-1}f(w))^\top$ the differential assigns to a vector $\xi\in\omega(\mathscr{S})$ the number

$$df(x)\xi=\sum_{j=1}^{n-1}\partial_jf(x)\xi_j,\ \ \partial_jf(x):=\partial_{dx_j}f(x),\ \ x\in\mathscr{S}_k, \tag{1.3.43}$$

where $\{dx_j = \partial_j\Theta_k\}_{j=1}^{n-1}$ is the covariant basis on $\mathscr{S}$ and $\Theta_k : \Omega_k \to \mathscr{S}_k$, $k = 1, \dots, N$, is the surface immersion.

From (1.3.30) and the definition of the derivative $\partial_j f(x) := \partial_{dx_j} f(x)$ in (1.3.43) it follows that (see for the differential matrix $D\Theta_k$)

$$\begin{aligned}
\partial_{\mathscr{S}} &:= (\partial_1, \dots, \partial_{n-1})^\top := (\partial_{dx_1}, \dots, \partial_{dx_{n-1}})^\top = (D\Theta_k)^\top \nabla_{\mathscr{S}},\\
\nabla_{\mathscr{S}} &:= (\mathscr{D}_1, \dots, \mathscr{D}_n)^\top \ \text{ or } \ \partial_{dx_j} = \sum_{m=1}^{n} (\partial_j \Theta_k^m)\mathscr{D}_m, \ \ m = 1, \dots, n-1.
\end{aligned} \tag{1.3.44}$$

Let $\mathscr{N}$ be a proper extension of the unit normal vector field $\boldsymbol{\nu}$ to $\mathscr{S}$ (cf. Definition 1.3.2). Then each operator $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ extends accordingly by setting

$$\mathscr{D}_j = \partial_j - \mathscr{N}_j\partial_{\mathscr{N}}, \ \ \mathscr{M}_{jk} := \mathscr{N}_j\partial_k - \mathscr{N}_k\partial_j, \ \ 1 \leqslant j,k \leqslant n. \tag{1.3.45}$$

In the sequel, we make no distinction between the operator $\mathscr{D}_j$ or $\mathscr{M}_{jk}$ on $\mathscr{S}$ and the extended one in $\mathbb{R}^n$ given by (1.3.45). Note that the extended operators $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ become even *strongly tangent.*

For further reference, below we collect some of the most basic properties of this system of differential operators.

Lemma 1.3.3. *Let $\mathscr{N}$ be a proper extension of the unit vector field of normal vectors $\boldsymbol{\nu}$ to $\mathscr{S}$. The following relations are valid for $j,k = 1, \dots, n$:*

(i) $\mathscr{M}_{jj} = 0$, $\mathscr{M}_{jk} = -\mathscr{M}_{kj}$;

(ii) $\partial_k = \sum_{j=1}^{n} \mathscr{N}_j\mathscr{M}_{jk} + \mathscr{N}_k\partial_{\mathscr{N}} = -\sum_{k=1}^{n} \mathscr{N}_k\mathscr{M}_{jk} + \mathscr{N}_j\partial_{\mathscr{N}}$;

(iii) $\sum_{k=1}^{n} \mathscr{M}_{jk}\mathscr{N}_k = -\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}}$, *where* $\mathscr{H}^0_{\mathscr{S}}(\mathscr{x}) = -\,\mathbf{div}_{\mathscr{S}}\,\boldsymbol{\nu}(\mathscr{x})$ *and* $\mathscr{H}_{\mathscr{S}}(\mathscr{x}) := (n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{x})$ *is the mean curvature at* $\mathscr{x} \in \mathscr{S}$ *(see* (0.0.15)*);*

(iv) $\mathscr{D}_j = \sum_{k=1}^{n} \mathscr{N}_k\mathscr{M}_{kj}$;

(v) $\mathscr{M}_{jk} = \mathscr{N}_j\mathscr{D}_k - \mathscr{N}_k\mathscr{D}_j$;

(vi) $\sum_{j=1}^{n} \mathscr{N}_j\mathscr{D}_j = 0$;

(vii) $\sum_{r,j,k=m-1}^{m+1} \sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk} = 2 \sum_{\{r,j,k\}\subset\{(m-1),m,(m+1)\}} \sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk} = 0$ *for* $m = 2, \dots, n-1$, *where $\sigma(r,j,k)$ is the permutation sign:*

$$\sigma(j_1, \dots, j_k) = \begin{cases} +1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of the strongly} \\ & \qquad \text{ordered row } (m_1, \dots, m_k),\ m_1 < \cdots < m_k, \\ 0 & \text{if } j_r = j_s \text{ for some } r,s = 1, \dots, k \text{ and } r \neq s, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of the strongly} \\ & \qquad \text{ordered row } (m_1, \dots, m_k),\ m_1 < \cdots < m_k; \end{cases} \tag{1.3.46}$$

(viii) $[\mathscr{D}_j, \mathscr{D}_k] = \sum_{r=1}^{n} (\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r + [\mathscr{N}_j\partial_{\mathscr{N}}\mathscr{N}_k - \mathscr{N}_k\partial_{\mathscr{N}}\mathscr{N}_j]\partial_{\mathscr{N}}$;

(ix) $[\mathscr{D}_j, \mathscr{D}_k] = \sum_{r=1}^{n} (\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r = \mathscr{N}_k[\mathscr{D}_{\mathscr{N}}, \partial_j] - \mathscr{N}_j[\mathscr{D}_{\mathscr{N}}, \partial_k]$;

(x) $\partial_j\mathscr{N}_k = \mathscr{D}_j\mathscr{N}_k = \mathscr{D}_k\mathscr{N}_j$.

Proof. The identities (i)–(ii) and (iv)–(vii) are simple consequences of the definitions. For the equality (iii) we have

$$\sum_{k=1}^{n}\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^{n}\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^{n}(\mathscr{N}_j\partial_k-\mathscr{N}_k\partial_j)\mathscr{N}_k=\mathscr{N}_j\,\mathbf{div}\,\mathscr{N}-\frac{1}{2}\,\partial_j(\|\mathscr{N}\|^2)=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}},$$

as claimed.

To prove (viii) we calculate

$$\begin{aligned}\mathscr{D}_j\mathscr{D}_k&=(\partial_j-\mathscr{N}_j\partial_{\mathscr{N}})(\partial_k-\mathscr{N}_k\partial_{\mathscr{N}})\\&=\partial_j\partial_k-(\partial_j\mathscr{N}_k)\partial_{\mathscr{N}}\\&\qquad-\sum_{r=1}^{n}\big[\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r+\mathscr{N}_k\mathscr{N}_r\partial_r\partial_j+\mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\big]+\mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}}+\mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}\\&=-\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r+\mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}}+B_{jk}\\&=-\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{D}_r+\mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}}+B_{jk},\end{aligned}\tag{1.3.47}$$

since

$$\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{N}_r\partial_{\mathscr{N}}=\frac{1}{2}\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r^2)\partial_{\mathscr{N}}=\frac{1}{2}\,\mathscr{N}_k(\partial_j1)\partial_{\mathscr{N}}=0.$$

The operator

$$B_{jk}=\partial_j\partial_k-(\partial_j\mathscr{N}_k)\partial_{\mathscr{N}}-\sum_{r=1}^{n}\big[\mathscr{N}_k\mathscr{N}_r\partial_r\partial_j+\mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\big]+\mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}$$

is symmetric, $B_{jk}=B_{kj}$, and the desired commutator identity in (viii) follows from (1.3.47).

The first commutator identity in (ix) utilizes the facts that $\partial_{\mathscr{N}}\mathscr{N}_k=0$ (cf. Lemma (1.3.10)) and follows from the identity in (viii). The second commutator identity in (ix) applies the same identity $\partial_{\mathscr{N}}\mathscr{N}_k=0$, the identity $\partial_j\mathscr{N}_k=\partial_k\mathscr{N}_j$ (cf. (1.3.13)), and follows by a routine calculations.

The identities in (x) are already proved in (1.3.10) and (1.3.15). □

The next Lemma 1.3.4 provides an useful and interesting example of restriction of the differential form to hypersurface and to its boundary.

Lemma 1.3.4. *Let $\Theta:\Omega\to\mathscr{S}$ be a smooth orientable hypersurface in $\mathbb{R}^n$ and with a smooth boundary $\Gamma:=\partial\mathscr{S}$, while $d\sigma$ and $d\mathfrak{s}$ designate the respective volume elements on $\mathscr{S}$ and on Γ. Let $\boldsymbol{\nu}(\mathscr{X})=\big(\nu_1(\mathscr{X}),\dots,\nu_n(\mathscr{X})\big)^\top$ be the (outer) unit normal vector to $\mathscr{S}$ at $\mathscr{X}\in\mathscr{S}$ compatible with a chosen orintation and $\boldsymbol{\nu}_\Gamma(s)=(\nu^1_\Gamma(s),\dots,\nu^n_\Gamma(s))^\top$ be the unit tangent vector to $\mathscr{S}$ at the boundary point $s\in\Gamma$, which is outward (unit) normal vector to the boundary $\mathscr{S}$. Then*

$$\nu_j dS=\beta_j\big|_{\mathscr{S}},\tag{1.3.48}$$

$$\big[\nu_j\nu^k_\Gamma-\nu_k\nu^j_\Gamma\big]\,d\mathfrak{s}=\beta_{jk}\big|_{\Gamma},\tag{1.3.49}$$

where

$$\begin{aligned}\beta_j&:=\big|dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\big|=(-1)^{j-1}dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n,\\\beta_{jk}&:=\big|dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n\big|=(-1)^{j+k-1}dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n\end{aligned}$$

and $\widehat{dx_m}$ denotes that the factor dx_m is dropped.

The next theorem generalizes Stokes' formulae (see [130, § 2.2, Theorem 2.1] for the version on compact Riemannian manifolds).

Theorem 1.3.2. *For any real-valued function $\varphi \in C^1(\mathscr{S})$ and any $1 \leqslant j < k \leqslant n$, there hold*

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi\, d\sigma = \oint_{\Gamma} \left[\nu_j \nu_\Gamma^k - \nu_k \nu_\Gamma^j\right]\varphi\, d\mathfrak{s}, \tag{1.3.50}$$

where $\boldsymbol{\nu}_\Gamma(\xi) = (\nu_\Gamma^1(\xi), \dots, \nu_\Gamma^n(\xi))^\top$ is the unit tangent vector to $\mathscr{S}$ at the boundary point $\xi \in \Gamma := \partial\mathscr{S}$ and outward (unit) normal vector to the boundary $\Gamma = \partial\mathscr{S}$.

Proof. With formula (1.3.48) at hand, the integrand in (1.3.50) can be represented as a total differential

$$(\mathscr{M}_{jk}\varphi)\, d\sigma = (\partial_k\varphi)\omega_j\big|_{\mathscr{S}} - (\partial_j\varphi)\omega_k\big|_{\mathscr{S}} = d[\varphi\omega_{jk}]\big|_{\mathscr{S}}.$$

Applying the well known Stokes' formula

$$\int_{\mathscr{S}} d\beta := \int_{\Gamma} \beta\, d\mathfrak{s} \tag{1.3.51}$$

(see, e.g., [58]) and formula (1.3.49) we get

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi\, d\sigma = \int_{\mathscr{S}} d\big[\varphi\omega_{jk}\big]\big|_{\mathscr{S}} = \int_{\Gamma} \varphi\omega_{jk}\big|_{\Gamma} = \int_{\Gamma} \left[\nu_j\nu_\Gamma^k - \nu_k\nu_\Gamma^j\right]\varphi\, d\mathfrak{s}$$

and (1.3.50) is proved. □

The formal adjoint (in $\mathbb{R}^n$) to P is defined by

$$P^* u = -\sum_j \partial_j a_j^\top u + b^\top u.$$

If $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, and if P is a first-order operator, weakly tangent to $\partial\Omega$, then, applying (1.3.58) (cf. Section 1.6), P can be integrated by parts over Ω without boundary terms, i.e.,

$$(Pu, v)_\Omega := \int_\Omega \langle Pu, v\rangle\, dx = \int_\Omega \langle u, P^*v\rangle\, dx =: (u, P^*v)_\Omega \tag{1.3.52}$$

for all vector-valued sections of vector fields $u, v \in C^1(\overline{\Omega})$.

For a weakly tangent differential operator Q on a closed hypersurface $\mathscr{S}$ let $Q^*_{\mathscr{S}}$ denote the "surface" adjoint:

$$(Q_{\mathscr{S}}\varphi, \psi)_{\mathscr{S}} := \oint_{\mathscr{S}} \langle Q_{\mathscr{S}}\varphi, \psi\rangle\, d\sigma = \oint_{\mathscr{S}} \langle \varphi, Q^*_{\mathscr{S}}\psi\rangle\, d\sigma = (\varphi, Q^*_{\mathscr{S}}\psi)_{\mathscr{S}} \tag{1.3.53}$$

for all vector-valued sections of vector fields $\varphi, \psi \in C^1(\overline{\Omega})$.

Corollary 1.3.3. *The surface-adjoint operator $P^*_{\mathscr{S}}$ to the weakly tangent differential operator P in* (1.3.22) *coincides with the formally adjoint one*

$$P^*_{\mathscr{S}}\varphi = P^*\varphi = -\sum_{j=1}^n \partial_j a_j^\top \varphi + b^\top \varphi. \tag{1.3.54}$$

In particular, this is true for the Stokes' derivatives and, moreover, Stokes' derivatives are skew-symmetric

$$(\mathscr{M}^*_{jk})_{\mathscr{S}} = \mathscr{M}^*_{jk} = -\mathscr{M}_{jk} = \mathscr{M}_{kj}, \quad \forall\, j, k = 1, \dots, n. \tag{1.3.55}$$

The adjoint operator to the operator $\mathscr{D}_j$ is

$$(\mathscr{D}_j)^*_{\mathscr{S}}\varphi = \mathscr{D}^*_j\varphi = -\mathscr{D}_j\varphi + \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi, \quad \varphi \in C^1(\mathscr{S}), \tag{1.3.56}$$

where $(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{X})=\mathscr{H}_{\mathscr{S}}(\mathscr{X})$ *is the mean curvature of the surface* $\mathscr{S}$ *(cf. (0.0.15)).*

For any real-valued function $\varphi\in C^1(\mathscr{S})$*, any* $1\leqslant j<k\leqslant n$ *and for* $\boldsymbol{\nu}_\Gamma=(\nu^1_\Gamma,\dots,\nu^n_\Gamma)^\top$ *being the the same as in Theorem 1.3.2 the following integration by parts formula*

$$\int\limits_{\mathscr{S}}\big[(\mathscr{D}_j\varphi)\psi-\varphi(\mathscr{D}^*_j\psi)\big]\,d\sigma=\oint\limits_{\Gamma}\nu^j_\Gamma\varphi\psi\,d\mathfrak{s} \tag{1.3.57}$$

holds. It is an analogue of the classical Gaußian ***integration by parts*** *formula*

$$\int\limits_{\Omega}\partial_j f(y)g(y)\,dy=\oint\limits_{\mathscr{S}}\nu_j(\tau)f(\tau)g(\tau)\,d\sigma-\int\limits_{\Omega}f(y)\partial_j g(y)\,dy, \tag{1.3.58}$$

which holds for arbitrary $f,\ g\in\mathbb{W}^1(\mathscr{S})$*.*

In particular, the following ***Gauß formulae for open surfaces*** *is valid:*

$$\int\limits_{\mathscr{S}}\mathscr{D}_j\varphi\,d\sigma=\oint\limits_{\Gamma}\nu^j_\Gamma\varphi\,d\mathfrak{s}+\int\limits_{\mathscr{S}}\nu_j\mathscr{H}^0_{\mathscr{S}}\varphi\,d\sigma. \tag{1.3.59}$$

Proof. Let us first prove that $\mathscr{M}_{jk}$ is skew symmetric on the surface. Indeed, by applying Stokes' formulae

$$\oint\limits_{\mathscr{S}}(\mathscr{M}_{jk}f)(\tau)\,d\sigma=0,\quad j,k=1,\dots,n,\quad f\in C^1(\mathscr{S}), \tag{1.3.60}$$

we get

$$\oint\limits_{\mathscr{S}}(\mathscr{M}_{jk}\varphi)\psi\,d\sigma=\oint\limits_{\mathscr{S}}(\mathscr{M}_{jk}\varphi\psi)\,d\sigma-\oint\limits_{\mathscr{S}}\varphi(\mathscr{M}_{jk}\psi)\,d\sigma=-\oint\limits_{\mathscr{S}}\varphi(\mathscr{M}_{jk}\psi)\,d\sigma,$$

and this proves skew-symmetry of $\mathscr{M}_{jk}$. On the other hand, the formal adjoint to $\mathscr{M}_{jk}=\mathscr{N}_j\mathscr{D}_k-\mathscr{N}_k\mathscr{D}_j$ is

$$\begin{aligned}\mathscr{M}^*_{jk}\varphi&=(\mathscr{N}_j\partial_k-\mathscr{N}_k\partial_j)^*\varphi=-\partial_j(\mathscr{N}_k\varphi)+\partial_k(\mathscr{N}_j\varphi)\\&=\mathscr{N}_k\partial_j\varphi-\mathscr{N}_j\partial_k\varphi+(\partial_j\mathscr{N}_k)\varphi-(\partial_k\mathscr{N}_j)\varphi=-\mathscr{M}_{jk}\varphi\end{aligned}$$

(cf. (1.3.10)), where $\varphi\in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$. Thus, formal adjoint to $\mathscr{M}_{jk}$ coincides with the surface adjoint and (1.3.55) is proved completely.

Now let us prove (1.3.54). To this end, note that on $\mathscr{S}$,

$$\begin{aligned}P\varphi&=\sum_{j=1}^n a_j\partial_j\varphi+b\varphi=\sum_j a_j\big[\mathscr{D}_j+\nu_j\partial_{\boldsymbol{\nu}}\big]\varphi\\&=\sum_{j=1}^n a_j\mathscr{D}_j\varphi+b\varphi+\sigma(P;\boldsymbol{\nu})\partial_{\boldsymbol{\nu}}\varphi=\sum_{j=1}^n a_j\mathscr{D}_j\varphi\\&=\sum_{j,k=1}^n a_j\nu_k\mathscr{M}_{kj}\varphi,\end{aligned} \tag{1.3.61}$$

due to Lemma 1.3.3(iv) and since P is weakly tangent. Now the property postulated in (1.3.54) follows from (1.3.61) and from (1.3.55):

$$P^*_{\mathscr{S}}\varphi=\sum_{j,k=1}^n(\mathscr{M}_{kj})^*_{\mathscr{S}}a_j^\top\nu_k\varphi+b^\top\varphi=\sum_{j,k=1}^n(\mathscr{M}_{kj})^*a_j^\top\nu_k\varphi+b^\top\varphi=P^*\varphi.$$

With (1.3.54) and (1.3.10) we get

$$\begin{aligned}&(\mathscr{D}_j)^*_{\mathscr{S}}\varphi=\mathscr{D}^*_j\varphi=-\partial_j\varphi+\sum_{k=1}^n\partial_k(\mathscr{N}_j\mathscr{N}_k\varphi)\\&=-\partial_j\varphi+\sum_{k=1}^n\big[\mathscr{N}_j\mathscr{N}_k\partial_k\varphi+(\mathscr{N}_k\partial_k\mathscr{N}_j)\varphi+\mathscr{N}_j(\partial_k\mathscr{N}_k)\varphi\big]=-\mathscr{D}_j\varphi-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}}\varphi+(\partial_{\mathscr{N}}\mathscr{N}_j)\varphi,\end{aligned} \tag{1.3.62}$$

where $\varphi \in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$ and

$$\mathscr{H}^0_{\mathscr{S}} := -\sum_{k=1}^{n} \mathscr{D}_k \mathscr{N}_k, \quad \mathscr{H}^0_{\mathscr{S}}(\mathscr{x}) = -\sum_{k=1}^{n} \mathscr{D}_k \nu_k(\mathscr{x}) \ \text{ for } \ \mathscr{x} \in \mathscr{S}. \tag{1.3.63}$$

Hence (1.3.56) follows, since (cf. (1.3.13)) $\partial_{\mathscr{N}} \mathscr{N}_j = 0$.

To prove (1.3.63) we apply

$$\partial_{\mathscr{N}} \mathscr{N}\big|_{\mathscr{S}} = \left\{ \sum_{j=1}^{n} \mathscr{N}_j \partial_j \mathscr{N}_k \right\}_{k=1}^{n} \Bigg|_{\mathscr{S}} = \left\{ \sum_{j=1}^{n} \mathscr{N}_j \partial_k \mathscr{N}_j \right\}_{k=1}^{n} \Bigg|_{\mathscr{S}} = \frac{1}{2} \nabla_x |\mathscr{N}|^2 \Big|_{\mathscr{S}} = \frac{1}{2} \nabla_x 1 = 0 \tag{1.3.64}$$

and proceed as follows:

$$\sum_{k=1}^{n} \mathscr{D}_k \nu_k = \sum_{k=1}^{n} \Big(\partial_k \nu_k - \nu_k \sum_{j=1}^{n} \nu_j \partial_j \nu_k \Big) = -\mathscr{H}^0_{\mathscr{S}} - \sum_{j=1}^{n} \frac{\nu_j}{2} \partial_j 1 = -\mathscr{H}^0_{\mathscr{S}}.$$

To prove formula (1.3.57), we apply Lemma 1.3.3(iv), (1.3.55), the equalities $\sum\limits_{k=1}^{n} \nu_k^2 = 1$, $\sum\limits_{k=1}^{n} \nu_k \nu_\Gamma^k = 0$ and proceed as follows:

$$\begin{aligned}\oint_{\mathscr{S}} (\mathscr{D}_j \varphi) \psi \, d\sigma &= \sum_{k=1}^{n} \oint_{\mathscr{S}} \nu_k (\mathscr{M}_{jk} \varphi) \psi \, d\sigma \\ &- \sum_{k=1}^{n} \oint_{\mathscr{S}} \psi (\mathscr{M}_{jk} \nu_k \psi) \, d\sigma + \sum_{k=1}^{n} \oint_{\Gamma} (\nu_k^2 \nu_\Gamma^j - \nu_k \nu_j \nu_\Gamma^k) \varphi \psi \, d\mathfrak{s} = \oint_{\mathscr{S}} \psi (\mathscr{D}_j^* \psi) \, d\sigma + \oint_{\Gamma} \nu_\Gamma^j \varphi \psi \, d\mathfrak{s}.\end{aligned}$$

And, finally, formula (1.3.59) follows from formulae (1.3.57) and (1.3.56), if we insert $\psi(t) \equiv 1$ in (1.3.57) and note that $\mathscr{D}_j 1 = 0$. □

Lemma 1.3.5. *Let P be, as in* (1.3.22)*, a first-order differential operator with C^1-smooth coefficients. P is weakly/strongly tangent if and only if the formally adjoint P^* is so.*

If P is weakly tangent to $\mathscr{S}$ and P is defined in a neighborhood of $\mathscr{S}$, then

$$(P\varphi)\big|_{\mathscr{S}} = P\big(\varphi\big|_{\mathscr{S}}\big) \tag{1.3.65}$$

for every C^1-function φ defined in a neighborhood of $\mathscr{S}$. In particular,

$$\mathscr{D}_j \varphi\big|_{\mathscr{S}} = \mathscr{D}_j\big(\varphi\big|_{\mathscr{S}}\big), \quad \mathscr{M}_{jk} \varphi\big|_{\mathscr{S}} = \mathscr{M}_{jk}\big(\varphi\big|_{\mathscr{S}}\big), \ \ j,k = 1, \dots, n. \tag{1.3.66}$$

Furthermore, (1.3.65) *is valid for the adjoint operator P^* and*

$$\int_{\mathscr{S}} \langle P\varphi, \psi \rangle \, d\sigma = \int_{\mathscr{S}} \langle \varphi, P^* \psi \rangle \, d\sigma + \oint_{\Gamma} \langle \sigma(P; \boldsymbol{\nu}_\Gamma) \varphi, \psi \rangle \, d\mathfrak{s} \tag{1.3.67}$$

for any vector-valued functions $\varphi, \psi \in \mathscr{S}$, where $\sigma(P;\xi)$ is the symbol of P (cf. (1.3.23)*).*

Proof. The first assertion follows, since $\sigma(P^*;\xi) = -\sigma(P;\xi)^\top$ for each $\xi \in \mathbb{R}^n$.

Due to the representation (1.3.26), it suffices to prove (1.3.65) only for the operator $\mathscr{D}_j = \boldsymbol{d}^j \cdot \nabla$, where $\boldsymbol{d}^j = \pi_{\mathscr{S}} \boldsymbol{e}^j = \mathscr{N} \wedge (\mathscr{N} \wedge \boldsymbol{e}^j)$ is at least C^1-smooth vector field in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, tangent to the surface $\mathscr{S}$ at surface points (cf. (1.3.27)). Thus, we have to justify the following equality:

$$\mathscr{D}_j \varphi\big|_{\mathscr{S}} = (\boldsymbol{d}^j \cdot \nabla) \varphi \Big|_{\mathscr{S}} = \boldsymbol{d}^j \cdot \nabla\big(\varphi\big|_{\mathscr{S}}\big) = \mathscr{D}_j\big(\varphi\big|_{\mathscr{S}}\big). \tag{1.3.68}$$

The vector field $\boldsymbol{d}^j(x) = \boldsymbol{d}^j(\theta, \mathscr{x})$ depends on the signed distance $\theta = \mathbf{dist}(x, \mathscr{S}) = \pm|x - \mathscr{x}|$ continuously ($\theta > 0$ for the outer domain and $\theta < 0$ for the inner one). Let $\mathscr{F}^t_{\boldsymbol{d}^j(\,\cdot\,)}$ be the integral curve of the vector field $\boldsymbol{d}^j$ and

$$\mathscr{F}^t_{\boldsymbol{d}^j(\,\cdot\,)} : \Omega_{\mathscr{S}} \to \Omega_{\mathscr{S}}, \quad \mathscr{F}^t_{\boldsymbol{d}^j(0,\,\cdot\,)} = \mathscr{F}^t_{\boldsymbol{d}^j}(\,\cdot\,) : \mathscr{S} \to \mathscr{S} \tag{1.3.69}$$

be the flow generated by this vector field ℓ_θ in the neighborhood $\Omega_{\mathscr{S}}$ (cf. Section 1.3). Since the flow depends continuously on the parameter θ, we get

$$\big(\boldsymbol{d}^j(\theta,\mathscr{X})\cdot\nabla\big)\varphi\Big|_{\mathscr{S}}=\lim_{\theta\to 0}\frac{d}{dt}\,\varphi(\mathscr{F}^t_{\boldsymbol{d}^j(\theta,\mathscr{X})})\Big|_{t=0}=\frac{d}{dt}\,\varphi(\mathscr{F}^t_{\boldsymbol{d}^j})\Big|_{t=0}=\boldsymbol{d}^j\cdot\nabla\big(\varphi\big|_{\mathscr{S}}\big),=\mathscr{D}_j\big(\varphi\big|_{\mathscr{S}}\big)$$

and (1.3.68) is proved.

Next, we prove (1.3.67) by using formulae (1.3.26), (1.3.57) (integrating by parts) and get

$$\begin{aligned}\int\limits_{\mathscr{S}}\langle P\varphi,\psi\rangle\,d\sigma&=\sum_{j=1}^n\int\limits_{\mathscr{S}}\langle a_j\mathscr{D}_j\varphi,\psi\rangle\,d\sigma+\int\limits_{\mathscr{S}}\langle b\varphi,\psi\rangle\,d\sigma\\&=\sum_{j=1}^n\int\limits_{\mathscr{S}}\langle\varphi,\mathscr{D}_j^*a_j^\top\psi\rangle\,d\sigma+\int\limits_{\mathscr{S}}\langle\varphi,b^\top\psi\rangle\,d\sigma+\sum_{j=1}^n\oint\limits_{\Gamma}\langle\nu^j_\Gamma a_j^\top\varphi,\psi\rangle\,d\sigma\\&=\int\limits_{\mathscr{S}}\langle\varphi,P^*\psi\rangle\,d\sigma+\oint\limits_{\Gamma}\langle\sigma(P;\boldsymbol{\nu}_\Gamma)\varphi,\psi\rangle\,d\mathfrak{s}.\end{aligned}$$

This completes the proof. □

Remark 1.3.1. By iteration, an identity similar in spirit to (1.3.67) holds for high order weakly tangent differential operators (i.e., for polynomials of Günter's or Stokes' derivatives; cf. Lemma 1.3.6).

In this connection, let us also point out that the strongly tangent operator, Stokes' gradient

$$\mathscr{M}_{\mathscr{S}}:=\mathscr{N}\wedge\nabla_{\mathscr{S}}=\mathscr{N}\wedge\nabla=\big\{\mathscr{M}_{23},-\mathscr{M}_{13},\mathscr{M}_{12}\big\},\quad\mathscr{M}_{\mathscr{S}}\big|_{\mathscr{S}}=\boldsymbol{\nu}\wedge\nabla_{\mathscr{S}}\tag{1.3.70}$$

in $\mathbb{R}^3$ acting on scalar functions on $\mathscr{S}$, is naturally identified with the skew-symmetric matrix whose entries are Stokes' derivatives in the sense that

$$\boldsymbol{\nu}\wedge\nabla_{\mathscr{S}}=\frac12\sum_{j,k=1}^3\mathscr{M}_{jk}\,dx_j\wedge dx_k=\sum_{1\leqslant j<k\leqslant 3}\mathscr{M}_{jk}\,dx_j\wedge dx_k.\tag{1.3.71}$$

A further important example of the strongly tangent, first-order differential operator is

$$P\boldsymbol{U}:=\mathbf{div}\,\boldsymbol{U}-\langle\boldsymbol{\nu},\boldsymbol{U}\rangle\,\partial_{\boldsymbol{\nu}}\boldsymbol{U}=\mathbf{div}_{\mathscr{S}}\,\pi_{\mathscr{S}}\boldsymbol{U},\ \text{ with }\ P_1^*\varphi=-\nabla\varphi+(\partial_{\boldsymbol{\nu}}\varphi+\mathscr{H}^0_{\mathscr{S}}\varphi)\boldsymbol{\nu}.\tag{1.3.72}$$

Indeed, $\sigma(P_1;\xi)=\xi-\langle\boldsymbol{\nu},\xi\rangle\,\boldsymbol{\nu}$ and, obviously, $\sigma(P_1;\boldsymbol{\nu})\equiv 0$.

In the sequel, we use the following standard notation

$$\begin{aligned}&\nabla^\alpha_{\mathscr{S}}:=\mathscr{D}_1^{\alpha_1}\dots\mathscr{D}_n^{\alpha_n},\ \ \alpha\in\mathbb{N}_0^n,\\&\mathscr{M}^\beta_{\mathscr{S}}:=\mathscr{M}_1^{\beta_1}\cdots\mathscr{M}_m^{\beta_m},\ \ \beta\in\mathbb{N}_0^m,\ \ m=\frac{n(n-1)}{2}\,,\end{aligned}\tag{1.3.73}$$

where

$$\nabla_{\mathscr{S}}:=(\mathscr{D}_1,\dots,\mathscr{D}_n)^\top,\ \ \mathscr{M}_{\mathscr{S}}:=(\mathscr{M}_{12},\dots,\mathscr{M}_{n-1,n})^\top\tag{1.3.74}$$

and the selected Stokes' derivatives $\mathscr{M}_1:=\mathscr{M}_{1,2},\dots,\mathscr{M}_m:=\mathscr{M}_{n-1,n}$ are non-vanishing, while the remaining non-vanishing Stokes' derivatives differ from the selected ones only by the sign. In contrast to the case of the usual derivatives ∂^α, it really matters in which sequence we apply the derivatives $\mathscr{D}_j^{\alpha_j}$ and $\mathscr{M}_k^{\beta_k}$ in (1.3), because they have variable coefficients. In this connection, let us write precisely what is meant under the dual operators:

$$\begin{aligned}&(\mathscr{D}^*_x)^\alpha:=(\mathscr{D}^*_n)^{\alpha_n}\cdots(\mathscr{D}^*_1)^{\alpha_1},\ \ \alpha\in\mathbb{N}_0^n,\\&(\mathscr{M}^*_x)^\beta:=(-1)^{|\beta|}(\mathscr{M}_m)^{\beta_m}\cdots(\mathscr{M}_1)^{\beta_1},\ \ \beta\in\mathbb{N}_0^m.\end{aligned}\tag{1.3.75}$$

Note that we use the same operators $\mathscr{M}_1^*=-\mathscr{M}_1=-\mathscr{M}_{1,2},\dots,\mathscr{M}_m^*=-\mathscr{M}_m:=-\mathscr{M}_{n-1,n}$ for the adjoint operators to Stokes' derivatives, because these operators are skew-symmetric $(\mathscr{M}_{j,k})^*=-\mathscr{M}_{j,k}$ (cf. (1.3.55)).

Lemma 1.3.6. *Let* $\mathbf{G}(\mathscr{D})$ *be a tangent differential operator of the form*

$$\mathbf{G}(\mathscr{D}) = \sum_{|\alpha|\leqslant k} \boldsymbol{g}_\alpha(t)\mathscr{D}_t^\alpha = \sum_{|\beta|\leqslant k} f_\beta(t)\mathscr{M}_t^\beta, \quad t \in \mathscr{S}. \tag{1.3.76}$$

Then

$$\oint_{\mathscr{S}} \langle \mathbf{G}(\mathscr{D})\varphi, \psi\rangle \, d\sigma = \oint_{\mathscr{S}} \langle \varphi, \mathbf{G}^*(\mathscr{D})\psi\rangle \, d\sigma, \tag{1.3.77}$$

where

$$\mathbf{G}^*(\mathscr{D}) = \sum_{|\alpha|\leqslant k} (\mathscr{D}_t^*)^\alpha g_\alpha^\top(t) I = \sum_{|\beta|\leqslant k} (-1)^{|\beta|}\mathscr{M}_t^\beta f_\beta^\top(t) I \tag{1.3.78}$$

and $\mathscr{D}^*$ *and* $\mathscr{M}^*$ *are the adjoint operators (cf.* (1.3.56) *and* (1.3.55)*).*

Remark 1.3.2. Note that the operators $i\mathscr{M}_j, \ j = 1, \dots, m$, with variable coefficients

$$\mathbf{A}(x, \mathscr{M}_x)u = \sum_{j=1}^{M} b_j(x)(i\mathscr{M}_j)^{m_j}\overline{b_j^\top(x)}u, \quad b_j \in [C^\infty(\mathscr{S})]^{N\times N} \tag{1.3.79}$$

and polynomials with constant self-adjoint $N \times N$ matrix coefficients

$$\mathbf{B}(\mathscr{M}_x)u = \sum_{j=1}^{M} a_j(i\mathscr{M})_j^{m_j}u, \quad \overline{a_j^\top} = a_j = const, \ \ \forall\, j = 1, \dots, M, \ \ \forall\, m_j \in \mathbb{N}_0, \tag{1.3.80}$$

are all self-adjoint on the hypersurface

$$\mathbf{A}^*_{\mathscr{S}}(\mathscr{M}_x) = \mathbf{A}(\mathscr{M}_x), \ \ \mathbf{B}^*_{\mathscr{S}}(\mathscr{M}_x) = \mathbf{B}(\mathscr{M}_x).$$

1.4 Equation of elastic hypersurface

One way of understanding the genesis of the Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ on the surface $\mathscr{S}$ (see (1.3.38)) is to consider the energy functional

$$\mathscr{E}[u] := \int_{\mathscr{S}} \|\nabla u\|^2 \, d\sigma, \quad u \in C^\infty(\mathscr{S}). \tag{1.4.1}$$

Then any minimizer u of functional (1.4.1) should satisfy

$$0 = \frac{d}{dt}\,\mathscr{E}[u+tv]\Big|_{t=0} = \int_{\mathscr{S}} \big[\langle \nabla u, \nabla v\rangle + \langle \nabla v, \nabla u\rangle\big]\, d\sigma$$
$$= 2\,\mathrm{Re}\int_{\mathscr{S}} \langle \nabla u, \nabla v\rangle \, d\sigma, \quad u \in C^\infty(\mathscr{S}), \ \ \forall\, v \in C_0^\infty(\mathscr{S}), \tag{1.4.2}$$

which implies

$$\Delta u = 0 \ \text{ on } \ \mathscr{S}. \tag{1.4.3}$$

In other words, (1.4.3) is the Euler–Lagrange equation associated with the integral functional (1.4.1).

Similarly, minimizers of the energy functional

$$\mathscr{E}[\boldsymbol{U}] := \int_{\mathscr{S}} \big[\, \|dU\|^2 + \|\delta \boldsymbol{U}\|^2\big]\, d\sigma, \quad \boldsymbol{U} \in \Lambda^\ell\omega(\mathscr{S}), \tag{1.4.4}$$

where $\Lambda^\ell\omega(\mathscr{S})$ is the space of differential ℓ forms on tangent space $\omega(\mathscr{S})$, are null-solutions to the Hodge–Laplacian (cf. later (1.5.16)), while minimizers of the energy functional

$$\mathscr{E}[\boldsymbol{U}] := \int\limits_{\mathscr{S}} \|\nabla \boldsymbol{U}\|^2 \, d\sigma, \;\; \boldsymbol{U} \in \omega(\mathscr{S}), \tag{1.4.5}$$

are null-solutions to the Bochner–Laplacian (cf. later (1.5.17)).

Our aim is to adopt a similar point of view in the case of anisotropic and isotropic (Lamé) system of elasticity on $\mathscr{S}$.

Günter's derivatives $\{\mathscr{D}_j\}_{j=1}^n$ are tangent and represent a full system (cf. (1.3.27)–(1.3.29)). But the derivative $\mathscr{D}_j\boldsymbol{V}$ is not covariant and maps the tangent vectors to non-tangent ones $\mathscr{D}_j : \omega(\mathscr{S}) \not\to \omega(\mathscr{S})$. To improve this, we just eliminate the normal component of the vector by applying the canonical orthogonal projection $\pi_{\mathscr{S}}$ onto $\omega(\mathscr{S})$ (cf. (0.0.8))

$$\mathscr{D}_j^{\mathscr{S}}\boldsymbol{V} := \pi_{\mathscr{S}}\mathscr{D}_j\boldsymbol{V} = \mathscr{D}_j\boldsymbol{V} - \langle\boldsymbol{\nu}, \mathscr{D}_j\boldsymbol{V}\rangle\boldsymbol{\nu} = \mathscr{D}_j\boldsymbol{V} + (\partial_{\boldsymbol{V}}\nu_j)\boldsymbol{\nu}, \tag{1.4.6}$$

where

$$\partial_{\boldsymbol{V}}\varphi := \sum_{k=1}^n V_k^0\partial_k\varphi = \sum_{k=1}^n V_k^0\mathscr{D}_k\varphi,$$

and obtain an automorphisms of the space of tangent vector fields

$$\mathscr{D}_j^{\mathscr{S}} : \omega(\mathscr{S}) \to \omega(\mathscr{S}). \tag{1.4.7}$$

The starting point is to consider the total free (elastic) energy as integral of stored energy density $E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x))$

$$\mathscr{E}[\boldsymbol{U}] := \int\limits_{\mathscr{S}} E(y, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y)) \, d\sigma, \quad \mathscr{D}^{\mathscr{S}}\boldsymbol{U} := \big[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0\big]_{n\times n}, \;\; \boldsymbol{U} \in \omega(\mathscr{S}) \tag{1.4.8}$$

(cf. (1.4.6), (1.4.7)), ignoring at the moment the displacement boundary conditions (Koiter's model). As before, equilibria states correspond to minimizers of the above variational integral (see [115, § 5.2]). First, we should identify the correct form of the stored energy density $E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x))$. We shall restrict attention to the case of linear elasticity. In this scenario, $E = (\mathfrak{S}_{\mathscr{S}}, \mathrm{Def}_{\mathscr{S}})$ depends bi-linearly on the stress tensor $\mathfrak{S}_{\mathscr{S}} = [\mathfrak{S}^{jk}]_{n\times n}$ and the deformation (strain) tensor

$$\mathrm{Def}_{\mathscr{S}} = [\mathfrak{D}_{jk}]_{n\times n}, \quad \mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\big[(\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j + (\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k\big], \;\; j,k = 1,\dots,n, \tag{1.4.9}$$

which, according to Hooke's law, satisfy $\mathfrak{S}_{\mathscr{S}} = \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}$ for some linear fourth-order tensor $\mathbb{T}$. If the medium is also homogeneous (i.e., the density and elastic parameters are position-independent), it follows that E *depends quadratically on the covariant derivative* $\mathscr{D}^{\mathscr{S}}\boldsymbol{U}$, i.e.,

$$E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) = \langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\rangle \tag{1.4.10}$$

for a linear operator

$$\mathbb{T} : \mathbb{M}^{n\times n}(\mathbb{R}) \to \mathbb{M}^{n\times n}(\mathbb{R}), \tag{1.4.11}$$

where $\mathbb{M}^{n\times n}(\mathbb{R})$ stands for the vector space of all $n\times n$ matrices with real entries. Hereafter, we organize $\mathbb{M}^{n\times n}(\mathbb{R})$ as a real Hilbert space with respect to the inner product

$$\langle A, B\rangle := \mathrm{Tr}(AB^\top) = \sum_{i,j} a_{ij}b_{ij}, \;\; \forall\, A = [a_{ij}]_{i,j}, \;\; B = [b_{ij}]_{i,j} \in \mathbb{M}^{n\times n}(\mathbb{R}), \tag{1.4.12}$$

where $B^\top$ denotes transposed matrix, and Tr is the usual trace operator for square matrices:

$$\mathrm{Tr}([g_{ij}]_{i,j=1}^n) = \sum_{i=1}^n g_{ii}, \quad G = [g_{ij}]_{i,j=1}^n.$$

The linear operator (1.4.11) is a tensor of order 4, i.e., $\mathbb{T} = [c_{ijk\ell}]_{ijk\ell}$, and

$$\mathbb{T}A = \Big[\sum_{k,\ell} c_{ijk\ell}a_{k\ell}\Big]_{ij} \quad \text{for} \quad A = [a_{k\ell}]_{k\ell} \in \mathbb{M}^{n\times n}(\mathbb{R}). \tag{1.4.13}$$

$\mathbb{T}$ will be referred to in the sequel as the **elasticity tensor**. It is customary to assume that the elasticity tensor (1.4.11) is self-adjoint

$$\langle \mathbb{T}A, B\rangle = \langle A, \mathbb{T}B\rangle, \quad A, B \in \mathbb{M}^{n\times n}(\mathbb{R}). \tag{1.4.14}$$

Condition (1.4.14), written in coordinate notation, is equivalent to the equality

$$c_{ijk\ell} = c_{k\ell ij}, \quad \forall\, i, j, k, \ell. \tag{1.4.15}$$

Indeed, the equality

$$\mathrm{Tr}((\mathbb{T}A)B^\top) = \sum_{i,j,k,\ell} c_{ijk\ell}a_{k\ell}b_{ij} = \sum_{i,j,k,\ell} c_{k\ell ij}a_{k\ell}b_{ij} = \mathrm{Tr}(A(\mathbb{T}B)^\top)$$

holds, for arbitrary $A = [a_{k\ell}]_{k\ell}$ and $B = [b_{k\ell}]_{k\ell}$, if and only if (1.4.15) holds: by inserting the delta functions $a_{k\ell} = \delta_{k\ell}$, $b_{ij} = \delta_{ij}$ we get equality (1.4.15).

It is also customary to impose a symmetry condition presented with two natural options:

$$\mathbb{T}(A^\top) = \mathbb{T}A \quad \text{and} \quad (\mathbb{T}A)^\top = \mathbb{T}A, \quad \forall\, A \in \mathbb{M}^{n\times n}(\mathbb{R}). \tag{1.4.16}$$

Then (1.4.16) amounts to the following symmetry in the indices of the elastic tensor:

$$c_{ijk\ell} = c_{ij\ell k} \quad \text{and} \quad c_{ijk\ell} = c_{jik\ell}, \quad \forall\, i, j, k, \ell. \tag{1.4.17}$$

Remark 1.4.1. Conditions (1.4.14) and the first equality in (1.4.16) imply the second equality in (1.4.16) as well as conditions (1.4.14) and the second equality in (1.4.16) imply the first equality in (1.4.16). This is evident if we apply an equivalent formulation for corresponding tensors and matrices: (1.4.15) and (1.4.17).

A linear operator $\mathbb{T}$ in the energy functional of anisotropic elasticity (1.4.10) satisfies the symmetry conditions (1.4.14) and (1.4.16). Equivalently, the corresponding elasticity tensor $\mathbb{T} = \big[c_{ijk\ell}\big]_{ijk\ell}$ has symmetries (1.4.15), (1.4.17) and, therefore, might have only $n + n^2(n-1)^2/2$ different entries.

Remark 1.4.2. It is rather natural to introduce the **deformation tensor** as the symmetrized covariant derivative (cf., e.g., [130, Volume I, Chapter 5, § 12])

$$(\mathrm{Def}_{\mathscr{S}}\, \boldsymbol{U})(\boldsymbol{V}, \boldsymbol{W}) = \frac{1}{2}\Big\{\langle \partial_{\boldsymbol{V}}\boldsymbol{U}, \boldsymbol{W}\rangle + \langle \partial_{\boldsymbol{W}}\boldsymbol{U}, \boldsymbol{V}\rangle\Big\} = \frac{1}{2}\Big\{\langle \partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U}, \boldsymbol{W}\rangle + \langle \partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U}, \boldsymbol{V}\rangle\Big\}, \quad \forall\, \boldsymbol{V}, \boldsymbol{W} \in \omega(\mathscr{S}). \tag{1.4.18}$$

It is also worth mentioning that the antisymmetric part of the covariant derivative $\partial^{\mathscr{S}}_{\boldsymbol{U}}$

$$d\boldsymbol{U}(\boldsymbol{V}, \boldsymbol{W}) = \langle d\boldsymbol{U}, \boldsymbol{V}\wedge\boldsymbol{W}\rangle = \frac{1}{2}\Big\{\langle \partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U}, \boldsymbol{W}\rangle - \langle \partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U}, \boldsymbol{V}\rangle\Big\}, \quad \forall\, \boldsymbol{V}, \boldsymbol{W} \in \omega(\mathscr{S}), \tag{1.4.19}$$

is the exterior differential.

By inserting value (1.4.9) of deformation tensor $\mathrm{Def}_{\mathscr{S}}\, \boldsymbol{U}$ and applying the symmetry properties (1.4.17), we obtain

$$4\langle \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\, \boldsymbol{U}(x), \mathrm{Def}_{\mathscr{S}}\, \boldsymbol{U}(x)\rangle = \langle \mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\rangle = E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) \tag{1.4.20}$$

(cf. (1.4.10)), which means that *the density of the elastic energy functional depends quadratically also on the deformation tensor*.

The density of the potential energy of an elastic medium should be strictly positive for the non-vanishing deformation tensor $\mathrm{Def}_{\mathscr{S}}\, \boldsymbol{U} \neq 0$. This leads to the following

Lemma 1.4.1. *There exists a constant* $C_0 > 0$ *such that*

$$\langle \mathbb{T}\zeta, \zeta\rangle := \sum_{i,j,k,\ell} c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell} \geqslant C_0 \sum_{i,j} |\zeta_{i,j}|^2 := C_0|\zeta|^2 \tag{1.4.21}$$

for all symmetric and complex-valued $\zeta_{ij} = \zeta_{ji} \in \mathbb{C}$ *matrices* (*tensors of order* 2) $\zeta := [\zeta_{ij}]_{n\times n}$.

Proof. The sum in the left-hand side of (1.4.21) is real $\langle \mathbb{T}\zeta, \zeta\rangle = \overline{\langle \mathbb{T}\zeta, \zeta\rangle}$ (easy to check applying the symmetry properties (1.4.17) of the real-valued coefficients). Dividing equality in (1.4.21) by $|\zeta|^2 = \sum\limits_{lm} |\zeta_{lm}|^2$ we find that it suffices to prove

$$\inf_{|\zeta|=1} \sum_{i,j,k,\ell} c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell} \geqslant C_0 > 0. \tag{1.4.22}$$

If otherwise $C_0 = 0$, we select a sequence $\zeta^{(q)}_{jk} = \zeta^{(q)}_{kj} \in \mathbb{C}$, $q = 1, 2, \dots$, such that

$$\lim_{q\to\infty} \sum_{i,j,k,\ell} c_{ijk\ell}\zeta^{(q)}_{ij}\overline{\zeta^{(q)}_{k\ell}} = 0, \ \ |\zeta^{(q)}| = 1.$$

Since the space of tensors $[\zeta^{(q)}_{jk}]_{n\times n}$ is finite-dimensional, there exists a convergent subsequence $\zeta^{(q_r)}_{k\ell} \to \zeta^{(0)}_{k\ell}$ as $r \to \infty$. Then we get an obvious contradiction

$$\sum_{i,j,k,\ell} c_{ijk\ell}\zeta^{(0)}_{ij}\overline{\zeta^{(0)}_{k\ell}} = 0, \ \ |\zeta^{(0)}| = 1,$$

which proves that $C_0 > 0$. □

Theorem 1.4.1. *The total free* (*elastic*) *energy functional* (*cf.* (1.4.8)) *acquires the form*

$$\mathscr{E}[\boldsymbol{U}] := \int\limits_{\mathscr{S}} \langle \mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y)\rangle\, d\sigma = 4\int\limits_{\mathscr{S}} \langle \mathbb{T}\operatorname{Def}_{\mathscr{S}}\boldsymbol{U}(y), \operatorname{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\, d\sigma, \ \ \boldsymbol{U} \in \omega(\mathscr{S}), \tag{1.4.23}$$

and the Euler–Lagrange equation associated with the energy functional (1.4.23) *for a linear anisotropic elastic medium reads:*

$$\mathscr{L}_{\mathscr{S}}\boldsymbol{U} = \operatorname{Def}^*_{\mathscr{S}}\mathbb{T}\operatorname{Def}_{\mathscr{S}}\boldsymbol{U}, \ \ \boldsymbol{U} \in \omega(\mathscr{S}). \tag{1.4.24}$$

Here again $\mathbb{T} = [c_{ijk\ell}]^n_{ijk\ell=1}$ *is the elasticity tensor which is positive definite* (*cf.* (1.4.22)) *and has the symmetry properties* (1.4.17).

Proof. Representation (1.4.23) follows from (1.4.8) and (1.4.20).

The Euler–Lagrange equation (1.4.24) is derived from (1.4.23) as a similar equation (1.4.3) is derived from (1.4.1):

$$\mathscr{E}[\boldsymbol{U}] := 4\int\limits_{\mathscr{S}} \langle \mathbb{T}\operatorname{Def}_{\mathscr{S}}\boldsymbol{U}(y), \operatorname{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\, d\sigma = 4\int\limits_{\mathscr{S}} \langle \operatorname{Def}^*_{\mathscr{S}}\mathbb{T}\operatorname{Def}_{\mathscr{S}}\boldsymbol{U}(y), \boldsymbol{U}(y)\rangle\, d\sigma = 0$$

if and only if $\boldsymbol{U} \in \omega(\mathscr{S})$ is a solution of equation (1.4.24) due to the positive definiteness of the elasticity tensor $\mathbb{T}$ (cf. (1.4.21)). □

Next, we will find the Euler–Lagrange equation associated with the energy functional (1.4.8) for a linear isotropic elastic medium (Lamé equation), which is simpler. Such energy functional should be invariant with respect to any rotation. For the elasticity tensor $\mathbb{T}$, this results into the requirement that

$$\mathbb{T}(BAB^{-1}) = B(\mathbb{T}A)B^{-1}, \ \ \forall\, A, B \in \mathbb{M}^{n\times n}(\mathbb{R}) \ \text{and unitary} \ \ B^\top = B^{-1}. \tag{1.4.25}$$

Examples of linear operators (1.4.11) satisfying (1.4.16) and (1.4.25) include

$$\mathbb{T} = \mathbb{T}A := (\operatorname{Tr} A)I \quad \text{and} \quad \mathbb{T}A := A + A^\top, \tag{1.4.26}$$

where I denotes the identity. The decisive step in the direction of identifying all such operators is the observation that any other operator of the type is a linear combination of these two. Namely, we have the following

Lemma 1.4.2. *Let a linear operator* $\mathbb{T}$ *in* (1.4.11) *be frame indifferent* (*cf.* (1.4.25))

$$\mathbb{T}(BAB^\top) = B(\mathbb{T}A)B^\top \quad \textit{for all} \ \ A \in \mathbb{M}^{n\times n} \ \ \textit{and for} \ \ B \in \mathbb{SO}(n)$$

and have the symmetry property: one of conditions in (1.4.16) *holds.*

Then $\mathbb{T}$ *has the form*

$$\mathbb{T}A = \lambda\,(\operatorname{Tr} A)I + \mu\,(A + A^\top), \quad A \in \mathbb{M}_{n,n}(\mathbb{R}), \tag{1.4.27}$$

where $\lambda, \mu \in \mathbb{R}$ *are some constants, and it has both symmetry properties from* (1.4.16).

Proof. Let us first show that any linear operator (1.4.11) satisfying (1.4.16), (1.4.25) is represented in form (1.4.27). By the previous discussion (cf. (1.4.26)), it suffices to prove that the space of linear operators (1.4.11) satisfying (1.4.16), (1.4.25) has dimension two when A is fixed and λ and μ are arbitrary parameters.

It suffices to show that

$$\mathbb{T}D = aD + b(I - D), \quad \text{where} \ \ D := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{1.4.28}$$

for the identity matrix I and two numbers $a, b \in \mathbb{R}$. Indeed, consider the following types of unitary matrices:

$$U_{j,k} := \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 \end{bmatrix}, \quad W_{j,k} := \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 \end{bmatrix},$$

where the only non-zero (equal 1), off-diagonal entries, are at (j, k) and (k, j). By multiplication $U_{j,k}A$ exchanges j-th with k-th rows in A, while $W_{j,k}A$, $j < k$, makes the same but changes the sign of j-th row before shifting it to k-th row.

By applying the unitary operator $U_{1,k}$, we get

$$\begin{aligned} \mathbb{T}E &= \sum_{j=1}^{n} \boldsymbol{e}^k \mathbb{T} U_{1,k} D U_{1,k}^{-1} = \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} (\mathbb{T}D) U_{1,k}^{-1} \\ &= \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} [aD - b(I - D)] U_{1,k}^{-1} = aE + b(I - E) \end{aligned} \tag{1.4.29}$$

for arbitrary diagonal matrix $E = [\delta_{jk}\boldsymbol{e}^k] = \sum\limits_{j=1}^{n} \boldsymbol{e}^k U_{1,k} D U_{1,k}^{-1}$. Since for any $A \in \mathbb{M}^{n\times n}(\mathbb{R})$ we have $\mathbb{T}A = \frac{1}{2}\,\mathbb{T}(A + A^\top)$, thanks to (1.4.16), and since a self-adjoint matrix can be diagonalized $\frac{1}{2}\,(A + A^\top) = UEU^{-1}$ with a suitable unitary matrix $\boldsymbol{U}$, equality (1.4.29) holds for arbitrary A:

$$\mathbb{T}A = \mathbb{T}UEU^{-1} = \boldsymbol{U}(\mathbb{T}E)U^{-1} = \boldsymbol{U}[aE + b(I - E)]U^{-1} = aA + b(I - A).$$

To check (1.4.28) we again apply the unitary matrices U_{i_0,j_0} and W_{i_0,j_0}. Set

$$A := \mathbb{T}D, \quad A = \big[a_{ij}\big]_{1\leqslant i,j\leqslant n},$$

and observe that D is invariant under conjugation by W_{i_0,j_0}, i.e., $W_{i_0,j_0} D W_{i_0,j_0}^\top = D$, as long as $i_0 \neq 1$ and $j_0 \neq 1$. Thus, by (1.4.25), the same is true for $A = \mathbb{T}D$ which, in turn, eventually implies that

$$a_{i_0 i_0} = a_{j_0 j_0}, \quad \forall\, i_0, j_0 \neq 1. \tag{1.4.30}$$

The next observation is that D is invariant under conjugation by the product $U_{i_0j_0}W_{i_0,j_0}$, i.e., $U_{i_0j_0}W_{i_0,j_0}DW^\top_{i_0,j_0}U^\top_{i_0j_0}=D$, this time for every $1\leqslant i_0\neq j_0\leqslant n$. Hence, by (1.4.25), the same holds for $A=\mathbb{T}D$, which ultimately implies that $a_{i_0j_0}=-a_{j_0i_0}$ for every pair of indices $1\leqslant i_0\neq j_0\leqslant n$. Consequently,

$$a_{i_0j_0}=0, \ \text{ for every } \ 1\leqslant i_0\neq j_0\leqslant n. \tag{1.4.31}$$

Under the current assumptions, that is (1.4.25), the first condition in (1.4.16), the desired conclusion, that is, that $\mathbb{T}D$ has the two-parameter diagonal form indicated above, now readily follows from (1.4.30) and (1.4.31).

Let us analyze the case where the linear operator $\mathbb{T}$ satisfies (1.4.25) along with the second condition in (1.4.16). In this situation, let us consider the adjoint $\mathbb{T}^*$ to the tensor $\mathbb{T}$ with respect to the inner product (1.4.12), $\langle\mathbb{T}A,B\rangle=\langle A,\mathbb{T}^*B\rangle$. It can be easily checked that the adjoint $\mathbb{T}^*$ satisfies (1.4.25) and the first condition in (1.4.16), so the previous reasoning applies. Consequently, $\mathbb{T}^*$ can be represented in form (1.4.27), which is invariant under taking the adjoint. Hence $\mathbb{T}$ can be written in form (1.4.27), too. In particular, (1.4.27) holds in this case as well.

Concerning the equivalence of the first and the second condition in (1.4.16), each of two conditions in (1.4.16) along with the condition (1.4.25) imply that the linear operator (1.4.11) has form (1.4.27). Then, in particular, $\mathbb{T}$ is self-adjoint. Since conditions in (1.4.16) are obtained by taking the adjoint, they are equivalent and the proof is completed. □

Remark 1.4.3. A posteriori, conditions (1.4.16) and (1.4.25) imply that the linear operator (1.4.11) has form (1.4.27) and, in particular, is self-adjoint, i.e., imply condition (1.4.14).

Remark 1.4.4. The above proof can be modified to hold in the case when (1.4.25) is (seemingly) weakened to allow only *orientation preserving* unitary matrices $\boldsymbol{U}$. All that needs to be done in the latter case is to employ the invariance of D under conjugation by $U_{k_0\ell_0}U_{i_0j_0}W_{i_0,j_0}$ (with $k_0,\ell_0\neq 1$) in place of conjugation by (the inversion) $U_{i_0j_0}W_{i_0,j_0}$ as in the original proof.

We are now ready to derive the Lamé equations of elasticity on a hypersurface.

Theorem 1.4.2. *On a smooth, closed hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, modeling a homogeneous, linear, isotropic, elastic medium, the Lamé operator $\mathscr{L}_\mathscr{S}$ is given by*

$$\mathscr{L}_\mathscr{S}=-\lambda\,\nabla_\mathscr{S}\,\mathbf{div}_\mathscr{S}+2\mu\,\mathrm{Def}^*_\mathscr{S}\mathrm{Def}_\mathscr{S}=\lambda\,\mathbf{div}^*_\mathscr{S}\,\mathbf{div}_\mathscr{S}+2\mu\,\mathrm{Def}^*_\mathscr{S}\,\mathrm{Def}_\mathscr{S}\,. \tag{1.4.32}$$

In particular, $\mathscr{L}_\mathscr{S}$ is a formally self-adjoint differential operator of second order.

Proof. According to the discussion in the first part of this section, the elasticity tensor in the case of linear, isotropic, elastic medium is given by (1.4.27), where λ,μ are the Lamé moduli. Applying the following properties of the trace

$$\mathrm{Tr}(A+B)=\mathrm{Tr}(A)+\mathrm{Tr}(B),\quad \mathrm{Tr}(A^\top)=\mathrm{Tr}(A),\quad \mathrm{Tr}(AB)=\mathrm{Tr}(A)\,\mathrm{Tr}(B),$$
$$\langle A+A^\top,A\rangle=\mathrm{Tr}\left[(A+A^\top)A^\top\right]=\frac{1}{2}\,\mathrm{Tr}\left[A^2+2AA^\top+(A^\top)^2\right]^2=\frac{1}{2}\,\mathrm{Tr}(A+A^\top)^2,$$

which are easy to verify directly, due to (1.4.10) the stored energy density is of the form

$$\begin{aligned}E(A)=\langle\mathbb{T}A,A\rangle&=\langle\lambda\,\mathrm{Tr}(A)I+\mu(A+A^\top),A\rangle\\&=\lambda\,\mathrm{Tr}(A)\langle I,A\rangle+\mu\langle A+A^\top,A\rangle=\lambda\,(\mathrm{Tr}\ A)^2+\frac{\mu}{2}\,\mathrm{Tr}\,\big((A+A^\top)^2\big).\end{aligned} \tag{1.4.33}$$

Further, by inserting $A:=\mathscr{D}^\mathscr{S}\boldsymbol{U}$ in (1.4.33) and recalling (1.3.37), we get

$$E(x,\mathscr{D}^\mathscr{S}\boldsymbol{U}(x))=\lambda\,(\mathbf{div}_\mathscr{S}\,\boldsymbol{U})^2(x)+2\mu\,\big\langle(\mathrm{Def}_\mathscr{S}\,\boldsymbol{U})(x),(\mathrm{Def}_\mathscr{S}\boldsymbol{U})(x)\big\rangle, \tag{1.4.34}$$

by (1.4.18) and since the trace

$$\mathrm{Tr}(\nabla_\mathscr{S}\boldsymbol{U})=\sum_{j=1}^n\mathscr{D}_jU_j,h_j=\mathbf{div}_\mathscr{S}\,\boldsymbol{U} \tag{1.4.35}$$

is the divergence and is independent of a basis $\{h_j\}_{j=1}^n$. Thus, we are led to the variational integral

$$\mathscr{E}[\boldsymbol{U}] = \int\limits_{\mathscr{S}} \Big[\lambda\,(\mathbf{div}_{\mathscr{S}}\,\boldsymbol{U})^2 + 2\mu\left\langle \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}, \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}\right\rangle\Big]\,d\sigma,\quad \boldsymbol{U}\in\omega(\mathscr{S}). \tag{1.4.36}$$

To determine the associated Euler–Lagrange equation, for a smooth and compactly supported vector field $\boldsymbol{V}\in\omega(\mathscr{S})\bigcap C_0^1(\mathscr{S})$ we compute

$$\frac{d}{dt}\,\mathscr{E}[\boldsymbol{U}+tV]\Big|_{t=0} = 2\int\limits_{\mathscr{S}} \Big[\lambda\;\mathbf{div}_{\mathscr{S}}\,\boldsymbol{U}\,\mathbf{div}_{\mathscr{S}}\,\boldsymbol{V} + 2\mu\left\langle \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}, \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{V}\right\rangle\Big]\,d\sigma.$$

By applying the formulae $\mathbf{div}^*_{\mathscr{S}} = -\nabla_{\mathscr{S}}$ (see (1.3.37)), we get

$$\frac{d}{dt}\,\mathscr{E}[\boldsymbol{U}+tV]\Big|_{t=0} = 2\int\limits_{\mathscr{S}}\langle(-\lambda\,\nabla_{\mathscr{S}}\,\mathbf{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}})\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma = 2\int\limits_{\mathscr{S}}\langle\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma = 0. \tag{1.4.37}$$

Since the vector field $\boldsymbol{V}\in\omega(\mathscr{S})\bigcap C_0^1(\mathscr{S})$ is arbitrary, from (1.4.37) it follows that the displacement vector field $\boldsymbol{U}$ satisfies the equality $\mathscr{L}_{\mathscr{S}}\boldsymbol{U}=0$.

The fact that the operator $\mathscr{L}_{\mathscr{S}} = \lambda\,\mathbf{div}^*_{\mathscr{S}}\,\mathbf{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\,\mathrm{Def}_{\mathscr{S}}$ is formally self-adjoint is obvious from its structure:

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V})_{\mathscr{S}} = \lambda(\,\mathbf{div}^*_{\mathscr{S}}\,\mathbf{div}_{\mathscr{S}}\,\boldsymbol{U},\boldsymbol{V})_{\mathscr{S}} + \mu(\,\mathrm{Def}^*_{\mathscr{S}}\,\mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U},\boldsymbol{V})_{\mathscr{S}} = (\boldsymbol{U},\mathscr{L}_{\mathscr{S}}\boldsymbol{V})_{\mathscr{S}}. \qquad \square$$

1.5 The surface Lamé operator and related PDO's

The present section deals mostly with the identification of the deformation tensor

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})(\boldsymbol{V},\boldsymbol{W}) := \frac{1}{2}\Big\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle + \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\},\quad \forall\,\boldsymbol{U},\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{1.5.1}$$

and the Lamé operator (1.4.32).

Theorem 1.5.1. *For the deformation tensor and the Lamé operator on $\mathscr{S}$ the following identities are valid:*

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{n\times n}, \tag{1.5.2}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\left[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k + (\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j\right] = \frac{1}{2}\left[\mathscr{D}_jU_k + \mathscr{D}_kU_j + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\right],\quad j,k=1,\dots,n, \tag{1.5.3}$$

$$\mathfrak{D}_{jj}(\boldsymbol{U}) = \frac{1}{2}\left[2\mathscr{D}_jU_j + \partial_{\boldsymbol{U}}\nu_j^2\right] = \mathscr{D}_jU_j + \nu_j\partial_{\boldsymbol{U}}\nu_j,,\quad j=1,\dots,n, \tag{1.5.4}$$

$$[\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})]^\top = \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\ \ \textit{and}\ \ \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{\nu} = 0, \tag{1.5.5}$$

$$\begin{aligned}\mathscr{L}_{\mathscr{S}} &= \mu\,\pi_{\mathscr{S}}\nabla^*_{\mathscr{S}}\,\nabla_{\mathscr{S}} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\,\nabla^*_{\mathscr{S}} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\\ &= -\mu\,\pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}} - (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}.\end{aligned} \tag{1.5.6}$$

Proof. Given the local nature of the identities we seek to prove, it suffices to work locally, in a small open subset $\mathscr{O}$ of $\mathscr{S}$, where an orthonormal basis $T_1,\dots,T_{n-1}$ to $\omega(\mathscr{S})$ has been fixed. We extend the basis by the outer unit normal vector field $T_n := \boldsymbol{\nu}$ so that $\{T_j\}_{1\leqslant j\leqslant n}$ becomes an orthonormal basis for $\mathbb{R}^n$, at points in $\mathscr{O}$.

Since $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ is a linear operator (see (1.5.1)), it is represented by an $n\times n$ matrix in the fixed basis $\{T_j\}_{1\leqslant j\leqslant n}$ and the first equality in (1.5.2) follows. The symmetry property of the matrix, recorded as the first equality in (1.5.5), follows from (1.5.1), since interchange of vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ does not affect definition (1.5.1).

For a tangent field $\boldsymbol{U}$ to $\mathscr{S}$ with supp $\boldsymbol{U}\subset\mathscr{O}$ and arbitrary $\boldsymbol{V},\boldsymbol{W}\in\mathbb{R}^n$ we have

$$\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U} = \partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U},\quad \langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle = \langle\partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U},\pi_{\mathscr{S}}\boldsymbol{W}\rangle$$

and, by the definition of the deformation tensor (cf. (1.5.1)), we obtain

$$\langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V}, \boldsymbol{W}\rangle := \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})(\pi_{\mathscr{S}}\boldsymbol{V}, \pi_{\mathscr{S}}\boldsymbol{W}), \quad \forall\, \boldsymbol{V}, \boldsymbol{W} \in \mathbb{R}^n. \tag{1.5.7}$$

Equality (1.5.7) implies the second equality in (1.5.5). Applying (1.4.18) and (1.4.6), we eventually obtain the second equality in (1.5.2):

$$\begin{aligned}\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\left[(\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j + (\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k\right] &= \frac{1}{2}\left[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\right]\\ &= \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \sum_{r=1}^n U_r(\mathscr{D}_r\nu_k)\nu_j + \sum_{r=1}^n U_r(\mathscr{D}_r\nu_j)\nu_k\Big].\end{aligned}$$

Equality (1.5.4) is a particular case of (1.5.3).

We proceed with the proof of the last remaining equality (1.5.6). If $\boldsymbol{V}$ is also a smooth vector field, tangent to $\mathscr{S}$, applying (1.5.2) we get

$$\begin{aligned}\int_{\mathscr{S}} \langle \mathrm{Def}^*_{\mathscr{S}}\,\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}), \boldsymbol{V}\rangle\, d\sigma &= \int_{\mathscr{S}} \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}), \mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})\rangle\, d\sigma\\ &= \sum_{j,k=1}^n \frac{1}{4}\int_{\mathscr{S}} \left[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\right]\left[\mathscr{D}_k V_j + \mathscr{D}_j V_k + \partial_{\boldsymbol{V}}(\nu_j\nu_k)\right] d\sigma.\end{aligned} \tag{1.5.8}$$

Next, consider

$$\begin{aligned}\sum_{j,k=1}^n \int_{\mathscr{S}} (\mathscr{D}_j U_k + \mathscr{D}_k U_j)(\mathscr{D}_j V_k + \mathscr{D}_k V_j)\, d\sigma &= 2\sum_{j,k=1}^n \int_{\mathscr{S}} \mathscr{D}_j^*(\mathscr{D}_j U_k + \mathscr{D}_k U_j)V_k\, d\sigma\\ &= 2\sum_{j,k=1}^n \int_{\mathscr{S}} \Big[-V_k\mathscr{D}_j^2 U_k - V_k\mathscr{D}_j\mathscr{D}_k U_j - \mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_j U_k)V_k - \mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_k U_j)V_k\Big]\, d\sigma\\ &= -2\int_{\mathscr{S}} \langle \boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\, d\sigma - 2\sum_{j,k=1}^n \int_{\mathscr{S}} \Big[V_k\mathscr{D}_j\mathscr{D}_k U_j + \mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_k U_j)V_k\Big]\, d\sigma,\end{aligned} \tag{1.5.9}$$

since $\sum\limits_{j=1}^n \nu_j\mathscr{D}_j = 0$ on $\mathscr{S}$.

To proceed in the second integrand in (1.5.9) we employ the commutator identity from Lemma 1.3.3.ix and recall that the fields $\boldsymbol{U}$ and $\boldsymbol{V}$ are tangent to write

$$\begin{aligned}\sum_{j,k=1}^n \int_{\mathscr{S}} V_k\mathscr{D}_j\mathscr{D}_k U_j\, d\sigma &= \sum_{j,k=1}^n \int_{\mathscr{S}} \left[V_k\mathscr{D}_k\mathscr{D}_j U_j + V_k[\mathscr{D}_j, \mathscr{D}_k]U_j\right] d\sigma\\ &= \int_{\mathscr{S}} \langle \nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\, d\sigma + \sum_{j,k,l=1}^n \int_{\mathscr{S}} \left[V_k\nu_j\mathscr{D}_k\nu_l - \nu_k V_k\mathscr{D}_j\nu_l\right]\mathscr{D}_l U_j\, dS\\ &= \int_{\mathscr{S}} \langle \nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\, d\sigma + \sum_{j,k,l=1}^n \int_{\mathscr{S}} V_k(\mathscr{D}_k\nu_l)\left[\mathscr{D}_l(\nu_j U_j) - (\mathscr{D}_l\nu_j)U_j\right] d\sigma\\ &= \int_{\mathscr{S}} \langle \nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\, d\sigma - \sum_{j,k,l=1}^n \int_{\mathscr{S}} (\partial_k\nu_l)(\partial_l\nu_j)U_j V_k\, d\sigma\\ &= \int_{\mathscr{S}} \langle \nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\, d\sigma - \int_{\mathscr{S}} \langle \mathscr{W}^2_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\, d\sigma\end{aligned} \tag{1.5.10}$$

on $\mathscr{S}$, because $\sum_{j=1}^{n}\nu_jU_j=\sum_{k=1}^{n}\nu_kV_k=0$ and, due to (1.3.19),

$$\sum_{j,l,k=1}^{n}(\partial_k\nu_l)(\partial_l\nu_j)U_jV_k=\sum_{j,l,k=1}^{n}(\partial_l\nu_j)U_j(\partial_j\nu_k)V_k=\langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\mathscr{W}_{\mathscr{S}}\boldsymbol{V}\rangle=\langle\mathscr{W}^2_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle.$$

For the third integrand in (1.5.9) we use Lemma 1.3.1(i) and the fact that the field $\boldsymbol{U}$ is tangent:

$$\begin{aligned}\sum_{j,k=1}^{n}\int_{\mathscr{S}}&\mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_kU_j)V_k\,d\sigma\\&=\mathscr{H}^0_{\mathscr{S}}\sum_{j,k=1}^{n}\int_{\mathscr{S}}V_k\big[\mathscr{D}_k(\nu_jU_j)-(\mathscr{D}_k\nu_j)U_j\big]\,d\sigma=\int_{\mathscr{S}}\mathscr{H}^0_{\mathscr{S}}\langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma.\end{aligned}\tag{1.5.11}$$

At this point we may, therefore, conclude that

$$\begin{aligned}\sum_{j,k=1}^{n}\int_{\mathscr{S}}&(\mathscr{D}_jU_k+\mathscr{D}_kU_j)(\mathscr{D}_jV_k+\mathscr{D}_kV_j)\,d\sigma\\&=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}+\mathscr{W}^2_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma.\end{aligned}\tag{1.5.12}$$

We now proceed to analyze the remaining terms in (1.5.8). More precisely, we still have to take into account the terms containing either $\partial_{\boldsymbol{U}}(\nu_j\nu_k)$ or $\partial_{\boldsymbol{V}}(\nu_j\nu_k)$. We start with the identity

$$\begin{aligned}\sum_{j,k=1}^{n}(\mathscr{D}_kU_j)\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)&=\sum_{j,k=1}^{n}\nu_k(\mathscr{D}_kU_j)\mathscr{D}_{\boldsymbol{V}}\nu_j+\sum_{j,k=1}^{n}(\mathscr{D}_{\boldsymbol{V}}\nu_k)\big[\mathscr{D}_k(\nu_jU_j)-U_j\mathscr{D}_k\nu_j\big]\\&=-\sum_{k,j=1}^{n}(\mathscr{D}_{\boldsymbol{V}}\nu_k)(\mathscr{D}_{\boldsymbol{U}}\nu_k)=-\langle\mathscr{W}^2_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle,\end{aligned}\tag{1.5.13}$$

valid at points on $\mathscr{S}$, because $\sum_k\nu_k\mathscr{D}_k=0$, $\sum_j\nu_jU_j=0$ and $\mathscr{D}_k\nu_j=\mathscr{D}_j\nu_k$. There are four such terms in (1.5.8), i.e., containing either $\mathscr{D}_{\boldsymbol{U}}(\nu_j\nu_k)$ or $\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)$. An inspection of the above calculation shows that, on $\mathscr{S}$, they are all equal to $-\langle\mathscr{W}^2_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle$.

We still have to compute the last integrand in (1.5.8):

$$\begin{aligned}\sum_{j,k=1}^{n}\mathscr{D}_{\boldsymbol{U}}(\nu_j\nu_k)\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)&=\sum_{j,k,r,l=1}^{n}\big[U_r(\mathscr{D}_r\nu_j)\nu_k+U_r(\mathscr{D}_r\nu_k)\nu_j\big]\big[V_l(\mathscr{D}_l\nu_j)\nu_k+V_l(\mathscr{D}_l\nu_k)\nu_j\big]\\&=2\langle\mathscr{W}^2_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle+2\sum_{k,r,l=1}^{n}U_r(\mathscr{D}_r\nu_k)V_l\nu_k\,\frac{1}{2}\,\mathscr{D}_l\Big(\sum_{j=1}^{n}(\nu_j)^2\Big)=2\langle\mathscr{W}^2_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\end{aligned}$$

on $\mathscr{S}$. At this point we combine all the above to get

$$\begin{aligned}4\int_{\mathscr{S}}\langle\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\boldsymbol{V}\rangle\,d\sigma&=4\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\,d\sigma\\&=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma,\end{aligned}\tag{1.5.14}$$

since $\langle\boldsymbol{W},\boldsymbol{V}\rangle=\langle\pi_{\mathscr{S}}\boldsymbol{W},\boldsymbol{V}\rangle$ for a tangent vector field $\boldsymbol{V}$ and an arbitrary vector field $\boldsymbol{W}$. Also we have applied that the vectors $\nabla_{\mathscr{S}}\mathbf{div}_{\mathscr{S}}\boldsymbol{U}$ and $\mathscr{W}_{\mathscr{S}}\boldsymbol{U}$ are tangent. Thus,

$$4\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}=-2\pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}-2\nabla_{\mathscr{S}}\,\mathbf{div}_{\mathscr{S}}-2\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}},\tag{1.5.15}$$

since the tangent vectors fields $\boldsymbol{U}$, $\boldsymbol{V}$ are arbitrary.

The first identity in (1.5.6) now easily follows from (1.5.15) and (1.4.32). Then the remaining identity in (1.5.6) follows from what we have just proved and from Theorem 1.3.1. □

Next, recall the definition of the Hodge–Laplacian acting on 1-forms, i.e.,

$$\boldsymbol{\Delta}_{HL} := -\boldsymbol{d}^{\mathscr{S}}\boldsymbol{d}^*_{\mathscr{S}} - \boldsymbol{d}^*_{\mathscr{S}}\boldsymbol{d}^{\mathscr{S}} : \Lambda^1\omega(\mathscr{S}) \to \Lambda^1\omega(\mathscr{S}), \tag{1.5.16}$$

where $\boldsymbol{d}^{\mathscr{S}}$ is the exterior derivative operator on $\mathscr{S}$, and $\boldsymbol{d}^*_{\mathscr{S}}$ is its formal adjoint. As explained in Section 1.3, 1-forms on $\mathscr{S}$ are naturally identified with tangent fields to $\mathscr{S}$, so, from now on, we shall think of $\boldsymbol{\Delta}_{HL}$ as mapping $\omega(\mathscr{S})$ into itself.

As pointed out in Section 1.3, the Hodge–Laplacian (1.5.16) is related to the Bochner–Laplacian on $\mathscr{S}$

$$\boldsymbol{\Delta}_{BL} := -(\nabla^{\mathscr{S}})^*\nabla^{\mathscr{S}} \tag{1.5.17}$$

via the Weitzenbock identity

$$\boldsymbol{\Delta}_{BL} = \boldsymbol{\Delta}_{HL} + \mathrm{Ric}_{\mathscr{S}}\,. \tag{1.5.18}$$

Our aim is to find alternative expressions for all these objects, starting with the Ricci curvature tensor.

The Ricci curvature $\mathrm{Ric}_{\mathscr{S}}$ on $\mathscr{S}$ is a $(0,2)$-tensor defined as a contraction of Riemannian curvature tensor $\boldsymbol{R}_{\mathscr{S}}$:

$$\mathrm{Ric}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) := \sum_{j=1}^{n}\langle \boldsymbol{R}_{\mathscr{S}}(h_j,\boldsymbol{V})\boldsymbol{U},h_j\rangle = \sum_{j=1}^{n}\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{V},h_j)h_j,\boldsymbol{U}\rangle,\ \ \forall\,\boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S}), \tag{1.5.19}$$

where $h_1,\dots,h_n$ is an orthonormal basis (of unit vectors) in $\omega(\mathscr{S})$. Thus, $\mathrm{Ric}_{\mathscr{S}}$ is a symmetric bilinear form.

Theorem 1.5.2. *For the Ricci tensor* $\mathrm{Ric}_{\mathscr{S}}$ *(cf.* (1.5.19)*) on* $\mathscr{S}$ *there holds*

$$\mathrm{Ric}_{\mathscr{S}} = -\mathscr{W}^2_{\mathscr{S}} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}. \tag{1.5.20}$$

In particular, when $n=3$*, i.e., for a two-dimensional hypersurface* $\mathscr{S}$ *in* $\mathbb{R}^3$*, the above identity reduces to*

$$\mathrm{Ric}_{\mathscr{S}} = -\det\mathscr{W}_{\mathscr{S}} = -\mathscr{K}_{\mathscr{S}}, \tag{1.5.21}$$

where $\mathscr{K}_{\mathscr{S}}$ *is the Gaußian curvature of the hypersurface* $\mathscr{S}$*.*

Proof. The Riemannian curvature tensor $\boldsymbol{R}_{\mathscr{S}}$ of $\mathscr{S}$ is given by

$$\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W} = [\partial^{\mathscr{S}}_{\boldsymbol{U}},\partial^{\mathscr{S}}_{\boldsymbol{V}}]\boldsymbol{W} - \partial^{\mathscr{S}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{W},\quad \boldsymbol{U},\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{1.5.22}$$

where $[\boldsymbol{U},\boldsymbol{V}] := \partial_{\boldsymbol{U}}\boldsymbol{Y} - \partial_{\boldsymbol{Y}}\boldsymbol{U}$ is the usual commutator bracket. It is convenient to change this into a $(0,4)$-tensor by setting

$$\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}) := \langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle,\ \ \boldsymbol{U},\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}\in\omega(\mathscr{S}). \tag{1.5.23}$$

Since $\mathbb{R}^n$ has zero curvature, it follows from Gauß's Theorema Egregium that if $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, $\boldsymbol{W}$ are tangent vector fields to $\mathscr{S}$, then

$$\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle = \langle II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{Z}), II_{\mathscr{S}}(\boldsymbol{V},\boldsymbol{W})\rangle - \langle II_{\mathscr{S}}(\boldsymbol{V},\boldsymbol{Z}), II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{W})\rangle \tag{1.5.24}$$

(see, e.g., [130, Vol. II, p. 481]). By inserting the second fundamental form $II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) = \langle\partial_{\boldsymbol{U}}\boldsymbol{V} - \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V},\boldsymbol{\nu}\rangle = \langle\partial_{\boldsymbol{U}}\boldsymbol{V},\boldsymbol{\nu}\rangle$ (cf. (1.3.16)), we obtain

$$\begin{aligned}\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle &= \langle\partial_{\boldsymbol{U}}\boldsymbol{Z},\boldsymbol{\nu}\rangle\langle\partial_{\boldsymbol{V}}\boldsymbol{W},\boldsymbol{\nu}\rangle - \langle\partial_{\boldsymbol{V}}\boldsymbol{Z},\boldsymbol{\nu}\rangle\langle\partial_{\boldsymbol{U}}\boldsymbol{W},\boldsymbol{\nu}\rangle\\ &= \langle\boldsymbol{Z},\partial_{\boldsymbol{U}}\boldsymbol{\nu}\rangle\langle\boldsymbol{W},\partial_{\boldsymbol{V}}\boldsymbol{\nu}\rangle - \langle\boldsymbol{Z},\partial_{\boldsymbol{V}}\boldsymbol{\nu}\rangle\langle\boldsymbol{W},\partial_{\boldsymbol{U}}\boldsymbol{\nu}\rangle\\ &= \langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{Z},\boldsymbol{U}\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{W},\boldsymbol{V}\rangle - \langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{Z},\boldsymbol{V}\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{W},\boldsymbol{U}\rangle.\end{aligned} \tag{1.5.25}$$

For the second equality in (1.5.25) we have used the fact that $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$, and $\boldsymbol{Z}$ are tangent, so, in particular, $\partial_{\boldsymbol{U}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, $\partial_{\boldsymbol{V}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, $\partial_{\boldsymbol{V}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, and $\partial_{\boldsymbol{U}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$ on $\mathscr{S}$.

Next, recall from (1.5.19) the definition of the Ricci tensor, i.e.,

$$\mathrm{Ric}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) = \sum_{j=1}^{n-1}\langle\boldsymbol{R}_{\mathscr{S}}(h_j,\boldsymbol{V})\boldsymbol{U},h_j\rangle,$$

where $h_1,\dots,h_{n-1}$ is, locally, an orthonormal basis in $\omega(\mathscr{S})$, and $\boldsymbol{U}$, $\boldsymbol{V}$ are arbitrary tangent vector fields to $\mathscr{S}$. If we set $h_n := \boldsymbol{\nu}$ and employ (1.5.25) together with $\mathscr{W}_{\mathscr{S}}\boldsymbol{\nu} = 0$, we obtain

$$\sum_{j=1}^{n-1}\big\langle\boldsymbol{R}_{\mathscr{S}}(T_j,\boldsymbol{V})\boldsymbol{U},T_j\big\rangle = \sum_{j=1}^{n}\Big[\langle\boldsymbol{R}_{\mathscr{S}}T_j,T_j\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle - \langle\boldsymbol{R}_{\mathscr{S}}T_j,\boldsymbol{V}\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},T_j\rangle\Big]$$

$$= -\mathscr{H}^0_{\mathscr{S}}\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle - \Big\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{V},\sum_{j=1}^{n}\langle T_j,\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U}\rangle T_j\Big\rangle - \big\langle(\mathscr{W}^2_{\mathscr{S}} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}})\boldsymbol{U},\boldsymbol{V}\big\rangle, \quad (1.5.26)$$

which takes care of (1.5.20).

Finally, (1.5.21) is a consequence of what we have proved so far in Lemma 1.3.2(ii), and the elementary identity $A^2 - (\mathrm{Tr}\,A)A = -(\det A)I$, valid for any 2×2 matrix A. □

Lemma 1.5.1. *Let $H := \{h_j\}_{j=1}^n$, $|h_j| = 1$, be a basis in n-dimensional Banach space $\mathfrak{B}$. Consider the hyperspace $\mathfrak{B}_{\boldsymbol{\nu}} := \{u\in\mathfrak{B} : \langle u,\boldsymbol{\nu}\rangle = 0\}$, orthogonal to some vector $\boldsymbol{\nu}\in\mathfrak{B}$, $|\boldsymbol{\nu}|\neq 0$. Consider the system*

$$\widehat{h}_j := h_j - \nu_j\boldsymbol{\nu}, \quad \nu_j := \langle\boldsymbol{\nu},h_j\rangle, \;\; j = 1,\dots,n, \quad (1.5.27)$$

*which is full in $\mathfrak{B}_{\boldsymbol{\nu}}$ but **linearly dependent**, and thus cannot be a basis. Nevertheless, for a linear operator $A = \big[a_{jk}\big]_{n\times n} : \mathfrak{B}\to\mathfrak{B}$ with $A\boldsymbol{\nu} = 0$ and $A\mathfrak{B}_{\boldsymbol{\nu}}\subset\mathfrak{B}_{\boldsymbol{\nu}}$ (i.e., $\mathfrak{B}_{\boldsymbol{\nu}}$ is invariant under A), we have*

$$\widehat{A} := [\widehat{a}_{jk}]_{n\times n} = [a_{jk}]_{n\times n} := A, \quad (1.5.28)$$

where $\widehat{A} := \big[\widehat{a}_{jk}\big]_{n\times n}$ is the matrix representations of A in the linearly dependent systems $\widehat{H} := \{\widehat{h}_j\}_{j=1}^n\subset\mathfrak{B}_{\boldsymbol{\nu}}$.

Proof. Let us note that

$$\sum_{k=1}^{n} a_{jk}\nu_k = \sum_{k=1}^{n} a_{kj}\nu_k = 0 \;\text{ for all }\; j = 1,\dots,n,$$

where the first equality is equivalent to $A\boldsymbol{\nu} = 0$ and the second one to $\langle\boldsymbol{\nu},A\xi\rangle = 0$ for all $\xi\in\mathfrak{B}$. Applying the obtained equalities we find that

$$A\widehat{h}_j = Ah_j - \nu_j A\boldsymbol{\nu} = \sum_{k=1}^{n} a_{kj}h_k = \sum_{k=1}^{n} a_{kj}\widehat{h}_k + \sum_{k=1}^{n} a_{kj}\nu_k\boldsymbol{\nu} = \sum_{k=1}^{n} a_{kj}\widehat{h}_k,$$

which entails $\widetilde{a}_{kj} = a_{kj}$. □

Theorem 1.5.3. *The following identities are valid:*

$$\boldsymbol{\Delta}_{BL} = \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}} + \mathscr{W}^2_{\mathscr{S}}, \quad (1.5.29)$$

$$\boldsymbol{\Delta}_{HL} = \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}} + 2\mathscr{W}^2_{\mathscr{S}} - \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}. \quad (1.5.30)$$

Proof. In order to identify the Bochner–Laplacian operator $\boldsymbol{\Delta}_{BL}$ on $\mathscr{S}$, we observe that, with tangent field $\boldsymbol{U}$ fixed, if the matrix $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ satisfies $\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V},\boldsymbol{W}\rangle = \langle\partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U},\pi_{\mathscr{S}}\boldsymbol{W}\rangle$ for each $\boldsymbol{V},\boldsymbol{W}\in\mathbb{R}^n$, then, much as in the proof of Theorem 1.3.1,

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{e}^k,\boldsymbol{e}^j\rangle = \langle\partial_{\overline{e}_k}\boldsymbol{U},\overline{e}_j\rangle = \mathscr{D}_k U_j - \sum_{r=1}^{n}\nu_j\nu_r\mathscr{D}_k(U_r). \quad (1.5.31)$$

On account of this we can now write

$$\begin{aligned}
\int\limits_{\mathscr{S}} \big\langle (\nabla^{\mathscr{S}})^*\nabla^{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\big\rangle\,d\sigma &= \int\limits_{\mathscr{S}} \langle \nabla^{\mathscr{S}}\boldsymbol{U},\nabla^{\mathscr{S}}\boldsymbol{V}\rangle\,d\sigma = \sum_{j,k=1}^{n-1}\int\limits_{\mathscr{S}} \langle \nabla^{\mathscr{S}}_{T_j}\boldsymbol{U},T_k\rangle\langle \nabla^{\mathscr{S}}_{T_j}\boldsymbol{V},T_k\rangle\,d\sigma\\
&= \sum_{j,k=1}^{n}\int\limits_{\mathscr{S}} \big\langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})T_j,T_k\big\rangle\big\langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})T_j,T_k\big\rangle\,d\sigma = \sum_{j,k=1}^{n}\int\limits_{\mathscr{S}} \mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\,d\sigma\\
&= \sum_{j,k=1}^{n}\int\limits_{\mathscr{S}} \bigg[\mathscr{D}_kU_j\mathscr{D}_kV_j - \sum_{r=1}^{n}\nu_j\nu_r\mathscr{D}_jU_r\mathscr{D}_kV_j - \sum_{l=1}^{n}\nu_j\nu_l\mathscr{D}_kU_j\mathscr{D}_kV_l + \sum_{r,l=1}^{n}\nu_r\nu_l\mathscr{D}_kU_r\mathscr{D}_kV_l\bigg]\,d\sigma\\
&= \sum_{j,k=1}^{n}\int\limits_{\mathscr{S}} \Big[(\mathscr{D}_k^*\mathscr{D}_kU_j)V_j - \sum_{r=1}^{n}U_rV_j(\partial_k\nu_r)(\partial_k\nu_j)\Big]\,d\sigma = \int\limits_{\mathscr{S}} \big\langle -\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U} - \mathscr{W}^2_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\big\rangle\,d\sigma. \qquad (1.5.32)
\end{aligned}$$

In the next-to-the-last equality, we have applied the following identity to the terms under the integral sign in the fourth line above:

$$\sum_{r=1}^{n}\nu_r\mathscr{D}_sW_r = \mathscr{D}_s\Big(\sum_{r=1}^{n}\nu_rW_r\Big) - \sum_{r=1}^{n}W_r\mathscr{D}_s\nu_r = -\sum_{r=1}^{n}W_r\partial_s\nu_r \ \text{ on } \ \mathscr{S}, \qquad (1.5.33)$$

which is valid for any tangent vector field $\boldsymbol{W}$ and any index $s\in\{1,\dots,n\}$. In turn, identity (1.5.33) can be seen from a direct computation (recall that $\partial_{\boldsymbol{\nu}}\nu_r=0$ on $\mathscr{S}$). Finally, to justify the last equality in (1.5.32), it suffices to recall (1.3.38), (1.3.56) and the fact that $\sum\limits_{k=1}^{n}\nu_k\mathscr{D}_k=0$.

The conclusion is that (1.5.29) holds. Finally, identity (1.5.29) in concert with (1.5.18) and (1.5.20) implies (1.5.30). □

Recall now from [71, *Note Added in Proof*, pp. 161–162], [129] (cf. also the remark at the end of this paper), and [130, Volume III], that the Navier–Stokes system for a velocity field $\boldsymbol{U}$, tangent to $\mathscr{S}$, and a (scalar-valued) pressure function p on $\mathscr{S}$ reads:

$$\begin{aligned}
&\frac{\partial\boldsymbol{U}}{\partial t} - 2\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) + \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{U} - \nabla_{\mathscr{S}}p = f \ \text{ in } \ \mathscr{S}\times(0,\infty),\\
&\mathbf{div}_{\mathscr{S}}\boldsymbol{U} = 0 \ \text{ in } \ \mathscr{S}.
\end{aligned} \qquad (1.5.34)$$

If $\mathscr{S}$ is embedded in $\mathbb{R}^n$ and the Riemannian metric is inherited from $\mathbb{R}^n$, a directional derivative $\partial_{\boldsymbol{U}}$ along a tangent vector field $\boldsymbol{U}\in\omega(\mathscr{S})$ maps the space of tangent vector fields to the space of possibly non-tangent vector fields

$$\partial_{\boldsymbol{U}}:\omega(\mathscr{S})\to\omega(\mathscr{S}).$$

If composed with the projection

$$\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V} := \pi_{\mathscr{S}}\partial_{\boldsymbol{U}}\boldsymbol{V} = \partial_{\boldsymbol{U}}\boldsymbol{V} - \langle\boldsymbol{\nu},\partial_{\boldsymbol{U}}\boldsymbol{V}\rangle\boldsymbol{\nu} \qquad (1.5.35)$$

(cf. (0.0.8)), it becomes an automorphism of the space of tangent vector fields. Such derivatives are compatible with the Riemannian metric on $\mathscr{S}$ and are torsion-free as well. Therefore, they represent the natural Levi–Civita connection on $\mathscr{S}$.

Theorem 1.5.4. *The Navier–Stokes system* (1.5.34) *is equivalent to*

$$\begin{aligned}
&\frac{\partial\boldsymbol{U}}{\partial t} + \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{U} + \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U} - \nabla_{\mathscr{S}}p = f \ \textit{ in } \ \mathscr{S}\times(0,\infty),\\
&\mathbf{div}_{\mathscr{S}}\,\boldsymbol{U} = 0 \ \textit{ in } \ \mathscr{S}.
\end{aligned} \qquad (1.5.36)$$

Proof. This is a direct consequence of (1.5.15) and (1.5.35). □

1.6 Lions' lemma and Korn's inequalities

For $1 \leqslant p < \infty$, an integer $m = 1, 2, \dots$ and a closed C^{m+1}-smooth hypersurface $\mathscr{S}$, by $\mathbb{W}^m_p(\mathscr{S})$ $\mathbb{W}^m(\mathscr{S}) := \mathbb{W}^m_2(\mathscr{S})$ for $p = 2$) we denote the Sobolev spaces. The space $\mathbb{W}^{-m}_p(\mathscr{S})$ is defined as the dual to $\mathbb{W}^m_{p'}(\mathscr{S})$, $p' := \frac{p}{p-1}$, with respect to the sesquilinear form $(\varphi, \psi)_{\mathscr{S}}$ (cf. (1.3.53)) on functions $\varphi, \psi \in C^m(\mathscr{S})$ and extended by continuity to pairs $\varphi \in \mathbb{W}^m_{p'}(\mathscr{S})$ and $\psi \in \mathbb{W}^{-m}_p(\mathscr{S})$.

The embeddings $\mathbb{W}^m_p(\mathscr{S}) \subset \mathbb{L}_p(\mathscr{S}) \subset \mathbb{W}^{-m}_p(\mathscr{S})$ are continuous, even compact, and

$$\mathbb{W}^{-m}_p(\mathscr{S}) := \Big\{\mathscr{D}^\alpha\varphi : \ \ \varphi \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } \ \mathscr{D}^\alpha = \mathscr{D}_1^{\alpha_1} \cdots \mathscr{D}_n^{\alpha_n}, \ \ |\alpha| = m\Big\}.$$

If $\mathscr{S}$ is an open surface with the Lipschitz boundary $\Gamma = \partial\mathscr{S} \neq \varnothing$, $\widetilde{\mathbb{W}}^m_p(\mathscr{S})$ denotes the space of functions obtained by closing the space $C_0^\infty(\mathscr{S})$ of smooth functions with compact support in the norm of $\mathbb{W}^m_p(\widetilde{\mathscr{S}})$, where $\widetilde{\mathscr{S}} \supset \mathscr{S}$ is a closed surface which extends the surface $\mathscr{S}$. The notation $\mathbb{W}^m_p(\mathscr{S})$ is used for the factor space $\mathbb{W}^m_p(\widetilde{\mathscr{S}})/\widetilde{\mathbb{W}}^m_p(\widetilde{\mathscr{S}} \setminus \mathscr{S})$; the space $\mathbb{W}^m_p(\mathscr{S})$ can also be viewed as the restriction of all functions $\varphi\big|_{\mathscr{S}}$ of the space $\mathbb{W}^m_p(\widetilde{\mathscr{S}})$ to the subsurface $\mathscr{S}$ (cf. [133] and [67] for details about these spaces).

The following generalizes essentially J. L. Lions' Lemma (cf. [129], [4, Proposition 2.10], [23, § 1.7], [110]).

Lemma 1.6.1. *Let $\mathscr{S}$ be a 2-smooth closed hypersurface in $\mathbb{R}^n$. Then the inclusions $\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$ for all $j = 1, \dots, n$ imply $\varphi \in \mathbb{L}_p(\mathscr{S})$.*

Moreover, the assertion holds for a hypersurface $\mathscr{S}$ with the Lipschitz boundary $\Gamma := \partial\mathscr{S}$ and the spaces $\mathbb{W}^{-1}_p(\mathscr{S})$ and $\widetilde{\mathbb{W}}^{-1}_p(\mathscr{S})$.

Proof. First, we assume that $\mathscr{S}$ is a closed surface. The proof is based on the following facts from [67, 88, 130], which we recall without proofs.

A. There exists a "lifting operator" (a Bessel potential operator) $\Lambda(\mathscr{x}, D)$, which maps isometrically the spaces

$$\Lambda^{-1}(\mathscr{x}, D) : \mathbb{W}^{m-1}_p(\mathscr{S}) \to \mathbb{W}^m_p(\mathscr{S}), \quad \Lambda(\mathscr{x}, D) : \mathbb{W}^m_p(\mathscr{S}) \to \mathbb{W}^{m-1}_p(\mathscr{S}) \tag{1.6.1}$$

for arbitrary $m = 0, \pm 1, \dots$ and has the inverse $\Lambda^{-1}(\mathscr{x}, D)$.

B. $\Lambda^{-1}(\mathscr{x}, D)$ is a pseudodifferential operator of order -1 and the commutant

$$[\mathscr{D}_j, \Lambda^{-1}(\mathscr{x}, D)] := \mathscr{D}_j\Lambda^{-1}(\mathscr{x}, D) - \Lambda^{-1}(\mathscr{x}, D)\mathscr{D}_j \tag{1.6.2}$$

with the pseudodifferential operator $\mathscr{D}_j$ has order -1, i.e., maps continuously the spaces

$$[\mathscr{D}_j, \Lambda^{-1}(\mathscr{x}, D)] : \mathbb{W}^{-1}_p(\mathscr{S}) \to \mathbb{L}_p(\mathscr{S}).$$

Let $\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$ for all $j = 1, \dots, n$. Then, due to (1.6.1), $\psi := \Lambda^{-1}(\mathscr{x}, D)\varphi \in \mathbb{L}_p(\mathscr{S})$ and, due to (1.6.2), $\mathscr{D}_j\psi = [\mathscr{D}_j, \Lambda^{-1}(\mathscr{x}, D)]\varphi + \Lambda^{-1}(\mathscr{x}, D)\mathscr{D}_j\varphi \in \mathbb{L}_p(\mathscr{S})$ for all $j = 1, \dots, n$. From the definition of the space $\mathbb{W}^1_p(\mathscr{S})$ it follows that $\psi \in \mathbb{W}^1_p(\mathscr{S})$. Due to (1.6.2), we finally get $\varphi = \Lambda(\mathscr{x}, D)\psi \in \mathbb{L}_p(\mathscr{S})$.

If $\mathscr{S}$ has non-empty Lipschitz boundary $\Gamma \neq \varnothing$, there exist pseudodifferential operators

$$\begin{aligned} &\Lambda^{-1}_-(\mathscr{x}, D) : \mathbb{W}^m_p(\mathscr{S}) \to \mathbb{W}^{m+1}_p(\mathscr{S}), \\ &\Lambda^{-1}_+(\mathscr{x}, D) : \widetilde{\mathbb{W}}^m_p(\mathscr{S}) \to \widetilde{\mathbb{W}}^{m+1}_p(\mathscr{S}) \end{aligned} \tag{1.6.3}$$

arranging isomorphisms between the indicated spaces and having the inverses $\Lambda^{+1}_-(\mathscr{x}, D)$, $\Lambda^{+1}_+(\mathscr{x}, D)$ (cf. [67]).

Moreover, the pseudodifferential operators $\Lambda^{-1}_\pm(\mathscr{x}, D)$ have order -1 and the commutants $[\mathscr{D}_j, \Lambda^{-1}_\pm(\mathscr{x}, D)] := \mathscr{D}_j\Lambda^{-1}_\pm(\mathscr{x}, D) - \Lambda^{-1}_\pm(\mathscr{x}, D)\mathscr{D}_j$ have order -1, i.e., map continuously the spaces $\mathbb{W}^{-1}_p(\mathscr{S}) \to \mathbb{L}_p(\mathscr{S})$.

By using the formulated assertions the proof is completed as in the case of a closed surface $\mathscr{S}$. $\square$

The foregoing Lemma 1.6.1 has the following generalization for the Bessel potential spaces $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$ and $\mathbb{H}^s_p(\mathscr{S})$ (see [133] and [67] for details about these spaces).

Lemma 1.6.2. *If $\mathscr{S}$ is closed, sufficiently smooth, $1<p<\infty$, $s\in\mathbb{R}$, $m=1,2,\dots$ and*

$$\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S}),\quad \mathscr{D}^\alpha\varphi=\mathscr{D}^{\alpha_1}_1\cdots\mathscr{D}^{\alpha_n}_n\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})\ \text{ for all }\ |\alpha|\leqslant m,$$

then $\varphi\in\mathbb{H}^s_p(\mathscr{S})$.

Moreover, the assertion holds for a hypersurface $\mathscr{S}$ with the Lipschitz boundary $\Gamma:=\partial\mathscr{S}$ and the spaces $\mathbb{H}^s_p(\mathscr{S})$ and $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$.

Proof. Assume first that $\mathscr{S}$ has no boundary. The proof is based, as in the foregoing case, on the following facts from [88, 130, 133], which we recall without proofs.

A. There exists a "lifting operator" (the Bessel potential operator),

$$\Lambda^r(\mathscr{X},D):\mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-r}_p(\mathscr{S}),\ \ r\in\mathbb{R},\tag{1.6.4}$$

which arranges isomorphism between the indicated spaces and having the inverse $\Lambda^{-r}(\mathscr{X},D)$.

B. $\Lambda^r(\mathscr{X},D)$ is a pseudodifferential operator of order $-r$ and the commutant

$$[\mathscr{D}^\alpha,\Lambda^r(\mathscr{X},D)]:=\mathscr{D}^\alpha\Lambda^r(\mathscr{X},D)-\Lambda^r(\mathscr{X},D)\mathscr{D}^\alpha\tag{1.6.5}$$

with the pseudodifferential operator $\mathscr{D}^\alpha=\mathscr{D}^{\alpha_1}_1\cdots\mathscr{D}^{\alpha_n}_n$ has order $|\alpha|+r-1$, i.e., maps continuously the spaces $\mathbb{H}^\gamma_p(\mathscr{S})\to\mathbb{H}^{\gamma-|\alpha|-r+1}_p(\mathscr{S})$, $\forall\,\gamma\in\mathbb{R}$.

Assume that $m=1$. Then $\varphi\in\mathbb{H}^{s-1}_p(\mathscr{S})$ and, due to (1.6.4), (1.6.5), it follows that $\psi:=\Lambda^{s-1}_{\mathscr{S}}(\mathscr{X},D)\varphi\in\mathbb{L}_p(\mathscr{S})$, $\mathscr{D}_j\psi=[\mathscr{D}_j,\Lambda^{s-1}_{\mathscr{S}}(\mathscr{X},D)]\varphi+\Lambda^{s-1}_{\mathscr{S}}(\mathscr{X},D)\mathscr{D}_j\varphi\in\mathbb{L}_p(\mathscr{S})$ for all $j=1,\dots,n$. By the definition of the space $\mathbb{W}^1_p(\mathscr{S})$ we conclude that $\psi\in\mathbb{W}^1_p(\mathscr{S})$. Due to (1.6.2), we finally get $\varphi=\Lambda^{1-s}(\mathscr{X},D)\psi\in\mathbb{H}^s_p(\mathscr{S})$.

Now assume $m=2,3,\dots$ and the assertion is valid for $m-1$. Then, due to the hypothesis, $\psi_j:=\mathscr{D}_j\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})$ for $j=1,\dots,n$. Moreover, due to the same hypothesis,

$$\mathscr{D}^\alpha\psi_j:=\mathscr{D}^\alpha\mathscr{D}_j\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})\ \text{ for all }\ |\alpha|\leqslant m-1\ \text{ and all }\ j=1,\dots,n.$$

Hence the induction hypothesis implies that $\psi_j:=\mathscr{D}_j\varphi\in\mathbb{H}^{s-1}_p(\mathscr{S})$ for $j=1,\dots,n$. Now from the already considered case $m=1$ it follows that $\varphi\in\mathbb{H}^s_p(\mathscr{S})$.

If $\mathscr{S}$ has non-empty Lipschitz boundary $\Gamma\neq\varnothing$, there exist pseudodifferential operators

$$\Lambda^r_-(\mathscr{X},D):\mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-r}_p(\mathscr{S}),\quad \Lambda^r_+(\mathscr{X},D):\widetilde{\mathbb{H}}^s_p(\mathscr{S})\to\widetilde{\mathbb{H}}^{s-r}_p(\mathscr{S})\tag{1.6.6}$$

arranging isomorphisms between the indicated spaces and having the inverses $\Lambda^{-r}_-(\mathscr{X},D)$, $\Lambda^{-r}_+(\mathscr{X},D)$ (cf. [67]).

Moreover, the pseudodifferential operators $\Lambda^{-r}_\pm(\mathscr{X},D)$ have order $-r$ and the commutants $[\mathscr{D}^\alpha,\Lambda^{-r}_\pm(\mathscr{X},D)]:=\mathscr{D}^\alpha\Lambda^{-1}_\pm(\mathscr{X},D)-\Lambda^{-r}_\pm(\mathscr{X},D)\mathscr{D}^\alpha$ have order $|\alpha|-r-1$, i.e., map continuously the spaces $\mathbb{H}^\gamma_p(\mathscr{S})\to\mathbb{H}^{\gamma+r+1-|\alpha|}_p(\mathscr{S})$.

By using the formulated assertions, the proof is completed, as in the case of closed surface. $\square$

Theorem 1.6.1 (Korn's I inequality "without boundary condition")**.** *Let $\mathscr{S}\subset\mathbb{R}^n$ be a Lipshitz hypersurface without boundary, $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}):=[\mathfrak{D}_{jk}(\boldsymbol{U})]_{n\times n}$ be the deformation tensor*

$$\mathfrak{D}_{jk}(\boldsymbol{U})=\frac{1}{2}\left[\mathscr{D}_kU_j+\mathscr{D}_jU_k+\partial_{\boldsymbol{U}}(\nu_j\nu_k)\right]=\frac{1}{2}\Big[\mathscr{D}_kU_j+\mathscr{D}_jU_k+\sum_{m=1}^n U_m\mathscr{D}_m\big(\nu_j\nu_k\big)\Big]$$

(*cf.* (1.5.2)) *and*

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\mid\mathbb{L}_p(\mathscr{S})\big\|:=\Big[\sum_{j,k=1}^n\big\|\mathfrak{D}_{jk}(\boldsymbol{U})\mid\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p},\ \ \boldsymbol{U}\in\mathbb{W}^1_p(\mathscr{S}),\tag{1.6.7}$$

for $1 < p < \infty$. *Then*

$$\big\|\boldsymbol{U} \mid \mathbb{W}^1_p(\mathscr{S})\big\| \leqslant M\Big[\big\|\boldsymbol{U} \mid |\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \mid \mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p} \tag{1.6.8}$$

for some constant $M > 0$ *or, equivalently, the mapping*

$$\boldsymbol{U} \longmapsto \Big[\big\|\boldsymbol{U} \mid \mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \mid \mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p} \tag{1.6.9}$$

defines an equivalent norm on the space $\mathbb{W}^1_p(\mathscr{S})$.

Proof. Consider the space

$$\widehat{\mathbb{W}}^1_p(\mathscr{S}) := \Big\{\boldsymbol{U} = (U_1, \dots, U_n)^\top : \ U_j, \mathfrak{D}_{jk}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } \ j,k = 1,\dots n\Big\}, \tag{1.6.10}$$

which is obtained by closing the space of smooth functions $C^1(\mathscr{S})$ with respect to the norm (cf. (1.6.8) and (1.6.9)):

$$\big\|\boldsymbol{U}\big|\widehat{\mathbb{W}}^1_p(\mathscr{S})\big\| := \big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\big]^{1/p}. \tag{1.6.11}$$

It is obviously sufficient to prove that the spaces $\mathbb{W}^1_p(\mathscr{S})$ and $\widehat{\mathbb{W}}^1_p(\mathscr{S})$ are identical, which means that the norms in these spaces are equivalent.

The inclusion $\mathbb{W}^1_p(\mathscr{S}) \subset \widehat{\mathbb{W}}^1_p(\mathscr{S})$ is trivial, because the inclusion $\boldsymbol{U} \in \mathbb{W}^1_p(\mathscr{S})$ (which means $U_j, \mathscr{D}_k U_j \in \mathbb{L}_p(\mathscr{S})$ for all $j,k = 1,\dots,n$) and the equalities (see (1.5.4))

$$\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\left[\mathscr{D}_k U_j + \mathscr{D}_j U_k\right] + \frac{1}{2}\sum_{r=1}^n \partial_r(\nu_j\nu_k)U_r \in \mathbb{L}_p(\mathscr{S}) \quad j,k = 1,\dots,n \tag{1.6.12}$$

imply that $\mathfrak{D}_{jk}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S})$ for all $j,k = 1,\dots,n$ and validate the inclusion $\mathbb{W}^1_p(\mathscr{S}) \subset \widehat{\mathbb{W}}^1_p(\mathscr{S})$.

To prove the inverse inclusion $\widehat{\mathbb{W}}^1_p(\mathscr{S}) \subset \mathbb{W}^1_p(\mathscr{S})$ we assume $\boldsymbol{U} \in \widehat{\mathbb{W}}^1_p(\mathscr{S})$, apply the formulae for the commutators $[\mathscr{D}_j, \mathscr{D}_k]$ from [64, Proposition 4.4.iv]

$$\big[\mathscr{D}_j, \mathscr{D}_k\big]U_m = \sum_{r=1}^n \big[\nu_j\mathscr{D}_k\nu_r - \nu_k\mathscr{D}_j\nu_r\big]\mathscr{D}_r U_m \in \mathbb{H}^{-1}_p(\mathscr{S}), \quad j,k = 1,\dots,n$$

and find out that

$$\mathscr{D}_j U_k \in \mathbb{H}^{-1}_p(\mathscr{S}), \quad \mathscr{D}_k\mathscr{D}_j U_m = \mathscr{D}_j\widetilde{\mathfrak{D}}_{km}(\boldsymbol{U}) + \mathscr{D}_k\widetilde{\mathfrak{D}}_{jm}(\boldsymbol{U}) - \mathscr{D}_m\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) - \frac{1}{2}\big[\mathscr{D}_j, \mathscr{D}_k\big]U_m$$
$$-\frac{1}{2}\big[\mathscr{D}_j, \mathscr{D}_m\big]U_k - \frac{1}{2}\big[\mathscr{D}_k, \mathscr{D}_m\big]U_j \in \mathbb{H}^{-1}_p(\mathscr{S}) \quad \text{for all} \quad j,k,m = 1,\dots,n\,,$$

because, by the assumption, $U_j, \mathfrak{D}_{jk}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S})$ for all $j,k = 1,\dots,n$. Due to Lemma 1.6.1 of J. L. Lions this implies $\mathscr{D}_j U_m \in \mathbb{L}_p(\mathscr{S})$ for all $j,m = 1,\dots,n$ and the claimed inclusion $\boldsymbol{U} \in \mathbb{W}^1_p(\mathscr{S})$ follows. □

Remark 1.6.1. The foregoing Theorem 1.6.1 is proved by P. Ciarlet in [23] for the case $p = 2$, for curvilinear coordinates and covariant derivatives.

A remarkable consequence of Korn's inequality (1.6.8) is that the space

$$\mathbb{W}^1_p(\mathscr{S}) := \Big\{\boldsymbol{U} = (U_1, \dots, U_n)^\top : \ U_j, \mathscr{D}_k U_j \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } \ j,k = 1,\dots,n\Big\}$$

and the space $\widehat{\mathbb{W}}^1_p(\mathscr{S})$ (cf. (1.6.10)) are isomorphic (i.e., can be identified), although only $\frac{n(n+1)}{2} < n^2$ linear combinations of the n^2 derivatives $\mathscr{D}_j U_k,\ j,k = 1,\dots,n$, participate in the definition of the space $\widehat{\mathbb{H}}^1_p(\mathscr{S})$.

1.7 Killing's vector fields and further Korn's inequalities

Definition 1.7.1. Let $\mathscr{S}$ be a hypersurface in the Euclidean space $\mathbb{R}^n$. The space $\mathscr{R}(\mathscr{S})$ of solutions to the deformation equations

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\left[\mathscr{D}_j^{\mathscr{S}} U_k + \mathscr{D}_k^{\mathscr{S}} U_j\right] = \frac{1}{2}\left[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \sum_{m=1}^n U_m \mathscr{D}_m(\nu_j \nu_k)\right] = 0, \tag{1.7.1}$$
$$\boldsymbol{U} = \sum_{j=1}^n U_j \boldsymbol{d}^j \in \omega(\mathscr{S}), \ \ j,k = 1,\dots,n,$$

is called the space of **Killing's vector fields**.

Killing's vector fields on a domain in the Euclidean space $\Omega \subset \mathbb{R}^n$ are known as the **rigid motions** and we start with this simplest class.

The space of rigid motions $\mathscr{R}(\Omega)$ extends naturally to the entire $\mathbb{R}^n$ and consists of affine vector-functions

$$\boldsymbol{V}(x) = a + \mathscr{B}x, \quad \mathscr{B} = [b_{jk}]_{n\times n}, \ \ a \in \mathbb{R}^n, \ \ x \in \mathbb{R}^n, \tag{1.7.2}$$

where the matrix $\mathscr{B}$ is skew symmetric

$$\mathscr{B} := \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1(n-2)} & b_{1(n-1)} \\ -b_{12} & 0 & b_{21} & \cdots & b_{1(n-3)} & b_{2(n-2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{1(n-2)} & -b_{2(n-3)} & -b_{3(n-4)} & \cdots & 0 & b_{(n-1)1} \\ -b_{1(n-1)} & -b_{2(n-2)} & -b_{3(n-3)} & \cdots & -b_{(n-1)1} & 0 \end{bmatrix} = -\mathscr{B}^\top \tag{1.7.3}$$

with real-valued entries $b_{jk} \in \mathbb{R}$. For $n = 3, 4, \dots$, the space $\mathscr{R}(\mathbb{R}^n)$ is finite-dimensional and $\dim \mathscr{R}(\mathbb{R}^n) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Note that for $n = 3$ the vector field $\boldsymbol{V} \in \mathscr{R}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is the classical rigid displacement

$$\boldsymbol{V}(x) = a + \mathscr{B}x = a + b \wedge x, \ \ b := (b_1, b_2, b_3)^\top \in \mathbb{R}^3, \ \ x \in \Omega, \ \ \mathscr{B} := \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{1.7.4}$$

Definition 1.7.2. We call a subset $\mathscr{M} \subset \mathbb{R}^n$ **essentially** m**-dimensional** and write $\operatorname{ess\,dim} \mathscr{M} = m$ if there exist $m+1$ points $\mathscr{x}^0, \mathscr{x}^1, \dots, \mathscr{x}^m \in \mathscr{M}$ such that the vectors $\{\mathscr{x}^j - \mathscr{x}^0\}_{j=1}^m$ are linearly independent.

Note that any m-dimensional subset $\mathscr{M} \subset \mathbb{R}^m$ is essentially m-dimensional, because contains m linearly independent vectors. Moreover, any collection of $m+1$ points in $\mathbb{R}^m$ is essentially m-dimensional, provided that these points do not belong to any $m-1$-dimensional hyperplane.

Lemma 1.7.1. *The operator*

$$\operatorname{Def}(\boldsymbol{U}) := \left[\mathfrak{D}_{jk}(\boldsymbol{U})\right]_{n\times n}, \ \ \mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\left[\partial_k U_j + \partial_j U_k\right], \ \ \boldsymbol{U} = \sum_{j=1}^n U_j \boldsymbol{e}^j, \tag{1.7.5}$$

is the deformation tensor in Cartesian coordinates.

The linear space $\mathscr{R}(\mathbb{R}^n)$ of rigid motions (of Killing's vector fields) in $\mathbb{R}^n$ consists of vector fields $\mathbf{K} = (K_1, \dots, K_n)^\top$ which are solutions to the system

$$2\mathfrak{D}_{jk}(\mathbf{K})(x) = \partial_k K_j(x) + \partial_j K_k(x) = 0, \ \ x \in \mathscr{S} \ \ \text{for all } \ j,k = 1,\dots,n. \tag{1.7.6}$$

If a rigid motion vanishes on an essentially $(n-1)$-dimensional subset $\mathbf{K}(\mathscr{x}) = 0$ for all $\mathscr{x} \in \mathscr{M}$, $\operatorname{ess\,dim} \mathscr{M} = n-1$, or at infinity $\mathbf{K}(x) = \mathscr{o}(1)$ as $|x| \to \infty$, then $\mathbf{K}$ vanishes identically on $\mathbb{R}^n$, $\mathbf{K}(x) \equiv 0$.

Proof. By differentiating (1.7.6) and recalling that $\partial_k\partial_l K_j = \partial_l\partial_k K_j$, we get

$$\partial_j\partial_k K_m = \partial_j\mathfrak{D}_{km}(\mathbf{K}) + \partial_k\mathfrak{D}_{jm}(\mathbf{K}) - \partial_m\mathfrak{D}_{jk}(\mathbf{K}) = 0 \ \text{ for all } \ j,k,m = 1,2,\dots,n-1.$$

Therefore,

$$K_j(x) = a_j + b_{j1}x_1 + \cdots + b_{jn}x_n, \ \ j = 1,2,\dots,n,$$

or

$$\mathbf{K}(x) = a + \mathscr{B}\,x \ \text{ with } \ \mathscr{B} = [b_{jk}]_{n\times n}. \tag{1.7.7}$$

From (1.7.6) we derive that $\mathscr{B}$ is a skew symmetric matrix (cf. (1.7.3)):

$$\partial_j K_k(x) = -\partial_k K_j(x) \equiv 0 \implies b_{jk} = -b_{kj}, \ \ j,k = 1,2,\dots,n \implies \mathscr{B} = -\mathscr{B}^\top.$$

The inclusion $\mathbf{K} \in \mathscr{R}(\mathbb{R}^n)$ is proved.

The inverse statement that any vector field $\mathbf{K} \in \mathscr{R}(\mathbb{R}^n)$ (of the form (1.7.2)) is a solution of the system (1.7.6), is easy to verify.

Let us prove the second assertion: for any linearly independent vectors $x^0,\dots,x^{n-1}$ the condition

$$\mathbf{K}(x^k) = 0 \implies a + \mathscr{B}x^k = \mathbf{K}(x^k) = 0 \tag{1.7.8}$$

implies $a = 0$ and $\mathscr{B} = 0$, i.e., $\mathbf{K}(x) = 0$ for all $x \in \mathbb{R}^n$. Indeed, if $\mathscr{B} = 0$, then, obviously, $a = 0$. Accepting $\mathscr{B} \neq 0$, for rank of $\mathscr{B}$ we have the estimate $2 \leqslant \operatorname{rank}\mathscr{B} < n$ (if $\mathscr{B} \neq 0$, then, due to the symmetry $\mathscr{B} = -\mathscr{B}^\top$, there exists a non-degenerate minor of order at least 2). On the other hand, from (1.7.8) it follows

$$\mathscr{B}(x^k - x^0) = 0, \ \ \forall\, k = 1,\dots,n-1,$$

which contradicts the estimate $2 \leqslant \operatorname{rank}\mathscr{B} < n$, since $\{x^1-x^0,\dots,x^{n-1}-x^0\}$ are linearly independent.

If a rigid motion $\mathbf{K}(x)$ in (1.7.7) vanishes at infinity $\mathbf{K}(x) = \mathscr{o}(1)$ as $|x| \to \infty$, then, obviously, $a = 0$, $\mathscr{B} = 0$ and, therefore, $\mathbf{K}(x) = 0$ for all $x \in \mathbb{R}^n$. □

Remark 1.7.1. For the deformation tensor in Cartesian coordinates $\operatorname{Def}(\boldsymbol{U})$ (cf. (1.7.5)) in a domain $\Omega \subset \mathbb{R}^n$ Korn's inequality

$$\big\|\boldsymbol{U} \mid \mathbb{W}^1_p(\Omega)\big\| \leqslant M\Big[\big\|\boldsymbol{U} \mid \mathbb{L}_p(\Omega)\big\|^P + \big\|\operatorname{Def}(\boldsymbol{U}) \mid \mathbb{L}_p(\Omega)\big\|^p\Big]^{1/p}, \ \ 1 < p < \infty, \tag{1.7.9}$$

with some constant $M > 0$ is well known and is proved e.g. in [21] (cf. (1.6.7) for a similar norm).

In contrast to the rigid motions in $\mathbb{R}^n$, Killing's vector fields on hypersurfaces nobody can identify explicitly so far. The next Theorem 1.7.1 underlines importance of Killing's vector fields for the Lamé equation on hypersurfaces. Later, we investigate properties of Killing's vector fields to prepare tools for investigations of boundary value problems for the Lamé equation.

Theorem 1.7.1. *Let $\mathscr{S}$ be an ℓ-smooth closed hypersurface in $\mathbb{R}^n$ and $\ell \geqslant 2$. The Lamé operator $\mathscr{L}_\mathscr{S}$ for an isotropic hypersurface*

$$\mathscr{L}_\mathscr{S} : \mathbb{H}^{s+1}_p(\mathscr{S}) \to \mathbb{H}^{s-1}_p(\mathscr{S}), \tag{1.7.10}$$
$$\mathscr{L}_\mathscr{S}\boldsymbol{U} = \mu\pi_\mathscr{S}\,\mathbf{div}_\mathscr{S}\,\nabla_\mathscr{S}\boldsymbol{U} + (\lambda+\mu)\,\nabla_\mathscr{S}\,\mathbf{div}_\mathscr{S}\,\boldsymbol{U} + \mu\,\mathscr{H}^0_\mathscr{S}\mathscr{W}_\mathscr{S}\boldsymbol{U},$$

is self-adjoint $\mathscr{L}^*_\mathscr{S} = \mathscr{L}_\mathscr{S}$, elliptic, Fredholm and has the trivial index $\mathbf{Ind}\,\mathscr{L}_\mathscr{S} = 0$ for all $1 < p < \infty$ and all $s \in \mathbb{R}$, provided that $|s| \leqslant \ell$.

The kernel of the operator $\mathbf{Ker}\,\mathscr{L}_\mathscr{S} \subset \mathbb{H}^s_p(\mathscr{S})$ is independent of the parameters p and s, coincides with the space of Killing's vector fields

$$\mathbf{Ker}\,\mathscr{L}_\mathscr{S} = \big\{\boldsymbol{U} \in \omega(\mathscr{S}) : \ \mathscr{L}_\mathscr{S}\boldsymbol{U} = 0\big\} = \mathscr{R}(\mathscr{S}), \tag{1.7.11}$$

and is finite-dimensional $\dim\mathscr{R}(\mathscr{S}) = \dim\mathbf{Ker}\,\mathscr{L}_\mathscr{S} < \infty$.

If $\mathscr{S}$ is C^∞-smooth, then Killing's vector fields are smooth as well, $\mathscr{R}(\mathscr{S}) \subset C^\infty(\mathscr{S})$.

$\mathscr{L}_{\mathscr{S}}$ is non-negative on the space $\mathbb{H}^1(\mathscr{S})$ and positive definite on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ to $\mathscr{R}(\mathscr{S})$ in $\mathbb{H}^1(\mathscr{S})$:

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geqslant 0 \ \text{ for all } \boldsymbol{U}\in\mathbb{H}^1(\mathscr{S}), \tag{1.7.12}$$

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geqslant C\big\|\boldsymbol{U} \mid \mathbb{H}^1(\mathscr{S})\big\|^2 \ \text{ for all } \boldsymbol{U}\in\mathbb{H}^1_{\mathscr{R}}(\mathscr{S}),\ \ C>0. \tag{1.7.13}$$

Moreover, the following Gåarding's inequality

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geqslant C_1\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\|^2 - C_0\|\boldsymbol{U}|\mathbb{H}^{-r}(\mathscr{S})\|^2 \tag{1.7.14}$$

holds for all $\boldsymbol{U}\in\mathbb{H}^1(\mathscr{S})$, with any $-1<r\leqslant\ell$ and some positive constants $C_0>0$, $C_1>0$.

Proof. (cf. [57, Theorem 3.5]). Let us check the ellipticity of $\mathscr{L}_{\mathscr{S}}$. The operator $\mathscr{L}_{\mathscr{S}}$ maps the tangent spaces and the principal symbol is defined on the cotangent space. The cotangent space is orthogonal to the normal vector and, therefore,

$$\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi)\eta = \mu|\xi|^2(1-\boldsymbol{\nu}\boldsymbol{\nu}^\top)\eta + (\lambda+\mu)\xi\xi^\top\eta = \mu|\xi|^2\eta + (\lambda+\mu)\xi\xi^\top\eta\,, \quad \forall\,\xi,\,\eta\perp\boldsymbol{\nu}.$$

Thus, while considering the principal symbol $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi)$, we can ignore the projection $\pi_{\mathscr{S}}$. With this assumption, the principal symbol of $\mathscr{L}_{\mathscr{S}}$ reads as

$$\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi) = \mu|\xi|^2 + (\lambda+\mu)\xi\xi^\top \quad \text{for} \quad (\mathscr{X},\xi)\in\mathbb{T}^*(\mathscr{S})\,. \tag{1.7.15}$$

The matrix $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi)$ has eigenvalue $(\lambda+2\mu)|\xi|^2$ (the corresponding eigenvector is ξ) and eigenvalue $\mu|\xi|^2$ with multiplicity $n-1$ (the corresponding eigenvectors θ^j are orthogonal to ξ: $\xi^\top\theta^j=\langle\xi,\theta^j\rangle=0$, $j=1,\dots,n-1$). Then

$$\det\,\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi) = (\lambda+2\mu)|\xi|^2\big[\mu|\xi|^2\big]^{n-1} = \mu^{n-1}(\lambda+2\mu)>0 \quad \text{for } \ (\mathscr{X},\xi)\in\mathbb{T}^*(\mathscr{S})\,,\ \ |\xi|=1$$

and the ellipticity is proved.

The ellipticity of the differential operator $\mathscr{L}_{\mathscr{S}}$ in (1.7.10) on a manifold without boundary $\mathscr{S}$, proved above, implies Fredholm property for all $1<p<\infty$ and all $s\in\mathbb{R}$. Indeed, $\mathscr{L}_{\mathscr{S}}$ has a parametrix $\boldsymbol{R}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$, which is a pseudodifferential operator (ΨDO) with the symbol $R_{\mathscr{S}}(\mathscr{X},\xi):=\chi(\xi)\mathscr{L}^{-1}_{\mathscr{S}}(\mathscr{X},\xi)$, where $\mathscr{L}^{-1}_{\mathscr{S}}(\mathscr{X},\xi)$ is the inverse symbol of $\mathscr{L}_{\mathscr{S}}$ and $\chi\in C^\infty(\mathbb{R}^n)$ is a smooth function, $\chi(\xi)=1$ for $|\xi|>2$ and $\chi(\xi)=0$ for $|\xi|<1$. ΨDO $\boldsymbol{R}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ is a bounded operator between the spaces

$$\boldsymbol{R}_{\mathscr{S}}(\mathscr{X},\mathscr{D})\ :\ \mathbb{H}^{s-2}_p(\mathscr{S})\to\mathbb{H}^s_p(\mathscr{S}) \quad \text{for all} \quad 1<p<\infty,\ \ s\in\mathbb{R},$$

because the symbol $R_{\mathscr{S}}(\mathscr{X},\xi)=\mathscr{L}^{-1}_{\mathscr{S}}(\mathscr{X},\xi)$ belongs to the Hörmander class $S^{-2}(\mathscr{S},\mathbb{R}^n)$

$$\Big|\mathscr{D}^\alpha\partial^\beta_\xi R_{\mathscr{S}}(\mathscr{X},\xi)\Big| \le C_{\alpha,\beta}|\xi|^{-2-|\beta|}$$

for all multi-indices $\alpha,\beta\in\mathbb{Z}^n_+$ (cf. [88, 126, 130] for details).

The Fredholm property for the case $p=2$ and $s=1$ follows from Gårding's inequality (1.7.14) as well (cf. [89, Thorem 5.3.10] and [110, Thorem 2.33]).

The Fredholm property implies that the kernel is finite-dimensional dim **Ker** $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})<\infty$.

To prove that the index is trivial **Ind** $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})=0$ for all $1<p<\infty$, $s\in\mathbb{R}$, note that the symbol $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi)$ is positive definite (cf. (1.7.15))

$$\begin{aligned}\langle\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi)\eta,\eta\rangle &= \mu|\xi|^2|\eta|^2 + (\lambda+\mu)\langle\xi\xi^\top\eta,\eta\rangle = \mu|\xi|^2|\eta|^2 + (\lambda+\mu)\sum_{j=1^n}(\xi_j\eta_j)^2\\ &\ge \mu|\xi|^2|\eta|^2, \quad \forall\,\mathscr{X}\in\mathscr{S}, \quad \forall\,\xi,\,\eta\in\mathbb{R}^n.\end{aligned} \tag{1.7.16}$$

Further, recall that the Bessel potential operator $\Lambda^2_{\mathscr{S}}(\mathscr{X},D)\ :\ \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-2}_p(\mathscr{S})$ (cf. (1.6.1)) lifting the Bessel potential spaces, also has positive definite symbol

$$\langle\Lambda^2_{\mathscr{S}}(\mathscr{X},\xi)\eta,\eta\rangle \ge C|\xi|^2|\eta|^2 \quad \forall\,\mathscr{X}\in\mathscr{S}, \quad \forall\,\xi,\,\eta\in\mathbb{R}^n \tag{1.7.17}$$

(cf. [67]). Now consider the symbols $\mathscr{B}_\tau(\mathscr{X},\xi)=(1-\tau)\mathscr{L}_{\mathscr{S}}(\mathscr{X},\xi)+\tau\Lambda^2_{\mathscr{S}}(\mathscr{X},\xi)$ and the corresponding ΨDO

$$\mathbf{B}_\tau(\mathscr{X},\mathscr{D})=(1-\tau)\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\tau\Lambda^2_{\mathscr{S}}(\mathscr{X},\mathscr{D})\ :\ \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-2}_p(\mathscr{S}). \tag{1.7.18}$$

Obviously, $\mathbf{B}_\tau(\mathscr{X},\mathscr{D})$ is a continuous (with respect to $0\le\tau\le1$) homotopy connecting the operator $\mathbf{B}_0(\mathscr{X},\mathscr{D})=\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ with $\mathbf{B}_1(\mathscr{X},\mathscr{D})=\Lambda^2_{\mathscr{S}}(\mathscr{X},\mathscr{D})$. Since the symbol $\mathscr{B}_\tau(\mathscr{X},\xi)$ is positive definite

$$\langle\mathscr{B}_\tau(\mathscr{X},\xi)\eta,\eta\rangle\ge[(1-\tau)\mu+\tau C]|\xi|^2|\eta|^2\quad\forall\,\xi,\,\eta\in\mathbb{R}^n$$

(cf. (1.7.16) and (1.7.17)), it is elliptic and the operator $\mathbf{B}_\tau(\mathscr{X},\mathscr{D})$ is then Fredholm for all $0\le\tau\le1$. Then $\mathbf{Ind}\ \mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})=\mathbf{Ind}\ \mathbf{B}_0(\mathscr{X},\mathscr{D})=\mathbf{Ind}\ \mathbf{B}_1(\mathscr{X},\mathscr{D})=\mathbf{Ind}\ \Lambda^2_{\mathscr{S}}(\mathscr{X},\mathscr{D})=0$, since the operator $\Lambda^2_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ is invertible.

From representation (1.4.32) it follows that the bilinear form $(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}$ is non-negative

$$\begin{aligned}(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}&=\lambda(\,\mathbf{div}^*_{\mathscr{S}}\ \mathbf{div}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}+2\mu(\,\mathrm{Def}^*_{\mathscr{S}}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}\\&=\lambda\big\|\,\mathbf{div}_{\mathscr{S}}\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2+2\mu\big\|\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2\ge0\end{aligned} \tag{1.7.19}$$

(cf. (1.7.12)) and vanishes (i.e., $\boldsymbol{U}\in\mathbf{Ker}\ \mathscr{L}_{\mathscr{S}}$) if and only if

$$\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}=0,\quad\mathbf{div}_{\mathscr{S}}\boldsymbol{U}. \tag{1.7.20}$$

Thus, $\mathbf{Ker}\ \mathscr{L}_{\mathscr{S}}\subset\mathbf{Ker}\ \mathrm{Def}_{\mathscr{S}}=\mathscr{R}(\mathscr{S})$.

But the first equality in (1.7.20) follows from the second one. Indeed, if $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}=0$, then, in particular, $\mathfrak{D}_{jj}(\boldsymbol{U})=\mathscr{D}_jU_j+\frac12\partial_{\boldsymbol{U}}\nu_j^2=0$, $j=1,\dots,n$ (cf. (1.5.4)), and

$$\mathbf{div}_{\mathscr{S}}\boldsymbol{U}=\sum_{j=1}^n\mathscr{D}_jU_j=-\frac12\sum_{j=1}^n\partial_{\boldsymbol{U}}(\nu_j)^2=\partial_{\boldsymbol{U}}|\boldsymbol{\nu}|^2=\partial_{\boldsymbol{U}}1=0,\quad\forall\,\boldsymbol{U}\in\mathscr{R}(\mathscr{S}), \tag{1.7.21}$$

since $|\boldsymbol{\nu}(\mathscr{X})|^2\equiv1$. Thence, due to (1.7.20), $\mathscr{R}(\mathscr{S})=\mathbf{Ker}\ \mathscr{L}_{\mathscr{S}}$. This accomplishes the proof of (1.7.11).

Estimate (1.7.13) is a direct consequence of (1.7.12) and of (1.7.11): since the operator $\mathscr{L}_{\mathscr{S}}$ is Fredholm, self-adjoint and $\mathbf{Ker}\ \mathscr{L}_{\mathscr{S}}=\mathscr{R}(\mathscr{S})$, then also $\mathbf{Coker}\ \mathscr{L}_{\mathscr{S}}=\mathscr{R}(\mathscr{S})$ and, therefore, the mapping

$$\mathscr{L}_{\mathscr{S}}\ :\ \mathbb{H}^1_{\mathscr{R}}(\mathscr{S})\longrightarrow\mathbb{H}^{-1}_{\mathscr{R}}(\mathscr{S})$$

is one-to-one, i.e., is invertible. The established invertibility implies the claimed inequality (1.7.13).

A priori regularity property of solutions to partial differential equations (cf. [88,130]) states that the ellipticity of $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ provides $C^\ell(\mathscr{S})$-smoothness of any solution $\boldsymbol{K}$ to the homogeneous equation $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})\boldsymbol{K}=0$, provided the hypersurface $\mathscr{S}$ is C^ℓ-smooth. Due to the embeddings $\mathbb{H}^r_q(\mathscr{S})\subset\mathbb{H}^s_p(\mathscr{S})$, $s\le r$, $p\le q$, the kernel $\mathbf{Ker}\ \mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ is independent of the space $\mathbb{H}^s_p(\mathscr{S})$ provided that the spaces are well defined, which is the case if $|s|\le\ell$ (cf. [1,44,65,92] for similar assertions).

In particular, the Killing's vector fields $\mathscr{R}(\mathscr{S})=\mathbf{Ker}\ \mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ are smooth $\mathscr{R}(\mathscr{S})\subset C^\infty(\mathscr{S})$, provided the hypersurface $\mathscr{S}$ is C^∞-smooth.

Let $\{\boldsymbol{K}_j\}_{j=1}^m$ be an orthogonal basis $(\boldsymbol{K}_j,\boldsymbol{K}_k)_{\mathscr{S}}=\delta_{jk}$ in the finite-dimensional space of Killing's vector fields $\mathscr{R}(\mathscr{S})$. Let

$$\mathbf{T}\boldsymbol{U}(\mathscr{X}):=\sum_{j=1}^m(\boldsymbol{K}_j,\boldsymbol{U})_{\mathscr{S}}\ \boldsymbol{K}_j(\mathscr{X}),\quad\mathscr{X}\in\mathscr{S}. \tag{1.7.22}$$

Due to the proved part, $\{\boldsymbol{K}_j\}_{j=1}^m\subset C^\ell(\mathscr{S})$ and the operator $\mathbf{T}$ is smoothing $\mathbf{T}\ :\ \mathbb{H}^{-r}(\mathscr{S})\to\mathbb{H}^r(\mathscr{S})$ (is infinitely smoothing if $\ell=\infty$). Then the operator

$$\mathscr{L}_{\mathscr{S}}+\mathbf{T}\ :\ \mathbb{H}^1(\mathscr{S})\to\mathbb{H}^{-1}(\mathscr{S})$$

is invertible and non-negative

$$((\mathscr{L}_{\mathscr{S}}+\mathbf{T})\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}=(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}+\sum_{j=1}^m(\boldsymbol{K}_j,\boldsymbol{U})^2_{\mathscr{S}}\ge0$$

(cf. (1.7.12)). This implies that $\mathscr{L}_{\mathscr{S}}+\mathbf{T}$ is positive definite

$$((\mathscr{L}_{\mathscr{S}}+\mathbf{T})\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geq C_1\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2$$

and we write

$$\begin{aligned}(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} &= ((\mathscr{L}_{\mathscr{S}}+\mathbf{T})\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} - (\mathbf{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geq C_1\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 - (\mathbf{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}\\ &\geq C_1\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 - C_2\big\|\boldsymbol{U}\big|\mathbb{H}^{-r}(\mathscr{S})\big\|^2,\end{aligned}$$

which proves (1.7.14). □

Corollary 1.7.1. *Let $\mathscr{S}\subset\mathbb{R}^n$ be a Lipshitz hypersurface without boundary,* $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := [\mathfrak{D}_{jk}(\boldsymbol{U})]_{n\times n}$ *be the deformation tensor* (*see* (1.4.9)). *The norm* $\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \mid \mathbb{L}_2(\mathscr{S})\|$ *is defined by* (1.6.7).

Then the following Korn's inequality

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \mid \mathbb{L}_2(\mathscr{S})\big\| \geqslant c\big\|\boldsymbol{U} \mid \mathbb{H}^1(\mathscr{S})\big\|, \quad \forall\, \boldsymbol{U}\in\mathbb{H}^1_{\mathscr{R}}(\mathscr{S}), \tag{1.7.23}$$

holds for some constant $c>0$ or, equivalently, the mapping $\boldsymbol{U}\mapsto\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \mid \mathbb{L}_2(\mathscr{S})\|$ is an equivalent norm on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ to the space of Killing's vector fields.

Proof. Due to Korn's inequality (1.6.8) for $p=2$

$$\big\|\boldsymbol{U} \mid \mathbb{L}_2(\mathscr{S})\big\|^2 \geqslant M_1\Big[\,\big\|\boldsymbol{U} \mid \mathbb{H}^1(\mathscr{S})\big\|^2 - \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \mid \mathbb{L}_2(\mathscr{S})\big\|^2\Big],$$

the mapping $\mathrm{Def}_{\mathscr{S}} : \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \to \mathbb{L}_2(\mathscr{S})$ is Fredholm and has index 0. Inequality (1.7.23) follows, since the mapping is injective (has an empty kernel). □

Let us recall some results related to the uniqueness of solutions to arbitrary elliptic equation.

Definition 1.7.3. Let Ω be an open subset with the Lipschitz boundary $\partial\Omega\neq\varnothing$ either on a Lipschitz hypersurface $\mathscr{S}\subset\mathbb{R}^n$ or in the Euclidean space $\mathbb{R}^{n-1}$.

A class of functions $\mathscr{U}(\Omega)$ defined in a domain Ω in $\mathbb{R}^n$ is said to have the **strong unique continuation property** if every $u\in\mathscr{U}(\Omega)$ in this class, which vanishes to infinite order at one point, must vanish identically.

If a surface $\mathscr{S}$ is C^∞-smooth, any elliptic operator on $\mathscr{S}$ has the strong unique continuation property due to Holmgren's theorem. But we can have more.

Lemma 1.7.2. *Let $\mathscr{S}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$. The class of solutions to a second order elliptic equation $\mathbb{A}(\mathscr{X},\mathscr{D})u=0$ with Lipschitz continuous top order coefficients on a surface $\mathscr{S}$ has the strong unique continuation property.*

In particular, if the solution $u(\mathscr{X})=0$ vanishes in any open subset of $\mathscr{S}$, it vanishes identically on entire $\mathscr{S}$.

Proof. The result was proved in [8] for a domain $\Omega\subset\mathbb{R}^n$ by the method of "Carleman estimates" (also see [88, Volume 3, Theorem 17.2.6]). Another proof, involving monotonicity of the frequency function, was discovered by N. Garofalo and F. Lin (see [78, 79]). A differential equation $\mathbb{A}(\mathscr{X},\mathscr{D})u(\mathscr{X})=0$ with Lipschitz continuous top order coefficients on a $\mathbb{W}^2_\infty$-smooth surface $\mathscr{S}$ is locally equivalent to a differential equation with Lipschitz continuous top order coefficients on a domain $\Omega\subset\mathbb{R}^{n-1}$. Therefore, a solution $u(\mathscr{X})$ has the strong unique continuation property locally (on each coordinate chart) on $\mathscr{S}$.

Since $\mathscr{S}$ is covered by a finite number of local coordinate charts which intersect on open neighborhoods, a solution $u(\mathscr{X})$ has the strong unique continuation property globally on $\mathscr{S}$. □

Remark 1.7.2. If the top order coefficients of a second order elliptic equation $\mathbb{A}(\mathscr{X},\mathscr{D})u=0$ in open subsets $\Omega\subset\mathbb{R}^n$, $n\geqslant 3$, are merely Hölder continuous, with exponent less than 1, examples due to A. Plis [116] and K. Miller [112] show that a solution $u(x)$ does not have the strong unique continuation property.

Lemma 1.7.3. *Let $\mathscr{C}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ and $\gamma \subset \Gamma$ be an open part of the boundary Γ. Let $\mathbb{A}(\mathscr{X}, \mathscr{D})$ be a second order elliptic system with Lipschitz continuous top order matrix coefficients on a surface $\mathscr{S}$.*

The Cauchy problem

$$\begin{cases} \mathbb{A}(\mathscr{X}, \mathscr{D})u = 0 & \text{on } \mathscr{C}, \quad u \in \mathbb{H}^1(\Omega), \\ u(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \\ (\partial_{\boldsymbol{V}} u)(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \end{cases} \tag{1.7.24}$$

where the vector field $\boldsymbol{V}$ is non-tangent to Γ, but tangent to $\mathscr{S}$, has only a trivial solution $u(\mathscr{X}) = 0$ on entire $\mathscr{S}$.

Proof. With a local diffeomorphism the Cauchy problem (1.7.24) is transformed into a similar problem on a domain $\Omega \subset \mathbb{R}^{n-1}$ with the Cauchy data vanishing on some open subset of the boundary $\gamma \subset \Gamma := \partial\Omega$.

Let us, for simplicity, use the same notation $\gamma \subset \Gamma = \partial\Omega$, $\boldsymbol{V}$ for a non-tangent vector field to γ, the function u and the differential operator $\mathbb{A}(x, \mathscr{D})$ for the transformed Cauchy problem in the transformed domain Ω. Moreover, we will suppose that γ is a part of the hypersurface $x_1 = 0$ (otherwise we can transform the domain Ω again). We also use new variables $t = x_1$ and $x := (x_2, \dots, x_{n-1})$. Then $(0, x) \in \gamma$, while $(t, x) \in \Omega$ for all small $0 < t < \varepsilon$ and some $x \in \Omega'$.

Thus, the natural basis element $\boldsymbol{e}^1$ (cf. (0.0.7)) is orthogonal to γ and, therefore, $\boldsymbol{e}^1 = c_1(x)\boldsymbol{V}(x) + c_2(x)\boldsymbol{g}(x)$ for some unit tangent vector $\boldsymbol{g}(x)$ to γ for some scalar functions $c_1(x)$, $c_2(x)$ and all $x \in \Omega'$. Then, due to the third line in (1.7.24),

$$(\partial_t u)(0, x) = \partial_{\boldsymbol{e}^j} u(0, x) = c_1(x)\partial_{\boldsymbol{V}} u(0, x) + c_2(x)\partial_{\boldsymbol{g}} u(0, x) = 0,$$

since any derivative along tangent vector to γ vanishes $\partial_{\boldsymbol{g}} u(0, x) = 0$ due to the second line in (1.7.24).

The second order equation $\mathbb{A}(t, x; \mathscr{D})$ can be written in the form

$$\mathbb{A}(t, x, D)u = \mathbb{A}(t, x; \boldsymbol{e}^1)\partial_t^2 u + \mathbb{A}_1(t, x; D)\partial_t u + \mathbb{A}_2(t, x; D)u, \quad D := -i\partial_x,$$

where $\mathbb{A}(t, x; \boldsymbol{e}^1)$ is the (invertible) matrix function, $\mathbb{A}_1(t, x; D)$ and $\mathbb{A}_2(t, x; D)$ are differential operators of orders 1 and 2, respectively, compiled of derivatives ∂_x, $x \in \Omega'$. Therefore, if $\mathbb{A}^0_j(t, x; D) := \mathbb{A}^{-1}(t, x; \boldsymbol{e}^1)\mathbb{A}_j(t, x; D)$, $j = 1, 2$, the Cauchy problem (1.7.24) transforms into

$$\begin{cases} \partial_t^2 u(t, x) + \mathbb{A}^0_1(t, x; D)\partial_t u(t, x) + \mathbb{A}^0_2(t, x; D)u(t, x) = 0 & \text{on } (t, x) \in \Omega_\varepsilon, \\ u(0, x) = 0 & \text{for all } x \in \Omega', \\ (\partial_t u)(0, x) = 0 & \text{for all } x \in \Omega', \end{cases} \tag{1.7.25}$$

where $\Omega_\varepsilon := (0, \varepsilon) \times \Omega' \subset \Omega$, $u \in \mathbb{H}^1(\Omega_\varepsilon)$ and $\gamma := \{(0, x) : \ x \in \Omega'\}$.

Now let us recall the inequality (see [113, § 4.3, Theorem 4.3, § 6.14], [123, § 4–7, Lemma 4–21]): there is a constant C which depends only on ε and $\mathbb{A}(t, x; D)$ and such that the inequality

$$\int\limits_{\Omega_\varepsilon} e^{-\lambda t}|v(t, x)|^2 \, dt \, dx \leqslant C \int\limits_{\Omega_\varepsilon} e^{-\lambda t}|(\mathbb{A}(t, x; D)v)(t, x)|^2 \, dt \, dx \tag{1.7.26}$$

holds for $\mathbb{A}(t, x; D)v \in \mathbb{L}_2(\Omega_\varepsilon)$, $v \in C^\infty(\Omega_\varepsilon)$; moreover, $v(t, x)$ should vanish near $t = \varepsilon$ and should have vanishing Cauchy data $v(0, x) = (\partial_t v)(0, x) = 0$ for all $x \in \Omega'$.

Let $\rho \in C^2(0, \varepsilon)$ be a cut-off function: $\rho(t) = 1$ for $0 \leqslant t < \varepsilon/2$ and $\rho(t) = 0$ for $3\varepsilon/4 \leqslant t < \varepsilon$. Then $v := \rho u \in \mathbb{H}^1(\Omega_\varepsilon)$ and, since $\mathbb{A}(t, x; D)u = 0$ on Ω_ε, we get

$$\begin{aligned} \mathbb{A}(t, x; D)(\rho u) &= \rho\mathbb{A}(t, x; D)u + (\partial_t^2 \rho)u + (\partial_t \rho)\partial_t u + (\partial_t \rho)\mathbb{A}^0_1(t, x; D)u \\ &= (\partial_t^2 \rho)u + (\partial_t \rho)\partial_t u + (\partial_t \rho)\mathbb{A}^0_1(t, x; D)u. \end{aligned}$$

We have asserted $u \in \mathbb{H}^1(\Omega_\varepsilon)$, $\rho \in C^2$, and this implies $(\partial_t^2 \rho)u \in \mathbb{L}_2(\Omega_\varepsilon)$, $(\partial_t \rho)\partial_t u \in \mathbb{L}_2(\Omega_\varepsilon)$. Note that $\partial_t \rho(t)$ vanishes for $0 < t < \varepsilon/2$. Therefore, $(\partial_t \rho)\mathbb{A}^0_1(t, x; D)u$ vanishes in a neighborhood of the boundary $\gamma \subset \Gamma$. Due to a priori regularity result (cf. [105, Chapter 2, § 3.2, § 3.3]), a solution to an elliptic equation in (1.7.25) has additional regularity $u \in \mathbb{H}^2(\Omega^0_\varepsilon)$ for arbitrary Ω^0_ε properly imbedded into Ω_ε. This implies $(\partial_t\rho)\mathbb{A}^0_1(t,x;D)u \in \mathbb{L}_2(\Omega_\varepsilon)$ and we conclude

$$\mathbb{A}(t,x;D)(\rho u) \in \mathbb{L}_2(\Omega_\varepsilon). \tag{1.7.27}$$

Introducing $v = \rho u$ into inequality (1.7.26) we get

$$\int\limits_{\Omega'}\int\limits_0^{\varepsilon/4} e^{-\lambda t}|\rho(t)u(t,x)|^2\,dt\,dx$$
$$\leqslant \int\limits_{\Omega_\varepsilon} e^{-\lambda t}|\rho(t)u(t,x)|^2\,dt\,dx \leqslant C\int\limits_{\Omega'}\int\limits_{\varepsilon/2}^{3\varepsilon/4} e^{-\lambda t}|(\mathbb{A}(t,x;D))\rho(t)u(t,x)|^2\,dt\,dx.$$

This implies, for $\lambda > 0$,

$$\int\limits_{\Omega'}\int\limits_0^{\varepsilon/4} |\rho(t)u(t,x)|^2\,dt\,dx \leqslant e^{-\lambda\varepsilon/4}\int\limits_{\Omega_\varepsilon} \big|(\mathbb{A}(t,x;D))\rho(t)u(t,x)\big|^2\,dt\,dx \leqslant C_1 e^{-\lambda\varepsilon/4},$$

where, due to (1.7.24), $C_1 > 0$ is a finite constant. By sending $\lambda \to \infty$, we get the desired result $u(t,x) = 0$ for all $0 \leqslant t \leqslant \varepsilon/4$ and all $x \in \Omega'$. Since $u(x)$ vanishes in a subset of the domain Ω bordering γ, due to Lemma 1.7.2 the solution vanishes on entire Ω (on entire $\mathscr{C}$). □

Due to our specific interest (see the next Lemma 1.7.4) and many applications, for example, to control theory, the following boundary unique continuation property is of special interest.

Definition 1.7.4. Let $\mathscr{S}$ be a Lipschitz hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open subsurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

We say that a class of functions $\mathscr{U}(\Omega)$ has the **strong unique continuation property from the boundary** if a vector-function $\boldsymbol{U} \in \mathscr{U}(\Omega)$ which vanishes on an open subset of the boundary $\gamma \subset \Gamma$, vanishes on the entire $\mathscr{C}$.

Lemma 1.7.4. *Let $\mathscr{S}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open $\mathbb{W}^2_\infty$-smooth subsurface.*

The set of Killing's vector fields $\mathscr{R}(\mathscr{S})$ on the open surface $\mathscr{C}$ has the strong unique continuation property from the boundary.

Proof. Let $\gamma \subset \Gamma := \partial\mathscr{C}$, $\mathbf{mes}\,\gamma > 0$ and $\boldsymbol{U}(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \gamma \subset \Gamma := \partial\mathscr{C}$. Then (cf. (1.3.22))

$$\begin{cases} (\mathscr{D}_j U^0_k)(\mathfrak{s}) + (\mathscr{D}_k U^0_j)(\mathfrak{s}) = -\displaystyle\sum_{m=1}^n U^0_m(\mathfrak{s})\mathscr{D}_m\big(\nu_j(\mathfrak{s})\nu_k(\mathfrak{s})\big) = 0, \\ U^0_k(\mathfrak{s}) = 0, \ \ \forall\,\mathfrak{s} \in \gamma, \ \ j,k = 1,\dots,n. \end{cases} \tag{1.7.28}$$

Among tangent vector fields generating Gunter's derivatives $\{\boldsymbol{d}^j(\mathfrak{s})\}_{j=1}^n$ only $n-1$ are linearly independent. One of vectors might collapse at a point $\boldsymbol{d}^j(\mathfrak{s}) = 0$ if the corresponding basis vector $\boldsymbol{e}^j$ is orthogonal to the surface at $\mathfrak{s} \in \mathscr{S}$, while others might be tangent to the subsurface Γ, except at least one, say $\boldsymbol{d}^n(\mathfrak{s})$, which is non-tangent to γ. Then from (1.7.28) it follows

$$2(\mathscr{D}_n U^0_n)(\mathfrak{s}) = 0 \ \text{ and implies } \ (\mathscr{D}_j U^0_n)(\mathfrak{s}) = 0 \ \text{ for all } \ \mathfrak{s} \in \gamma \ \text{ and all } \ j = 1,\dots,n. \tag{1.7.29}$$

Indeed, the vector $\boldsymbol{d}^j$, $1 \leqslant j \leqslant n-1$, is a linear combination $\boldsymbol{d}^j(\mathfrak{s}) = c_1(\mathfrak{s})\boldsymbol{d}^n(\mathfrak{s}) + c_2(\mathfrak{s})\boldsymbol{\tau}^j(\mathfrak{s})$ of the non-tangent vector $\boldsymbol{d}^n(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^j(\mathfrak{s}) := \pi_\gamma\boldsymbol{d}^j(\mathfrak{s})$ of $\boldsymbol{d}^j(\mathfrak{s})$ to the subsurface γ at the point $\mathfrak{s} \in \gamma$. Since U^0_n vanishes identically on γ, the derivative $(\partial_{\boldsymbol{\tau}^j} U^0_n)(\mathfrak{s}) = 0$ vanishes as well and (1.7.29) follows:

$$(\mathscr{D}_j U^0_n)(\mathfrak{s}) = c_1(\mathfrak{s})(\partial_{\boldsymbol{d}^n} U^0_n)(\mathfrak{s}) + c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^j} U^0_n)(\mathfrak{s}) = c_1(\mathfrak{s})(\mathscr{D}_n U^0_n)(\mathfrak{s}) = 0, \ \ \forall\,\mathfrak{s} \in \gamma.$$

Equalities (1.7.28) and (1.7.29) imply

$$(\mathscr{D}_n U_j^0)(\mathfrak{s}) = -(\mathscr{D}_j U_n^0)(\mathfrak{s}) = 0, \ \ \forall\, \mathfrak{s} \in \gamma, \ \ \forall\, j = 1, \ldots, n. \tag{1.7.30}$$

Thus, we have the following Cauchy problem

$$\begin{cases} \mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\boldsymbol{U}(\mathscr{X}) = 0 & \text{on } \mathscr{C}, \\ \boldsymbol{U}(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \\ (\mathscr{D}_n \boldsymbol{U})(\mathfrak{s}) = (\partial_{\boldsymbol{d}^n} \boldsymbol{U})(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \end{cases} \tag{1.7.31}$$

where $\boldsymbol{d}^n$ is a vector field non-tangent to Γ. Due to Lemma 41.7.3, $\boldsymbol{U}(\mathscr{X}) = 0$ for all $\mathscr{X} \in \mathscr{C}$. □

Before we draw some consequences from the proved unique continuation property, we should make some comments. The finite dimensionality of the linear space $\mathscr{R}(\mathscr{C})$ when the surface $\mathscr{C}$ is 2-smooth, was proved in the papers [29, 80].

The foregoing Lemma 1.7.4 generalizes essentially the "infinitesimal rigid displacement lemma" (see [23, Theorem 2.7-2]). The following conditions are imposed:

(i) $\mathscr{C} \subset \mathscr{S}$ is C^3-smooth, **elliptic** in $\mathbb{R}^3$, i.e.,

$$\sum_{k=1}^{2} |\xi^k|^2 \leqslant C \sum_{k,j=1}^{2} |b_{jk}(\mathscr{X})\xi^j\xi^k|, \ \ \forall\, \mathscr{X} \in \mathscr{S}, \ \ \forall\, (\xi^1, \xi^2)^\top \in \mathbb{R}^2, \tag{1.7.32}$$

where $b_{jk}(\mathscr{X}) : \mathscr{S} \to \mathbb{R}$ are the covariant components of the curvature tensor of $\mathscr{S}$; the equivalent condition is that the Gaußian curvature is positive on the entire surface $\mathscr{S}$ or that the principal curvatures of the surface $\mathscr{S}$ have the same sign everywhere on $\mathscr{S}$.

(ii) Killing's vector field $\boldsymbol{U}$ vanishes on the entire boundary $\partial\mathscr{S}$, i.e.,

$$\mathscr{R}_0(\mathscr{C}) = \big\{\boldsymbol{U} \in \mathscr{R} : \ \boldsymbol{U}\big|_{\partial\mathscr{C}} = 0\big\} = \{0\}. \tag{1.7.33}$$

A similar assertion is proved by V. Lods and C. Mardare in [107], but for $C^{2,1}$-smooth hypersurface with the Lipschitz boundary $\partial\mathscr{S}$ and when Killing's vector field expires on the entire boundary $\partial\mathscr{S}$. An earlier version of the "infinitesimal rigid displacement lemma" is due to I. Vekua [134], who proved it using the theory of "generalized analytic functions".

Corollary 1.7.2 (Korn's I inequality "with boundary condition")**.** *Let $\mathscr{C} \subset \mathbb{R}^n$ be a C^ℓ-smooth hypersurface with the Lipschitz boundary $\Gamma := \partial\mathscr{C} \neq \varnothing$ and $\ell \geqslant 2$, $|s| \leqslant \ell$. Then*

$$\big\|\boldsymbol{U} \mid \mathbb{H}^s_p(\mathscr{C})\big\| \leqslant M\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}) \mid \mathbb{H}^{s-1}_p(\mathscr{C})\big\|, \ \ \forall\, \boldsymbol{U} \in \widetilde{\mathbb{H}}^s_p(\mathscr{C}),$$

for some constant $M > 0$. In other words, the mapping

$$\boldsymbol{U} \longmapsto \big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}) \mid \mathbb{H}^{s-1}_p(\mathscr{C})\big\| \tag{1.7.34}$$

is an equivalent norm on the space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$.

Proof. If the claimed inequality (1.7.34) is false, there exists a sequence $\boldsymbol{U}^j \in \widetilde{\mathbb{H}}^s_p(\mathscr{C})$, $j = 1, 2, \ldots$, such that

$$\big\|\boldsymbol{U}^j \mid \mathbb{H}^s_p(\mathscr{C})\big\| = 1, \ \ \forall j = 1, 2, \ldots, \quad \lim_{j\to\infty} \big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^j) \mid \mathbb{H}^{s-1}_p(\mathscr{C})\big\| = 0.$$

Due to the compact embedding $\widetilde{\mathbb{H}}^s_p(\mathscr{C}) \subset \mathbb{H}^s_p(\mathscr{C}) \subset \mathbb{H}^{s-1}_p(\mathscr{C})$, a convergent subsequence $\boldsymbol{U}^{j_1}, \boldsymbol{U}^{j_2}, \ldots$ in $\mathbb{H}^{s-1}_p(\mathscr{C})$ can be selected. Let $\boldsymbol{U}^0 = \lim\limits_{k\to\infty} \boldsymbol{U}^{j_k}$. Then

$$\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^0) \mid \mathbb{H}^{s-1}_p(\mathscr{C})\big\| = \lim_{k\to\infty} \big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^{j_k}) \mid \mathbb{H}^{s-1}_p(\mathscr{C})\big\| = 0$$

and $\boldsymbol{U}^0$ is a Killing's vector field. Since $\boldsymbol{U}(x) = 0$ on Γ, due to Lemma 1.7.4, $\boldsymbol{U}^0(x) = 0$ for all $x \in \mathscr{C}$, which contradicts to $\|\boldsymbol{U}^0 \mid \mathbb{H}^s_p(\mathscr{C})\| = \lim\limits_{k\to\infty} \|\boldsymbol{U}^{j_k} \mid \mathbb{H}^s_p(\mathscr{C})\| = 1$. □

Let us check the following equalities for a later use:

$$\nabla_{\Omega^\varepsilon}\boldsymbol{U}=\big[\mathscr{D}_jU^0_k\big]_{n+1\times n+1}+\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{W}_{\Omega^\varepsilon},\tag{1.7.35}$$

where

$$\boldsymbol{U}:=\sum_{m=1}^{n+1}U^0_m\boldsymbol{d}^m=\sum_{m=1}^{n}U_m\boldsymbol{e}^m,\ \ U^0_{n+1}=\sum_{m=1}^{n}\mathscr{N}_mU_m,\ \ \mathscr{D}_{n+1}:=\partial_{\mathscr{N}},\ \ \boldsymbol{d}^{n+1}:=\mathscr{N}.$$

$\mathscr{W}_{\Omega^\varepsilon}$ is the extended Weingarten matrix

$$\mathscr{W}_{\Omega^\varepsilon}:=\big[\mathscr{D}_j\mathscr{N}_k\big]_{n+1\times n+1}\tag{1.7.36}$$

and its last column and last row are 0, because $\mathscr{D}_j\mathscr{N}_{n+1}=\mathscr{D}_{n+1}\mathscr{N}_j=\mathscr{D}_{n+1}\mathscr{N}_{n+1}=0$ for $j=1,\dots,n$.

In fact (see (2.2.12) for some further details of calculation),

$$\begin{aligned}
\nabla_{\Omega^\varepsilon}\boldsymbol{U}&:=\big[\partial_jU_k\big]_{n\times n}=\sum_{j,ky=1}^{n}\partial_jU_k\boldsymbol{e}^j\otimes\boldsymbol{e}^k\\
&:=\sum_{j,k=1}^{n}\big[\mathscr{D}_j+\mathscr{N}_j\partial_{\mathscr{N}}\big]\big[U^0_k+\mathscr{N}_k\langle\mathscr{N},\boldsymbol{U}\rangle\big]\,[\boldsymbol{d}^j+\mathscr{N}_j\mathscr{N}]\otimes[\boldsymbol{d}^k+\mathscr{N}_k\mathscr{N}]\\
&=\sum_{j,k=1}^{n}(\mathscr{D}_jU^0_k)\boldsymbol{d}^j\otimes[\boldsymbol{d}^k+\mathscr{N}_k\mathscr{N}]+\sum_{j,k=1}^{n}\mathscr{D}_j\big[\mathscr{N}_k\langle\mathscr{N},\boldsymbol{U}\rangle\big]\boldsymbol{d}^j\otimes[\boldsymbol{d}^k+\mathscr{N}_k\mathscr{N}]\\
&\quad+\sum_{j,k=1}^{n}\mathscr{N}_j^2(\partial_{\mathscr{N}}U^0_k)\mathscr{N}\otimes[\boldsymbol{d}^k+\mathscr{N}_k\mathscr{N}]+\sum_{j,k=1}^{n}\mathscr{N}_j^2\mathscr{N}_k^2\partial_{\mathscr{N}}\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{N}\otimes\mathscr{N}\\
&=\sum_{j,k=1}^{n}(\mathscr{D}_jU^0_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^k+\sum_{j,k=1}^{n}\mathscr{N}_k(\mathscr{D}_jU^0_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^{n+1}\\
&\quad+\sum_{j,k=1}^{n}\langle\mathscr{N},\boldsymbol{U}\rangle(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j\otimes[\boldsymbol{d}^k+\mathscr{N}_k\mathscr{N}]+\sum_{j,k=1}^{n}\mathscr{N}_k^2\mathscr{D}_j\langle\mathscr{N},\boldsymbol{U}\rangle\boldsymbol{d}^j\otimes\boldsymbol{d}^{n+1}\\
&\quad+\sum_{k=1}^{n}(\mathscr{D}_{n+1}U^0_k)\boldsymbol{d}^{n+1}\otimes\boldsymbol{d}^k+\sum_{k=1}^{n}\big[\mathscr{N}_k\mathscr{D}_{n+1}U^0_k+\mathscr{D}_{n+1}U^0_{n+1}\big]\boldsymbol{d}^{n+1}\otimes\boldsymbol{d}^{n+1}\\
&=\sum_{j,k=1}^{n}(\mathscr{D}_jU^0_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^k+\sum_{j,k=1}^{n}\big[\mathscr{D}_j(\mathscr{N}_kU^0_k)-U^0_k\mathscr{D}_j\mathscr{N}_k\big]\boldsymbol{d}^j\otimes\boldsymbol{d}^{n+1}\\
&\quad+\langle\mathscr{N},\boldsymbol{U}\rangle\sum_{j,k=1}^{n}(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^k+\sum_{j=1}^{n}\mathscr{D}_j\langle\mathscr{N},\boldsymbol{U}\rangle\boldsymbol{d}^j\otimes\boldsymbol{d}^{n+1}\\
&\quad+\sum_{k=1}^{n}(\mathscr{D}_{n+1}U^0_k)\boldsymbol{d}^{n+1}\otimes\boldsymbol{d}^k+(\mathscr{D}_{n+1}U^0_{n+1})\boldsymbol{d}^{n+1}\otimes\boldsymbol{d}^{n+1}\\
&=\sum_{j,k=1}^{n+1}(\mathscr{D}_jU^0_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^k-\sum_{j,k=1}^{n}U^0_k(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^{n+1}+\langle\mathscr{N},\boldsymbol{U}\rangle\sum_{j,k=1}^{n}(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^k\\
&=\big[\mathscr{D}_jU_k\big]_{(n+1)\times(n+1)}+\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{W}_{\Omega^\varepsilon}-\sum_{j,k=1}^{n}U^0_k(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j\otimes\boldsymbol{d}^{n+1}\\
&=\big[\mathscr{D}_jU_k\big]_{(n+1)\times(n+1)}+\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{W}_{\Omega^\varepsilon}-\big[(\mathscr{W}_{\Omega^\varepsilon}\boldsymbol{U}^0)_j\delta_{j,n+1}\big]_{(n+1)\times(n+1)},
\end{aligned}$$

since

$$\partial_{\mathscr{N}}\mathscr{N}_j=0,\quad\sum_{j,k=1}^{n}\mathscr{N}_j^2=1,\quad\sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j=0,\quad\sum_{j=1}^{n}\mathscr{N}_j\boldsymbol{d}^j=0,$$

$$\sum_{k=1}^{n} \mathscr{N}_k U_k^0 = 0, \quad \sum_{k=1}^{n} \mathscr{N}_k \mathscr{D}_j \mathscr{N}_k = \frac{1}{2}\,\mathscr{D}_j \sum_{k=1}^{n} \mathscr{N}_k^2 = \frac{1}{2}\,\mathscr{D}_j 1 = 0, \ \ j = 1, 2, \ldots, n+1.$$

Let $\Omega \subset \mathbb{R}^n$ be domain with a smooth boundary $\mathscr{M} := \partial\Omega$, $\mathscr{M}_0 \subset \mathscr{M}$ be a subsurface of non-zero measure and $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ denote a subspace of functions $\varphi \in \mathbb{W}^1(\Omega, \mathscr{M}_0)$ which is the closure of the set $C^\infty(\Omega, \mathscr{M}_0)$ of smooth functions $\varphi(x)$ which have vanishing traces on $\mathscr{M}_0$, i.e., $\varphi^+(\mathscr{X}) = 0$ for all $\mathscr{X} \in \mathscr{M}_0$. The space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ inherits the standard norm from $\mathbb{W}^1(\Omega)$:

$$\|\varphi \mid \widetilde{\mathbb{W}}^1(\Omega)\| := \|\varphi \mid \mathbb{W}^1(\Omega)\| = \Big[\, \big\|\varphi \mid \mathbb{L}_p(\Omega)\big\| + \sum_{j=1}^{n} \big\|\partial_j\varphi \mid \mathbb{L}_p(\Omega)\big\|\Big]^{1/p}.$$

Since the space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ does not contain constants, it is easy to prove the following

Lemma 1.7.5. *The formula*

$$\big\|\varphi \mid \widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)\big\| := \Big[\sum_{j=1}^{n} \big\|\partial_j\varphi \mid \mathbb{L}_p(\Omega)\big\|\Big]^{1/p} \tag{1.7.37}$$

defines an equivalent norm in the space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$.

If ε is sufficiently small, the boundary $\mathscr{M}_\varepsilon := \partial\Omega_\varepsilon$ is represented as the union of three C^1-smooth surfaces $\mathscr{M}_\varepsilon = \mathscr{M}_{\varepsilon,D} \cup \mathscr{M}^-_{\varepsilon,N} \cup \mathscr{M}^+_{\varepsilon,N}$, where $\mathscr{M}_{\varepsilon,D} = \partial\mathscr{C} \times [-\varepsilon, \varepsilon]$ is the lateral surface, $\mathscr{M}^+_{\varepsilon,N} = \mathscr{C} \times \{+\varepsilon\}$ is the upper surface and $\mathscr{M}^-_{\varepsilon,N} = \mathscr{C} \times \{-\varepsilon\}$ is the lower surface of the boundary $\mathscr{M}_\varepsilon$ of the layer domain Ω_ε.

The next Lemma 1.7.6 is proved for a later use in Section 3.

Lemma 1.7.6. *Let $\mathscr{M}_0 := \gamma \times [-\varepsilon, \varepsilon]$, where $\gamma \subset \Gamma := \partial\mathscr{C}$ is a subset of the boundary of the surface $\mathscr{C}$ of non-trivial measure. If $g \in \mathbb{L}_2(\Omega_\varepsilon)$, for the linear functional*

$$E_\varepsilon(u) = \int\limits_{\Omega_\varepsilon} g(x)u(x)\,dx, \quad u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0), \tag{1.7.38}$$

we have the following estimate

$$E_\varepsilon(u) \leqslant C\|g|\mathbb{L}_2(\Omega_\varepsilon)\|\ \|\mathscr{D}_\mathscr{C} u|\mathbb{L}_2(\Omega_\varepsilon)\| \tag{1.7.39}$$

with a constant $C > 0$ independent of $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$.

Proof. To prove (1.7.39) we recall that $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$ vanishes on the lateral subsurface $\mathscr{X} \in \mathscr{M}_0 \subset \mathscr{M}_D := \partial\mathscr{C} \times (-\varepsilon, \varepsilon)$.

Let $\mathscr{C}_t$ be the "parallel" surface to the mid-surface $\mathscr{C}$ on a distance $|t|$ and for negative $t < 0$ the surface $\mathscr{C}_t$ is "below" $\mathscr{C}$, while for positive $t > 0$ is "above" $\mathscr{C}$ with respect to the direction of the normal vector field $\boldsymbol{\nu}(\mathscr{X})$, $\mathscr{X} \in \mathscr{C}$. Note that $\mathscr{C}_{\pm\varepsilon} = \mathscr{M}^\pm_D$. Taking $u(\mathscr{X}, t)$, $\mathscr{X} \in \mathscr{C}$, $-\varepsilon < t < \varepsilon$, from a dense subset of the space $\widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$ we can assume that $u(\,\cdot\,, t) \in \widetilde{\mathbb{W}}^1(\mathscr{C}_t)$ for all fixed $-\varepsilon \leqslant t \leqslant \varepsilon$. Since $u(\mathscr{X}, t)$ vanishes on the part of the boundary $\mathscr{M}_0 \cap \partial\mathscr{C}_t$, the Sobolev semi-norm

$$\big\|u(\,\cdot\,, t) \mid \mathbb{W}^1(\mathscr{C}_t)\big\|_0 := \big\|\mathscr{D}_\mathscr{C} u(\,\cdot\,, t) \mid \mathbb{L}_2(\mathscr{C}_t)\big\| = \Bigg[\sum_{j=1}^{3} \int\limits_{\mathscr{C}_t} |\mathscr{D}_j u(\mathscr{X}, t)|^2\,d\sigma\Bigg]^{1/2}$$

turns into the norm and is equivalent to the standard Sobolev norm

$$\big\|u(\,\cdot\,, t) \mid \mathbb{W}^1(\mathscr{C}_t)\big\| := \Bigg[\int\limits_{\mathscr{C}_t} |u(\mathscr{X}, t)|^2\,d\sigma + \sum_{j=1}^{3} \int\limits_{\mathscr{C}_t} |\mathscr{D}_j u(\mathscr{X}, t)|^2\,d\sigma\Bigg]^{1/2}$$

for all $t \in [-\varepsilon, \varepsilon]$, which means

$$M\big\|u(\,\cdot\,,t) \mid \mathbb{W}^1(\mathscr{C}_t)\big\| \leqslant \big\|u(\,\cdot\,,t) \mid \mathbb{W}^1(\mathscr{C}_t)\big\|_0 \leqslant \big\|u(\,\cdot\,,t) \mid \mathbb{W}^1(\mathscr{C}_t)\big\|$$

for some constant $0 < M < 1$, independent of t and u. From this equivalence we get the estimate

$$\big\|u(\,\cdot\,,t) \mid \mathbb{L}_2(\mathscr{C}_t)\big\|^2 \leqslant \frac{1-M^2}{M^2}\,\big\|\mathscr{D}_{\mathscr{C}}u(\,\cdot\,,t) \mid \mathbb{L}_2(\mathscr{C}_t)\big\|^2. \tag{1.7.40}$$

By integrating the obtained inequality with respect to the variable t we get the final estimate

$$\big\|u \mid \mathbb{L}_2(\Omega_\varepsilon)\big\| \leqslant \frac{\sqrt{1-M^2}}{M}\,\big\|\mathscr{D}_{\mathscr{C}}u \mid \mathbb{L}_2(\Omega_\varepsilon)\big\|, \ \ \forall\, u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0). \tag{1.7.41}$$

The estimate in (1.7.39) follows with the help of the Cauchy inequality and inequality (1.7.41):

$$\int\limits_{\Omega_\varepsilon} g(x)u(x)\,dx \leqslant \big\|g \mid \mathbb{L}_2(\Omega_\varepsilon)\big\|\,\big\|u \mid \mathbb{L}_2(\Omega_\varepsilon)\big\| \leqslant \frac{\sqrt{1-M^2}}{M}\,\big\|g \mid \mathbb{L}_2(\Omega_\varepsilon)\big\|\,\big\|\mathscr{D}_{\mathscr{C}}u \mid \mathbb{L}_2(\Omega_\varepsilon)\big\|. \qquad \square$$

Remark 1.7.3. Let us underline that in estimate (1.7.39) we only need the surface derivatives $\mathscr{D}_1, \mathscr{D}_2$ and $\mathscr{D}_3$. If we would have $g \in \mathbb{W}^{-1}(\Omega_\varepsilon)$, then we should assume $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon)$. These spaces are dual and, therefore, if the integral in the functional E_ε in (1.7.38) is understood as the duality, the functional E_ε is bounded, but then we have the estimate

$$E_\varepsilon(u) \leqslant C\big\|g \mid \mathbb{L}_2(\Omega_\varepsilon)\big\|\,\big\|\mathscr{D}_{\Omega_\varepsilon}u \mid \mathbb{L}_2(\Omega_\varepsilon)\big\|, \ \ \mathscr{D}_{\Omega_\varepsilon} = (\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3, \mathscr{D}_4)^\top. \tag{1.7.42}$$

Note that all derivatives $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$ and the transversal $\partial_t = \partial_{\boldsymbol{\nu}} = \mathscr{D}_4$ (the normal to the surface $\mathscr{C}$) derivative appear in the latter inequality.

1.8 Geometric rigidity

The basic rigidity result relevant to passage to the thin plate limit is the following statement.

Proposition 1.8.1 (see [77])**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$ and $1 < p < \infty$. There exists a constant $C(\Omega)$ with the following property: for each $U \in \mathbb{W}^1(\Omega)$ there is an associated rotation $R_U \in \mathbb{SO}(n) := \mathbb{SO}(\mathbb{R}^n)$ such that*

$$\big\|\nabla U - R_U \mid \mathbb{L}_p(\Omega)\big\| \leqslant C(\Omega)\big\|\,\mathbf{dist}(\nabla U, \mathbb{SO}(n)) \mid \mathbb{L}_p(\Omega)\big\|. \tag{1.8.1}$$

The result is sharp in the sense that neither the norm on the right-hand side nor the power with which it appears can be improved.

By considering the special case when the right-hand side in (1.8.1) is zero, Proposition 1.8.1 reduces to the following

Corollary 1.8.1 (Liouville theorem)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. If U is a $\mathbb{W}^1(\Omega)$ map which satisfies the partial differential equation*

$$\nabla U(x) = V(x) \quad \text{a.e. in } \Omega, \quad V \in \mathbb{SO}(n), \tag{1.8.2}$$

then it is affine $U(x) = Rx + c$, $R \in \mathbb{SO}(n)$, $c = const$ or, equivalently, $\nabla U = R \in \mathbb{SO}(n)$.

Proof. In the setting of Sobolev maps, this was first proved by Reshetnyak [118]. A short modern proof belongs to G. Friesecke, R. D. James and S. Müller [77] and consists of three observations.

First, for $n \times n$ matrix $A = [a_{jk}]_{n\times n}$ let $\operatorname{cof} A$ denote the matrix of cofactors of A, i.e.,

$$\operatorname{cof} A = \big[(-1)^{j+k}\det A_{jk}\big]_{n\times n}, \tag{1.8.3}$$

where A_{jk} is the $(n-1)\times(n-1)$-matrix obtained from A by deleting the j-th row and the k-th column (so called (j,k)-minor). It is well known that

$$\operatorname{div}\operatorname{cof}\nabla U = 0 \ \text{ for all } \ U \in \mathbb{W}^1(\Omega). \tag{1.8.4}$$

Note first that if equality (1.8.4) is proved for $U \in C^2(\Omega)$, it can be extended to arbitrary $U \in \mathbb{W}^1(\Omega)$.

We have to prove

$$C_i := \sum_{k=1}^{n} \partial_k (\operatorname{cof}\nabla U)_{ki} = 0, \ \ i = 1, \dots, n. \tag{1.8.5}$$

Note that C_i can be formally written as

$$C_i = \det \begin{pmatrix} \partial_1 & \partial_2 & \dots & \partial_n \\ \partial_1 v_1^{(i)} & \partial_2 v_1^{(i)} & \dots & \partial_n v_1^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 v_{n-1}^{(i)} & \partial_2 v_{n-1}^{(i)} & \dots & \partial_n v_{n-1}^{(i)} \end{pmatrix}, \tag{1.8.6}$$

where $v^{(i)} = (U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_n)$. Equality (1.8.5) follows from the following assertion: for any $u = (u_1, \dots, u_{n-1}) \in C^2(\mathbb{R}^{n-1})$,

$$\begin{pmatrix} \partial_1 & \partial_2 & \dots & \partial_n \\ \partial_1 u_1 & \partial_2 u_1 & \dots & \partial_n u_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 u_{n-1} & \partial_2 u_{n-1} & \dots & \partial_n u_{n-1} \end{pmatrix} = 0, \tag{1.8.7}$$

which can be easily proved by induction, expanding the determinant with respect to the last row.

Second, (1.8.2) implies that U is harmonic and, in particular, smooth. To prove this, recall that if $A \in \mathbb{GL}(n)$ is an invertible matrix, then $A^{-1} = \det A(\operatorname{cof} A)^\top$. In particular, for $B \in \mathbb{SO}(n)$, which means $B^{-1} = B^\top$, $\det B = 1$, we get $\operatorname{cof} B = B$. Then from the asserted inclusion $\nabla U \in \mathbb{SO}(n)$ we get $\nabla(U)(x) = \operatorname{cof}\nabla U(x)$ and, by taking the divergence, we obtain

$$\Delta\, U = \operatorname{div}\nabla \boldsymbol{U} = \operatorname{div}\operatorname{cof}\nabla U(x) = 0.$$

Third, the second gradient squared of any harmonic map can be expressed pointwise via derivatives of the inner products,

$$\frac{1}{2}\left(|\nabla U|^2 - n\right) = \langle \nabla U, \Delta\,\nabla U\rangle + |\nabla^2 U|^2 = |\nabla^2 U|^2; \tag{1.8.8}$$

but $|\nabla U|^2 - n = 0$ when U satisfies (1.8.2). □

An estimate in terms of $\varepsilon + \sqrt{\varepsilon}$, where $\varepsilon := \|\,\mathbf{dist}(\nabla U, \mathbb{SO}(n))\ ||\ \mathbb{L}_p(\Omega)\|$, is much easier to prove, but it is insufficient for the application to plate theory, where one needs to sum the estimate over many small cubes of size h.

Corollary 1.8.2 (see [118])**.** *If $U_j \to U$ in $\mathbb{W}^1(\Omega)$ and $\mathbf{dist}(\nabla U_j, \mathbb{SO}(n)) \to 0$ in measure, then $\nabla U_j \to R$ in $\mathbb{L}_2(\Omega)$ for some constant rotation matrix $R \in \mathbb{SO}(n)$).*

Chapter 2

Γ-convergence of heat transfer equation

In the present chapter, we investigate a mixed boundary value problem for the stationary heat transfer equation in a thin layer around a surface $\mathscr{C}$ with the boundary. The main objective is to trace what happens in Γ-limit when the thickness of the layer converges to zero. The limit Dirichlet BVP for the Laplace–Beltrami equation on the surface is described explicitly and we show how the Neumann boundary conditions in the initial BVP transform in the Γ-limit. For this, we apply the variational formulation and the calculus of Günter's tangent differential operators on a hypersurface and layers, which allow global representation of basic differential operators and of corresponding boundary value problems in terms of the standard Euclidean coordinates of the ambient space $\mathbb{R}^n$.

The exposition follows mostly the paper of T. Buchukuri, R. Duduchava and G. Tephnadze [16].

2.1 Introduction

The main objective of the present chapter is to demonstrate what happens with a boundary value problem for the Laplace equation in a thin layer Ω^ε around a surface $\mathscr{C}$ in $\mathbb{R}^3$ when the thickness of the layer ε diminishes to zero: $\varepsilon \to 0$. We impose the Neumann boundary conditions on the upper and lower faces of the layer $\mathscr{C} \times \{\pm\varepsilon\}$ and the Dirichlet boundary conditions on the lateral surface $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$.

The limit of the associated functionals is understood in the sense of Γ-convergence and the main tool is the representation of differential operators with the help of Günter's derivatives – the system of tangent derivatives on the surface $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$ and the normal derivative $\mathscr{D}_4 = \partial_{\boldsymbol{\nu}} := \sum\limits_{j=1}^{3} \nu_j \partial_j$, where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ is the unit normal vector field on the mid-surface $\mathscr{C}$.

We consider heat conduction by an "isotropic" medium, governed by the Laplace equations, with the classical mixed Dirichlet–Neumann boundary conditions on the boundary in the layer domain $\Omega^\varepsilon := \mathscr{C} \times (-\varepsilon, \varepsilon)$ of thickness 2ε, where $\mathscr{C} \subset \mathscr{S}$ is a smooth subsurface of a closed hypersurface $\mathscr{S}$ with smooth nonempty boundary $\partial\mathscr{C}$. In particular, we confine ourselves with zero Dirichlet and non-zero Neumann data (see Remark 2.4.1 for the case of non-zero Dirichlet data):

$$\begin{aligned}
\boldsymbol{\Delta}_{\Omega^\varepsilon} \widetilde{T}(\mathscr{x}, t) &= f(\mathscr{x}, t), \quad (\mathscr{x}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon), \\
\widetilde{T}^+(\mathscr{x}, t) &= 0, \quad (\mathscr{x}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon), \\
(\partial_t \widetilde{T})^+(\mathscr{x}, \pm\varepsilon) &= q_\varepsilon^\pm(\mathscr{x}), \quad \mathscr{x} \in \mathscr{C}.
\end{aligned} \tag{2.1.1}$$

In the investigation we apply the fact that the Laplace operator $\boldsymbol{\Delta}_{\Omega^\varepsilon} = \partial_1^2 + \partial_2^2 + \partial_3^2$ is represented as the sum of the Laplace–Beltrami operator on the mid-surface, the square of the transversal derivative and the lower order term

$$\boldsymbol{\Delta}_{\Omega^\varepsilon} \widetilde{T} = \boldsymbol{\Delta}_{\mathscr{C}} \widetilde{T} + \partial_t^2 \widetilde{T} + 2\mathscr{H}_{\mathscr{C}} \partial_t \widetilde{T}, \tag{2.1.2}$$

where $\mathscr{D}_4 = \partial_t$. The Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}}$ defined in (0.0.12) and the mean curvature $\mathscr{H}_{\mathscr{C}}(\mathscr{X}) = \sum_{k=1}^{3} \mathscr{D}_k \mathscr{N}_k(\mathscr{X})$ of the surface are extended properly from $\mathscr{C}$ (see the forthcoming Lemma 2.2.2).

Introducing the function $G(\mathscr{X}, t)$ which has the same Dirichlet and Neumann traces as T on the $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$ and on $\mathscr{C} \times \{\pm\varepsilon\}$, respectively,

$$G(\mathscr{X}, t) = \frac{1}{4\varepsilon}(t+\varepsilon)^2 q_\varepsilon^+(\mathscr{X}) - \frac{1}{4\varepsilon}(t-\varepsilon)^2 q_\varepsilon^-(\mathscr{X}), \tag{2.1.3}$$

we can reduce problem (2.1.1) to the following boundary value problem with respect to unknown function $T = \widetilde{T} - G$:

$$\boldsymbol{\Delta}_{\Omega^\varepsilon} T(\mathscr{X}, t) = F(\mathscr{X}, t), \quad (\mathscr{X}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon), \tag{2.1.4}$$

$$T^+(\mathscr{X}, t) = 0, \quad (\mathscr{X}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon), \tag{2.1.5}$$

$$(\partial_t T)^+(\mathscr{X}, \pm\varepsilon) = 0, \quad \mathscr{X} \in \mathscr{C}, \tag{2.1.6}$$

where

$$\begin{aligned} F(\mathscr{X}, t) &:= f(\mathscr{X}, t) - \frac{1}{4\varepsilon}\Big((t+\varepsilon)^2 \boldsymbol{\Delta}_{\mathscr{C}} q_\varepsilon^+(\mathscr{X}) - (t-\varepsilon)^2 \boldsymbol{\Delta}_{\mathscr{C}} q_\varepsilon^-(\mathscr{X})\Big) \\ &- \frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{X})}{2\varepsilon}\big((t+\varepsilon) q_\varepsilon^+(\mathscr{X}) - (t-\varepsilon) q_\varepsilon^-(\mathscr{X})\big) - \frac{1}{2\varepsilon}\big(q_\varepsilon^+(\mathscr{X}) - q_\varepsilon^-(\mathscr{X})\big), \quad (\mathscr{X}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon). \end{aligned} \tag{2.1.7}$$

The BVP (2.1.4)–(2.1.6) is reformulated as the minimization problem for the functional which, after scaling (stretching the variable $t = \varepsilon\tau$ and dividing the entire functional by ε), has the form

$$E_\varepsilon(T_\varepsilon) := \int_{-1}^{1} \int_{\mathscr{C}} \Big[\frac{1}{2}(\mathscr{D}_{\mathscr{C}} T_\varepsilon)^2(\mathscr{X}, \tau) + \frac{1}{2\varepsilon^2}(\partial_\tau T_\varepsilon)^2(\mathscr{X}, \tau) + F_\varepsilon(\mathscr{X}, \tau) T_\varepsilon(\mathscr{X}, \tau)\Big]\, d\sigma\, d\tau, \tag{2.1.8}$$

$$\begin{aligned} F_\varepsilon(\mathscr{X}, t) &:= F(\mathscr{X}, \varepsilon t) = f(\mathscr{X}, \varepsilon t) - \frac{\varepsilon}{4}\Big((t+1)^2 \boldsymbol{\Delta}_{\mathscr{C}} q_\varepsilon^+(\mathscr{X}) - \frac{\varepsilon}{4}(t-1)^2 \boldsymbol{\Delta}_{\mathscr{C}} q_\varepsilon^-(\mathscr{X})\Big) \\ &- \frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{X})}{2}\big((t+1) q_\varepsilon^+(\mathscr{X}) - (t-1) q_\varepsilon^-(\mathscr{X})\big) - \frac{1}{2\varepsilon}\big(q_\varepsilon^+(\mathscr{X}) - q_\varepsilon^-(\mathscr{X})\big), \end{aligned} \tag{2.1.9}$$

$$T_\varepsilon(\mathscr{X}, \tau) := T(\mathscr{X}, \varepsilon\tau), \quad T_\varepsilon \in \widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1, 1)),$$
$$F_\varepsilon \in \widetilde{\mathbb{H}}^{-1}(\Omega^1), \quad q_\varepsilon^\pm \in \widetilde{\mathbb{H}}^2(\mathscr{C}), \quad (\mathscr{X}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon).$$

(For the definition of $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1, 1))$, see (2.4.9).)

Let

$$\mathscr{P}(\mathscr{C}) := \Big\{T \in \mathbb{H}^1(\Omega^1) : \ T(\mathscr{X}, \tau) = T_{\mathscr{C}}(\mathscr{X}), \ T_{\mathscr{C}} \in \widetilde{\mathbb{H}}^1(\mathscr{C}), \ \tau \in [-1, 1]\Big\}. \tag{2.1.10}$$

The main result of the present investigation is the following statement.

Theorem 2.1.1. *Let*

$$f_\varepsilon(\mathscr{X}, t) := f(\mathscr{X}, \varepsilon t) \underset{\varepsilon \to 0}{\to} f^0(\mathscr{X}) \ \ in \ \mathbb{L}_2(\Omega^1),$$

$q_\varepsilon^\pm \in \widetilde{\mathbb{H}}^2(\mathscr{C})$ *be uniformly bounded (with respect to* ε*) in* $\mathbb{H}^2(\mathscr{C})$*, and*

$$\lim_{\varepsilon \to 0} q_\varepsilon^+ = \lim_{\varepsilon \to 0} q_\varepsilon^- = q_0, \quad q_0 \in \mathbb{L}_2(\mathscr{C}),$$
$$\frac{1}{2\varepsilon}(q_\varepsilon^+ - q_\varepsilon^-) \underset{\varepsilon \to 0}{\longrightarrow} q_1 \ \ in \ \mathbb{L}_2(\mathscr{C}).$$

Then the functional in (2.1.8) Γ*-converges to the functional*

$$E^{(0)}(T) = \begin{cases} \displaystyle\int_{\mathscr{C}} \Big[\big\langle \mathscr{D}_{\mathscr{C}} T_{\mathscr{C}}(\mathscr{X}), \mathscr{D}_{\mathscr{C}} T_{\mathscr{C}}(\mathscr{X})\big\rangle \\ \qquad + 2\big(f^0(\mathscr{X}) - \mathscr{H}^0_{\mathscr{C}} q_0(\mathscr{X}) - q_1(\mathscr{X})\big) T_{\mathscr{C}}(\mathscr{X})\Big]\, d\sigma & \text{if } T \in \mathscr{P}(\mathscr{C}); \\ +\infty & \text{if } T \notin \mathscr{P}(\mathscr{C}). \end{cases} \tag{2.1.11}$$

The following Dirichlet boundary value problem for Laplace–Beltrami equation on the mid-surface $\mathscr{C}$

$$\begin{gathered}\boldsymbol{\Delta}_{\mathscr{C}}T(\mathscr{X})=f^0(\mathscr{X})-\mathscr{H}^0_{\mathscr{C}}q_0(\mathscr{X})-q_1(\mathscr{X}),\quad \mathscr{X}\in\mathscr{C},\\ T^+(\mathscr{X})=0,\quad \mathscr{X}\in\partial\mathscr{C},\\ T\in\mathbb{H}^1(\mathscr{C}),\quad f^0,q_0,q_1\in\mathbb{L}_2(\mathscr{C}),\end{gathered}\tag{2.1.12}$$

is an equivalent reformulation of the minimization problem with the energy functional (2.1.11).

Remark 2.1.1. The BVP (2.1.12) is the "Γ-limit" of the initial BVP (2.1.1) in the following sense: the corresponding functional (2.1.11) is the Γ-limit of functional (2.1.8), corresponding to the BVP (2.1.4)–(2.1.6).

It is remarkable to note that the weak derivative q^0 of the Neumann condition from the initial BVP (2.1.1) migrated into the right-hand side of the limit equation.

Note as well that the Γ-limit $T_{\mathscr{C}}(\mathscr{X})$ of a solution $T(\mathscr{X},\varepsilon\tau)$, $T\in\mathbb{H}^1(\Omega_\varepsilon)$, to the BVP (2.1.4)–(2.1.6) has better smoothness $T_{\mathscr{C}}\in\mathbb{H}^1(\mathscr{C})$ than expected.

Γ-limits of boundary value problems in thin structures, reformulated as a minimization problem for the associated energy functional, were studied by many authors (see, e.g., [13, 76, 77, 134] and the references therein). But mostly the Lamé equations for elastic plates $\mathscr{C}\subset\mathbb{R}^2$ and the zero boundary conditions were treated (the Laplace equation for a plate is studied in [13]). In the papers [76, 134], the case of shells is treated, but with a different technique. Our approach is based on the calculus of Günter's derivatives, which we find more appropriate for such problems.

These results are useful in numerical and engineering applications (cf. [6, 9, 20, 32, 34, 128]) and the results exposed here allow to treat cases of special surfaces in greater detail.

The layout of the chapter is as follows. In Section 2.2, we identify the most important partial differential operators on hypersurfaces such as gradient, divergence, Laplace–Beltrami operator. In Section 2.3, we consider the energy functional (2.3.3) and the associated Euler–Lagrange equation (2.3.4). In Sections 2.4 and 2.5, the aforementioned approach is applied and main theorems of the present chapter, including Theorem 2.1.1, are proved.

2.2 Laplace operator in curvilinear coordinates

We will keep the notation of Chapter 1: Θ, ω, $\mathscr{S}$ and $\mathscr{C}$. We consider a **layer domain**

$$\Omega^\varepsilon:=\Big\{\mathscr{X}_t\in\mathbb{R}^n:\quad \mathscr{X}_t=\mathscr{X}+t\boldsymbol{\nu}(\mathscr{X})=\Theta(x)+t\boldsymbol{\nu}(\Theta(x)),\ \ x\in\omega,\ \ -\varepsilon<t<\varepsilon\Big\}=\mathscr{C}\times(-\varepsilon,\varepsilon),\tag{2.2.1}$$

where $\boldsymbol{\nu}(\mathscr{X})=\boldsymbol{\nu}(\Theta(y))$ for $\mathscr{X}=\Theta(y)\in\mathscr{S}$ is the outer unit normal vector field (see (1.3.7) and (1.3.9)). The surface $\mathscr{C}$ is a mid-surface for the layer domain.

We will also use the notation $\boldsymbol{\nu}(y):=\boldsymbol{\nu}(\Theta(y))$ for brevity, unless this leads to a confusion. The coordinate t will be referred to as the **transverse variable**.

Without going into details, let us only remark that if the hypersurface $\mathscr{S}$ is C^2-smooth and $1/\varepsilon$ is more than the maximum of modules of all principal curvatures of the surface $\mathscr{S}$ (i.e., of all eigenvalues $|\lambda_1(\mathscr{X})|,\dots,|\lambda_{n-1}(\mathscr{X})|,\lambda_n(\mathscr{X})\equiv0$ of the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(\mathscr{X})$, $\mathscr{X}\in\mathscr{S}$), then the mapping

$$\begin{gathered}\Theta^\varepsilon:\omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)\to\Omega^\varepsilon,\ \ \omega^\varepsilon\subset\mathbb{R}^n,\\ \Theta^\varepsilon(y,t):=\Theta(y)+t\boldsymbol{\nu}(y),\ \ (y,t)\in\omega^\varepsilon,\end{gathered}\tag{2.2.2}$$

is a diffeomorphism.

We will also suppose that $\mathscr{N}$ is a proper extension of the outer unit normal vector field $\boldsymbol{\nu}$ into the layer neighborhood Ω^ε (cf. Definition 1.3.2).

The n-tuple $\boldsymbol{g}_1:=\partial_1\Theta,\dots,\boldsymbol{g}_{n-1}:=\partial_{n-1}\Theta,\boldsymbol{g}_n:=\mathscr{N}$, where $\mathscr{N}$ is the proper extension of $\boldsymbol{\nu}$ in the neighborhood Ω^ε, is a basis in Ω^ε, and an arbitrary vector field $\boldsymbol{U}=\sum\limits_{j=1}^n U^0_j\boldsymbol{e}^j$ on Ω^ε is represented with this basis in "curvilinear coordinates".

Let us consider the system of $(n+1)$-vectors

$$\boldsymbol{d}^{\,j} := \boldsymbol{e}^j - \mathscr{N}_j\mathscr{N}, \quad j = 1,\dots,n, \quad \text{and} \quad \boldsymbol{d}^{\,n+1} := \mathscr{N}, \tag{2.2.3}$$

where $\boldsymbol{e}^1,\dots,\boldsymbol{e}^n$ is the Cartesian basis in $\mathbb{R}^n$ (cf. (0.0.7)); the first n vectors $\boldsymbol{d}^{\,1},\dots,\boldsymbol{d}^{\,n}$ are tangent to the surface $\mathscr{C}$, while the last one $\boldsymbol{d}^{\,n+1} = \mathscr{N}$ is orthogonal to all $\boldsymbol{d}^{\,1},\dots,\boldsymbol{d}^{\,n}$. This system is, obviously, linearly dependent, but full and any vector field $\boldsymbol{U} \in \mathscr{W}(\Omega^\varepsilon)$ is written in the following form:

$$\boldsymbol{U} = \sum_{j=1}^{n} U_j\boldsymbol{e}^j = \sum_{j=1}^{n+1} U_j^0\boldsymbol{d}^{\,j}. \tag{2.2.4}$$

Since the system $\{\boldsymbol{d}^{\,j}\}_{j=1}^{n+1}$ is linearly dependent

$$\sum_{j=1}^{n} \mathscr{N}_j\boldsymbol{d}^{\,j} = 0, \quad \langle\mathscr{N}, \boldsymbol{d}^j\rangle = 0, \quad j = 1,\dots,n, \tag{2.2.5}$$

representation (2.2.4) is not unique. To fix the unique representation in (2.2.4) we will keep the following convention:

$$U_j^0 := U_j - \langle\mathscr{N}, \boldsymbol{U}\rangle\mathscr{N}_j, \quad j = 1,\dots,n, \quad U_{n+1}^0 = \langle\mathscr{N}, \boldsymbol{U}\rangle = \sum_{j=1}^{n} U_j\mathscr{N}_j. \tag{2.2.6}$$

Convention (2.2.6) is natural, because if the vector $\boldsymbol{U}(\mathscr{x})$ is tangent to $\mathscr{C}$ for $\mathscr{x} \in \mathscr{C}$, then $U_j^0(\mathscr{x}) := U_j(\mathscr{x})$ for $j = 1,\dots,n$ and $U_{n+1}^0(\mathscr{x}) = 0$.

Lemma 2.2.1. *Representation* (2.2.4) *is unique, provided that conditions* (2.2.6) *hold:*

$$\text{If } \boldsymbol{U}^0 = \sum_{j=1}^{4} U_j^0\boldsymbol{d}^{\,j} = 0, \quad \text{then } U_1^0 = U_2^0 = U_3^0 = U_4^0 = 0. \tag{2.2.7}$$

The scalar product and, consequently, the distance between two vectors does not change:

$$\langle\boldsymbol{U}^0, \boldsymbol{V}^0\rangle = \sum_{j=1}^{4} U_j^0V_j^0 = \sum_{j=1}^{3} U_jV_j = \langle\boldsymbol{U}, \boldsymbol{V}\rangle, \quad \|\boldsymbol{U}^0 - \boldsymbol{V}^0\| = \|\boldsymbol{U} - \boldsymbol{V}\| \tag{2.2.8}$$

for arbitrary vectors $\boldsymbol{U} = (U_1, U_2, U_3, U_4)^\top, \boldsymbol{V} = (V_1, V_2, V_3, V_4)^\top \in \mathbb{R}^3$.

Proof. If $\boldsymbol{U}^0 = \sum\limits_{j=1}^{4} U_j^0\boldsymbol{d}^{\,j} = 0$, then $U_4 = \langle\boldsymbol{U}, \mathscr{N}\rangle = 0$, since $\mathscr{N}$ is orthogonal to vectors $\boldsymbol{d}^{\,1}\boldsymbol{d}^{\,2}, \boldsymbol{d}^{\,3}$. On the other hand, using $U_j^0 = U_j - \mathscr{N}_j\langle\mathscr{N}, \boldsymbol{U}\rangle$, $j = 1, 2, 3$, and the obtained equality $U_4 = 0$ (cf. (2.2.6)), we get

$$0 = \langle\boldsymbol{U}^0, \boldsymbol{e}^k\rangle = \sum_{j=1}^{3} U_j^0\langle\boldsymbol{d}^{\,j}, \boldsymbol{e}^k\rangle = \sum_{j=1}^{3} U_j^0\big[\delta_{jk} - \mathscr{N}_j\langle\mathscr{N}, \boldsymbol{e}^k\rangle\big] = U_k^0 - \mathscr{N}_k\sum_{j=1}^{3} U_j^0\mathscr{N}_j = U_k^0, \quad k = 1, 2, 3,$$

since

$$\sum_{j=1}^{3} U_j^0\mathscr{N}_j = \sum_{j=1}^{3} \big(U_j - \mathscr{N}_j\langle\mathscr{N}, \boldsymbol{U}\rangle\big)\mathscr{N}_j \sum_{j=1}^{3} U_j\mathscr{N}_j - \sum_{j=1}^{n} \mathscr{N}_j^2\langle\mathscr{N}, \boldsymbol{U}\rangle = \langle\mathscr{N}, \boldsymbol{U}\rangle - \langle\mathscr{N}, \boldsymbol{U}\rangle = 0$$

and $U_1^0 = U_2^0 = U_3^0 = U_4^0 = 0$.

Let us prove the first equality in (2.2.8):

$$\langle \boldsymbol{U}^0, \boldsymbol{V}^0\rangle = \sum_{j=1}^{4} U_j^0 V_j^0 = \sum_{j=1}^{3} (U_j - \mathscr{N}_j\langle\mathscr{N},\boldsymbol{U}\rangle)\big(V_j - \mathscr{N}_j\langle\mathscr{N},\boldsymbol{V}\rangle\big) + \langle\mathscr{N},\boldsymbol{U}\rangle\langle\mathscr{N},\boldsymbol{V}\rangle$$
$$= \sum_{j=1}^{3}\Big[U_jV_j - \langle\mathscr{N},\boldsymbol{V}\rangle U_j\mathscr{N}_j - \langle\mathscr{N},\boldsymbol{U}\rangle V_j\mathscr{N}_j + \langle\mathscr{N},\boldsymbol{U}\rangle\langle\mathscr{N},\boldsymbol{V}\rangle\mathscr{N}_j^2\Big] = \sum_{j=1}^{3} U_jV_j = \langle\boldsymbol{U},\boldsymbol{V}\rangle$$

and the equality is proved.

The second equality in (2.2.8) is a simple consequence of the first one, since

$$\|\boldsymbol{U}^0 - \boldsymbol{V}^0\| = \sqrt{\langle \boldsymbol{U}^0 - \boldsymbol{V}^0, \boldsymbol{U}^0 - \boldsymbol{V}^0\rangle} = \sqrt{\langle \boldsymbol{U} - \boldsymbol{V}, \boldsymbol{U} - \boldsymbol{V}\rangle} = \|\boldsymbol{U} - \boldsymbol{V}\|. \qquad \square$$

Note for later use that due to equalities (2.2.5) and convention (2.2.6) we get

$$\partial_{\boldsymbol{U}} = \sum_{j=1}^{n} U_j\partial_j = \sum_{j=1}^{n} [U_j^0\partial_j + \langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{N}_j\partial_j]$$
$$= \sum_{j=1}^{n} U_j^0(\partial_j - \mathscr{N}_j\partial_{\mathscr{N}}) + \langle\mathscr{N},\boldsymbol{U}\rangle\partial_{\mathscr{N}} = \sum_{j=1}^{n} U_j^0\mathscr{D}_j + U_{n+1}^0\mathscr{D}_{n+1} = \sum_{j=1}^{n+1} U_j^0\mathscr{D}_j =: \mathscr{D}_{\boldsymbol{U}}.$$

Definition 2.2.1. For a function $\varphi \in \mathbb{H}^1(\Omega^\varepsilon)$ the extended gradient is

$$\nabla_{\Omega^\varepsilon}\varphi = \big\{\mathscr{D}_1\varphi, \dots, \mathscr{D}_n\varphi, \mathscr{D}_{n+1}\varphi\big\}^\top = \sum_{j=1}^{n+1} (\mathscr{D}_j\varphi)\boldsymbol{d}^j, \quad \mathscr{D}_{n+1}\varphi := \partial_{\mathscr{N}}\varphi, \tag{2.2.9}$$

and for a smooth vector field $\boldsymbol{U} = \sum\limits_{j=1}^{n+1} U_j^0\boldsymbol{d}^j \in \mathscr{W}(\Omega^\varepsilon)$ (see (2.2.4), (2.2.6)) the extended divergence is

$$\mathbf{div}_{\Omega^\varepsilon}\boldsymbol{U} := \sum_{j=1}^{n+1} \mathscr{D}_jU_j^0 + \mathscr{H}_{\mathscr{C}}^0\langle\mathscr{N},\boldsymbol{U}\rangle = -\nabla_{\Omega^\varepsilon}^*\boldsymbol{U}, \tag{2.2.10}$$

since

$$\mathscr{H}_{\Omega^\varepsilon}^0(x) := \sum_{j=1}^{n} \partial_j\mathscr{N}_j(x) = \sum_{j=1}^{n+1} \mathscr{D}_j\mathscr{N}_j(x) = \sum_{j=1}^{n} \mathscr{D}_j\nu_j(\mathscr{x}) = \mathscr{H}_{\mathscr{C}}^0(\mathscr{x}), \quad x \in \Omega^\varepsilon, \quad \mathscr{x} = \pi_{\mathscr{S}}x,$$

and $\mathscr{H}_{\mathscr{C}}^0(\mathscr{x})$ differs from the mean curvature $\mathscr{H}_{\mathscr{C}}(\mathscr{x})$ (see (1.3.63)) by the constant multiplier $\mathscr{H}_{\mathscr{C}}^0(\mathscr{x}) = (n-1)\mathscr{H}_{\mathscr{C}}(\mathscr{x})$.

Lemma 2.2.2. *The classical gradient* $\nabla\varphi := \{\partial_1\varphi, \dots, \partial_n\varphi\}^\top$ *written in the full system of vectors* $\{\boldsymbol{d}^j\}_{j=1}^{n+1}$ *in* (2.2.3) *coincides with the extended gradient* $\nabla_{\Omega^\varepsilon}\varphi$ *in* (2.2.9).

Similarly, the classical divergence $\mathbf{div}\,\boldsymbol{U} := \sum\limits_{j=1}^{n}\partial_jU_j$ *of a vector field* $\boldsymbol{U} := \sum\limits_{j=1}^{n} U_j\boldsymbol{e}^j$ *written in the full system* (2.2.3) *coincides with the extended divergence* $\mathbf{div}\,\boldsymbol{U} = \mathbf{div}_{\Omega^\varepsilon}\,\boldsymbol{U}$ *in* (2.2.10).

The extended gradient and the negative extended divergence are dual: $\mathscr{D}_{\Omega^\varepsilon}^* = -\,\mathbf{div}_{\Omega^\varepsilon}$ *and* $\mathbf{div}_{\Omega^\varepsilon}^* = -\nabla_{\Omega^\varepsilon}$.

The Laplace–Beltrami operator $\boldsymbol{\Delta}_{\Omega^\varepsilon} := \mathbf{div}_{\Omega^\varepsilon}\nabla_{\Omega^\varepsilon}\varphi = -\mathscr{D}_{\Omega^\varepsilon}^*(\nabla_{\Omega^\varepsilon}\varphi)$ *on* Ω^ε *written in the full system* (2.2.3) *acquires the form*

$$\boldsymbol{\Delta}_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n} \mathscr{D}_j^2\varphi + \partial_{\mathscr{N}}^2\varphi + \mathscr{H}_{\mathscr{C}}^0\partial_{\mathscr{N}}\varphi = \sum_{j=1}^{n+1} \mathscr{D}_j^2\varphi + \mathscr{H}_{\mathscr{C}}^0\mathscr{D}_{n+1}\varphi, \quad \varphi \in \mathbb{H}^2(\Omega^\varepsilon). \tag{2.2.11}$$

Proof. A similar lemma is proved in [57, Lemma 4.3], but the definition of divergence $\mathbf{div}_{\Omega^\varepsilon}$ is different there. Therefore, we expose the full proof below.

The fact that the gradients coincide follows from the choice of the full system (2.2.3):

$$\begin{aligned}\nabla\varphi &:= \left\{\partial_1\varphi, , \partial_n\varphi\right\}^\top = \sum_{j=1}^{n}(\partial_j\varphi)\boldsymbol{e}^j \\ &= \sum_{j=1}^{n}(\mathscr{D}_j\varphi + \mathscr{N}_j\mathscr{D}_{n+1}\varphi)\boldsymbol{e}^j = \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j + (\mathscr{D}_{n+1}\varphi)\mathscr{N} = \sum_{j=1}^{n+1}(\mathscr{D}_j\varphi)\boldsymbol{d}^j = \nabla_{\Omega^\varepsilon}\varphi,\end{aligned} \tag{2.2.12}$$

since

$$\boldsymbol{e}^j = \boldsymbol{d}^j + \mathscr{N}_j\mathscr{N}, \ \ \partial_j = \mathscr{D}_j + \mathscr{N}_j\partial_{\mathscr{N}}, \ \ \sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j = 0, \ \ \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{e}^j = \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j. \tag{2.2.13}$$

By applying (2.2.6) and (2.2.13) we proceed as follows:

$$\begin{aligned}\mathbf{div}\,\boldsymbol{U} &= \sum_{j=1}^{n}\partial_jU_j = \sum_{j=1}^{n}\mathscr{D}_jU_j + \sum_{j=1}^{n}\mathscr{N}_j\partial_{\mathscr{N}}U_j = \sum_{j=1}^{n}\mathscr{D}_j\big[U_j^0 + \mathscr{N}_j\langle\mathscr{N},\boldsymbol{U}\rangle\big] + \sum_{j=1}^{n}\partial_{\mathscr{N}}(\mathscr{N}_jU_j) \\ &= \sum_{j=1}^{n}\mathscr{D}_jU_j^0 + \sum_{j=1}^{n}(\mathscr{D}_j\mathscr{N}_j)\langle\mathscr{N},\boldsymbol{U}\rangle + \mathscr{D}_{n+1}U_{n+1}^0 = \sum_{j=1}^{n+1}\mathscr{D}_jU_j^0 + \mathscr{H}_{\mathscr{C}}^0\langle\mathscr{N},\boldsymbol{U}\rangle = \mathbf{div}_{\Omega^\varepsilon}\boldsymbol{U}.\end{aligned} \tag{2.2.14}$$

The proved equality and the classical equality $\nabla^* = -\,\mathbf{div}$ ensure the both claimed equalities $\mathscr{D}_{\Omega^\varepsilon}^* = -\,\mathbf{div}_{\Omega^\varepsilon}$ and $\mathbf{div}_{\Omega^\varepsilon}^* = -\nabla_{\Omega^\varepsilon}$:

$$(\nabla_{\Omega^\varepsilon}\varphi, \boldsymbol{U}) = (\nabla\varphi, \boldsymbol{U}) = -(\varphi, \mathbf{div}\ \boldsymbol{U}) = -(\varphi, \mathbf{div}_{\Omega^\varepsilon}\boldsymbol{U}).$$

Formula (2.2.11) for the Laplace–Beltrami operator is a direct consequence of equalities (2.2.12), (2.2.14) and definitions. Indeed, the first n components of the gradient

$$\nabla\varphi = \nabla_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j + (\mathscr{D}_{n+1}\varphi)\mathscr{N}$$

have the property $(\mathscr{D}_j\varphi)^0 = \mathscr{D}_j\varphi - \langle\mathscr{N}, \nabla_{\Omega^\varepsilon}\varphi\rangle\mathscr{N}_j = \mathscr{D}_j\varphi$ because (see the third formula in (2.2.13)) $\langle\mathscr{N}, \nabla_{\Omega^\varepsilon}\varphi\rangle = \sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j\varphi = 0$, and we can write

$$\boldsymbol{\Delta}\varphi = \mathbf{div}\ \nabla\varphi = \mathbf{div}_{\Omega^\varepsilon}\ \mathscr{D}_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi + \mathscr{H}_{\mathscr{C}}^0\langle\mathscr{N}, \nabla\varphi\rangle = \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi + \mathscr{H}_{\mathscr{C}}^0\mathscr{D}_{n+1}\varphi = \boldsymbol{\Delta}_{\Omega^\varepsilon}\varphi. \qquad \square$$

2.3 Convex energies

Let again Ω^ε be a layer domain of width 2ε in the direction transversal to the mid-surface $\mathscr{C}$ (cf. Section 2.2).

Any minimizer u of the energy functional

$$\mathscr{E}^\varepsilon(u) := \int_{\Omega^\varepsilon}\langle\nabla u, \nabla u\rangle\,dy, \ \ u \in \mathbb{H}^1(\Omega^\varepsilon), \tag{2.3.1}$$

should satisfy

$$\begin{aligned}0 = \frac{d}{dt}\,\mathscr{E}^\varepsilon(u+tv)\Big|_{t=0} &= \int_{\Omega^\varepsilon}\big[\langle\nabla u, \nabla v\rangle + \langle\nabla v, \nabla u\rangle\big]\,dy \\ &= 2\,\mathrm{Re}\int_{\Omega^\varepsilon}\langle\nabla u, \nabla v\rangle\,dy = -2\,\mathrm{Re}\int_{\Omega^\varepsilon}\langle\mathbf{div}\,\nabla u, v\rangle\,dy = -2\,\mathrm{Re}\int_{\Omega^\varepsilon}\langle\Delta\,u, v\rangle\,dy\end{aligned}$$

for arbitrary $v \in \widetilde{\mathbb{H}}^1(\Omega^\varepsilon)$, which implies

$$\Delta\, u = 0 \ \text{ on } \ \Omega^\varepsilon. \tag{2.3.2}$$

In other words, (2.3.2) is the Euler–Lagrange equation associated with the energy functional (2.3.1). Similarly, a minimizer of the energy functional

$$\mathscr{E}_0(u) := \int\limits_{\mathscr{C}} \langle \nabla_{\mathscr{C}} u, \nabla_{\mathscr{C}} u\rangle\, d\sigma, \quad u \in H^1(\mathscr{C}), \tag{2.3.3}$$

on the hypersurface $\mathscr{C}$ should satisfy the following Laplace–Beltrami equation

$$\boldsymbol{\Delta}_{\mathscr{C}} u := \mathbf{div}_{\mathscr{C}}\, \nabla_{\mathscr{C}} u = 0 \ \text{ on } \ \mathscr{C}. \tag{2.3.4}$$

To treat the dimension reduction problem for the Laplace equation (see [13] for a similar consideration in case of a flat 3D body), we assume, without restricting generality, that Ω^1 (i.e., for $\varepsilon = 1$) is still a layer domain. Otherwise, we can first change the variable $\mathscr{X}_n = \varepsilon_0 \overline{\mathscr{X}}_n$, $0 < \overline{\mathscr{X}}_n < 1$, where $0 < \varepsilon_0 < 1$ is such that Ω^{ε_0} is still a layer domain.

Next, we introduce a new coordinate system (cf. (2.2.6))

$$\begin{gathered} x := \sum_{m=1}^{n} x_m \boldsymbol{e}^m = \sum_{m=1}^{n} \mathscr{X}_m \boldsymbol{d}^{\,m} + t\boldsymbol{d}^{\,n+1}, \\ \mathscr{X}_k := x_k - \mathscr{N}_k \langle \mathscr{N}, x\rangle, \ \ k = 1, \dots, n, \ \ t = \mathscr{X}_{n+1} := \langle x, \mathscr{N}\rangle = \sum_{m=1}^{n} x_m \mathscr{N}_m \end{gathered} \tag{2.3.5}$$

and define the scalar product of elements as follows (cf. similar in (2.2.8)):

$$\langle \mathscr{X}, \mathscr{Y}\rangle := \sum_{j=1}^{n+1} \mathscr{X}_j \mathscr{Y}_j \ \text{ for } \ \mathscr{X} := \sum_{m=1}^{n+1} \mathscr{X}_m \boldsymbol{d}^{\,m}, \ \ \mathscr{Y} := \sum_{m=1}^{n+1} \mathscr{Y}_m \boldsymbol{d}^{\,m}.$$

Then (cf. (2.2.8))

$$\langle \mathscr{X}, \mathscr{Y}\rangle = \sum_{j=1}^{n+1} \mathscr{X}_j \mathscr{Y}_j = \sum_{j=1}^{n} \big(x_j - \mathscr{N}_j \langle \mathscr{N}, x\rangle\big)\big(y_j - \mathscr{N}_j \langle \mathscr{N}, y\rangle\big) + \langle \mathscr{N}, x\rangle \langle \mathscr{N}, y\rangle = \sum_{j=1}^{n} x_j \overline{y}_j = \langle x, y\rangle.$$

In particular,

$$\|\mathscr{X}\| := \sum_{j=1}^{n+1} |\mathscr{X}_j|^2 = \sum_{j=1}^{n} |x_j|^2 = \|x\|. \tag{2.3.6}$$

Due to Lemma 2.2.2, the classical gradient in the energy functional (2.3.1) can be replaced by the extended gradient

$$\mathscr{E}^\varepsilon(u) := \int\limits_{\Omega^\varepsilon} \langle \mathscr{D}_{\Omega^\varepsilon} u(y), \mathscr{D}_{\Omega^\varepsilon} u(y)\rangle\, dy = \int\limits_{-\varepsilon}^{\varepsilon} \int\limits_{\mathscr{C}} \big[\, |\mathscr{D}_{\mathscr{C}} u(\mathscr{X}, t)|^2 + |\partial_t u(\mathscr{X}, t)|^2 \big]\, d\sigma\, dt, \tag{2.3.7}$$

where $\mathscr{D}_{\mathscr{C}} := (\mathscr{D}_1, \dots, \mathscr{D}_n)^\top$ is the surface gradient and $u \in \mathbb{H}^1(\Omega^\varepsilon)$ is arbitrary, since $\mathscr{D}_{n+1} = \partial_{\mathscr{N}} = \partial_t$. Here $\mathscr{C}$ is the mid-surface of the layer domain $\Omega^\varepsilon = \mathscr{C} \times (-\varepsilon, \varepsilon)$ and $d\sigma$ is the surface measure on $\mathscr{C}$.

Due to representation (2.3.7) and the new coordinate system (2.3.5), we can apply the scaling with respect to the variable t and study the scaled energy. The approach is based on Γ-convergence (see [13,77]) and can be applied to a general energy functional which is convex and has square growth. The problem we have in mind is the following: *Do these energies defined on thin n-dimensional domains Ω^ε converge (and in which sense) to an energy defend on the $(n-1)$-dimensional Hypersurface $\mathscr{C}$ (the mid-surface of Ω^ε) when the domain Ω^ε is "squeezed" infinitely in the transversal direction to $\mathscr{C}$?*

In the next two sections, we apply the results developed in the present chapter to boundary value problems for the heat conduction by a hypersurface. In particular, we show that if the thickness of the layer domain Ω^ε, with the mid-surface $\mathscr{C}$, tends to zero, the sequence of functionals in variational formulation of the linear heat conduction equation Γ-converges to the functional corresponding to some explicit boundary value problem for the Laplace–Beltrami equation on the mid-surface $\mathscr{C}$.

2.4 Variational reformulation of heat transfer problems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with the piecewise smooth boundary $\partial\Omega = \overline{\mathscr{C}}_D \cup \overline{\mathscr{C}}_N$, where $\mathscr{C}_D$ and $\mathscr{C}_N$ are open non-intersecting surfaces, $\mathscr{C}_D \cap \mathscr{C}_N = \varnothing$, and their common boundary is a smooth arc. Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ the unit normal on $\mathscr{C}$, external with respect to Ω.

We consider the general steady-state, linear heat transfer problem for a medium occupying domain Ω. We assume that on the part $\mathscr{C}_D$ of the boundary $\partial\Omega$ the temperature g is prescribed, while on the part $\mathscr{C}_N$ of $\partial\Omega$ it is prescribed the heat flux q.

We look for a temperature distribution $T(x)$ in Ω, which satisfies the linear heat conduction equation

$$\mathbf{div}(\mathscr{A}(x)\nabla T)(x) = f(x), \quad x \in \Omega, \tag{2.4.1}$$

and the boundary conditions

$$T^+(y) = g(y) \quad \text{on } \mathscr{C}_D, \tag{2.4.2}$$

$$-\big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle = q(y) \quad \text{on } \mathscr{C}_N, \tag{2.4.3}$$

where $\mathscr{A}$ is the thermal conductivity, f is the heat source, g is the distribution of temperature and q is the heat flux. All these quantities are supposed to be known.

We assume that $\mathscr{A}(x)$ is a bounded measurable and positive definite 3×3 matrix-function (cf. the similar condition (1.4.37))

$$\langle \mathscr{A}(x)\xi, \xi\rangle \geqslant C\|\xi\|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^3.$$

The following inequality is an obvious consequence of the positive definiteness of $\mathscr{A}$:

$$(\mathscr{A}\boldsymbol{U}, \boldsymbol{U}) \geqslant C\|\boldsymbol{U}|\mathbb{L}_2(\Omega)\|^2$$

for all 3-vectors $\boldsymbol{U} = (U_1, U_2, U_3)^\top \in \mathbb{L}_2(\Omega)$. Further, we assume that the traces $\mathscr{A}^+(y)$ at the boundary $\mathscr{C}$ exist. Then $\mathscr{A}^+$ has the same properties as $\mathscr{A}$ on Ω, namely, it is a bounded, measurable positive definite matrix function.

We impose the following natural constraints on the solution T and on the prescribed data f, g, q:

$$T \in \mathbb{H}^1(\Omega), \quad f \in \widetilde{\mathbb{H}}^{-1}(\Omega), \quad g \in \mathbb{H}^{1/2}(\mathscr{C}_D), \quad q \in \mathbb{H}^{-1/2}(\mathscr{C}_N). \tag{2.4.4}$$

The existence of the traces $\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+\rangle \in \mathbb{H}^{-1/2}(\mathscr{C}_3)$, which is not ensured by the trace theorem, follows from the Green formula

$$\int_\Omega (\mathbf{div}\ \mathscr{A}(x)\nabla T)(x)\psi(x)\,dx = \int_{\mathscr{C}} \big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle \psi^+(y)\,d\sigma - \int_\Omega \big\langle \mathscr{A}(x)\nabla T(x), \nabla\psi(x)\big\rangle\,dx \tag{2.4.5}$$

by the duality between the spaces $\mathbb{H}^{1/2}(\mathscr{C})$ and $\widetilde{\mathbb{H}}^{-1/2}(\mathscr{C})$, due to the fact that T is a solution to equation (2.4.1). For this, we rewrite (2.4.5) in the form

$$\int_{\mathscr{C}} \big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle \psi^+(y)\,d\sigma = \int_\Omega f(x)\psi(x)\,dx + \int_\Omega \big\langle \mathscr{A}(x)\nabla T(x), \nabla\psi(x)\big\rangle\,dx$$

and note that $\psi \in \mathbb{H}^1(\Omega)$ is arbitrary and, therefore, $\psi^+ \in \mathbb{H}^{1/2}(\mathscr{C})$ is arbitrary, too.

First, we reduce the BVP (2.4.1)–(2.4.3) to the equivalent BVP with vanishing Dirichlet data.

Remark 2.4.1. Let us assume that the subsurface $\mathscr{C}_D$ is smooth and $g \in \mathbb{H}^s(\mathscr{C}_D)$, $s \geqslant \frac{1}{2}$. There exists a domain Ω' with a smooth boundary $\mathscr{C}' := \partial\Omega'$ with the properties: $\Omega \subset \Omega'$ and $\mathscr{C}_D \subset \mathscr{C}'$. Let $g^0 \in \mathbb{H}^s(\mathscr{C}')$ be such extension of g which maintains the space.

The Dirichlet BVP

$$\begin{aligned} \mathbf{div}(\mathscr{A}(x)\nabla G)(x) &= 0, \quad x \in \Omega', \\ G^+(y) &= g^0(y) \quad \text{on } \mathscr{C}' \end{aligned} \tag{2.4.6}$$

has a unique solution

$$G(x)=W\Big(\frac{1}{2}\,I+W_0\Big)^{-1}g^0(x),\ \ x\in\Omega',\ \ G\in\mathbb{H}^{s+1/2}(\Omega'),$$

where W is the double layer potential for the operator $\mathbf{div}\,\mathscr{A}(x)\nabla$ and W_0 is its direct value (a singular integral operator) on the surface $\mathscr{C}'$, $I:\mathbb{H}^s(\mathscr{C}')\to\mathbb{H}^s(\mathscr{C}')$ is a unit operator. Then the BVP

$$\begin{aligned}\mathbf{div}(\mathscr{A}(x)\nabla T_0)(x)&=f(x),\ \ x\in\Omega,\\ T_0^+(y)&=0\ \text{ on }\ \mathscr{C}_D,\\ -\big\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T_0)^+(y)\big\rangle&=q_0(y)\ \text{ on }\ \mathscr{C}_N\end{aligned}\tag{2.4.7}$$

is an equivalent reformulation of the BVP (2.4.1)–(2.4.3), now with vanishing Dirichlet traces. The solutions and Neumann data are related as follows:

$$\begin{aligned}&T_0(x):=T(x)-G(x),\ \ x\in\Omega,\\ &q_0(y):=q(y)-\Big(\partial_{\boldsymbol{\nu}}W\Big(\frac{1}{2}\,I+W_0\Big)^{-1}g^0\Big)^+(y),\ \ x\in\mathscr{C}.\end{aligned}\tag{2.4.8}$$

Note that if we require higher smoothness for the Neumann data $q\in\mathbb{H}^r(\mathscr{C}_N)$, $r>-1/2$, and take $g\in\mathbb{H}^{r+1}(\mathscr{C}_D)$ (i.e., $s=r+1$ in (2.4.6)), the Neumann data in the BVP (2.4.7) inherits the same smoothness $q_0\in\mathbb{H}^r(\mathscr{C}_N)$.

Let $\Omega\subset\mathbb{R}^n$ be a domain with a Lipshitz boundary $\mathscr{M}:=\partial\Omega$ and $\mathscr{M}_0\subset\partial\Omega$ be a subsurface of the boundary surface which has the non-zero measure. By $\widetilde{\mathbb{H}}^1(\Omega,\mathscr{M}_0)$ we denote a subspace of $\mathbb{H}^1(\Omega)$ of those functions which have vanishing traces on the part of the boundary

$$\widetilde{\mathbb{H}}^1(\Omega,\mathscr{M}_0):=\big\{\varphi\in\mathbb{H}^1(\Omega):\ \ \varphi^+(y)=0,\ \ \forall\,y\in\mathscr{M}_0\big\}.\tag{2.4.9}$$

This space inherits the standard norm from $\mathbb{H}^1(\Omega)$:

$$\big\|\varphi\mid\mathbb{H}^1(\Omega)\big\|:=\Big[\,\big\|\varphi\mid\mathbb{L}_2(\Omega)\big\|^2+\sum_{j=1}^n\big\|\partial_j\varphi\mid\mathbb{L}_2(\Omega)\big\|^2\Big]^{1/2}.$$

Consider the functional

$$\Phi(T)=\int\limits_{\Omega}\Big[\frac{1}{2}\,\big\langle\mathscr{A}(x)\nabla T(x),\nabla T(x)\big\rangle+f(x)T(x)\Big]\,dx+\int\limits_{\mathscr{C}_N}q(y)T^+(y)\,d\sigma,\tag{2.4.10}$$

where f and q satisfy conditions (2.4.4) and $T\in\mathbb{H}^1(\Omega)$ has vanishing traces on $\mathscr{C}_D$, i.e., $T\in\widetilde{\mathbb{H}}^1(\Omega,\mathscr{C}_D)$ (see (2.4.9)).

The second summand in the integral on Ω is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{-1}(\Omega)$ and $\mathbb{H}^1(\Omega)$. Concerning the integral on $\mathscr{C}_N$: it is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$ and $\mathbb{H}^{-1/2}(\mathscr{C}_N)$, since $q\in\mathbb{H}^{-1/2}(\mathscr{C}_N)$ and the conditions $T\in\widetilde{\mathbb{H}}^1(\Omega,\mathscr{C}_D)$, $\operatorname{supp}T^+\subset\mathscr{C}_N$ imply the inclusion $T^+\in\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$.

Theorem 2.4.1. *Problem* (2.4.1)–(2.4.3) *with vanishing Dirichlet condition $T^+(y)=g(y)=0$ for all $y\in\mathscr{C}_D$ is reformulated into the following equivalent variational problem: let f and q satisfy conditions* (2.4.4) *and look for a temperature distribution $T\in\widetilde{\mathbb{H}}^1(\Omega,\mathscr{C}_D)$ (see* (2.4.9)*) which is a stationary point of functional* (2.4.10).

Proof. Let $T(x)$ be a stationary point of functional (2.4.10). Consider the variation

$$\delta\Phi=\frac{d}{d\varepsilon}\,\Phi(T+\varepsilon\boldsymbol{V})|_{\varepsilon=0}=\int\limits_{\Omega}\Big[\big\langle\mathscr{A}(x)\nabla T(x),\nabla\boldsymbol{V}(x)\big\rangle+f(x)\boldsymbol{V}(x)\Big]\,dx+\int\limits_{\mathscr{C}_N}q(y)\boldsymbol{V}^+(y)\,d\sigma.\tag{2.4.11}$$

The trial function $\boldsymbol{V} \in \mathbb{H}^1(\Omega)$ is such that $T + \varepsilon\boldsymbol{V}$ satisfies the boundary conditions. Then from the equalities $T^+(y) + \boldsymbol{V}^+(y) = 0 = T^+(y)$ on $\mathscr{C}_D$ it follows that $T^+(y) = \boldsymbol{V}^+(y) = 0$ on $\mathscr{C}_D$, i.e., T and $\boldsymbol{V}$ have the traces vanishing on the part $\mathscr{C}_D$ of the boundary $\partial\Omega$.

It is clear that for those $\boldsymbol{V}$ for which the functional $\Phi(T + \varepsilon\boldsymbol{V})$ has a stationary point, we have $\delta\Phi = 0$. By applying the Gauß theorem to the first summand under the integral on Ω in (2.4.11), we obtain the associated Euler–Lagrange equation

$$\int\limits_{\Omega} \big[-\mathbf{div}\,\mathscr{A}(x)\nabla T(x) + f(x)\big]\boldsymbol{V}(x)\,dx + \int\limits_{\mathscr{C}_D} \big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle \boldsymbol{V}^+(y)\,d\sigma$$
$$+ \int\limits_{\mathscr{C}_N} \Big[q(y) + \big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle\Big]\boldsymbol{V}^+(y)\,d\sigma = 0. \quad (2.4.12)$$

Since the trial function $\boldsymbol{V}$ vanishes on $\mathscr{C}_D$ (see (2.4.9)), the integral on $\mathscr{C}_D$ in (2.4.12) vanishes. Now, taking arbitrary function $\boldsymbol{V} \in C_0^\infty(\Omega)$ (vanishing in the vicinity of the boundary $\mathscr{C}$), all summands in (2.4.12) except the first one vanish and we obtain

$$\int\limits_{\Omega} \big[-\mathbf{div}\,\mathscr{A}(x)\nabla T(x) + f(x)\big]\boldsymbol{V}(x)\,dx = 0,$$

which is equivalent to the basic differential equation in (2.4.1).

Therefore, it follows from (2.4.12) that

$$\int\limits_{\mathscr{C}_N} \Big[q(y) + \big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle\Big]\boldsymbol{V}^+(y)\,d\sigma = 0, \quad (2.4.13)$$

where the trace $\boldsymbol{V}^+$ of a trial function in (2.4.13) is arbitrary. Thus we derive the boundary condition (2.4.3).

Vice versa: Let T be a solution to the mixed problem (2.4.1)–(2.4.3) with vanishing Dirichlet traces $T^+(y) = g(y) = 0$ on $\mathscr{C}$. Taking the scalar product of the basic equation in (2.4.1) with the solution T, applying the Green formulae and the boundary conditions (2.4.2) with $g = 0$, we get the following equality:

$$0 = \int\limits_{\Omega} \big[-\mathbf{div}\,\mathscr{A}(x)\nabla T(x) + f(x)\big]T(x)\,dx = \int\limits_{\Omega} \big[\mathscr{A}(x)\nabla T(x) + f(x)\big]\nabla T(x)\,dx$$
$$+ \int\limits_{\mathscr{C}_D\cup\mathscr{C}_N} \big\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\big\rangle T^+(y)\,d\sigma = \int\limits_{\Omega} \big[\mathscr{A}(x)\nabla T(x) + f(x)\big]\nabla T(x)\,dx \int\limits_{\mathscr{C}_N} q(y)T^+(y)\,d\sigma.$$

Therefore, T is a stationary point of the functional Φ in (2.4.10). □

If $\mathscr{C}_D = \mathscr{C}$, $\mathscr{C}_N = \varnothing$, problem (2.4.1)–(2.4.3) reduces to the problem with a Dirichlet boundary condition

$$T^+(y) = 0 \ \text{ on } \ \mathscr{C},$$

and the corresponding functional Φ in variational formulation (see (2.4.10)) takes the form

$$\Phi_D(T) = \frac{1}{2}\int\limits_{\Omega} \Big[\big\langle \mathscr{A}(x)\nabla T(x), \nabla T(x)\big\rangle + f(x)T(x)\Big]\,dx.$$

If $\mathscr{C}_D = \varnothing$, $\mathscr{C}_N = \mathscr{C}$, from (2.4.1)–(2.4.3) we get the problem with Neumann boundary condition

$$-\big\langle \mathscr{A}^+(y)\boldsymbol{\nu}(y), (\nabla T)^+(y)\big\rangle = q(y) \ \text{ on } \ \mathscr{C},$$

and the corresponding functional in variational formulation (see (2.4.10)) takes the form

$$\Phi_N(T)=\frac{1}{2}\int\limits_{\Omega}\Big[\big\langle\mathscr{A}(x)\nabla T(x),\nabla T(x)\big\rangle+f(x)T(x)\Big]\,dx+\int\limits_{\mathscr{C}}q(y)T^+(y)\,d\sigma.$$

We conclude this section with some auxiliary results on Lebesgue points of integrable functions which is important in the next section.

Let $B(x)$ be a ball in the Euclidean space $B\subset\mathbb{R}^n$ centered at x. *The derivative of the integral at* x is defined to be

$$\lim_{B(x)\to x}\frac{1}{|B(x)|}\int\limits_{B(x)}f(y)\,\mathrm{d}y, \tag{2.4.14}$$

where $|B(x)|$ denotes the volume (i.e., the Lebesgue measure) of $B(x)$, and $B(x)\to x$ means that the diameter of $B(x)$ tends to 0. Note that

$$\bigg|\frac{1}{|B(x)|}\int\limits_{B(x)}f(y)\,\mathrm{d}y-f(x)\bigg|=\bigg|\frac{1}{|B(x)|}\int\limits_{B(x)}[f(y)-f(x)]\,\mathrm{d}y\bigg|\leqslant\frac{1}{|B(x)|}\int\limits_{B(x)}|f(y)-f(x)|\,\mathrm{d}y. \tag{2.4.15}$$

The points x for which the right-hand side tends to zero are called the *Lebesgue points of* f.

Theorem 2.4.2 (Lebesgue Differentiation Theorem, Lebesgue (1910))**.** *For an integrable function* $f\in\mathbb{L}_1(\Omega)$ *the derivative of integral* (2.4.14) *exists and is equal to* $f(x)$ *at almost every point* $x\in\Omega$.

Moreover, almost every point $x\in\Omega$ *is a Lebesgue point of* f *(see* (2.4.15)*).*

Corollary 2.4.1. *If* $g\in\mathbb{L}_2(\Omega)$, $f\in\mathbb{L}_2(\Omega\times(-1,1))$, *then*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int\limits_{t-\varepsilon}^{t+\varepsilon}\big(g(\,\cdot\,),f(\,\cdot\,,\tau)\big)_{\Omega}\,d\tau=\big(g(\,\cdot\,),f(\,\cdot\,,t)\big)_{\Omega} \tag{2.4.16}$$

for almost all $t\in(-1,1)$.

Proof. It is clear that $g\cdot f\in\mathbb{L}_1(\Omega\times(-1,1))$ and for the function $h(t):=\big(g(\,\cdot\,),f(\,\cdot\,,t)\big)_{\Omega}$ the inclusion $h\in\mathbb{L}_1((-1,1))$ is true. Thence we can apply Theorem 2.4.2 to the function $h(t)$ and get (2.4.16). □

2.5 Heat transfer in thin Layers

Let $\mathscr{C}$ be a C^2-smooth orientable surface in $\mathbb{R}^3$ given by a single chart (immersion)

$$\theta:\omega\to\mathscr{C},\ \ \omega\subset\mathbb{R}^2,$$

and let $\boldsymbol{\nu}(\mathscr{x})$, $\mathscr{x}\in\mathscr{C}$, be the unit normal vector field on $\mathscr{C}$ with the fixed orientation. The chart is supposed to be single just for convenience, and the multi-chart case can be considered similarly. Denote by Ω^ε the layer domain, i.e., the set of all points in $\mathbb{R}^3$ in the distance less than ε from $\mathscr{C}$. Then for sufficiently small ε the map $\Theta:\mathscr{C}\times(-\varepsilon,\varepsilon)\to\Omega^\varepsilon$,

$$\Theta(\mathscr{x},t)=\mathscr{x}+t\boldsymbol{\nu}(\mathscr{x})=\theta(x)+t\boldsymbol{\nu}(\theta(x)),\ \ x\in\omega,$$

is C^1-homeomorphism and $\Theta(\mathscr{C}\times\{0\})=\mathscr{C}$.

As noted above, we can extend unit normal vector field to the entire Ω^ε properly by assuming

$$\boldsymbol{\nu}(\mathscr{x}+t\boldsymbol{\nu}(\mathscr{x}))=\boldsymbol{\nu}(\mathscr{x}),\ \ \mathscr{x}\in\mathscr{C},\ \ -\varepsilon<t<\varepsilon.$$

If ε is sufficiently small, the boundary $\mathscr{M}^\varepsilon:=\partial\Omega^\varepsilon$ is represented as the union of three C^1-smooth surfaces $\mathscr{M}^\varepsilon=\mathscr{M}_{\varepsilon,D}\cup\mathscr{M}^-_{\varepsilon,N}\cup\mathscr{M}^+_{\varepsilon,N}$, where $\mathscr{M}_{\varepsilon,D}=\partial\mathscr{C}\times[-\varepsilon,\varepsilon]$ is the lateral surface, $\mathscr{M}^+_{\varepsilon,N}=\mathscr{C}\times\{+\varepsilon\}$ is the upper surface and $\mathscr{M}^-_{\varepsilon,N}=\mathscr{C}\times\{-\varepsilon\}$ is the lower surface of the boundary $\mathscr{M}^\varepsilon$ of layer domain Ω^ε.

In the present section, we will consider the heat conduction problem by an "isotropic" medium governed by the BVP (cf. (2.1.2) for $\boldsymbol{\Delta}_{\Omega^\varepsilon}$)

$$\begin{aligned}\boldsymbol{\Delta}_{\Omega^\varepsilon} T(\mathscr{X},t) &= f(\mathscr{X},t), \quad (\mathscr{X},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon),\\ T^+(\mathscr{X},t) &= 0, \quad (\mathscr{X},t)\in\partial\mathscr{C}\times(-\varepsilon,\varepsilon),\\ (\partial_t T)^+(\mathscr{X},\pm\varepsilon) &= q_\varepsilon^\pm(\mathscr{X}), \quad \mathscr{X}\in\mathscr{C}.\end{aligned} \tag{2.5.1}$$

The case of an "anisotropic" medium will be treated in a forthcoming publication.

We impose the following constraints

$$\begin{gathered} T\in\mathbb{H}^1(\Omega^\varepsilon), \quad q_\varepsilon^\pm\in\widetilde{\mathbb{H}}^2(\mathscr{C}), \quad f\in\mathbb{L}_2(\Omega^1),\\ 0 \text{ is the Lebesgue point for the function } \widetilde{f}(t) := \int_{\mathscr{C}} |f(\mathscr{X},t)|^2\, d\sigma \end{gathered} \tag{2.5.2}$$

(see (2.4.15) and note that $\|\widetilde{f}\mid\mathbb{L}_1(-1,1)\|\leqslant\|f\mid\mathbb{L}_2(\Omega^1)\|^2$). The latter constraint implies that $\widetilde{f}(0)$ exists and, due to Theorem 2.4.2,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\widetilde{f}(t)\,dt = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{\mathscr{C}}|f(\mathscr{X},t)|^2\,d\sigma\,dt = \widetilde{f}(0).$$

The formulated BVP (2.5.1) governs a heat transfer in the body Ω^ε when there are thermal sources or sinks in Ω^ε. The temperature on the lateral surface $\partial\mathscr{C}\times(-\varepsilon,\varepsilon)$ is zero, the heat fluxes are fixed on the upper and lower surfaces $\mathscr{C}^\pm := \mathscr{C}\times\{\pm\varepsilon)$. It is well known that the boundary value problem (2.5.1), as well as its equivalent problem (2.1.4)–(2.1.6), has a unique solution $T\in\mathbb{H}^1(\Omega^\varepsilon)$ (respectively, $T_0\in\mathbb{H}^1(\Omega^\varepsilon)$; see, e.g., [70]).

The energy functional associated with problem (2.5.1) reads (cf. Theorem 2.4.1):

$$E(T_\varepsilon) := \int_{-\varepsilon}^{\varepsilon}\int_{\mathscr{C}}\Big[\frac{1}{2}\left(\mathscr{D}_{\mathscr{C}}T^2(\mathscr{X},\tau)+\frac{1}{2\varepsilon^2}\left(\partial_\tau T^2(\mathscr{X},\tau)\right)+F(\mathscr{X},\tau)T_\varepsilon(\mathscr{X},\tau)\Big]\,d\sigma\,d\tau, \tag{2.5.3}$$

$$\begin{aligned} F(\mathscr{X},t) := f(\mathscr{X},t) &- \frac{1}{4\varepsilon}\left((t+\varepsilon)^2\boldsymbol{\Delta}_{\mathscr{C}}q_\varepsilon^+(\mathscr{X})-(t-\varepsilon)^2\boldsymbol{\Delta}_{\mathscr{C}}q_\varepsilon^-(\mathscr{X})\right)\\ &- \frac{\mathscr{H}_{\mathscr{C}}^0}{2\varepsilon}\left((t+\varepsilon)q_\varepsilon^+(\mathscr{X})-(t-\varepsilon)q_\varepsilon^-(\mathscr{X})\right)\\ &- \frac{1}{2\varepsilon}\left(q_\varepsilon^+(\mathscr{X})-q_\varepsilon^-(\mathscr{X})\right), \quad (\mathscr{X},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon).\end{aligned} \tag{2.5.4}$$

More generally, we consider the non-linear functional

$$E_\varepsilon(T) = \int_{\Omega^\varepsilon}\left[\mathscr{K}_0(\nabla_{\Omega^\varepsilon}T(x),\,T(x))+F_\varepsilon(x)T(x)\right]dx, \tag{2.5.5}$$

where $\mathscr{K}_0(\nabla_{\Omega^\varepsilon}T,\,T)$ is strictly convex and has quadratic estimate. In the case of functional (2.5.3),

$$\mathscr{K}_0(\nabla_{\Omega^\varepsilon}T,\,T) = \frac{1}{2}\left\langle\mathscr{D}_{\Omega^\varepsilon}T,\mathscr{D}_{\Omega^\varepsilon}T\right\rangle = \frac{1}{2}\left(\mathscr{D}_{\Omega^\varepsilon}T\right)^2 = \frac{1}{2}\left(\mathscr{D}_{\mathscr{C}}T_\varepsilon\right)^2(\mathscr{X},\tau)+\frac{1}{2\varepsilon^2}\left(\partial_\tau T_\varepsilon\right)^2(\mathscr{X},\tau), \tag{2.5.6}$$

and it is clear that the kernel is strictly convex because the quadratic function $F(x)=x^2$ is strictly convex: $[\theta x_1+(1-\theta)x_2]^2<\theta x_1^2+(1-\theta)x_2^2$ for all $x_1,x_2\in\mathbb{R}$, $x_1\neq x_2$, $0<\theta<1$. The kernel has a trivial quadratic estimate, since it is a quadratic function.

A nice proof of the next Lemma 1.7.5 is exposed in [2, Example 3.6].

Lemma 2.5.1. *Let Ω be a domain in $\mathbb{R}^n$ with the Lipshitz boundary $\mathscr{M}:=\partial\Omega$ and $\mathscr{M}_0\subset\mathscr{M}$ be a subsurface of non-zero measure. Then the inequality*

$$\big\|\varphi\mid\mathbb{L}_2(\Omega)\big\|\leqslant C\big\|\nabla\varphi\mid\mathbb{L}_2(\Omega)\big\| = C\Big[\sum_{j=1}^n\big\|\partial_j\varphi\mid\mathbb{L}_2(\Omega)\big\|^2\Big]^{1/2} \tag{2.5.7}$$

holds for all functions $\varphi\in\widetilde{\mathbb{H}}^1(\Omega,\mathscr{M}_0)$ and the constant C is independent of φ.

Now we perform the scaling of the variable $t=\varepsilon\tau$, $-1<\tau<1$, and study the following functionals in the scaled domain $\Omega^1=\mathscr{C}\times(-1,1)$:

$$E_\varepsilon(T_\varepsilon)=\int\limits_{-1}^{1}\int\limits_{\mathscr{C}}\Big[\mathscr{K}_0\Big(\mathscr{D}_{\mathscr{C}}T_\varepsilon,\frac{1}{\varepsilon}\,\partial_t T_\varepsilon,T_\varepsilon\Big)+F_\varepsilon T_\varepsilon\Big]\,d\sigma\,d\tau, \tag{2.5.8}$$

where $\mathscr{D}_{\mathscr{C}}=(\mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3)$, $\mathscr{D}_4=\partial_t$. The functionals $E_\varepsilon(T_\varepsilon)$ are related to the original functional $E(T)$ by the equality

$$E_\varepsilon(T_\varepsilon)=\frac{1}{\varepsilon}\,E(T),\ \text{ where }\ T_\varepsilon(\mathscr{x},t)=T(\mathscr{x}_1,\mathscr{x}_2,\mathscr{x}_3,\varepsilon t), \tag{2.5.9}$$

$$\begin{aligned}F_\varepsilon(\mathscr{x},t)=F(\mathscr{x},\varepsilon t)&=f(\mathscr{x},\varepsilon t)-\frac{\varepsilon}{4}\Big((t+1)^2\boldsymbol{\Delta}_{\mathscr{C}}q_\varepsilon^+(\mathscr{x})-\frac{\varepsilon}{4}\,(t-1)^2\boldsymbol{\Delta}_{\mathscr{C}}q_\varepsilon^-(\mathscr{x})\Big)\\&-\frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2}\,\big((t+1)q_\varepsilon^+(\mathscr{x})-(t-1)q_\varepsilon^-(\mathscr{x})\big)\\&-\frac{1}{2\varepsilon}\,\big(q_\varepsilon^+(\mathscr{x})-q_\varepsilon^-(\mathscr{x})\big),\ \ (\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon).\end{aligned} \tag{2.5.10}$$

Lemma 2.5.2. *Let F_ε be uniformly bounded in $\mathbb{L}_2(\Omega^1)$:*

$$\sup_{\varepsilon<\varepsilon_0}\|F_\varepsilon|\mathbb{L}_2(\Omega^1)\|<\infty. \tag{2.5.11}$$

Then the energy functional $E_\varepsilon(T)$ in (2.5.8) is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1,\partial\mathscr{C}\times(-1,1))$, is strictly convex and has the following quadratic estimate:

$$\begin{gathered}E_\varepsilon(\theta T_1+(1-\theta)T_2)<\theta E_\varepsilon(T_1)+(1-\theta)E_\varepsilon(T_2),\ \ 0<\theta<1,\\ C_1\int\limits_{\Omega^1}\mathscr{K}_0\Big(\mathscr{D}_{\mathscr{C}}T,\,\frac{1}{\varepsilon}\,\partial_t T,T\Big)\,d\sigma\,dt-C_2\leqslant E_\varepsilon(T)\\ \leqslant C_3\Big[1+\int\limits_{\Omega^1}\mathscr{K}_0\Big(\mathscr{D}_{\mathscr{C}}T,\frac{1}{\varepsilon}\,\partial_t T,T\Big)\,d\sigma\,dt\Big],\ \ \forall\,T_1,T_2,T\in\widetilde{\mathbb{H}}^1(\Omega^1,\partial\mathscr{C}\times(-1,1))\end{gathered} \tag{2.5.12}$$

for some positive constants C_1, C_2 and C_3 not depending on ε.

Proof. Let us decompose the functional $E_\varepsilon(T)$ in (2.5.8) into the sum of non-linear and linear parts

$$\begin{aligned}E_\varepsilon(T)\ &=E^{(1)}_\varepsilon(T)+E^{(2)}_\varepsilon(T),\\ E^{(1)}_\varepsilon(T)&:=\int\limits_{\Omega^1}\mathscr{K}_0\Big(\mathscr{D}_{\mathscr{C}}T,\frac{1}{\varepsilon}\,\partial_t T,T\Big)\,dx,\\ E^{(2)}_\varepsilon(T)&:=\int\limits_{\Omega^1}F_\varepsilon(x)T(x)\,dx.\end{aligned} \tag{2.5.13}$$

By the conditions imposed on $\mathscr{K}_0$ in (2.5.5), the first (non-linear) functional $E^{(1)}_\varepsilon(T)$ is strictly convex and has a quadratic estimate:

$$\begin{aligned}C^0_1\int\limits_{\Omega^1}\Big(\langle\mathscr{D}_{\mathscr{C}}T_j,\mathscr{D}_{\mathscr{C}}T_j\rangle+\frac{1}{\varepsilon_j^2}\,|\partial_t T_j|^2\Big)\,dx-C^0_2\leqslant E^{(1)}_\varepsilon(T)\\ \leqslant C^0_3\Big[1+\int\limits_{\Omega^1}\Big(\langle\mathscr{D}_{\mathscr{C}}T_j,\mathscr{D}_{\mathscr{C}}T_j\rangle+\frac{1}{\varepsilon_j^2}\,|\partial_t T_j|^2\Big)\,dx\Big].\end{aligned} \tag{2.5.14}$$

On the other hand, $E_\varepsilon^{(2)}(T)$ is linear and, therefore, strictly convex (see the first inequality in (2.5.12)). Thus, we only have to prove the two-sided quadratic estimate in (2.5.12) for the linear functional $E_\varepsilon^{(2)}(T)$. Due to Lemma 2.5.1 and equality (2.2.12), we can write

$$|E_\varepsilon^{(2)}(T)| \leqslant \Big|\int\limits_{\Omega^1} F_\varepsilon(x)T(x)\,dx\Big| \leqslant \big\|F_\varepsilon \mid \mathbb{L}_2(\Omega^1)\big\|\,\big\|T \mid \mathbb{L}_2(\Omega^1)\big\|$$

$$\leqslant M\big\|\nabla T \mid \mathbb{L}_2(\Omega^1)\big\| \leqslant M\Big(\frac{1}{\eta} + \eta\big\|\nabla T \mid \mathbb{L}_2(\Omega^1)\big\|^2\Big) \leqslant M\Big(\frac{1}{\eta} + \eta\big\|\mathscr{D}_{\Omega^1}T \mid \mathbb{L}_2(\Omega^1)\big\|^2\Big). \quad (2.5.15)$$

Choosing $\eta = 1$ in (2.5.15) and taking into account (2.5.14) we get the right inequality in the second line of (2.5.12), whereas taking η sufficiently small we obtain

$$E_\varepsilon(T) \geqslant |E_\varepsilon^{(1)}(T)| - |E_\varepsilon^{(2)}(T)| \geqslant C_1\big\|\mathscr{D}_{\Omega^\varepsilon}T \mid \mathbb{L}_2(\Omega^\varepsilon)\big\|^2 - C_2. \qquad \square$$

Let $F_j = F_{\varepsilon_j}$, $0 < \varepsilon_j \le 1$, $\lim\limits_{j\to\infty} \varepsilon_j = 0$ and F_{ε_j} be uniformly bounded (see (2.5.11)). Further, let $T_j = T_{\varepsilon_j} \in \widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1))$, $j = 1, 2, \dots$, be the sequence of functions with "finite energy":

$$\sup_j E_{\varepsilon_j}(T_j) < +\infty. \quad (2.5.16)$$

Then from (2.5.14)–(2.5.15) we get

$$C_1^0\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\|^2$$

$$= \int\limits_{\Omega^1} \Big(\frac{1}{2}\,\langle \mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{2\varepsilon_j^2}\,|\partial_t T_j|^2\Big)\,dx = C_1^0 E_{\varepsilon_j}(T_j) - C_1^0\int\limits_{\Omega^1} F_j(\mathscr{X}, t)T_j(\mathscr{X}, t)\,d\sigma\,dt$$

$$\leqslant C_2^0\Big(1 + \big\|F_j \mid \mathbb{L}_2(\Omega^1)\big\|\,\big\|T_j \mid \mathbb{L}_2(\Omega^1)\big\|\Big) \leqslant C_3^0\Big(1 + \big\|\mathscr{D}_{\Omega^1}T_j \mid \mathbb{L}_2(\Omega^\varepsilon)\big\|^2\Big)^{\frac{1}{2}}, \quad (2.5.17)$$

since, due to Lemma 2.5.1,

$$\big\|T_j \mid \mathbb{L}_2(\Omega^1)\big\| \leqslant C_0\big\|\mathscr{D}_{\Omega^1}T_j \mid \mathbb{L}_2(\Omega^1)\big\|. \quad (2.5.18)$$

Consequently,

$$\sup_j \big\|\mathscr{D}_{\Omega^1}T_j \mid \mathbb{L}_2(\Omega^1)\big\| = \sup_j \Bigg(\int\limits_{\Omega^1} \Big(\frac{1}{2}\,\langle \mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{2\varepsilon_j^2}\,|\partial_t T_j|^2\Big)\,dx\Bigg)^{1/2} < +\infty. \quad (2.5.19)$$

From (2.5.17)–(2.5.19) it follows

$$\sup_j \int\limits_{\Omega^1} |T_j|^2\,dx < \infty, \quad \sup_j \int\limits_{\Omega^1} |\mathscr{D}_{\mathscr{C}}T_j|^2\,dx < \infty, \quad \sup_j \frac{1}{\varepsilon_j^2}\int\limits_{\Omega^1} |\partial_t T_j|^2\,dx < \infty. \quad (2.5.20)$$

Note that if T_j are the scaled solutions to problem (2.1.1), then from the Euler–Lagrange equation associated with the functional (see (2.4.12)) it follows that $E_{\varepsilon_j}(T_j) = 0$ and, therefore, conditions (2.5.20) are satisfied.

Due to (2.5.20), the sequence $\{T_j\}_{j=1}^\infty$ is uniformly bounded in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1))$ and a weakly converging subsequence (say $\{T_j\}_{j=1}^\infty$ itself) to a function T in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1))$ can be extracted.

The functional

$$H(T) = \int\limits_{\Omega^1} |\partial_t T|^2\,dx$$

is convex and continuous in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1))$; then it is weakly lower semi-continuous and $\partial_t T = 0$ a.e., since

$$\int\limits_{\Omega^1} |\partial_t T|^2\,dx = H(T) \le \liminf_j H(T_j) = \liminf_j \int\limits_{\Omega^1} |\partial_t T_j|^2\,dx = 0$$

(see the last inequality in (2.5.20)). Hence $T(\mathscr{X},t)$ is independent of t, i.e.,

$$T(\mathscr{X},t)=T(\mathscr{X}),\quad \mathscr{X}\in\mathscr{C},\quad -1\le t\le 1. \tag{2.5.21}$$

Let the following conditions are fulfilled

$$f_\varepsilon(\mathscr{X},t):=f(\mathscr{X},\varepsilon t)\underset{\varepsilon\to 0}{\longrightarrow} f^0(\mathscr{X})\ \text{ in }\ \mathbb{L}_2(\Omega^1), \tag{2.5.22}$$

$q^\pm_\varepsilon\in\mathbb{H}^2(\mathscr{C})$ are uniformly bounded (with respect to ε) in $\mathbb{H}^2(\mathscr{C})$, and

$$\lim_{\varepsilon\to 0} q^+_\varepsilon=\lim_{\varepsilon\to 0} q^-_\varepsilon=q_0\ \text{ in }\ \mathbb{L}_2(\mathscr{C}) \tag{2.5.23}$$

and

$$\frac{1}{2\varepsilon}\,(q^+_\varepsilon-q^-_\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow} q_1\ \text{ in }\ \mathbb{L}_2(\mathscr{C}). \tag{2.5.24}$$

From (2.5.22)–(2.5.24) it follows, in particular, that

$$F_j(\mathscr{X},t)\to F(\mathscr{X},0)\ \text{ in }\ \mathbb{L}_2(\Omega^1). \tag{2.5.25}$$

Set

$$E^{(0)}(T)=\begin{cases} E^{(1)}(T)+E^{(2)}(T) & \text{for } T\in\mathscr{P}(\mathscr{C}),\\ +\infty, & \text{for } T\notin\mathscr{P}(\mathscr{C}),\end{cases} \tag{2.5.26}$$

where $\mathscr{P}(\mathscr{C})$ is defined in (2.1.10), and

$$E^{(1)}(T):=\frac12\int\limits_{\Omega^1}\big\langle(\mathscr{D}_{\Omega^1}T)(x,t),(\mathscr{D}_{\Omega^1}T)(x,t)\big\rangle\,d\sigma\,dt=\int\limits_{\mathscr{C}}\big\langle(\mathscr{D}_{\mathscr{C}}T_{\mathscr{C}})(\mathscr{X}),(\mathscr{D}_{\mathscr{C}}T_{\mathscr{C}})(\mathscr{X})\big\rangle\,d\sigma, \tag{2.5.27}$$

$$E^{(2)}(T):=\int\limits_{\Omega^1}F(\mathscr{X},0)T(\mathscr{X},t)\,d\sigma\,dt=2\int\limits_{\mathscr{C}}\big(f^0(\mathscr{X})-\mathscr{H}^0_{\mathscr{C}}q_0(\mathscr{X})-q_1(\mathscr{X})\big)T_{\mathscr{C}}(\mathscr{X})\,d\sigma. \tag{2.5.28}$$

Let us check that the sequence E_{ε_j} Γ-converges to $E^{(0)}$ in $\widetilde{\mathbb{H}}^1(\Omega^\varepsilon,\partial\mathscr{C}\times(-1,1))$. Indeed, we have

$$E_{\varepsilon_j}(T_j)=E^{(1)}_{\varepsilon_j}(T_j)+E^{(2)}_{\varepsilon_j}(T_j),$$

where

$$E^{(1)}_{\varepsilon_j}(T_j)=\int\limits_{\Omega^1}\Big(\frac12\,\langle\mathscr{D}_{\mathscr{C}}T_j,\mathscr{D}_{\mathscr{C}}T_j\rangle+\frac{1}{2\varepsilon_j^2}\,|\partial_tT_j|^2\Big)\,dx,\quad E^{(2)}_{\varepsilon_j}(T_j)=\int\limits_{\Omega^1}F_jT_j\,dx.$$

The functional $E^{(1)}(T)$ is convex and continuous and so it is weakly lower semicontinuous in $\widetilde{\mathbb{H}}^1(\Omega^\varepsilon,\partial\mathscr{C}\times(-1,1))$, therefore,

$$\liminf_j E^{(1)}_{\varepsilon_j}(T_j)\geqslant\liminf_j E^{(1)}(T_j)\geqslant E^{(1)}(T).$$

The sequence $E^{(2)}_{\varepsilon_j}(T_j)$ converges to $E^{(2)}(T)$, since $F_j(\mathscr{X},t)\to F(\mathscr{X},0)$ and $T_j\rightharpoonup T$ in $\mathbb{L}_2(\Omega^1)$. Consequently,

$$\liminf_j E_{\varepsilon_j}(T_j)\geqslant E^{(0)}(T).$$

This proves $\liminf$ inequality for the sequence E_{ε_j}.

Note that

$$E^{(2)}(T)=\int\limits_{\mathscr{C}}\int\limits_{-1}^{1}F(\mathscr{X},0)T(\mathscr{X},t)\,dt\,d\sigma=2\int\limits_{\mathscr{C}}F(\mathscr{X},0)T_{\mathscr{C}}(\mathscr{X})\,d\sigma.$$

To show that the lower bound is reached, i.e., to build a recovery sequence T_j, we fix $T_{\mathscr{C}}\in\mathbb{H}^1(\mathscr{C})$ and set $T(\mathscr{X},t)=T_{\mathscr{C}}(\mathscr{X})$, $\mathscr{X}\in\mathscr{C}$, $t\in(-1,1)$. Define recovery sequence as $T_j(x,t)=T(x,t)=T_{\mathscr{C}}(x)$. Then $\partial_tT_j=\partial_tT=0$ and

$$\lim_{j\to\infty}E_{\varepsilon_j}(T_j)=\lim_{j\to\infty}E^{(1)}_{\varepsilon_j}(T)+\lim_{j\to\infty}E^{(2)}_{\varepsilon_j}(T)=E^{(1)}(T)+E^{(2)}(T)=E^{(0)}(T).$$

We have proved the following result.

Theorem 2.5.1. *If conditions* (2.5.22)–(2.5.24) *are fulfilled, then the functional in* (2.5.8) Γ*-converges to the functional* $E^{(0)}(T)$ *defined in* (2.1.11) *as* $\varepsilon \to 0$*.*

Now we are able to prove the main Theorem 2.1.1 formulated in the introduction.

Proof of Theorem 2.1.1. The first part of the theorem, i.e., Γ-convergence of functional (2.1.11) to functional $E^{(0)}$ defined by (2.1.11), is proved in Theorem 2.5.1.

The concluding assertion that the BVP (2.1.12) is an equivalent reformulation of the minimization problem with the energy functional (2.1.11) is explained in Theorem 2.4.1. □

Chapter 3

Shell equations in terms of Günter's derivatives, derived by the Γ-convergence

In the present chapter, we expose results on mixed boundary value problems for the Lamé equation in a thin layer $\Omega^\varepsilon = \mathscr{C} \times (-\varepsilon, \varepsilon)$ around a surface $\mathscr{C}$ with the Lipshitz boundary (cf. (2.2.1)). The main objective is to find out what happens in Γ-limit when the thickness of the layer converges to zero. The limit BVP for the Lamé equation on the surface is derived in explicit form in terms of Günter's derivatives (see (0.0.9)) and it is shown how the Neumann boundary condition in the initial BVP on the upper and lower surfaces wanders into the right-hand side of the equation in the Γ-limit. For this, we apply the variational formulation and the calculus of Günter's tangent differential operators on a hypersurface and layers, which allow global representation of basic differential operators and of corresponding boundary value problems in terms of the curvilinear coordinates on the surface $\mathscr{C}$.

3.1 Introduction

Let $\mathscr{C} \subset \mathbb{R}^3$ be an open surface with the boundary $\Gamma = \partial\mathscr{C}$ in the Euclidean space $\mathbb{R}^3$, represented by a single coordinate function

$$\theta : \omega \to \mathscr{C}, \tag{3.1.1}$$

where ω is open simple connected domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$. Let $\zeta : \mathscr{S} \to \omega$ be the inverse mapping

$$\zeta : \mathscr{S} \to \omega, \quad \theta \circ \zeta = Id : \mathscr{S} \to \mathscr{S}, \quad \zeta \circ \theta = Id : \omega \to \omega \tag{3.1.2}$$

(the case of multiple coordinate function is similar and we skip this case for simplicity).

Denote by $\boldsymbol{\nu}(\mathscr{X}) = (\nu_1(\mathscr{X}), \nu_2(\mathscr{X}), \nu_3(\mathscr{X}))^\top$, $\mathscr{X} \in \mathscr{C}$, the normal vector field on $\mathscr{C}$ and let $\mathscr{N}(x) = (\mathscr{N}_1(x), \mathscr{N}_2(x), \mathscr{N}_3(x))^\top$ be its extension in the neighbourhood $U_\mathscr{C}$ of the surface $\mathscr{C}$. It is well known that such an extension is unique under some natural constraints (see [66] for details).

The equations of three-dimensional linearized elasticity have been extensively studied, but mostly in Cartesian coordinates. The linear shell theories justified in this Chapter from three-dimensional elasticity require, however, that these equations are recorded rather in terms of curvilinear coordinates that "follow the geometry" of the shell in a most natural way. Accordingly, the purpose of this preliminary section is to provide a thorough derivation and a mathematical treatment of the equations of linearized three dimensional elasticity in terms of special curvilinear coordinates.

The 3-tuple of tangent vector fields to the surface $\boldsymbol{g}_1 := \partial_1\Theta$, $\boldsymbol{g}_2 := \partial_2\Theta$ (the covariant metric tensor) and the proper extension $\boldsymbol{g}_3 := \mathscr{N}$ of normal vector field $\boldsymbol{\nu}$ from the surface $\mathscr{C}$ into the neighborhood Ω^h depends only on the variable $x' \in \mathscr{C}$ and constitutes a basis in Ω^h. That means that an arbitrary vector field $\boldsymbol{U} = \sum\limits_{j=1}^{3} U_j \boldsymbol{e}^j$ can also be represented with this basis in "curvilinear coordinates".

Along with the covariant metric tensor it is used the contravariant metric tensor $\boldsymbol{g}^j,\dots,\boldsymbol{g}^{n-1}$ which is a bi-orthogonal system to the system of covariant metric tensors $\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}$, where δ_{jk} denotes the Kroneker's symbol, $j,k=1,2$ (see, e.g., [22, 23]). For example, using the Christofells symbols $\Gamma^i_{jk} := \langle \boldsymbol{g}^i, \partial_j \boldsymbol{g}_k\rangle$, covariant derivatives are defined $v_{i\|j} := \partial_j v_i - \sum\limits_{k=1}^{2} \Gamma^k_{ij} v_k$.

Consider the problem of deformation of an isotropic layer domain $\Omega^h := \mathscr{C} \times (-h,h)$ of thickness $2h$ around the mid-surface $\mathscr{C}$ which has the nonempty Lipschitz boundary $\partial\mathscr{C}$. The deformation is governed by the Lamé equation, with the classical mixed boundary conditions, the Dirichlet conditions on the lateral surface $\Gamma^h_L := \partial\mathscr{C} \times (-h,h)$ and the Neumann conditions on the upper and lower surfaces $\Gamma^\pm := \mathscr{C} \times \{\pm h\}$:

$$\begin{aligned} \mathscr{L}_{\Omega^h}\boldsymbol{U}(x) &= \boldsymbol{F}(x), \ \ x \in \Omega^h := \mathscr{C} \times (-h,h),\\ \boldsymbol{U}^+(t) &= \boldsymbol{G}(t), \ \ t \in \Gamma^h_L := \partial\mathscr{C} \times (-h,h),\\ (\mathfrak{T}(\mathscr{x},\nabla)\boldsymbol{U})^+(\mathscr{x}) &= \boldsymbol{H}(\mathscr{x},\pm h), \ \ (\mathscr{x},t) \in \Gamma^\pm \ = \mathscr{C} \times \{\pm h\}. \end{aligned} \tag{3.1.3}$$

Here $\boldsymbol{U}(x) = (U_1(x),U_2(x),U_3(x))^\top$ is the displacement vector, $\mathscr{L}_{\Omega^h}$ is the Lamé differential operator and $(\mathfrak{T}(\mathscr{x},\nabla)$ is the traction operator

$$\begin{aligned} \mathscr{L}_{\Omega^h}\boldsymbol{U} &= -\mu\,\boldsymbol{\Delta U} - (\lambda+\mu)\,\nabla\,\mathbf{div}\,\boldsymbol{U},\\ [\mathfrak{T}(\mathscr{x},\nabla)\boldsymbol{U}]_j &= \lambda\,\nu_j\partial_k\boldsymbol{U}_k + \mu\,\nu_k\partial_j\boldsymbol{U}_k + \mu\,\partial_{\boldsymbol{\nu}}\boldsymbol{U}_j, \ \ j=1,2,3. \end{aligned} \tag{3.1.4}$$

We consider the BVP (3.1.3) in the following weak classical setting:

$$\boldsymbol{U} \in \mathbb{H}^1(\Omega^h), \ \ \boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\Omega^h), \ \ \boldsymbol{G} \in \mathbb{H}^{\frac{1}{2}}(\Gamma^h_L), \ \ \boldsymbol{H}(\,\cdot\,,\pm h) \in \mathbb{H}^{-\frac{1}{2}}(\mathscr{C}). \tag{3.1.5}$$

For definitions of Bessel potential spaces $\mathbb{H}^s$, $\widetilde{\mathbb{H}}^s$, see, e.g., [133].

Let us consider the following subspace of $\mathbb{H}^1(\Omega^h)$:

$$\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L) := \Big\{\boldsymbol{V} \in \mathbb{H}^1(\Omega^h): \ \ \boldsymbol{V}^+(t) = 0 \ \text{ for all } \ t \in \Gamma^h_L\Big\}. \tag{3.1.6}$$

Theorem 3.1.1. *The BVP* (3.1.3) *in the weak classical setting* (3.1.5) *has a unique solution.*

Proof. Since the Lamé operator $\mathscr{L}_{\Omega^h}$ is strictly positive on the subspace $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$,

$$\langle \mathscr{L}_{\Omega^h}\boldsymbol{V},\boldsymbol{V}\rangle \geqslant M\|\boldsymbol{V}\|^2, \ \ \forall\,\boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L), \tag{3.1.7}$$

the proof easily follows from the Lax–Milgram Lemma (see, e.g., [70] for similar proofs). □

3.2 Lamé operator in curvilinear coordinates

In the present section, we use the notation from Section 2.2 and will represent Lamé and traction operators in curvilinear coordinate system introduced in Section 2.2.

Lemma 3.2.1. *A matrix-operator* $\boldsymbol{A} = [\boldsymbol{A}_{jk}]_{3\times 3}$ *written in curvilinear coordinates* (2.2.3)–(2.2.6) *acquires the form*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \langle \boldsymbol{A}_{1,\,\cdot\,},\boldsymbol{\nu}\rangle\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} & \langle \boldsymbol{A}_{2,\,\cdot\,},\boldsymbol{\nu}\rangle\\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & \langle \boldsymbol{A}_{3,\,\cdot\,},\boldsymbol{\nu}\rangle\\ \langle \boldsymbol{A}_{\,\cdot\,,1},\boldsymbol{\nu}\rangle & \langle \boldsymbol{A}_{\,\cdot\,,2},\boldsymbol{\nu}\rangle & \langle \boldsymbol{A}_{\,\cdot\,,3},\boldsymbol{\nu}\rangle & \langle \boldsymbol{A\nu},\boldsymbol{\nu}\rangle \end{bmatrix}, \tag{3.2.1}$$

$$\boldsymbol{A}_{j,\,\cdot} := (\boldsymbol{A}_{j,1},\boldsymbol{A}_{j,2},\boldsymbol{A}_{j,3})^\top, \ \ \boldsymbol{A}_{\,\cdot\,,j} := (\boldsymbol{A}_{1,j},\boldsymbol{A}_{2,j},\boldsymbol{A}_{3,j})^\top, \ \ j=1,2,3.$$

Proof. Indeed, take the vector-function $\boldsymbol{U}=(U_1,U_2,U_3)$ and proceed as follows:

$$\begin{aligned}\boldsymbol{A}\boldsymbol{U}=\sum_{j,k=1}^{3}\boldsymbol{A}_{jk}U_k\boldsymbol{e}^j&=\sum_{j,k=1}^{3}\boldsymbol{A}_{jk}(U_k^0+\nu_kU_4^0)(\boldsymbol{d}^j+\nu_j\boldsymbol{d}^4)\\&=\sum_{j,k=1}^{3}\boldsymbol{A}_{jk}U_k^0\boldsymbol{d}^j+\sum_{j,k=1}^{3}\boldsymbol{A}_{jk}\nu_jU_k^0\boldsymbol{d}^4+\sum_{j,k=1}^{3}\boldsymbol{A}_{jk}\nu_kU_4^0\boldsymbol{d}^j+\sum_{j,k=1}^{3}\boldsymbol{A}_{jk}\nu_j\nu_kU_4^0\boldsymbol{d}^4\\&=\begin{bmatrix}\boldsymbol{A}_{11}&\boldsymbol{A}_{12}&\boldsymbol{A}_{13}&\langle\boldsymbol{A}_{1,\,\cdot\,},\boldsymbol{\nu}\rangle\\\boldsymbol{A}_{21}&\boldsymbol{A}_{22}&\boldsymbol{A}_{23}&\langle\boldsymbol{A}_{2,\,\cdot\,},\boldsymbol{\nu}\rangle\\\boldsymbol{A}_{31}&\boldsymbol{A}_{32}&\boldsymbol{A}_{33}&\langle\boldsymbol{A}_{3,\,\cdot\,},\boldsymbol{\nu}\rangle\\\langle\boldsymbol{A}_{\,\cdot\,,1},\boldsymbol{\nu}\rangle&\langle\boldsymbol{A}_{\,\cdot\,,2},\boldsymbol{\nu}\rangle&\langle\boldsymbol{A}_{\,\cdot\,,3},\boldsymbol{\nu}\rangle&\langle\boldsymbol{A}\boldsymbol{\nu},\boldsymbol{\nu}\rangle\end{bmatrix}\begin{bmatrix}U_1^0\\U_2^0\\U_3^0\\U_4^0\end{bmatrix}\end{aligned}$$

and (3.2.1) is proved. □

The Lamé operator

$$\begin{aligned}\mathscr{L}\boldsymbol{U}=-\mu\,\boldsymbol{\Delta}\boldsymbol{U}-(\lambda+\mu)\,\nabla\,\mathbf{div}\,\boldsymbol{U}&=-\big[\mu\delta_{jk}\partial_k^2+(\lambda+\mu)\partial_j\partial_k\big]_{3\times3}\boldsymbol{U}\\&=\Big[-\sum_{k,\ell=1}^{3}c_{ijk\ell}\partial_j\partial_\ell\Big]_{3\times3}\boldsymbol{U},\quad c_{ijk\ell}=\lambda\delta_{ij}\delta_{k\ell}+\mu(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\end{aligned}\tag{3.2.2}$$

is formally self-adjoint differential operator of the second order and, written in the full system (2.2.3), acquires the form

$$\mathscr{L}_{\Omega^h}\boldsymbol{U}^0=-\mu\,\boldsymbol{\Delta}_{\Omega^h}\boldsymbol{U}^0-(\lambda+\mu)\,\nabla_{\Omega^h}\,\mathbf{div}_{\Omega^h}\,\boldsymbol{U}^0.\tag{3.2.3}$$

To reformulate the BVP (3.1.3) in curvilinear coordinates we also need to represent the traction operator (cf. (3.1.4))

$$\begin{aligned}\mathfrak{T}(x,\partial)\boldsymbol{U}&=\sum_{j,k=1}^{3}\big(\mathfrak{T}_{jk}(x,\partial)\boldsymbol{U}_k\big)e^j\\&=\sum_{j,k=1}^{3}\Big(\big\{\lambda\nu_j\partial_k+\mu\nu_k\partial_j+\delta_{kj}\mu\,\partial_{\boldsymbol{\nu}}\big\}U_k\Big)\boldsymbol{e}^j,\quad\boldsymbol{U}=(U_1,U_2,U_3)^\top=\sum_{j=1}^{3}U_k\boldsymbol{e}^k\end{aligned}$$

in Günter's derivatives:

$$\begin{aligned}\mathfrak{T}(\mathscr{X},\mathscr{D})&=\sum_{j,k=1}^{3}\boldsymbol{e}^j\otimes\boldsymbol{e}^k\big\{\lambda\nu_j\partial_k+\mu\nu_k\partial_j+\delta_{kj}\mu\,\partial_{\boldsymbol{\nu}}\big\}\\&=\lambda\sum_{k=1}^{3}\boldsymbol{d}^4\otimes(\boldsymbol{d}^k+\nu_k\boldsymbol{d}^4)(\mathscr{D}_k+\nu_k\mathscr{D}_4)+\mu\sum_{k=1}^{3}(\boldsymbol{d}^k+\nu_k\boldsymbol{d}^4)\otimes(\boldsymbol{d}^k+\nu_k\boldsymbol{d}^4)\mathscr{D}_4\\&\quad+\mu\sum_{j=1}^{3}(\boldsymbol{d}^j+\nu_j\boldsymbol{d}^4)\otimes\boldsymbol{d}^4(\mathscr{D}_j+\nu_j\mathscr{D}_4)=\begin{bmatrix}\mu\mathscr{D}_4&0&0&\mu\mathscr{D}_1\\0&\mu\mathscr{D}_4&0&\mu\mathscr{D}_2\\0&0&\mu\mathscr{D}_4&\mu\mathscr{D}_3\\\lambda\mathscr{D}_1&\lambda\mathscr{D}_2&\lambda\mathscr{D}_3&(\lambda+2\mu)\mathscr{D}_4\end{bmatrix},\end{aligned}\tag{3.2.4}$$

since

$$\partial_{\boldsymbol{\nu}}=\mathscr{D}_4,\quad\sum_{j=1}^{3}\nu_j\boldsymbol{e}^j=\boldsymbol{\nu}=\boldsymbol{d}^4,\quad\sum_{k=1}^{3}\nu_k\boldsymbol{d}^k=0,\quad\sum_{k=1}^{3}\nu_k^2=1.$$

3.3 Convex energies

In the present section, we expose some results about convex energies and energy functionals from [77] and endow it with description of similar results in curvilinear coordinates introduced in Section 2.2.

Let an elastic body occupy a thin domain Ω^h (see (2.2.1)) in a reference configuration and let a three-dimensional vector $\boldsymbol{U} : \Omega^h \to \mathbb{R}^3$ represent the deformation of the body subject to the action of internal and external forces. We assume that $\boldsymbol{U}$ is sufficiently smooth mapping, $\boldsymbol{U} \in \mathbb{H}^2(\mathscr{C})$, and the elastic energy of the deformation is represented by a non-linear functional

$$\mathscr{E}^{(h)}(\boldsymbol{U}) := \int\limits_{\Omega^h} W(\nabla \boldsymbol{U}(x))\, dx = \int\limits_{\Omega^h} W\big([\partial_j U_k]_{3\times 3}\big)\, dx. \tag{3.3.1}$$

The non-linear stored energy function $W : \mathbb{M}^{3\times 3} \to \mathbb{R}$ has a single energy well at the group of orthogonal matrices

$$\mathbb{SO}(3) := \big\{A \in \mathbb{M}^{3\times 3} : \quad A^\top A = AA^\top = I\big\}.$$

The stored energy function W is subject to the following constraints:

$$\begin{aligned}
&1. \quad W \in C(\mathbb{M}^{3\times 3}),\ \ W \in C^2 \ \text{ in a neighbourhood of } \ \mathbb{SO}(3);\\
&2. \quad W \ \text{ is frame-indifferent: } W(F) = W(RF),\ \ \forall\, F \in \mathbb{M}^{3\times 3},\ \ \forall\, R \in \mathbb{SO}(3);\\
&3. \quad W(F) \geq C\, \mathbf{dist}^2(F, \mathbb{SO}(3)),\ \ W(F) = 0 \ \text{ if } \ F \in \mathbb{SO}(3).
\end{aligned} \tag{3.3.2}$$

The condition $W \in C(\mathbb{M}^{3\times 3})$ in (3.3.2) can be weakened to include energy functions W which become $+\infty$ outside an open neighbourhood of $\mathbb{SO}(3)$, such as the following model functional for isotropic materials which goes back to St. Venant and Kirchhoff:

$$W(F) = \begin{cases} \mu\Big|\sqrt{F^\top F} - I\Big|^2 + \dfrac{\lambda}{2}\Big(\operatorname{Tr}\Big(\sqrt{F^\top F} - I\Big)\Big)^2, & \det F > 0,\\ +\infty & \text{otherwise.}\end{cases}$$

Let us rewrite the functional of non-linear elastic energy of a deformation $\mathscr{E}^{(h)}(\boldsymbol{U})$ in (3.3.1) in curvilinear coordinates (2.2.3)–(2.2.6):

$$\mathscr{E}^{(h)}_0(\boldsymbol{U}^0) :=: \int\limits_{\mathscr{C}} \int\limits_{-h}^{h} W_0\big(\mathscr{D}_{\Omega^h}\boldsymbol{U}^0(x', t)\big)\, d\sigma\, dt, \tag{3.3.3}$$

$$W_0\big(\mathscr{D}_{\Omega^h}\boldsymbol{U}^0(x', t)\big) := W\Big(\Big[(\mathscr{D}_j + \mathscr{N}_j(x')\mathscr{D}_4)(U^0_k(x', t) + \mathscr{N}_k(x')U^0_4(x', t))\Big]_{3\times 3}\Big).$$

Lemma 3.3.1. *The non-linear stored energy function $W_0 : \mathbb{M}^{4\times 4} \to \mathbb{R}$ has a single energy well at the set of matrices $\mathbb{A}(4)$, which consists of matrices of the form*

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \langle \boldsymbol{\nu}, \boldsymbol{a}_{1,\,\cdot}\rangle\\ a_{21} & a_{22} & a_{23} & \langle \boldsymbol{\nu}, \boldsymbol{a}_{2,\,\cdot}\rangle\\ a_{31} & a_{32} & a_{33} & \langle \boldsymbol{\nu}, \boldsymbol{a}_{3,\,\cdot}\rangle\\ \langle \boldsymbol{\nu}, \boldsymbol{a}_{\cdot\,,1}\rangle & \langle \boldsymbol{\nu}, \boldsymbol{a}_{\cdot\,,2}\rangle & \langle \boldsymbol{\nu}, \boldsymbol{a}_{\cdot\,,3}\rangle & \langle \boldsymbol{A}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle \end{bmatrix} = \boldsymbol{V}^\top \boldsymbol{A}_0 \boldsymbol{V}, \tag{3.3.4}$$

$$\boldsymbol{a}_{j,\,\cdot} := (a_{j,1}, a_{j,2}, a_{j,3})^\top,\ \ \boldsymbol{a}_{\cdot\,,j} := (a_{1,j}, a_{2,j}, a_{3,j})^\top,\ \ j = 1, 2, 3,$$

where $\boldsymbol{A} = [a_{jk}]_{3\times 3} \in \mathbb{SO}(3)$ is an orthogonal matrix and $\boldsymbol{A}_0$, $\boldsymbol{V}$ are given by the formulae

$$\boldsymbol{A}_0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0\\ a_{21} & a_{22} & a_{23} & 0\\ a_{31} & a_{32} & a_{33} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad \boldsymbol{V} := \begin{bmatrix} 1 & 0 & 0 & \nu_1\\ 0 & 1 & 0 & \nu_2\\ 0 & 0 & 1 & \nu_3\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{3.3.5}$$

The matrix $\boldsymbol{V}$ is invertible and the inverse is

$$\boldsymbol{V}^{-1} := \begin{bmatrix} 1 & 0 & 0 & -\nu_1\\ 0 & 1 & 0 & -\nu_2\\ 0 & 0 & 1 & -\nu_3\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

The stored energy function W_0 is subject to the following constraints:

$$\begin{aligned}
&1.\quad W_0\in C^2(\mathbb{M}^{4\times4})\ \ \textit{in a neighbourhood of}\ \mathbb{A}(4);\\
&2.\quad W_0(\boldsymbol{R}_1\boldsymbol{V}^{-1}(\boldsymbol{V}^\top)^{-1}\boldsymbol{G}\boldsymbol{V}^{-1}(\boldsymbol{V}^\top)^{-1}\boldsymbol{R}_2)=W_0(\boldsymbol{G}),\ \ \forall\,\boldsymbol{G}\in\mathbb{M}^{3\times3},\ \ \forall\,\boldsymbol{R}_1,\boldsymbol{R}_2\in\mathbb{A}(4);\\
&3.\quad W_0(\boldsymbol{G})\geq C\mathrm{dist}^2(\boldsymbol{G},\mathbb{A}(4)),\ \ W_0(\boldsymbol{G})=0\ \ \textit{if}\ \boldsymbol{G}\in\mathbb{A}(4).
\end{aligned}\tag{3.3.6}$$

Proof. From Lemma 3.2.1 we know that a matrix function $\boldsymbol{A}=[a_{jk}]_{3\times3}$ in curvilinear coordinates (2.2.3)–(2.2.6) acquires form (3.2.1). Since the functions a_{jk} and components of the normal vector field ν_1, ν_2, ν_3 commute, we can represent the matrix $\boldsymbol{A}_0$ as follows:

$$\begin{aligned}
\boldsymbol{A}_0=&\begin{bmatrix} a_{11} & a_{12} & a_{13} & \langle\boldsymbol{\nu},\boldsymbol{a}_{1,\,\cdot}\rangle\\ a_{21} & a_{22} & a_{23} & \langle\boldsymbol{\nu},\boldsymbol{a}_{2,\,\cdot}\rangle\\ a_{31} & a_{32} & a_{33} & \langle\boldsymbol{\nu},\boldsymbol{a}_{3,\,\cdot}\rangle\\ \langle\boldsymbol{\nu},\boldsymbol{a}_{\,\cdot\,,1}\rangle & \langle\boldsymbol{\nu},\boldsymbol{a}_{\,\cdot\,,2}\rangle & \langle\boldsymbol{\nu},\boldsymbol{a}_{\,\cdot\,,3}\rangle & \langle\boldsymbol{A}\boldsymbol{\nu},\boldsymbol{\nu}\rangle\end{bmatrix}\\
&=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\ \nu_1&\nu_2&\nu_3&0\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}&0\\a_{21}&a_{22}&a_{23}&0\\a_{31}&a_{32}&a_{33}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&\nu_1\\0&1&0&\nu_2\\0&0&1&\nu_3\\0&0&0&1\end{bmatrix}=\boldsymbol{V}^\top\boldsymbol{A}_0\boldsymbol{V},
\end{aligned}$$

where $\boldsymbol{A}_0$ and the invertible matrix $\boldsymbol{V}$ with its inverse $\boldsymbol{V}^{-1}$ are defined above. Therefore, the algebra of orthogonal matrices $\mathbb{SO}(3)$ in curvilinear coordinates transforms into the set $\mathbb{A}(4)$ and the energy integral $\mathscr{E}_0^{(h)}$ has a single energy well on $\mathbb{A}(4)$.

Properties (3.3.6) of the energy function $W_0(\boldsymbol{G})$ follow from (3.3.2) with the help of representation (3.3.4) and the last part of Lemma 2.2.1 (see (2.2.8)) asserting that the distance is invariant under the change of Euclidean coordinates to the curvilinear ones. □

Remark 3.3.1. From representation (3.3.4) it follows that if an initial matrix-function $\boldsymbol{A}=[a_{jk}]_{3\times3}$ is skew symmetric

$$\boldsymbol{A}=\begin{bmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{13}\\-a_{13}&-a_{23}&0\\0&0&0\end{bmatrix}=-\boldsymbol{A}^\top,$$

it maintains the skew symmetry in curvilinear coordinates (2.2.3)–(2.2.6):

$$\boldsymbol{A}^\top=(\boldsymbol{V}^\top\boldsymbol{A}_0\boldsymbol{V})^\top=\boldsymbol{V}^\top\boldsymbol{A}_0^\top(\boldsymbol{V}^\top)^\top=-\boldsymbol{V}^\top\boldsymbol{A}_0\boldsymbol{V}=-\boldsymbol{A}.$$

But if the initial matrix $\boldsymbol{A}=[a_{jk}]_{3\times3}$ is orthogonal $\boldsymbol{A}\in\mathbb{SO}(3)$, which implies the equalities

$$\sum_{j=1}^{3}a_{kj}a_{jm}=\delta_{km}a,\ \ k,m=1,2,3,$$

in curvilinear coordinates (2.2.3)–(2.2.6), it looses the orthogonality: $\boldsymbol{A}^\top=\boldsymbol{V}^\top(\boldsymbol{A}^0)^\top\boldsymbol{V}=\boldsymbol{V}^\top(\boldsymbol{A}^0)^{-1}\boldsymbol{V}$ is not the inverse to $\boldsymbol{A}$ (moreover, $\boldsymbol{A}$ is not invertible at all).

3.4 Variational reformulation of the problem

To apply the method of Γ-convergence, we have to reformulate the BVP (3.1.3) in an equivalent variational problem for the energy functional. For this, note that it is sufficient to consider the BVP with wanishing Dirichlet condition on the lateral surface:

$$\begin{aligned}
\mathscr{L}_{\Omega^h}\boldsymbol{U}(x)&=\boldsymbol{F}(x),\ \ x\in\Omega^h:=\mathscr{C}\times(-h,h),\\
\boldsymbol{U}^+(t)&=0,\ \ t\in\Gamma^h_L:=\partial\mathscr{C}\times(-h,h),\\
(\mathfrak{T}(\mathscr{x},\nabla)\boldsymbol{U})^+(\mathscr{x},\pm h)&=\boldsymbol{H}(\mathscr{x},\pm h),\ \ \mathscr{x}\in\mathscr{C}.
\end{aligned}\tag{3.4.1}$$

Indeed, consider the BVP

$$\begin{aligned} \mathscr{L}_{\Omega^h}\boldsymbol{V}(x)&=0, \ \ x\in\Omega^h:=\mathscr{C}\times(-h,h),\\ \boldsymbol{V}^+(t)&=G, \ \ t\in\Gamma^h_L,\\ (\mathfrak{T}((\mathscr{x},\nabla)\boldsymbol{V})^+(\mathscr{x},\pm h)&=0, \ \ (\mathscr{x},\pm h)\in\Gamma^\pm=\mathscr{C}\times\{\pm h\}, \end{aligned} \tag{3.4.2}$$

which has a unique solution $\boldsymbol{V}\in\mathbb{W}^1(\Omega^h)$ (see Theorem 3.1.1), and note that the difference $\boldsymbol{U}-\boldsymbol{V}$ of solutions to BVPs (3.1.3) and (3.4.2) satisfies the BVP (3.4.1). Thus, solution to the BVP (3.1.3) is recovered as the sum of solutions $\boldsymbol{U}+\boldsymbol{V}$ of the BVPs (3.4.1) and (3.4.2).

Theorem 3.4.1. *Problem* (3.4.1) *with the constraints*

$$\boldsymbol{U}\in\mathbb{H}^1(\Omega^h,\Gamma^h_L), \ \ \boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h), \ \ \boldsymbol{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-\frac12}(\mathscr{C}) \tag{3.4.3}$$

is reformulated into the following equivalent variational problem: under the same constraints (3.4.3)*, look for a displacement vector-function $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$, which is a stationary point of the following functional:*

$$\begin{aligned} \mathscr{E}_{\Omega^h}(\boldsymbol{U}):=&\frac12\int\limits_{\Omega^h}\Big[\mu|\nabla_{\Omega^h}\boldsymbol{U}(x)|^2+(\lambda+\mu)\big\{\,\mathbf{div}_{\Omega^h}\boldsymbol{U}(x)\big\}^2+2\langle\boldsymbol{F}(x),\boldsymbol{U}(x)\rangle\Big]\,dx\\ &+\int\limits_{\mathscr{C}}\Big[\big\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{U}^+(\mathscr{x},+h)\big\rangle-\big\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{U}^+(\mathscr{x},-h)\big\rangle\Big]\,d\sigma\\ =&\frac12\int\limits_{-h}^{h}\int\limits_{\mathscr{S}}\Big[\mu|\nabla_{\Omega^h}\boldsymbol{U}(x)|^2+(\lambda+\mu)\big\{\,\mathbf{div}_{\Omega^h}\boldsymbol{U}(x)\big\}^2+2\langle\boldsymbol{F}(x),\boldsymbol{U}(x)\rangle\\ &+\frac1h\Big[\big\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{U}^+(\mathscr{x},+h)\big\rangle-\big\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{U}^+(\mathscr{x},-h)\big\rangle\Big]\Big]\,d\sigma\,dt. \end{aligned} \tag{3.4.4}$$

Remark 3.4.1. The integral on $\mathscr{C}$ in (3.4.4) is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{\frac12}(\mathscr{C})$ and $\mathbb{H}^{-\frac12}(\mathscr{C})$ because $\boldsymbol{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-\frac12}(\mathscr{C}_N)$ and the condition $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ implies the inclusion $\boldsymbol{U}^+(\,\cdot\,,\pm h)\in\widetilde{\mathbb{H}}^{\frac12}(\mathscr{C}_N)$.

Proof of Theorem 3.4.1. Let $\boldsymbol{U}$ be a solution to the mixed problem (3.4.1). By taking the scalar product of the first equation $\mathscr{L}_{\Omega^h}\boldsymbol{U}(x)=\boldsymbol{F}(x)$ in (3.4.1) with a function $\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ and applying the Green formulae we get the following equality:

$$\begin{aligned} \int\limits_{\Omega^h}\big\langle\boldsymbol{F}(x),\boldsymbol{V}(x)\big\rangle\,dx=&\int\limits_{\Omega^h}\big\langle\mathscr{L}_{\Omega^h}\boldsymbol{U}(x),\boldsymbol{V}(x)\big\rangle\,dx\\ =&-\int\limits_{\Omega^h}\Big[\mu\big\langle\nabla_{\Omega^h}\boldsymbol{U}(x),\nabla_{\Omega^h}\boldsymbol{V}(x)\big\rangle+(\lambda+\mu)\big\langle\,\mathbf{div}_{\Omega^h}\boldsymbol{U}(x),\mathbf{div}_{\Omega^h}\boldsymbol{V}(x)\big\rangle\Big]\,dx\\ &+\int\limits_{\Gamma^h_L}\big\langle(\mathfrak{T}(y,\nabla)\boldsymbol{U})^+(y),\boldsymbol{V}^+(y)\big\rangle\,d\sigma\\ &+\int\limits_{\mathscr{C}}\Big[\big\langle(\mathfrak{T}(y,\nabla)\boldsymbol{U})^+(y),\boldsymbol{V}^+(\mathscr{x},+h)\big\rangle-\big\langle(\mathfrak{T}(y,\nabla)\boldsymbol{U})^+(y),\boldsymbol{V}^+(\mathscr{x},-h)\big\rangle\Big]\,d\sigma. \end{aligned}$$

By inserting the boundary conditions from (3.4.1) we derive that the solution $\boldsymbol{U}$ to the BVP (3.4.1) solves the following variational problem for arbitrary trial function $\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$:

$$\begin{aligned} &\int\limits_{\Omega^h}\Big[\mu\big\langle\nabla_{\Omega^h}\boldsymbol{U}(x),\nabla_{\Omega^h}\boldsymbol{V}(x)\big\rangle+(\lambda+\mu)\big\langle\,\mathbf{div}_{\Omega^h}\boldsymbol{U}(x),\mathbf{div}_{\Omega^h}\boldsymbol{V}(x)\big\rangle\Big]\,dx\\ &\quad=-\int\limits_{\Omega^h}\langle\boldsymbol{F}(x),\boldsymbol{V}(x)\rangle\,dx+\int\limits_{\mathscr{C}}\Big[\big\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{V}^+(\mathscr{x},+h)\big\rangle-\big\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{V}^+(\mathscr{x},-h)\big\rangle\Big]\,d\sigma. \end{aligned} \tag{3.4.5}$$

Next, note that the quadratic form (i.e., when $\boldsymbol{V}=\boldsymbol{U}$) in the left-hand side of equality (3.4.5) is positive definite in the space $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ and, therefore, defines an equivalent norm in the Hilbert space $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$. On the other hand, the functional in the right-hand side with a fixed $\boldsymbol{U}$ is bounded in the same space $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$. Therefore, by the Riesz theorem on functionals in the Hilbert spaces, there exists a unique function $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ which defines the functional in (3.4.5).

Now let $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ be a solution to the variational problem (3.4.5) and $\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ is arbitrary. A direct verification shows that

$$\mathscr{E}_{\Omega_h}(\boldsymbol{U}+\boldsymbol{V})=\mathscr{E}_{\Omega_h}(\boldsymbol{U})+\big[\mathscr{A}(\boldsymbol{U},\boldsymbol{V})-\mathscr{F}(\boldsymbol{V})\big]+\frac{1}{2}\,\mathscr{A}(\boldsymbol{V},\boldsymbol{V}), \tag{3.4.6}$$

where $\mathscr{A}(\boldsymbol{U},\boldsymbol{V})$ is a bilinear form and $\mathscr{F}(\boldsymbol{V})$ is a functional

$$\begin{aligned}\mathscr{A}(\boldsymbol{U},\boldsymbol{V})&=\int\limits_{\Omega^h}\Big[\mu\big\langle\nabla_{\Omega^h}\boldsymbol{U}(x),\nabla_{\Omega^h}\boldsymbol{V}(x)\big\rangle+(\lambda+\mu)\big\langle\,\mathbf{div}_{\Omega^h}\boldsymbol{U}(x),\mathbf{div}_{\Omega^h}\boldsymbol{V}(x)\big\rangle\Big]\,dx,\\ \mathscr{F}(\boldsymbol{V})&=-\int\limits_{\Omega^h}\langle\boldsymbol{F}(x),\boldsymbol{V}(x)\rangle\,dx\\ &\quad+\int\limits_{\mathscr{C}}\Big[\big\langle\boldsymbol{H}(\mathscr{x},+h),\boldsymbol{V}^+(\mathscr{x},+h)\big\rangle-\big\langle\boldsymbol{H}(\mathscr{x},-h),\boldsymbol{V}^+(\mathscr{x},-h)\big\rangle\Big]\,d\sigma,\end{aligned} \tag{3.4.7}$$

and the equality

$$\mathscr{E}_{\Omega_h}(\boldsymbol{U})=\frac{1}{2}\,\mathscr{A}(\boldsymbol{U},\boldsymbol{U})-\mathscr{F}(\boldsymbol{U}) \tag{3.4.8}$$

holds. Then, due to (3.4.5)–(3.4.6),

$$\mathscr{A}(\boldsymbol{U},\boldsymbol{V})-\mathscr{F}(\boldsymbol{V})=0\quad\text{for all}\quad\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$$

implies

$$\mathscr{E}_{\Omega_h}(\boldsymbol{U}+\boldsymbol{V})-\mathscr{E}_{\Omega_h}(\boldsymbol{U})=\frac{1}{2}\,\mathscr{A}(\boldsymbol{V},\boldsymbol{V})\geqslant\frac{C}{2}\,\big\|\boldsymbol{V}\mid\widetilde{\mathbb{H}}^1(\Omega^h)\big\|^2,\ \ \forall\,\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$$

and, thus, $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ is the minimizer of the functional $\mathscr{E}_{\Omega_h}(\boldsymbol{U})$ in this case.

Conversely: Let $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ be the minimizer of $\mathscr{E}_{\Omega_h}(\boldsymbol{V})$ and $\boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ be arbitrary. The inequality (cf. (3.4.6))

$$0\leqslant\mathscr{E}_{\Omega_h}(\boldsymbol{U}+\varepsilon\boldsymbol{V})-\mathscr{E}_{\Omega_h}(\boldsymbol{U})=\varepsilon\big\{\mathscr{A}(\boldsymbol{U},\boldsymbol{V})-\mathscr{F}(\boldsymbol{V})\big\}+\frac{\varepsilon^2}{2}\mathscr{A}(\boldsymbol{V},\boldsymbol{V}),\ \ \forall\,\varepsilon\in\mathbb{R},$$

implies that $\mathscr{A}(\boldsymbol{U},\boldsymbol{V})=\mathscr{F}(\boldsymbol{V})$. Indeed, the first summand in the right-hand side of the equality dominates for small ε (positive and negative) and the second is non-negative. If we assume the contrary $\mathscr{A}(\boldsymbol{U},\boldsymbol{V})\neq\mathscr{F}(\boldsymbol{V})$, the difference $\mathscr{E}_{\Omega_h}(\boldsymbol{U}+\theta\boldsymbol{V})-\mathscr{E}_{\Omega_h}(\boldsymbol{U})$ would become negative for certain small ε, which is a contradiction. □

By using representations (2.2.9) and (2.2.10) of extended gradien and extended divergence, we rewrite the energy functional in the following form:

$$\begin{aligned}\mathscr{E}_{\Omega^h}(\boldsymbol{U}^0)=\frac{1}{2}\int\limits_{-h}^{h}\int\limits_{\mathscr{C}}\Bigg[\mu\sum_{j=1}^{4}\Bigg\{&\sum_{\alpha=1}^{3}(\mathscr{D}_\alpha U^0_j(\mathscr{x},t))^2+\Big(\frac{\partial U^0_j(\mathscr{x},t)}{\partial t}\Big)^2+2\langle\boldsymbol{F}^0(\mathscr{x},t),\boldsymbol{U}^0(\mathscr{x},t)\rangle\Bigg\}\\ &+(\lambda+\mu)\Bigg\{\sum_{\alpha=1}^{3}\mathscr{D}_\alpha\boldsymbol{U}^0_\alpha(x)+\frac{\partial U^0_4(\mathscr{x},t)}{\partial t}+2\mathscr{H}_{\mathscr{C}}(\mathscr{x})U^0_4(\mathscr{x},t)\Bigg\}^2\\ &+\frac{1}{h}\,\Big\{\big\langle\boldsymbol{H}^0(\mathscr{x},+h),\boldsymbol{U}^{0,+}(\mathscr{x},+h)\big\rangle-\big\langle\boldsymbol{H}^0(\mathscr{x},-h),\boldsymbol{U}^{0,+}(\mathscr{x},-h)\big\rangle\Big\}\Bigg]\,d\sigma\,dt,\end{aligned} \tag{3.4.9}$$

$$\boldsymbol{U}^0:=(U^0_1,U^0_2,U^0_3,U^0_4)^\top,\ \ \mathscr{D}_4=\frac{\partial}{\partial t}\,,\ \ x=(\mathscr{x},t),\ \ \mathscr{x}\in\mathscr{C},\ \ t\in(-h,h).$$

Next, we perform the scaling of the variable $t = h\tau$, $-1 < \tau < 1$, divide by h the scaled energy functional and study the following functionals in the scaled domain $\Omega^1 = \mathscr{C} \times (-1, 1)$:

$$\begin{gathered}
\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h) = \frac{1}{h}\,\mathscr{E}_{\Omega^h}(\boldsymbol{U}^0_h) = \frac{1}{2}\int\limits_{-1}^{1}\int\limits_{\mathscr{C}} Q_4\big(\nabla_{\mathscr{C}}\boldsymbol{U}^0(\mathscr{x}, h\tau), h^{-1}\partial_\tau \boldsymbol{U}^0(\mathscr{x}, h\tau)\big)\, d\sigma\, d\tau,\\
Q_4\big(\nabla_{\mathscr{S}}\boldsymbol{U}^0(\mathscr{x}, h\tau), h^{-1}\partial_t \boldsymbol{U}^0(\mathscr{x}, h\tau)\big) := \mu\sum_{j=1}^{4}\Big\{(\nabla_{\mathscr{C}} U^0_j(\mathscr{x}, h\tau))^2 + \Big(\frac{1}{h}\frac{\partial U^0_j(\mathscr{x}, h\tau)}{\partial\tau}\Big)^2\Big\}\\
+(\lambda+\mu)\Big\{\,\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x}, h\tau) + 2\mathscr{H}_{\mathscr{C}}(\mathscr{x})U^0_4(\mathscr{x}, h\tau) + \frac{1}{h}\,\frac{\partial U^0_4(\mathscr{x}, h\tau)}{\partial\tau}\Big\}^2\\
+2\big\langle \boldsymbol{F}^0(\mathscr{x}, h\tau), \boldsymbol{U}^0(\mathscr{x}, h\tau)\big\rangle\\
+\frac{1}{h}\Big[\big\langle \boldsymbol{H}^0(\mathscr{x}, +h), \boldsymbol{U}^{0,+}(\mathscr{x}, +h)\big\rangle - \big\langle \boldsymbol{H}^0(\mathscr{x}, -h), \boldsymbol{U}^{0,+}(\mathscr{x}, -h)\big\rangle\Big],
\end{gathered}\tag{3.4.10}$$

where

$$\begin{gathered}
\boldsymbol{U}^0 := (U^0_1, U^0_2, U^0_3, U^0_4)^\top = (\widehat{\boldsymbol{U}}^0, U^0_4)^\top,\ \ \widehat{\boldsymbol{U}}^0 := (U^0_1, U^0_2, U^0_3)^\top,\ \ \boldsymbol{U}^0_h(\mathscr{x}, \tau) := \boldsymbol{U}^0(\mathscr{x}, h\tau),\\
\boldsymbol{F}^0 := (F^0_1, F^0_2, F^0_3, F^0_4)^\top,\ \ F^0_j := F_j - \nu_j F^0_4,\ \ F^0_4 := \langle \boldsymbol{\nu}, \boldsymbol{F}\rangle,
\end{gathered}$$

and $\nabla_{\mathscr{C}}$, $\mathbf{div}_{\mathscr{C}}$ are the surface gradient and divergence:

$$\nabla_{\mathscr{C}}\varphi := (\mathscr{D}_1\varphi, \mathscr{D}_2\varphi, \mathscr{D}_3\varphi)^\top, \quad \mathbf{div}_{\mathscr{C}}\,\boldsymbol{V} := \sum_{\alpha=1}^{3}\mathscr{D}_\alpha V_\alpha.$$

Lemma 3.4.1. *The energy functional $\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)$ in (3.4.11) is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1, \Gamma^1_L)$ (see (3.1.6)) and is convex*

$$\mathscr{E}^0_{\Omega^h}(\theta\boldsymbol{U}^0_h + (1-\theta)\overline{\boldsymbol{V}}^0_h) \leqslant \theta\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h) + (1-\theta)\mathscr{E}^0_{\Omega^h}(\overline{\boldsymbol{V}}^0_h),\ \ 0 < \theta < 1. \tag{3.4.11}$$

Moreover, if $\boldsymbol{F}^0_h(\mathscr{x}, \tau) := \boldsymbol{F}^0(\mathscr{x}, h\tau)$ are uniformly bounded in $\mathbb{L}_2(\Omega^1)$,

$$\sup_{h<h_0}\|\boldsymbol{F}^0_h|\mathbb{L}_2(\Omega^1)\| < \infty. \tag{3.4.12}$$

for some $h_0 > 0$, the energy functional has the following quadratic estimate: there exist positive constants C_1, C_2 and C_3 independent of the parameter h such that

$$\begin{aligned}
C_1\int\limits_{\Omega^1}\Bigg(\sum_{j=1}^{4}\Big\{(\nabla_{\mathscr{C}} U^0_j(\mathscr{x}, h\tau))^2 + \Big(\frac{1}{h}\,\frac{\partial U^0_j(\mathscr{x}, h\tau)}{\partial\tau}\Big)^2\Big\}\Bigg)\,dx - C_2 \leqslant \mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)\\
\leqslant C_3\Bigg[1 + \int\limits_{\Omega^1}\Bigg(\sum_{j=1}^{4}\Big\{(\nabla_{\mathscr{C}} U^0_j(\mathscr{x}, h\tau))^2 + \Big(\frac{1}{h}\,\frac{\partial U^0_j(\mathscr{x}, h\tau)}{\partial\tau}\Big)^2\Big\}\Bigg)\,dx\Bigg]
\end{aligned}\tag{3.4.13}$$

for all $\boldsymbol{U}^0_h \in \widetilde{\mathbb{H}}^1(\Omega^1, \Gamma^1_L)$.

Proof. Let us decompose the energy functional into the sum of quadratic and linear functionals

$$\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h) = \mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h) + \mathscr{E}^L_{\Omega^h}(\boldsymbol{U}^0_h), \tag{3.4.14}$$

$$\begin{aligned}
\mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h) := \frac{1}{2}\int\limits_{-1}^{1}\int\limits_{\mathscr{C}}\Bigg[\mu\sum_{j=1}^{4}\Big\{(\nabla_{\mathscr{C}} U^0_j(\mathscr{x}, h\tau))^2 + \Big(\frac{1}{h}\,\frac{\partial U^0_j(\mathscr{x}, h\tau)}{\partial\tau}\Big)^2\Big\}\\
+ (\lambda+\mu)\Big\{\,\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x}, h\tau) + 2\mathscr{H}_{\mathscr{C}}(\mathscr{x})U^0_4(\mathscr{x}, h\tau) + \frac{1}{h}\,\frac{\partial U^0_4(\mathscr{x}, h\tau)}{\partial\tau}\Big\}^2\Bigg]\,d\sigma\,d\tau,
\end{aligned}$$

$$\mathscr{E}^L_{\Omega^h}(\boldsymbol{U}^0_h) := \frac{1}{2}\int\limits_{-1}^{1}\int\limits_{\mathscr{C}}\Big[2\big\langle\boldsymbol{F}^0(\mathscr{x},h\tau),\boldsymbol{U}^0(\mathscr{x},h\tau)\big\rangle + \frac{1}{h}\Big[\big\langle\boldsymbol{H}^0(\mathscr{x},+h),\boldsymbol{U}^{0,+}(\mathscr{x},+h)\big\rangle - \big\langle\boldsymbol{H}^0(\mathscr{x},-h),\boldsymbol{U}^{0,+}(\mathscr{x},-h)\big\rangle\Big]\Big]\,d\sigma\,d\tau\,.$$

The convexity of the linear part $\mathscr{E}^L_{\Omega^h}(\boldsymbol{U}^0_h)$ is trivially obvious. The convexity of the quadratic part $\mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h)$ is also rather trivial to prove if based on the well-known inequality

$$\begin{aligned}\big[\theta a+(1-\theta)b\big]^2 &= \theta^2a^2+2\theta(1-\theta)ab+(1-\theta)^2b^2\\ &\leqslant \theta^2a^2+\theta(1-\theta)(a^2+b^2)+(1-\theta)^2b^2=\theta a^2+(1-\theta)b^2.\end{aligned}$$

Thus, inequality (3.4.11) is proved.

Inequality (3.4.13) is trivial for the quadratic part $\mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h)$ of the energy functional (even with $C_2=0$) and since the quadratic part dominates the linear one $\mathscr{E}^L_{\Omega^h}(\boldsymbol{U}^0_h)\leqslant C_4\mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h)$, the proof for $\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)=\mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h)+\mathscr{E}^L_{\Omega^h}(\boldsymbol{U}^0_h)$ follows from the proved one for $\mathscr{E}^Q_{\Omega^h}(\boldsymbol{U}^0_h)$. □

Theorem 3.4.2. *Let the weak limits*

$$\lim_{h\to0}\boldsymbol{F}^0(\mathscr{x},h\tau)=\boldsymbol{F}^0(\mathscr{x}),\quad \lim_{h\to0}\frac{1}{2h}\big[\boldsymbol{H}^0(\mathscr{x},+h)-\boldsymbol{H}^0(\mathscr{x},-h)\big]=\boldsymbol{H}^{(1),0}(\mathscr{x}),\qquad \boldsymbol{F}^0,\boldsymbol{H}^{(1),0}\in\mathbb{L}_2(\mathscr{C}), \tag{3.4.15}$$

exist, respectively, in $\mathbb{L}_2(\Omega^h)$ and $\mathbb{L}_2(\mathscr{C})$. Then the Γ-limit of the energy functional $\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)$ exists:

$$\lim_{h\to0}\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)=\mathscr{E}^3_{\mathscr{C}}(\boldsymbol{U}^0):=\int\limits_{\mathscr{C}}Q_3(\boldsymbol{U}^0(\mathscr{x}))\,d\sigma, \tag{3.4.16}$$

where

$$Q_3(\boldsymbol{U}^0(\mathscr{x}))=\frac{\mu}{2}\sum_{j=1}^{3}(\nabla_{\mathscr{C}}U^0_j(\mathscr{x}))^2 + \frac{\mu}{2}\,\frac{\lambda+\mu}{\lambda+2\mu}\big\{\,\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x})+\mathscr{H}_{\mathscr{C}}(\mathscr{x})U^0_4(\mathscr{x})\big\}^2+2\big\langle\boldsymbol{F}^0(\mathscr{x})+\boldsymbol{H}^{(1),0}(\mathscr{x}),\boldsymbol{U}^0(\mathscr{x})\big\rangle \tag{3.4.17}$$

and

$$\begin{gathered}\boldsymbol{U}^0(\mathscr{x}):=(U^0_1(\mathscr{x}),U^0_2(\mathscr{x}),U^0_3(\mathscr{x}),U^0_4(\mathscr{x}))^\top,\quad \widehat{\boldsymbol{U}}^0(\mathscr{x}):=(U^0_1(\mathscr{x}),U^0_2(\mathscr{x}),U^0_3(\mathscr{x}))^\top,\\ U^0_j(\mathscr{x}):=U^0_j(\mathscr{x},0)=U_j(\mathscr{x})-\nu_j(\mathscr{x})U^0_4(\mathscr{x}),\ \ j=1,2,3,\\ U^0_4(\mathscr{x}):=(\boldsymbol{\nu}(\mathscr{x}),\boldsymbol{U}(\mathscr{x})),\quad \boldsymbol{U}(\mathscr{x})=(U_1(\mathscr{x},0),U_2(\mathscr{x},0),U_3(\mathscr{x},0))^\top,\\ \boldsymbol{F}^0:=(F^0_1,F^0_2,F^0_3,F^0_4)^\top,\quad F^0_j(\mathscr{x}):=F^0_j(\mathscr{x},0)=F_j(\mathscr{x},0)-\nu_j(\mathscr{x})\boldsymbol{F}^0_4(\mathscr{x}),\ \ j=1,2,3,\\ F^0_4(\mathscr{x}):=(\boldsymbol{\nu}(\mathscr{x}),\boldsymbol{F}(\mathscr{x})),\quad \boldsymbol{F}(\mathscr{x})=\boldsymbol{F}(\mathscr{x},0)=(F_1(\mathscr{x},0),F_2(\mathscr{x},0),F_3(\mathscr{x},0))^\top.\end{gathered} \tag{3.4.18}$$

Proof. To check the Γ-convergence (3.4.16), first we prove the estimate

$$\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)\geqslant\mathscr{E}^3_{\Omega^h}(\boldsymbol{U}^0_h):=\int\limits_{\mathscr{C}}Q_3(\boldsymbol{U}^0(\mathscr{x}))\,d\sigma. \tag{3.4.19}$$

For this, we rewrite the quadratic form

$$Q_4\big(\boldsymbol{U}^0(\mathscr{x},h\tau),h^{-1}\partial_t\boldsymbol{U}^0(\mathscr{x},h\tau)\big):=\frac{\mu}{2}\sum_{j=1}^{4}\big\{(U^0_\alpha(\mathscr{x},h\tau))^2+c_j^2(h)\big\}$$

$$+\frac{\lambda+\mu}{2}\left\{\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x},h\tau)+2\mathscr{H}_{\mathscr{C}}(\mathscr{x})U_4^0(\mathscr{x},h\tau)+c_4(h)\right\}^2+\left\langle \boldsymbol{F}^0(\mathscr{x},h\tau),\boldsymbol{U}^0(\mathscr{x},h\tau)\right\rangle$$
$$+\frac{1}{2h}\left[\left\langle \boldsymbol{H}^{0,+}(\mathscr{x}),\boldsymbol{U}^{0,+}(\mathscr{x},+h)\right\rangle-\left\langle \boldsymbol{H}^{0,-}(\mathscr{x}),\boldsymbol{U}^{0,+}(\mathscr{x},-h)\right\rangle\right]$$

and have to find the infimum with respect to 4 variables depending on h:

$$c_j(h):=\frac{1}{h}\,\frac{\partial U_j^0(\mathscr{x},h\tau)}{\partial\tau},\quad j=1,2,3,4. \tag{3.4.20}$$

From the extremum conditions

$$\begin{aligned}
&\frac{\partial Q_4(\boldsymbol{U}^0(\mathscr{x},h\tau),h^{-1}\partial_t\boldsymbol{U}^0(\mathscr{x},h\tau))}{\partial c_j(h)}=\mu c_j(h)=0,\quad j=1,2,3,\\
&\frac{\partial Q_4(\boldsymbol{U}^0(\mathscr{x},h\tau),h^{-1}\partial_t\boldsymbol{U}^0(\mathscr{x},h\tau))}{\partial c_4(h)}=\mu c_4(h)\\
&\qquad+(\lambda+\mu)\left\{\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x},h\tau)+\mathscr{H}_{\mathscr{C}}(\mathscr{x})U_4^0(\mathscr{x},h\tau)+c_4(h)\right\}=0
\end{aligned} \tag{3.4.21}$$

we find that

$$c_1(h)=c_2(h)=c_3(h)=0,$$
$$c_4(h)=-\frac{\lambda+\mu}{\lambda+2\mu}\left\{\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x},h\tau)+2\mathscr{H}_{\mathscr{C}}(\mathscr{x})U_4^0(\mathscr{x},h\tau)\right\}.$$

By introducing the obtained values into the quadratic form we get a new quadratic form $\mathscr{E}^3_{\Omega^h}(\overline{\boldsymbol{U}^0_h})$ which is minimum of $\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0_h)$ and, therefore, estimates this from below.

Thus, estimate (3.4.19) is proved.

To accomplish the proof of the Γ-convergence (3.4.16) it remains to build a recovery sequence

$$\boldsymbol{U}^0(\mathscr{x},h_kt)=\left(\boldsymbol{U}^0_1(\mathscr{x},h_kt),\boldsymbol{U}^0_2(\mathscr{x},h_kt),\boldsymbol{U}^0_3(\mathscr{x},h_kt),\boldsymbol{U}^0_4(\mathscr{x},h_kt)\right)^\top$$
$$\to\boldsymbol{U}^0(\mathscr{x})=\left(\boldsymbol{U}^0_1(\mathscr{x}),\boldsymbol{U}^0_2(\mathscr{x},),\boldsymbol{U}^0_3(\mathscr{x}),\boldsymbol{U}^0_4(\mathscr{x})\right)^\top$$

along which the quadratic form reaches its minimum

$$\lim_{h_k\to0}\mathscr{E}^0_{\Omega^h}(\boldsymbol{U}^0(\mathscr{x},h_kt))=\mathscr{E}^3_{\mathscr{C}}(\boldsymbol{U}^0(\mathscr{x})). \tag{3.4.22}$$

The minimizing sequence $\boldsymbol{U}^0(\mathscr{x},h_kt)$ should satisfy conditions (3.4.21) and, therefore (cf. (3.4.20)),

$$\frac{1}{h_k}\,\frac{\partial U^0(\mathscr{x},h_k\tau)}{\partial\tau}=0,\quad j=1,2,3,$$
$$\frac{1}{h_k}\,\frac{\partial \boldsymbol{U}^0_4(\mathscr{x},h_k\tau)}{\partial\tau}=-\frac{\lambda+\mu}{\lambda+2\mu}\left\{\mathbf{div}_{\mathscr{C}}\,\boldsymbol{U}^o(\mathscr{x},h_k\tau)+2\mathscr{H}_{\mathscr{C}}(\mathscr{x})U^o_4(\mathscr{x},h_k\tau)\right\}, \tag{3.4.23}$$
$$\lim_{h_k\to0}U^0_m(\mathscr{x},h_k\tau)=U^0_m(x),\quad m=1,2,3,4. \tag{3.4.24}$$

From (3.4.12) we derive that the first 3 components of the vector-function $\boldsymbol{U}^0(\mathscr{x},h_k\tau)$ is independent of the transversal variable to the surface, of $\tau\in(-1,1)$, i.e.,

$$\boldsymbol{U}^0_j(\mathscr{x},h_k\tau)=\boldsymbol{U}^0_j(\mathscr{x})\ \text{ for }\ j=1,2,3, \tag{3.4.25}$$

as well as its surface divergence

$$\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x},h_k\tau)=\mathscr{D}_1U^0_1(\mathscr{x})+\mathscr{D}_2U^0_2(\mathscr{x})+\mathscr{D}_3U^0_3(\mathscr{x})=\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x}).$$

Solution $U_4^0(\mathscr{x}, h_k\tau)$ to the Cauchy problem (3.4.23)–(3.4.24) for the first order differential equation depends on the surface variable $\mathscr{x}$ as a parameter:

$$U_4^0(\mathscr{x}, h_k\tau) = \frac{\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x})}{2\mathscr{H}_{\mathscr{C}}(\mathscr{x})}\,(e^{-A(x)h_k\tau} - 1) + e^{-A(x)h_k\tau}U_4^0(\mathscr{x}) \ \text{ if } \ \mathscr{H}_{\mathscr{C}}(\mathscr{x}) \neq 0, \tag{3.4.26}$$

$$U_4^0(\mathscr{x}, h_k\tau) = B(x)h_k\tau + U_4^0(\mathscr{x}) \ \text{ if } \ \mathscr{H}_{\mathscr{C}}(\mathscr{x}) = 0, \tag{3.4.27}$$

where

$$A(x) = \frac{2(\lambda+2\mu)\mathscr{H}_{\mathscr{C}}(\mathscr{x})}{\lambda+4\mu}, \quad B(x) = \frac{(\lambda+2\mu)\,\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x})}{\lambda+4\mu}.$$

Note that if $A(x) \to 0$, then the limit of (3.4.26) coincides to (3.4.27), so $U_4^0(\mathscr{x}, h_k\tau)$ is smooth with respect to $\mathscr{x}$.

Inserting equalities (3.4.25) and (3.4.26) into the quadratic form $\mathscr{E}^0_{\Omega^{h_k}}(\boldsymbol{U}^0_{h_k})$ (see (3.4.9)) and sending $h_k \to 0$ we prove that the limit in (3.4.16) is attained. □

Corollary 3.4.1. *The boundary value problem*

$$\begin{cases} \mu\boldsymbol{\Delta}_{\mathscr{C}}\widehat{\boldsymbol{U}}^0(\mathscr{x}) + \mu\,\dfrac{\lambda+\mu}{\lambda+2\mu}\Big\{\nabla_{\mathscr{C}}\,\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x}) + \nabla_{\mathscr{C}}\big[\mathscr{H}_{\mathscr{C}}(\mathscr{x})\boldsymbol{U}_4^0(\mathscr{x})\big]\Big\} & \\ \qquad\qquad = \boldsymbol{F}^0(\mathscr{x}) + \boldsymbol{H}^{(1),0}(\mathscr{x}) & \text{on } \mathscr{C}, \\ \mu\,\dfrac{\lambda+\mu}{\lambda+2\mu}\mathscr{H}_{\mathscr{C}}(\mathscr{x})\big(\mathbf{div}_{\mathscr{C}}\,\widehat{\boldsymbol{U}}^0(\mathscr{x}) + \mathscr{H}_{\mathscr{C}}(\mathscr{x})\boldsymbol{U}_4^0(\mathscr{x})\big) = -(F_4^0 + H_4^{(1),0}) & \text{on } \mathscr{C}, \\ \boldsymbol{U}^0(t) = 0 & \text{on } \Gamma = \partial\mathscr{C} \end{cases} \tag{3.4.28}$$

corresponds to the energy functional $\mathscr{E}^3_{\mathscr{C}}(\boldsymbol{U}^0)$ *in* (3.4.16) *and, therefore, can be considered as the* Γ*-limit of the BVP* (3.4.1).

Proof. Let $\boldsymbol{U}^0$ minimizes functional $\mathscr{E}^3_{\mathscr{C}}$. To determine the associated Euler–Lagrange equation, for an arbitrary $\boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\Omega^h, \Gamma^h_L)$, we should solve the variational equation

$$\frac{d}{dt}\,\mathscr{E}^3_{\mathscr{C}}(U^0 + tV^0)\Big|_{t=0} = \int\limits_{\mathscr{C}} \frac{d}{dt}\,Q_3\big(\boldsymbol{U}^0(\mathscr{x}) + tV^0(\mathscr{x})\big)\Big|_{t=0}\,d\sigma = 0.$$

Calculating integrand and applying Stokes' theorem we obtain

$$\begin{aligned} &\int\limits_{\mathscr{C}} \Bigg(2\mu\sum_{j=1}^{3}\nabla_{\mathscr{C}}U_j^0\cdot\nabla_{\mathscr{C}}V_j^0 \\ &\qquad + \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\big\{\,\mathbf{div}_{\mathscr{C}}\,\widehat{U}^0 + \mathscr{H}_{\mathscr{C}}U_4^0\big\}\big\{\,\mathbf{div}_{\mathscr{C}}\,\widehat{V}^0 + \mathscr{H}_{\mathscr{C}}V_4^0\big\} + \big\langle F^0 + H^{(1),0}, V^0\big\rangle\Bigg)\,d\sigma \\ &\qquad - \int\limits_{\mathscr{C}}\sum_{j=1}^{3}\Big\{2\mu\boldsymbol{\Delta}_{\mathscr{C}}U_j^0 + \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\mathscr{D}_j\big[\,\mathbf{div}\,U_j^0 + \mathscr{H}_{\mathscr{C}}U_4^0\big] - 2\big[F_j^0 + H_j^{(1),0}\big]\Big\}V_j^0\,d\sigma \\ &\qquad\qquad + \int\limits_{\mathscr{C}}\Big\{\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\big[\,\mathbf{div}\,U_j^0 + \mathscr{H}_{\mathscr{C}}U_4^0\big] + 2\big[F_4^0 + H_4^{(1),0}\big]\Big\}\mathscr{H}_{\mathscr{C}}V_4^0\,d\sigma = 0. \end{aligned} \tag{3.4.29}$$

Taking $V_4^0 = 0$ and an arbitrary $\widehat{V}^0$ in (3.4.29) we obtain the first equation of (3.4.28), while taking $\widehat{V}^0 = 0$ and an arbitrary $V_4^0 = 0$ we obtain the second equation. □

Remark 3.4.2. The boundary value problem for shell (3.4.28) is written in the new coordinate system (2.2.3)–(2.2.6) and first three components of the displacement vector

$$\boldsymbol{U}^0(\mathscr{x}) := (U_1^0(\mathscr{x}), U_2^0(\mathscr{x}), U_3^0(\mathscr{x}), U_4^0(\mathscr{x}))^\top$$

correspond to the displacements in the direction tangent to the mid-surface $\mathscr{C}$ vectors $\boldsymbol{d}^1$,$\boldsymbol{d}^2$, $\boldsymbol{d}^3$ (projections of the coordinate vectors $\boldsymbol{e}^1$, $\boldsymbol{e}^2$, $\boldsymbol{e}^3$ on the surface $\mathscr{C}$), while the fourth one gives the displacement in the direction of the normal vector field $\boldsymbol{d}^4 = \boldsymbol{\nu}$.

Note that components of the tangential part of the displacement vector

$$\boldsymbol{U}^0(\mathscr{x}) := \big(U_1^0(\mathscr{x}), U_2^0(\mathscr{x}), U_3^0(\mathscr{x}), U_4^0(\mathscr{x})\big)^\top$$

are linearly dependent:

$$\nu_1(\mathscr{x})U_1^0(\mathscr{x}) + \nu_2(\mathscr{x})U_2^0(\mathscr{x}) + \nu_3(\mathscr{x})U_3^0(\mathscr{x}) \equiv 0 \ \text{ for all } \ \mathscr{x} \in \mathscr{C}.$$

3.5 Shell operator is non-negative

Main theorem of the present paper, Theorem 3.5.2, will be proved later. Here we recall main results about Γ-limit of the energy functional $\mathscr{E}_{\Omega^h}(\boldsymbol{U})$ in (1.4.4).

Next, we perform the scaling of the variable $t = h\tau$, $-1 < \tau < 1$, in the modified kernel $Q_4(\nabla\boldsymbol{U})$ of the quadratic part of energy functional (1.4.4) and divide by h.

Lemma 3.5.1. *The scaled and divided by h energy functional*

$$\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) = \frac{1}{h}\,\mathscr{E}_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) = \frac{1}{2}\,\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h) - \mathscr{F}^0(\widetilde{\boldsymbol{U}}^0_h) \tag{3.5.1}$$

with the quadratic and linear parts

$$\mathscr{Q}^0_4(\widetilde{\boldsymbol{U}}^h) = \int_{-1}^{1}\int_{\mathscr{C}} Q^0_4\big(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau)\big)\,d\sigma\,d\tau$$

$$\mathscr{F}^0(\widetilde{\boldsymbol{U}}^0_h) = -\int_{-h}^{h}\int_{\mathscr{C}}\Big[\langle\widetilde{\boldsymbol{F}}^0_h,\boldsymbol{U}^0_h\rangle + \frac{1}{h}\Big[\big\langle\widetilde{\boldsymbol{H}}(\mathscr{x},+h),\widetilde{\boldsymbol{U}}^{0,+}(\mathscr{x},+h)\big\rangle - \big\langle\widetilde{\boldsymbol{H}}^0(\mathscr{x},-h),\widetilde{\boldsymbol{U}}^{0,+}(\mathscr{x},-h)\big\rangle\Big]\Big]\,d\sigma\,d\tau,$$

$$\widetilde{\boldsymbol{F}}^0_h(\mathscr{x},\tau) := \Big(F^0_1(\mathscr{x},h\tau), F^0_2(\mathscr{x},h\tau), F^0_3(\mathscr{x},h\tau), F^0_4(\mathscr{x},h\tau)\Big)^\top, \ \ F^0_4 = \mathscr{N}_\alpha F_\alpha,$$

$$\widetilde{\boldsymbol{H}}^0_h(\mathscr{x},\tau) := \Big(H^0_1(\mathscr{x},h\tau), H^0_2(\mathscr{x},h\tau), H^0_3(\mathscr{x},h\tau), H^0_4(\mathscr{x},h\tau)\Big)^\top, \ \ H^0_4 = \mathscr{N}_\alpha H_\alpha,$$

is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$ (*see* (3.5.1)) *and is convex*

$$\mathscr{E}^0_{\Omega^h}\big(\theta\widetilde{\boldsymbol{U}}^h + (1-\theta)\widetilde{\boldsymbol{V}}^h\big) \leqslant \mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) + (1-\theta)\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{V}}^h), \ \ 0<\theta<1, \tag{3.5.2}$$

for arbitrary vector $\widetilde{\boldsymbol{V}}^h(\mathscr{x},\tau) := (V_1(\mathscr{x},h\tau), V_2(\mathscr{x},h\tau), V^0_3(\mathscr{x},h\tau), V_4(\mathscr{x},h\tau))^\top$, $\widetilde{\boldsymbol{V}}^h \in \widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$.

Moreover, if $\widetilde{\boldsymbol{F}}^0_h(\mathscr{x},\tau) := \boldsymbol{F}^0(\mathscr{x},h\tau)$ *is uniformly bounded in* $\mathbb{L}_2(\Omega^1)$,

$$\sup_{h<h_0}\big\|\widetilde{\boldsymbol{F}}^0_h \mid \mathbb{L}_2(\Omega^1)\big\| < \infty \tag{3.5.3}$$

for some $h_0 > 0$, *the energy functional has the following quadratic estimate: there exist positive constants* C_1, C_2 *and* C_3 *independent of the parameter h such that*

$$\begin{aligned} C_1\int_{\Omega^1}\Big[(\mathscr{D}_\alpha U^0_j(\mathscr{x},h\tau))^2 + \Big(\frac{1}{h}\frac{\partial U^0_j(\mathscr{x},h\tau)}{\partial\tau}\Big)^2\Big]\,dx - C_2 &\leqslant \mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h) \\ &\leqslant C_3\Big\{1 + \int_{\Omega^1}\Big[(\mathscr{D}_\alpha U^0_j(\mathscr{x},h\tau))^2 + \Big(\frac{1}{h}\,\frac{\partial U^0_j(\mathscr{x},h\tau)}{\partial\tau}\Big)^2\Big]\,dx\Big\} \end{aligned} \tag{3.5.4}$$

for all $\widetilde{\boldsymbol{U}}^h \in \widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$.

Proof see in [14, Lemma 3.1].

Theorem 3.5.1. *Let the weak limits*

$$\lim_{h\to 0}\boldsymbol{F}(\mathscr{x},h\tau)=\boldsymbol{F}(\mathscr{x}),\quad \lim_{h\to 0}\frac{1}{2h}\big[\boldsymbol{H}(\mathscr{x},+h)-\boldsymbol{H}(\mathscr{x},-h)\big]=\boldsymbol{H}^{(1)}(\mathscr{x}),\quad \boldsymbol{F},\boldsymbol{H}^{(1)}\in\mathbb{L}_2(\mathscr{C}) \tag{3.5.5}$$

exist, respectively, in $\mathbb{L}_2(\Omega^h)$ *and* $\mathbb{L}_2(\mathscr{C})$. *Then the* Γ*-limit of the energy functional* $\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h)$ *exists*

$$\Gamma-\lim_{h\to 0}\mathscr{E}^0_{\Omega^h}(\widetilde{\boldsymbol{U}}^h)=\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}}):=\int\limits_{\mathscr{C}}Q_3(\overline{\boldsymbol{U}}(\mathscr{x}))\,d\sigma, \tag{3.5.6}$$

where

$$Q_3(\overline{\boldsymbol{U}})=\frac{\mu}{2}\Big[\big[\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha\big]^2-2\nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\alpha\overline{U}_\gamma\Big] \\ +\frac{2\lambda\mu}{\lambda+2\mu}\,(\mathscr{D}_\alpha\overline{U}_\alpha)^2+\big\langle\boldsymbol{F}(\mathscr{x})+2\boldsymbol{H}^{(1)}(\mathscr{x}),\overline{U}(\mathscr{x})\big\rangle \tag{3.5.7}$$

and

$$\overline{\boldsymbol{U}}(\mathscr{x}):=\big(\overline{U}_1(\mathscr{x}),\overline{U}_2(\mathscr{x}),\overline{U}_3(\mathscr{x})\big)^\top,\quad \overline{U}_\alpha(\mathscr{x}):=U_\alpha(\mathscr{x},0),\ \ \alpha=1,2,3.$$

Proof see in [14, Theorem 3.2].

Theorem 3.5.2. *Let* $\boldsymbol{F},\boldsymbol{H}^{(1)}\in\mathbb{L}_2(\mathscr{C})$. *The vector-function* $\overline{\boldsymbol{U}}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$ *which minimizes the energy functional* $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ *in* (3.5.6)–(3.5.7) *is a solution to the following boundary value problem:*

$$\begin{cases}(\mathscr{L}_{\mathscr{C}}\overline{\boldsymbol{U}})_\alpha:=\mu\Big[\boldsymbol{\Delta}_{\mathscr{C}}\overline{U}_\alpha+\mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\beta-2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\beta-\mathscr{D}_\gamma(\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta)\Big]\\ \qquad +\dfrac{4\lambda\mu}{\lambda+2\mu}\big[\mathscr{D}_\alpha\mathscr{D}_\beta\overline{U}_\beta-2\mathscr{H}_{\mathscr{C}}\nu_\alpha\mathscr{D}_\beta\overline{U}_\beta\big]=\dfrac{1}{2}\,F_\alpha+H^{(1)}_\alpha & \text{on } \mathscr{C},\\ \overline{U}_\alpha(t)=0 & \text{on } \Gamma=\partial\mathscr{C},\\ & \alpha=1,2,3.\end{cases} \tag{3.5.8}$$

Vice versa: on the solution $\overline{\boldsymbol{U}}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$ *to the boundary value problem* (3.5.8) *under the condition* $\boldsymbol{F},\boldsymbol{H}^{(1)}\in\mathbb{L}_2(\mathscr{C})$, *the energy functional* $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ *in* (3.5.6)–(3.5.7) *attains the minimum.*

Moreover, the operator $\mathscr{L}_{\mathscr{C}}$ *in the left-hand side of the shell equation* (3.5.8) *is elliptic, positive definite and has finite-dimensional kernel, which consists of the solutions to the following system of equations:*

$$\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)\big]\equiv 0,\ \ \alpha,\beta=1,2,3. \tag{3.5.9}$$

The boundary value problem (3.5.8) *has a unique solution in the classical setting:*

$$\overline{\mathbf{U}}:=(\overline{U}_1,\overline{U}_2,\overline{U}_3)^\top\in\mathbb{H}^1(\mathscr{C}),\ \ \frac{1}{2}\,\mathbf{F}+\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathscr{C}). \tag{3.5.10}$$

Proof. The first part of the theorem that BVP (3.5.8) is the Γ-limit of the BVP (1.4.1) (i.e., the solution to the BVP (3.5.8) $\overline{\boldsymbol{U}}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$ minimizes the energy functional $\mathscr{E}^3_{\mathscr{C}}(\overline{\boldsymbol{U}})$ in (3.5.6)–(3.5.7)) is proved in [14, Theorem 3.3].

Ellipticity of the operator $\mathscr{L}_{\mathscr{C}}$ in the left-hand side of the shell equation (3.5.8) is checked directly and it is Fredholm operator in the setting $\mathscr{L}_{\mathscr{C}}:\mathbb{H}^{-1}(\mathscr{C})\to\mathbb{H}^1(\mathscr{C})$. This follows from the Lax–Milgram Lemma (see [70, Theorem 14] for a similar proof). Therefore, $\mathscr{L}_{\mathscr{C}}$ has the finite-dimensional kernel.

Let us start with the energy functional and recall the quadratic part of the energy functional (see (3.5.1) and formulae [14, (2.7)]):

$$\mathscr{Q}^0_4(\boldsymbol{U})=\int\limits_{-h}^{h}\int\limits_{\mathscr{C}}Q^0_4(\nabla\boldsymbol{U}(\mathscr{x},t))\,d\sigma\,dt,\ \ Q^0_4(\boldsymbol{F})=2\mu|\boldsymbol{E}|^2+\lambda(\mathrm{Trace}\,E)^2,\ \ \boldsymbol{E}=\frac{1}{2}\,(\boldsymbol{F}+\boldsymbol{F}^\top), \tag{3.5.11}$$

where $\boldsymbol{F} = [F_{\alpha\beta}]_{3\times3}$ and $\boldsymbol{E} = [E_{\alpha\beta}]_{3\times3}$ are 3×3 matrix and $|\boldsymbol{E}|^2 = \mathrm{Trace}(\boldsymbol{E}^\top\boldsymbol{E}) = \sum\limits_{\alpha,\beta} E^2_{\alpha\beta}$. From Lemma 3.5.1 it follows that the kernel $Q^0_4(\boldsymbol{F})$ is non-negative:

$$Q^0_4(\boldsymbol{F}) = 2\mu\sum_{\alpha\neq\beta} E^2_{\alpha\beta} + (\mu+\lambda)\Big(\sum_\alpha E_{\alpha\alpha}\Big)^2 + \mu\sum_{\alpha\neq\beta}(E_{\alpha\alpha}-E_{\beta\beta})^2 \geq 0. \tag{3.5.12}$$

Let us rewrite the kernel $Q^0_4(\nabla\boldsymbol{U})$ of the quadratic part $\mathscr{Q}^0_4(\boldsymbol{U})$ of the energy functional in (1.4.4), (3.5.11), (3.5.12) by using the equalities

$$\boldsymbol{F} = \nabla\boldsymbol{U} = [\partial_\alpha U_\beta]_{3\times3}, \quad (\mathrm{Def}\,\boldsymbol{U}) := \frac{1}{2}\left((\nabla\boldsymbol{U}) + (\nabla\boldsymbol{U})^\top\right) = \Big[\frac{1}{2}\left(\partial_\alpha U_\beta + \partial_\beta U_\alpha\right)\Big]_{3\times3}$$

and (5.4.5) as follows:

$$\begin{aligned} Q_4(\nabla\boldsymbol{U}) &= 2\mu\sum_{\alpha\neq\beta}(\mathrm{Def}\,\boldsymbol{U})^2_{\alpha\beta} + (\mu+\lambda)\Big(\sum_\alpha \partial_\alpha U_\alpha\Big)^2 + \mu\sum_{\alpha\neq\beta}\big[\partial_\alpha U_\alpha - \partial_\beta U_\beta\big]^2 \qquad (3.5.13)\\ &= 2\mu\sum_{\alpha\neq\beta}\Big[(\mathscr{D}\mathrm{ef}\,\boldsymbol{U})_{\alpha\beta} + \frac{\nu_\alpha\mathscr{D}_4 U_\alpha + \nu_\beta\mathscr{D}_4 U_\beta}{2}\Big]^2 + (\mu+\lambda)\Big(\sum_\alpha \mathscr{D}_\alpha U_\alpha + \mathscr{D}_4 U_4\Big)^2\\ &\quad + \mu\sum_{\alpha\neq\beta}\Big[\mathscr{D}_\alpha U_\alpha - \mathscr{D}_\beta U_\beta + \nu_\alpha\mathscr{D}_4 U_\alpha - \nu_\beta\mathscr{D}_4 U_\beta\Big]^2\\ &= 2\mu\sum_{\alpha\neq\beta}\Big[(\mathscr{D}\mathrm{ef}\,\boldsymbol{U})_{\alpha\beta} + \frac{\nu_\alpha\mathscr{D}_4 U_\alpha + \nu_\beta\mathscr{D}_4 U_\beta}{2}\Big]^2 + (\mu+\lambda)\Big(\sum_\alpha \mathscr{D}_\alpha U_\alpha + \mathscr{D}_4 U_4\Big)^2\\ &\quad + \mu\sum_{\alpha,\beta}\Big[\mathscr{D}_\alpha U_\alpha - \mathscr{D}_\beta U_\beta + \nu_\alpha\mathscr{D}_4 U_\alpha - \nu_\beta\mathscr{D}_4 U_\beta\Big]^2, \qquad (3.5.14) \end{aligned}$$

where

$$(\mathscr{D}\mathrm{ef}\,\boldsymbol{U})_{\alpha\beta} := \frac{\mathscr{D}_\alpha U_\beta + \mathscr{D}_\beta U_\alpha}{2}, \quad \alpha,\beta = 1,2,3.$$

Next, we perform the scaling of the variable $t = h\tau$, $-1 < \tau < 1$, in the modified kernel $Q_4(\nabla\boldsymbol{U})$ of the quadratic part of energy functional (3.5.13), divide by h and study the following kernel in the scaled domain $\Omega^1 = \mathscr{C}\times(1,1)$:

$$\begin{aligned} Q^0_4\big(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau)\big) &= \frac{1}{h}\,Q_4\big(\nabla\boldsymbol{U}(\mathscr{x},h\tau)\big)\\ &= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau) + \mathscr{D}_\beta U_\alpha(\mathscr{x},h\tau) + \frac{\nu_\alpha}{h}\,\frac{\partial U_\beta(\mathscr{x},h\tau)}{\partial\tau} + \frac{\nu_\beta}{h}\,\frac{\partial U_\alpha(\mathscr{x},h\tau)}{\partial\tau}\Big]^2\\ &\qquad + (\mu+\lambda)\Big(\sum_\alpha \mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau) + \frac{1}{h}\,\frac{\partial U_4(\mathscr{x},h\tau)}{\partial\tau}\Big)^2\\ &\quad + \mu\sum_{\alpha,\beta}\Big[\mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau) - \mathscr{D}_\beta U_\beta(\mathscr{x},h\tau) + \frac{\nu_\alpha}{h}\,\frac{\partial U_\alpha(\mathscr{x},h\tau)}{\partial\tau} - \frac{\nu_\beta}{h}\,\frac{\partial U_\beta(\mathscr{x},h\tau)}{\partial\tau}\Big]^2, \qquad (3.5.15) \end{aligned}$$

where

$$\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau) := \big(U^0_1(\mathscr{x},h\tau), U^0_2(\mathscr{x},h\tau), U^0_3(\mathscr{x},h\tau), U^0_4(\mathscr{x},h\tau)\big)^\top, \ \ U^0_4 = \mathscr{N}_\alpha U_\alpha. \tag{3.5.16}$$

For this, let us rewrite Q^0_4 in (3.5.15) in the form

$$\begin{aligned} Q^0_4\big(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{x},\tau)\big) &= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathscr{D}_\alpha U_\beta(\mathscr{x},h\tau) + \mathscr{D}_\beta U_\alpha(\mathscr{x},h\tau) + \mathscr{N}_\alpha\xi_\beta + \mathscr{N}_\beta\xi_\alpha\Big]^2\\ + (\mu+\lambda)\Big(\sum_\alpha \mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau) + \xi_4\Big)^2 &+ \mu\sum_{\alpha,\beta}\Big[\mathscr{D}_\alpha U_\alpha(\mathscr{x},h\tau) - \mathscr{D}_\beta U_\beta(\mathscr{x},h\tau) + \mathscr{N}_\alpha\xi_\alpha - \mathscr{N}_\beta\xi_\beta\Big]^2 \end{aligned}$$

$$= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathscr{D}_\alpha U_\beta(\mathscr{X},h\tau)+\mathscr{D}_\beta U_\alpha(\mathscr{X},h\tau)+\mathscr{N}_\alpha\xi_\beta+\mathscr{N}_\beta\xi_\alpha\Big]^2+(\mu+\lambda)\big(\mathscr{D}\mathrm{iv}\,\boldsymbol{U}(\mathscr{X},h\tau)+\xi_4\big)^2$$

$$+\,\mu\sum_{\alpha,\beta}\Big[\mathscr{D}_\alpha U_\alpha(\mathscr{X},h\tau)-\mathscr{D}_\beta U_\beta(\mathscr{X},h\tau)+\mathscr{N}_\alpha\xi_\alpha-\mathscr{N}_\beta\xi_\beta\Big]^2, \quad (3.5.17)$$

where the variables

$$\xi_\alpha=\xi_\alpha(\mathscr{X},h\tau):=\frac{1}{h}\,\frac{\partial U_\alpha(\mathscr{X},h\tau)}{\partial\tau},\ \ \alpha=1,2,3,\ \ \xi_4=\mathscr{N}_\alpha\xi_\alpha \quad (3.5.18)$$

depend on h, and we will find minimum of the kernel $Q_4^0(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}(\mathscr{X},\tau))$ with respect to the variables $\xi_1,\ \xi_2,\ \xi_3$. It was shown in [14] that by $Q_4^0(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h(\mathscr{X},\tau))$ the Γ-limit is attained on the following values of the variables:

$$\xi_4=-\frac{\lambda}{\lambda+2\mu}\,\mathscr{D}_\beta U_\beta=-\frac{\lambda}{\lambda+2\mu}\,\mathscr{D}\mathrm{iv}\,\boldsymbol{U}, \quad (3.5.19)$$

$$\xi_\alpha=-\mathscr{N}_\gamma(\mathscr{D}_\alpha U_\gamma)-\frac{\lambda}{\lambda+2\mu}\,\mathscr{N}_\alpha\mathscr{D}\mathrm{iv}\,\boldsymbol{U},\ \ \alpha=1,2,3, \quad (3.5.20)$$

where we remind $\mathscr{D}\mathrm{iv}\,\boldsymbol{U}=\mathscr{D}_\alpha U_\alpha$. From (3.5.19), (3.5.21) and (3.5.17) we find the Γ-limit $Q_3^0(\overline{\boldsymbol{U}})$ (the same as in [14], but written in a different form):

$$\begin{aligned}
Q_3^0(\overline{\boldsymbol{U}}) &= \min_{\xi_1,\xi_2,\xi_3} Q_4^0(\nabla_{\Omega^h}\widetilde{\boldsymbol{U}}^h)\\
&=\frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)\big]-\frac{2\lambda}{\lambda+2\mu}\,\nu_\alpha\nu_\beta\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}\Big]^2\\
&\quad+(\mu+\lambda)\Big(\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}-\frac{\lambda}{\lambda+2\mu}\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}\Big)^2\\
&\quad+\mu\sum_{\alpha,\beta}\Big[\mathscr{D}_\alpha\overline{U}_\alpha-\mathscr{D}_\beta\overline{U}_\beta-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)-\nu_\beta\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)\big]\\
&\qquad\qquad-\frac{\lambda}{\lambda+2\mu}\,\nu_\alpha^2\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}+\frac{\lambda}{\lambda+2\mu}\,\nu_\beta^2\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}\Big]^2\\
&=\frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha-\sum_\gamma\Big[\nu_\alpha\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\gamma\Big]-\frac{2\lambda}{\lambda+2\mu}\,\nu_\alpha\nu_\beta\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}\Big]^2\\
&\quad+\frac{4\mu^2(\mu+\lambda)}{(\lambda+2\mu)^2}\,\big[\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}\big]^2\\
&\quad+\mu\sum_{\alpha,\beta}\Big[\mathscr{D}_\alpha\overline{U}_\alpha-\mathscr{D}_\beta\overline{U}_\beta-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)-\nu_\beta\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)\big]\Big]^2.
\end{aligned}\quad (3.5.21)$$

From (3.5.21) it follows that $Q_3^0(\overline{\boldsymbol{U}})$ is a nonnegative quadratic form $Q_3^0(\overline{\boldsymbol{U}})\geqslant 0$ for all $\boldsymbol{U}\in\mathbb{H}^1(\mathscr{C},\Gamma)$, $\Gamma:=\partial\mathscr{C}$. □

3.6 Shell operator is positive definite

If $Q_3^0(\overline{\boldsymbol{U}})\equiv 0$, from (3.5.21) we get

$$\begin{aligned}
&\mathscr{D}\mathrm{iv}\,\overline{\boldsymbol{U}}\equiv 0,\\
&\mathscr{D}_\alpha\overline{U}_\alpha-\mathscr{D}_\beta\overline{U}_\beta-\sum_\gamma\Big[\nu_\alpha\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)-\nu_\beta\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)\Big]\equiv 0,\ \ \alpha\neq\beta=1,2,3,\\
&\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha-\sum_\gamma\Big[\nu_\alpha\nu_\gamma(\mathscr{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)\Big]\equiv 0,\ \ \alpha\neq\beta=1,2,3.
\end{aligned}\quad (3.6.1)$$

Taking the sum over β in the second equality in (3.6.1), we get

$$\mathscr{D}_\alpha\overline{U}_\alpha=\sum_\gamma \nu_\alpha\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma),\quad \alpha=1,2,3. \tag{3.6.2}$$

Note that the obtained equality implies both, the first and the second equalities from (3.6.1). Moreover, it coincides with the third equality in (3.6.1) if we allow there $\alpha=\beta=1,2,3$. Thus, equation (3.5.9) implies all three equalities in (3.6.1) and describes the kernel $\mathbf{Ker}\,\mathscr{L}_\mathscr{C}$ of the shell equation $\mathscr{L}_\mathscr{C}$ in (3.5.8).

Now we rewrite the obtained equation in the following form:

$$\begin{aligned}\mathscr{D}_\alpha\overline{U}_\alpha&=\sum_\gamma \nu_\alpha\nu_\gamma(\mathscr{D}_\alpha\overline{U}_\gamma)=\nu_\alpha\mathscr{D}_\alpha\Big(\sum_\gamma\nu_\gamma\overline{U}_\gamma\Big)-\sum_\gamma\nu_\alpha(\mathscr{D}_\alpha\nu_\gamma)\overline{U}_\gamma\\&=\nu_\alpha(\mathscr{D}_\alpha\overline{U}_4)-\sum_\gamma\nu_\alpha(\mathscr{D}_\alpha\nu_\gamma)\overline{U}_\gamma,\quad \overline{U}_4=\sum_\gamma\nu_\gamma\overline{U}_\gamma,\quad \alpha=1,2,3.\end{aligned} \tag{3.6.3}$$

Similarly to (3.6.3), from equality (3.5.9) (see the third equality in (3.6.1)) we derive

$$\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha=\nu_\alpha\mathscr{D}_\beta\overline{U}_4+\nu_\beta\mathscr{D}_\alpha\overline{U}_4-\sum_\gamma\big[\nu_\alpha(\mathscr{D}_\beta\nu_\gamma)+\nu_\beta(\mathscr{D}_\alpha\nu_\gamma)\big]\overline{U}_\gamma,\quad \alpha,\beta=1,2,3. \tag{3.6.4}$$

To equalities (3.5.9), (3.6.3), (3.6.4) we add the following:

$$\begin{aligned}&\sum_{\alpha,\beta}\Big[\big[\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha\big]^2-2\sum_\gamma\nu_\beta\nu_\gamma\mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\alpha\overline{U}_\gamma\Big]\\&\quad=\sum_{\alpha,\beta}\Big[\big[\mathscr{D}_\alpha\overline{U}_\beta+\mathscr{D}_\beta\overline{U}_\alpha\big]^2\Big]-2\sum_\alpha(\mathscr{D}_\alpha\overline{U}_4)^2-2\sum_{\alpha,\beta,\gamma}(\mathscr{D}_\alpha\nu_\beta)(\mathscr{D}_\alpha\nu_\gamma)\overline{U}_\beta\overline{U}_\gamma\\&\qquad+2\sum_{\alpha,\beta}(\mathscr{D}_\alpha\nu_\beta)(\mathscr{D}_\alpha\overline{U}_4)\overline{U}_\beta-2\sum_{\alpha,\gamma}(\mathscr{D}_\alpha\nu_\gamma)(\mathscr{D}_\alpha\overline{U}_4)\overline{U}_\gamma\equiv0,\end{aligned} \tag{3.6.5}$$

which follows from (3.5.7) if we apply the first equality from (3.6.1) and recall that $Q^0_3(\overline{\boldsymbol{U}})=0$.

If $\overline{U}_\alpha(\mathfrak{s})=0$, $\alpha=1,2,3$, equalities (3.6.3)–(3.6.5) simplify:

$$\begin{aligned}\mathscr{D}_\alpha(\mathfrak{s})\overline{U}_\alpha(\mathfrak{s})&=\nu_\alpha(\mathfrak{s})\mathscr{D}_\alpha\overline{U}_4(\mathfrak{s}),\\ \mathscr{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathscr{D}_\beta\overline{U}_\alpha(\mathfrak{s})&=\nu_\alpha(\mathfrak{s})\mathscr{D}_\beta\overline{U}_4(\mathfrak{s})+\nu_\beta(\mathfrak{s})\mathscr{D}_\alpha\overline{U}_4(\mathfrak{s}),\quad\alpha,\beta=1,2,3,\\ \sum_{\alpha,\beta}\Big[\big[\mathscr{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathscr{D}_\beta\overline{U}_\alpha(\mathfrak{s})\big]^2\Big]&=2\sum_\alpha(\mathscr{D}_\alpha\overline{U}_4(\mathfrak{s}))^2,\quad\mathfrak{s}\in\partial\mathscr{C}.\end{aligned} \tag{3.6.6}$$

We can see that not only the first equality in (3.6.6) is the consequence of the second one (by taking $\alpha=\beta$), but also the third equality follows from the second one if we take into account that $\sum\limits_\alpha\nu_\alpha^2=1$ and $\sum\limits_\alpha\nu_\alpha\mathscr{D}_\alpha=0$.

By inserting the first equality from (3.6.6) into the second one we get

$$\mathscr{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathscr{D}_\beta\overline{U}_\alpha(\mathfrak{s})=\frac{\nu_\alpha(\mathfrak{s})}{\nu_\beta(\mathfrak{s})}\,\mathscr{D}_\beta\overline{U}_\beta(\mathfrak{s})+\frac{\nu_\beta(\mathfrak{s})}{\nu_\alpha(\mathfrak{s})}\,\mathscr{D}_\alpha\overline{U}_\alpha(\mathfrak{s}),\quad\alpha,\beta=1,2,3.$$

If we succeed to prove that

$$\mathscr{D}_\alpha\overline{U}_4(\mathfrak{s})\equiv0,\quad\mathfrak{s}\in\partial\mathscr{C},\quad\alpha=1,2,3, \tag{3.6.7}$$

from (3.6.6) and (3.6.7) follow

$$\mathscr{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathscr{D}_\beta\overline{U}_\alpha(\mathfrak{s})\equiv0,\quad\mathfrak{s}\in\partial\mathscr{C},\quad\alpha,\beta=1,2,3. \tag{3.6.8}$$

The latter implies that

$$\mathscr{D}_\alpha \overline{U}_\beta(\mathfrak{s}) \equiv 0, \ \ \forall\, \alpha, \beta = 1, 2, 3, \ \ \forall\, \mathfrak{s} \in \partial\mathscr{C}. \tag{3.6.9}$$

Indeed, among directing tangent vector fields $\{\boldsymbol{d}^k(\mathfrak{s})\}_{k=1}^3$ generating Günter's derivatives $\mathscr{D}_k = \partial_{\boldsymbol{d}^k}$, $k = 1, 2, 3$, only two are linearly independent (one of these vectors might even collapse at a point $\boldsymbol{d}^k(\mathfrak{s}) = 0$ if the corresponding basis vector $\boldsymbol{e}^k$ is orthogonal to the surface at $\mathfrak{s} \in \mathscr{C}$). One of these vectors might be tangent to the boundary curve $\partial\mathscr{C}$ and, at least one, say $\boldsymbol{d}^3(\mathfrak{s})$, is non-tangential to $\partial\mathscr{C}$. The vector $\boldsymbol{d}^\alpha$ for $\alpha = 1, 2, 3$ is a linear combination $\boldsymbol{d}^\alpha(\mathfrak{s}) = c_1(\mathfrak{s})\boldsymbol{d}^3(\mathfrak{s}) + c_2(\mathfrak{s})\boldsymbol{\tau}^\alpha(\mathfrak{s})$ of the non-tangential vector $\boldsymbol{d}^3(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^\alpha(\mathfrak{s}) := \pi_{\partial\mathscr{C}}\boldsymbol{d}^\alpha(\mathfrak{s})$ of the vector $\boldsymbol{d}^\alpha(\mathfrak{s})$ to the boundary curve $\partial\mathscr{C}$ at the point $\mathfrak{s} \in \partial\mathscr{C}$. Then

$$(\mathscr{D}_\alpha U_3)(\mathfrak{s}) = c_1(\mathfrak{s})(\partial_{\boldsymbol{d}^3} U_3)(\mathfrak{s}) + c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^\alpha} U_3)(\mathfrak{s}) = c_1(\mathfrak{s})(\mathscr{D}_3 U_3)(\mathfrak{s}) \tag{3.6.10}$$

for all $\mathfrak{s} \in \gamma$ and all $\alpha = 1, 2, 3$, because $(\mathscr{D}_{\boldsymbol{d}^3} U_3)(\mathfrak{s}) = (\mathscr{D}_3 U_3)(\mathfrak{s})U_3$, U_3 vanishes identically on $\partial\mathscr{C}$ and the derivative $(\partial_{\boldsymbol{\tau}^j} U_3^0)(\mathfrak{s}) = 0$ vanishes, as well.

On the other hand, from (3.6.8) for $\beta = \alpha = 3$ it follows $2\mathscr{D}_3 U_3(\mathfrak{s}) = 0$ and, together with (3.6.10), this gives $(\mathscr{D}_\alpha U_3)(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \gamma$, $\beta = 1, 2, 3$. Then, due to (3.6.8), $(\mathscr{D}_3 U_\alpha)(\mathfrak{s}) = (\mathscr{D}_\alpha U_3)(\mathfrak{s}) = 0$ and, due to (3.6.8), $(\mathscr{D}_\alpha U_\alpha)(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \gamma$, $\alpha = 1, 2, 3$. Applying again the above arguments exposed for U_3, we prove equalities (3.6.9).

3.7 Numerical approximation of the shell equation

Consider the boundary value problem (3.5.8)

$$\begin{cases} (\mathscr{L}_{\mathscr{C}}\overline{\boldsymbol{U}})_\alpha := \mu\Big[\Delta_{\mathscr{C}}\overline{U}_\alpha + \mathscr{D}_\beta\mathscr{D}_\alpha\overline{U}_\beta - 2\mathscr{H}_{\mathscr{C}}\nu_\beta\mathscr{D}_\alpha\overline{U}_\beta - \mathscr{D}_\gamma(\nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta)\Big] & \\ \qquad\qquad + \dfrac{4\lambda\mu}{\lambda+2\mu}\Big[\mathscr{D}_\alpha\mathscr{D}_\beta\overline{U}_\beta - 2\mathscr{H}_{\mathscr{C}}\nu_\alpha\mathscr{D}_\beta\overline{U}_\beta\Big] = \dfrac{1}{2}\,G_\alpha, & \text{on } \mathscr{C}, \\ \overline{U}_\alpha(t) = 0, & \text{on } \Gamma = \partial\mathscr{C}, \\ & \alpha = 1, 2, 3, \end{cases} \tag{3.7.1}$$

where $G_\alpha = F_\alpha + 2H_\alpha^{(1)} \in [\mathbb{L}_2(\mathscr{C})]$, $\alpha = 1, 2, 3$.

In [14, Theorem 5.1], it is proved that if $\overline{\boldsymbol{U}} \in [\widetilde{\mathbb{H}}^1(\mathscr{C})]^3$ is a solution of the BVP (3.7.1) and $\overline{\boldsymbol{V}} \in [\widetilde{\mathbb{H}}^1(\mathscr{C})]^3$, then

$$\int\limits_{\mathscr{C}} \bigg\{2\mu\Big[\mathscr{D}_\beta\overline{U}_\alpha\mathscr{D}_\beta\overline{V}_\alpha + \mathscr{D}_\alpha\overline{U}_\beta\mathscr{D}_\beta\overline{V}_\alpha - \nu_\alpha\nu_\beta\mathscr{D}_\gamma\overline{U}_\beta\mathscr{D}_\gamma\overline{V}_\alpha\Big] + \frac{4\lambda\mu}{\lambda+2\mu}\,\mathscr{D}_\beta\overline{U}_\beta\mathscr{D}_\alpha\overline{V}_\alpha\bigg\}\,d\sigma$$

$$= \int\limits_{\mathscr{C}} \langle\overline{G}_\alpha, \overline{V}_\alpha\rangle\,d\sigma. \tag{3.7.2}$$

Therefore, the BVP (3.7.1) can be reformulated in the following way.

Find a vector $\overline{\boldsymbol{U}} \in [\widetilde{\mathbb{H}}^1(\mathscr{C})]^3$ satisfying equation (3.7) for any $\overline{\boldsymbol{V}} \in [\widetilde{\mathbb{H}}^1(\mathscr{C})]^3$

$$\big(c_{\alpha\beta\gamma\zeta}(x)\mathscr{D}_\beta U_\alpha, \mathscr{D}_\zeta V_\gamma\big) = \big(G_\alpha, V_\alpha\big), \ \ \forall\,\boldsymbol{V} \in [\widetilde{\mathbb{H}}^1(\mathscr{C})]^3, \tag{3.7.3}$$

where

$$c_{\alpha\beta\gamma\zeta}(x) = \frac{4\lambda\mu}{\lambda+2\mu}\,\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\gamma}\delta_{\beta\zeta} + \delta_{\alpha\zeta}\delta_{\beta\gamma} - \nu_\alpha\nu_\gamma\delta_{\beta\zeta})$$

and $(\,\cdot\,,\,\cdot\,)$ denotes an inner product

$$(f, g) = \int\limits_{\mathscr{C}} \langle f, g\rangle\,d\sigma.$$

Due to (3.5.21), the sesquilinear form

$$a(U, V) := \big(c_{\alpha\beta\gamma\zeta}\mathscr{D}_\beta U_\alpha, \mathscr{D}_\zeta V_\gamma\big) \tag{3.7.4}$$

is bounded and coercive in $\mathbb{H}^1_0(\mathscr{C})$

$$M_1\big\|U \mid \mathbb{H}^1(\mathscr{C})\big\|^2 \geq a(U,U) \geq M\big\|U \mid \mathbb{H}^1(\mathscr{C})\big\|^2, \ \ \forall\, U \in [\mathbb{H}^1_0(\mathscr{C})]^3, \tag{3.7.5}$$

for some $M > 0$, $M_1 > 0$. Therefore, by Lax–Milgram theorem, problem (3.7.3) possesses a unique solution.

Now, let us consider the discrete counterpart of the problem.

Let X_h be a family of finite-dimensional subspaces approximating $[\mathbb{H}^1(\mathscr{C})]^3$, i.e., such that $\bigcup\limits_h X_h$ is dense in $[\mathbb{H}^1(\mathscr{C})]^3$.

Consider equation (3.7.3) in the finite-dimensional space X_h

$$a(U_h, V_h) = \widetilde{g}(V_h), \ \ \forall\, V \in X_h, \tag{3.7.6}$$

where $\widetilde{g}(V_h) = -\big(G, V_h\big)_{\mathscr{C}}$.

Theorem 3.7.1. *Equation* (3.7.6) *has a unique solution* $U_h \in X_h$ *for all* $h > 0$. *This solution converges in* $[\mathbb{H}^1(\mathscr{C})]^3$ *to the solution* U *of* (3.7.3) *as* $h \to 0$.

Proof. Immediately follows from the coercivity of sesquilinear form $a(\cdot,\cdot)$

$$c_1\big\|U_h \mid [\mathbb{H}^1(\mathscr{C})]^3\big\|^2 \leq a(U_h, U_h) = |\widetilde{f}(U_h)| \leq c_2\big\|U_h \mid [\mathbb{H}^1(\mathscr{C})]^3\big\| \ \text{ for all } \ h. \tag{3.7.7}$$

Let U_h be a unique solution of the homogeneous equation

$$a(U_h, \psi_h) = 0 \ \text{ for all } \ \psi_h \in X_h. \tag{3.7.8}$$

Then (3.7.7) implies $\|U_h \mid [\mathbb{H}^1(\mathscr{C})]^3\| = 0$ and, consequently, $U_h = 0$. Therefore, equation (3.7.6) has a unique solution. From (3.7.7) it also follows that

$$\big\|U_h \mid [\mathbb{H}^1(\mathscr{C})]^3\big\|^2 \leq \frac{c_2}{c_1}\,\big\|U_h \mid [\mathbb{H}^1(\mathscr{C})]^3\big\|.$$

Hence, the sequence $\{\|U_h \mid [\mathbb{H}^1(\mathscr{C})]^3\|\}$ is bounded and we can extract a subsequence $\{U_{h_k}\}$ which converges weakly to some $U \in \mathbb{H}^1(\mathscr{C})$.

Let us take an arbitrary $V \in [\mathbb{H}^1(\mathscr{C})]^3$ and for each $h > 0$ choose $V_h \in X_h$ such that $V_h \to V$ in $[\mathbb{H}^1(\mathscr{C})]^3$. Then from (3.7.6) we have

$$a(U, V) = \widetilde{g}(V), \ \ \forall\, V \in [\mathbb{H}^1(\mathscr{C})]^3.$$

Hence, U solves (3.7.3). Note that since (3.7.3) is uniquely solvable, each subsequence $\{U_{h_k}\}$ converges weakly to the same solution U and, consequently, the whole sequence $\{U_h\}$ also converges weakly to U. Now let us prove that it converges in the space $[\mathbb{H}^1(\mathscr{C})]^3$.

Indeed, due to (3.7.7) we have

$$\begin{aligned} c_1\|U_h - U\|^2 &\leq \big|a(U_h - U, U_h - U)\big| \leq \big|a(U_h, U_h - U) - a(U, U_h - U)\big| \\ &= c_1\big|\widetilde{g}(U_h) - a(U_h, U) - \widetilde{g}(U_h - U)\big| \to c_1\big|\widetilde{g}(U) - a(U,U)\big| = 0, \end{aligned}$$

which completes the proof. □

We can choose spaces X_h in different ways.

In particular, consider a case, when ω in parametrization (3.1.1) is a square part of $\mathbb{R}^2$

$$\omega = \big\{(x_1, x_2): \ \ 0 < x_1 < 1, \ \ 0 < x_2 < 1\big\}, \ \ \vartheta(\omega) = \mathscr{C}.$$

Allocate N^2 nodes $P_{ij} = (\frac{i}{N+1}\,, \frac{j}{N+1})$, $i, j = 1, \dots, N$, on ω.

Let α_k, $k=1,\dots,N$, be piecewise linear functions defined on segment $[0,1]$ as follows:

$$\alpha_k(x)=\begin{cases}0, & 0\le x\le \dfrac{k-1}{N+1},\\ (N+1)\Big(x-\dfrac{k-1}{N+1}\Big), & \dfrac{k-1}{N+1}<x\le\dfrac{k}{N+1}\,,\\ (N+1)\Big(\dfrac{k+1}{N+1}-x\Big), & \dfrac{k}{N+1}<x\le\dfrac{k+1}{N+1}\,,\\ 0, & \dfrac{k+1}{N+1}<x\le 1\,.\end{cases} \tag{3.7.9}$$

Denote by φ_{ij}, $i,j=1,\dots,N$, the functions

$$\varphi_{ij}(x_1,\,x_2)=\alpha_i(x_1)\alpha_j(x_2),\ \ i,j=1,\dots,N,\ \ (x_1,\,x_2)\in\omega. \tag{3.7.10}$$

Evidently, φ_{ij} are continuous functions, which take their maximal value $\varphi_{ij}(P_{ij})=1$ at point P_{ij} and vanish outside the set

$$\omega_{ij}=\omega\cap\Big\{(x_1,x_2):\ \ 0\le\Big|x_1-\frac{i}{N+1}\Big|\le1,\ \ 0\le\Big|x_2-\frac{j}{N+1}\Big|\le1\Big\}, \tag{3.7.11}$$

consequently, they belong to $\mathbb{H}^1(\omega)$ and are linearly independent.

Denote by X_N the linear span of the functions $\widehat{\varphi}_{ij}=\varphi_{ij}\circ\zeta$, $i,j=1,\dots,N$. The space X_N is N^2-dimensional space contained into $\mathbb{H}^1(\mathscr{C})$.

Let $\widetilde{\varphi}^{(k)}_{ij}=(\delta_{1k},\delta_{2k},\delta_{3k})\widehat{\varphi}_{ij}\in[X_N]^3$, $k=1,2,3;\ \ i,j=1,\dots,N$.

Consider equation (3.7.6) in the space $[X_N]^3$,

$$a(U,V)=\widetilde{g}(V),\ \ \forall\,V\in[X_N]^3. \tag{3.7.12}$$

We are looking for a solution $U\in[X_N]^3$ of equation (3.7.12) in the form

$$U=\sum_{m=1}^{3}\sum_{i,j=1}^{N}C^{(m)}_{ij}\widetilde{\varphi}^{(m)}_{ij}, \tag{3.7.13}$$

where $C^{(m)}_{ij}$ are unknown coefficients. Substituting U in (3.7.12) and replacing V successively by $\widetilde{\varphi}^{(m)}_{ij}$, $m=1,2,3$, $i,j=1,\dots,N$, we get the equivalent system of $3N^2$ linear algebraic equations

$$\sum_{m=1}^{3}\sum_{i,j=1}^{N}A^{(m,n)}_{ijkl}C^{(m)}_{ij}=g^{(n)}_{kl},\ \ n=1,2,3,\ \ k,l=1,\dots,N, \tag{3.7.14}$$

where

$$A^{(m,n)}_{ijkl}=a\big(\widetilde{\varphi}^{(m)}_{ij},\widetilde{\varphi}^{(n)}_{kl}\big),\quad g^{(n)}_{kl}=\widetilde{g}(\widetilde{\varphi}^{(n)}_{kl}). \tag{3.7.15}$$

The matrix $A=A^{(m,n)}_{(ijkl)}$ is Gram's matrix defined by the positive semidefinite bilinear form $a(\cdot,\cdot)$ attached on basis vectors $\widetilde{\varphi}^{(m)}_{ij}$, $m=1,2,3$, $i,j=1,\dots,N$, of $[X_N]^3$, therefore, it is a nonsingular matrix and equation (3.7.14) has a unique solution

$$U=\sum_{i,j,k,l=1}^{N}(A^{-1})^{(m,n)}_{ijkl}\widetilde{\varphi}^{(m)}_{ij}g^{(n)}_{kl}. \tag{3.7.16}$$

To calculate explicitly $A^{(m,n)}_{ijkl}$ and $g^{(n)}_{kl}$ note that

$$\begin{aligned}\mathscr{D}_r\widetilde{\varphi}^{(m)}_{ij}(y)&=\partial_{y_r}\widetilde{\varphi}^{(m)}_{ij}(y)+\nu_r\partial_\nu\widetilde{\varphi}^{(m)}_{ij}(y)\\ &=\sum_{p=1}^{2}\partial_p\varphi_{ij}(\zeta(y))\big(\partial_r\zeta_p(y)+\nu_r\nu_l\partial_l\zeta_p(y)\big)(\delta_{m1},\delta_{m2},\delta_{m3})\end{aligned}$$

$$= \sum_{p=1}^{2} \partial_p \varphi_{ij}(\zeta(y)) \mathscr{D}_r \zeta_p(y)(\delta_{m1}, \delta_{m2}, \delta_{m3}), \tag{3.7.17}$$

$$A_{ijkl}^{(m,n)} = a\big(\widetilde{\varphi}_{ij}^{(m)}, \widetilde{\varphi}_{kl}^{(n)}\big) = (c_{qrst}\delta_{rm}\delta_{tn}\mathscr{D}_q\varphi_{ij}, \mathscr{D}_s\varphi_{kl})$$

$$= \sum_{\alpha,\beta=1}^{2} \int\limits_{\omega_{ij}\cap\omega_{kl}} c_{qmsn}(x)\big(\partial_\alpha\varphi_{ij}(x)\big)\big(\partial_\beta\varphi_{kl}(x)\big)\mathscr{D}_q\zeta_\alpha(\vartheta(x))\mathscr{D}_s\zeta_\beta(\vartheta(x))|\sigma'(x)|\,dx, \tag{3.7.18}$$

$$g_{kl}^{(n)} = -(g, \widetilde{\varphi}_{kl}^{(n)})_{\mathscr{C}} = -\int\limits_{\omega_{ij}\cap\omega_{kl}} g(\vartheta(x))\,\widetilde{\varphi}_{kl}^{(n)}(\vartheta(x))|\sigma'(x)|\,dx, \tag{3.7.19}$$

where $|\sigma'(x)|$ is a surface element of $\mathscr{C}$

$$|\sigma'(x)| = |\partial_1\vartheta(x) \times \partial_2\vartheta(x)|.$$

Chapter 4

Mellin convolution equations in the Bessel potential spaces

In the present chapter, we expose investigations of Mellin convolution equations in the Bessel potential spaces published in the papers [37, 59]. Such equations are important while investigating boundary value problems (BVPs) for elliptic equations on surfaces and domains with Lipschitz boundary and will be applied in the next chapter to the investigations of BVPs for the Laplace–Beltrami and Lamé equations on surfaces.

4.1 Introduction

It is well-known that various boundary value problems for PDE in planar domains with angular points on the boundary, e.g., Lamé systems in elasticity (cracks in elastic media, reinforced plates), Maxwell's system and Helmholtz equation in electromagnetic scattering, Cauchy–Riemann systems, Carleman–Vekua systems in generalized analytic function theory, etc., can be studied with the help of the Mellin convolution equations of the form

$$\mathbf{A}\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau} = f(t) \tag{4.1.1}$$

with the kernel $\mathscr{K}$ satisfying the condition

$$\int_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt < \infty, \quad 0 < \beta < 1, \tag{4.1.2}$$

which makes it a bounded operator in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$, provided $1 \leqslant p \leqslant \infty$, $-1 < \gamma < p - 1$, $\beta := (1 + \gamma)/p$ (cf. [47]).

In particular, integral equations with fixed singularities in the kernel

$$c_0(t)\varphi(t) + \frac{c_1(t)}{\pi i}\int_0^\infty \frac{\varphi(\tau)}{\tau - t}\,dt + \sum_{k=0}^n \frac{c_{k+2}(t)t^{k-r}}{\pi i}\int_0^\infty \frac{\tau^r\varphi(\tau)}{(\tau + t)^{k+1}}\,d\tau = f(t), \quad 0 \leqslant t \leqslant 1, \tag{4.1.3}$$

where $0 \leqslant r \leqslant k$, are of type (4.1.1) after localization, i.e., after "freezing" the coefficients.

The Fredholm theory and the unique solvability of equations (4.1.1) in the weighted Lebesgue spaces were accomplished in [47]. This investigation was based on the following observation: if $1 < p < \infty$, $-1 < \gamma < p - 1$, $\beta := (1 + \gamma)/p$, the following mutually invertible exponential transformations

$$\begin{aligned} &Z_\beta : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}^+), \quad Z_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R} := (-\infty, \infty), \\ &Z_\beta^{-1} : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma), \quad Z_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in \mathbb{R}^+ := (0, \infty), \end{aligned} \tag{4.1.4}$$

transform equation (4.1.1) from the weighted Lebesgue space $f, \varphi \in \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ into the Fourier convolution equation $W^0_{\mathscr{A}_\beta}\psi = g$, $\psi = Z_\beta\varphi$, $g = Z_\beta f \in \mathbb{L}_p(\mathbb{R})$ of the form

$$W^0_{\mathscr{A}_\beta}\psi(x) = c_0\psi(x) + \int\limits_{-\infty}^{\infty} \mathscr{K}_1(x-y)\varphi(y)\,dy, \quad \mathscr{K}_1(x) = e^{-\beta x}\Big[\frac{c_1}{1-e^{-x}} + \mathscr{K}(e^{-x})\Big].$$

Note that the symbol of the operator $W^0_{\mathscr{A}_\beta}$, viz. the Fourier transform of the kernel

$$\mathscr{A}_\beta(\xi) := c_0 + \int\limits_{-\infty}^{\infty} e^{i\xi x}\mathscr{K}_1(x)\,dx := c_0 - ic_1 \cot\pi(\beta - i\xi) + \int\limits_{-\infty}^{\infty} e^{(i\xi-\beta)x}\mathscr{K}(e^{-x})\,dx, \quad \xi \in \mathbb{R}, \tag{4.1.5}$$

is a piecewise continuous function. Let us recall that the theory of Fourier convolution operators with discontinuous symbols is well developed, cf. [42, 43, 45, 46, 132]. This allows one to investigate various properties of operators (4.1.1), (4.1.3). In particular, Fredholm criteria, index formula and conditions of unique solvability of equations (4.1.1) and (4.1.3) have been established in [47].

Similar integral operators with fixed singularities in kernel arise in the theory of singular integral equations with the complex conjugation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}\,dt + \frac{e(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\overline{\varphi(\tau)}}{\tau - t}\,dt} = f(t), \quad t \in \Gamma,$$

and in more general R-linear equations

$$a(t)\varphi(t) + b(t)\overline{\varphi(t)} + \frac{c(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}\,dt + \frac{d(t)}{\pi i}\int\limits_\Gamma \frac{\overline{\varphi(\tau)}}{\tau - t}\,dt +$$
$$+ \frac{e(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}\,dt} + \frac{g(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\overline{\varphi(\tau)}}{\tau - t}\,dt} = f(t), \quad t \in \Gamma,$$

if the contour Γ possesses corner points. Note that a complete theory of such equations is presented in [62, 63].

Let $t_1, \ldots, t_n \in \Gamma$ be the corner points of a piecewise-smooth contour Γ, and let $\mathbb{L}_p(\Gamma, \rho)$ denote the weighted $\mathbb{L}_p$-space with a power weight $\rho(t) := \prod\limits_{j=1}^n |t - t_j|^{\gamma_j}$. Assume that the parameters p and $\beta_j := (1 + \gamma_j)/p$ satisfy the conditions

$$1 < p < \infty, \quad 0 < \beta_j < 1, \quad j = 1, \ldots, n.$$

If the coefficients of the above equations are piecewise-continuous matrix functions, one can construct a function $\mathscr{A}_{\vec{\beta}}(t, \xi)$, $t \in \Gamma$, $\xi \in \mathbb{R}$, $\vec{\beta} := (\beta_1, \ldots, \beta_n)$, called the symbol of the equation (of the related operator). It is possible to express various properties of the equation in terms of $\mathscr{A}_{\vec{\beta}}$:

- The equation is Fredholm in $\mathbb{L}_p(\Gamma, \rho)$ if and only if its symbol is elliptic., i.e., if and only if

$$\inf_{(t,\xi)\in\Gamma\times\mathbb{R}} |\,\mathscr{A}_{\vec{\beta}}(t, \xi)| > 0;$$

- To an elliptic symbol $\mathscr{A}_{\vec{\beta}}(t, \xi)$ there corresponds an integer valued index $\mathbf{ind}\,\mathscr{A}_{\vec{\beta}}(t, \xi)$, the winding number, which coincides with the Fredholm index of the corresponding operator modulo a constant multiplier.

For more detailed survey of the theory and various applications to the problems of elasticity we refer the reader to [42, 43, 45, 47–51, 124].

Similar approach to boundary integral equations on curves with corner points based on Mellin transformation has been exploited by M. Costabel and E. Stephan [31, 33].

However, one of the main problems in boundary integral equations for elliptic partial differential equations is the absence of appropriate results for Mellin convolution operators in the Bessel potential spaces, cf. [48, 50, 51] and recent publications on nano-photonics [10, 11, 85]. Such results are needed to obtain an equivalent reformulation of boundary value problems into boundary integral equations in the Bessel potential spaces. Nevertheless, numerous works on Mellin convolution equations seem to pay almost no attention to the mentioned problem.

The first arising problem is the boundedness results for Mellin convolution operators in the Bessel potential spaces. The conditions on kernels known so far are very restrictive. The following boundedness result for the Mellin convolution operator can be proved.

Proposition 4.1.1. *Let $1<p<\infty$ and let $m=1,2,\dots$ be an integer. If a function $\mathscr{K}$ satisfies the condition*

$$\int\limits_0^1 t^{\frac{1}{p}-m-1}|\mathscr{K}(t)|\,dt+\int\limits_1^\infty t^{\frac{1}{p}-1}|\mathscr{K}(t)|\,dt<\infty, \tag{4.1.6}$$

then the Mellin convolution operator (*see* (4.1.1))

$$\boldsymbol{A}=\mathfrak{M}^0_{\mathscr{A}_{1/p}}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+)$$

with the symbol (*see* (4.1.5))

$$\mathscr{A}_{1/p}(\xi):=c_0+c_1\coth\pi\Big(\frac{i}{p}+\xi\Big)+\int\limits_0^\infty t^{1/p-i\xi}\mathscr{K}(t)\,\frac{dt}{t}\,,\quad \xi\in\mathbb{R}, \tag{4.1.7}$$

is bounded for any $0\leqslant s\leqslant m$.

Note that the condition

$$K_\beta:=\int\limits_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt<\infty \tag{4.1.8}$$

ensures that the operator

$$\mathfrak{M}^0_a:\mathbb{L}_p(\mathbb{R}^+,t^\gamma)\to\mathbb{L}_p(\mathbb{R}^+,t^\gamma)$$

is bounded, while the norm of the Mellin convolution

$$\mathfrak{M}^0_{a_\beta}\varphi(t):=\int\limits_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau} \tag{4.1.9}$$

admits the estimate $\|\mathfrak{M}^0_{a_\beta}\|\leqslant K_\beta$.

The above-formulated result has very restricted application. For example, the operators

$$\begin{aligned}
\boldsymbol{N}_\alpha\varphi(t)&=\frac{\sin\alpha}{\pi}\int\limits_0^\infty\frac{t\,\varphi(\tau)}{t^2+\tau^2-2t\tau\cos\alpha}\,d\tau,\\
\boldsymbol{N}^*_\alpha\varphi(t)&=\frac{\sin\alpha}{\pi}\int\limits_0^\infty\frac{\tau\,\psi_j(\tau)}{t^2+\tau^2-2t\tau\cos\alpha}\,d\tau,\qquad -\pi<\alpha<\pi,\\
\boldsymbol{M}_\alpha\varphi(t)&=\frac{1}{2\pi}\int\limits_{\mathbb{R}^+}\frac{[\tau\,\cos\alpha-t]\varphi(\tau)}{t^2+\tau^2-2t\,\tau\cos\alpha}\,d\tau,
\end{aligned} \tag{4.1.10}$$

which we encounter in boundary integral equations for elliptic boundary value problems (see [15]), as well as the operators

$$\boldsymbol{N}_{m,k}\varphi(t) := \frac{t^k}{\pi i}\int_0^\infty \frac{\tau^{m-k}\varphi(\tau)}{(\tau+t)^{m+1}}\,d\tau, \quad k = 0,\dots,m,$$

represented in (4.1.3), do not satisfy conditions (4.1.6). In particular, $\boldsymbol{N}_\alpha$ satisfies condition (4.1.6) only for $m = 1$ and $\boldsymbol{N}_{m,k}$ only for $m = k$. Although, as we will see below in Theorem 4.3.1, all operators $\boldsymbol{N}_\alpha$, $\boldsymbol{N}^*_\alpha$ and $\boldsymbol{N}_{m,k}$ are bounded in the Bessel potential spaces in setting (5.6.2) for all $s \in \mathbb{R}$.

Here we introduce *admissible kernels*, which are meromorphic functions on the complex plane $\mathbb{C}$ vanishing at infinity

$$\mathscr{K}(t) := \sum_{j=0}^{\ell} \frac{d_j}{t-c_j} + \sum_{j=\ell+1}^{\infty} \frac{d_j}{(t-c_j)^{m_j}}\,, \quad j = 0,1,\dots, \tag{4.1.11}$$
$$c_0,\dots,c_\ell \in \mathbb{R}, \quad 0 < \alpha_k := \arg c_k < 2\pi, \quad k = \ell+1, \ell+2, \dots .$$

$\mathscr{K}(t)$ have poles at $c_0, c_1, \dots \in \mathbb{C}\setminus\{0\}$ and complex coefficients $d_j \in \mathbb{C}$. The Mellin convolution operator

$$\mathbf{K}^m_c\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\tau^{m-1}\varphi(\tau)}{(t-c\,\tau)^m}\,d\tau \tag{4.1.12}$$

with the kernel

$$\mathscr{K}^m_c(t) := \frac{1}{(t-c)^m}\,, \quad 0 < \arg c < 2\pi$$

(see (4.1.1)) turns out to be bounded in the Bessel potential spaces (see Theorem 4.3.1).

In order to study Mellin convolution operators in the Bessel potential spaces, we use the “lifting” procedure, performed with the help of the Bessel potential operators $\mathbf{\Lambda}^s_+$ and $\mathbf{\Lambda}^{s-r}_-$, which transform the initial operator $\mathfrak{M}^0_a$ into the lifted operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}^0_a\mathbf{\Lambda}^{-s}_+$ acting already on a Lebesgue $\mathbb{L}_p$ spaces. However, the lifted operator is neither Mellin nor Fourier convolution and to describe its properties, one has to study the commutants of the Bessel potential operators and Mellin convolutions with meromorphic kernels. It turns out that the Bessel potentials alter after commutation with Mellin convolutions and the result depends essentially on poles of the meromorphic kernels. These results allow us to show that the lifted operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}_a\mathbf{\Lambda}^{-s}_+$ belongs to the Banach algebra of operators generated by Mellin and Fourier convolution operators with discontinuous symbols. Since such algebras have been studied before [52], one can derive various information (Fredholm properties, index, the unique solvability) about the initial Mellin convolution equation $\mathfrak{M}^0_a\varphi = g$ in the Bessel potential spaces in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+)$ and in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \mathbb{H}^{s-r}_p(\mathbb{R}^+)$.

The results of the present work are already applied in [70] to the investigation of some boundary value problems studied before by Lax–Milgram Lemma in [10,11]. Note that the present approach is more flexible and provides better tools for analyzing the solvability of the boundary value problems and the asymptotic behavior of their solutions.

It is worth noting that the obtained results can also be used to study Schrödinger operator on combinatorial and quantum graphs. Such a problem recently has attracted a lot of attention, since the operator mentioned above possesses interesting properties and has various applications, in particular, in nano-structures (see [98,99] and the references therein). Another area for application of the present results are Mellin pseudodifferential operators on graphs. This problem has been studied in [117], but only in the periodic case. Moreover, some of the results can be applied in the study of stability of approximation methods for Mellin convolution equations in the Bessel potential spaces.

The present chapter is organized as follows. In Section 4.2, we observe Mellin and Fourier convolution operators with discontinuous symbols acting on Lebesgue spaces. Most of these results are well known and we recall them for convenience. In Section 4.3, we define Mellin convolutions with admissible meromorphic kernels and prove their boundedness in the Bessel potential spaces. In Section 4.4, we present local principle of I. Gohbrg and N. Krupnik – a key toolkit for the investigation.

In Section 4.5, we recall results of R. Duduchava on Banach algebra of operators generated by Mellin and Fourier (Winer-Hopf) operators, which play key role in the investigation. We enhance results on Banach algebra generated by Mellin and Fourier convolution operators by adding explicit definition of the symbol of a Hankel operator, which belong to this algebra. In Section 4.6, it is proved the key result on commutants of the Mellin convolution operator (with admissible meromorphic kernel) and a Bessel potential operators. In Section 4.7, the exposed results are applied to find the Fredholm criteria and the index of Mellin convolution operators with admissible meromorphic kernels in the Bessel potential spaces.

4.2 Mellin convolution and the Bessel potential operators

Let N be a positive integer. If there arises no confusion, we write $\mathfrak{A}$ for both scalar and matrix $N\times N$ algebras with entries from $\mathfrak{A}$. Similarly, the same notation $\mathfrak{B}$ is used for the set of N-dimensional vectors with entries from $\mathfrak{B}$. It will be usually clear from the context what kind of space or algebra is considered.

The integral operator (4.1.1) is called Mellin convolution. More generally, if $a\in\mathbb{L}_\infty(\mathbb{R})$ is an essentially bounded measurable $N\times N$ matrix function, the Mellin convolution operator $\mathfrak{M}^0_a$ is defined by

$$\mathfrak{M}^0_a\varphi(t):=\mathscr{M}^{-1}_\beta a\mathscr{M}_\beta\varphi(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}a(\xi)\int\limits_0^\infty\Big(\frac{t}{\tau}\Big)^{i\xi-\beta}\varphi(\tau)\,\frac{d\tau}{\tau}\,d\xi,\ \ \varphi\in\mathbb{S}(\mathbb{R}^+),$$

where $\mathbb{S}(\mathbb{R}^+)$ is the Schwartz space of fast decaying functions on $\mathbb{R}^+$, whereas $\mathscr{M}_\beta$ and $\mathscr{M}^{-1}_\beta$ are the Mellin transform and its inverse, i.e.,

$$\mathscr{M}_\beta\psi(\xi):=\int\limits_0^\infty t^{\beta-i\xi}\psi(t)\,\frac{dt}{t},\ \ \xi\in\mathbb{R},$$

$$\mathscr{M}^{-1}_\beta\varphi(t):=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}t^{i\xi-\beta}\varphi(\xi)\,d\xi,\ \ t\in\mathbb{R}^+.$$

The function $a(\xi)$ is usually referred to as a symbol of the Mellin operator $\mathfrak{M}^0_a$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}^0_a$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+,t^\gamma)$ of N-vector functions endowed with the norm

$$\big\|\varphi\mid\mathbb{L}_p(\mathbb{R}^+,t^\gamma)\big\|:=\bigg[\int\limits_0^\infty t^\gamma|\varphi(t)|^p\,dt\bigg]^{1/p},$$

then the symbol $a(\xi)$ is called a Mellin $\mathbb{L}_{p,\gamma}$ *multiplier.*

The transformations

$$\mathbf{Z}_\beta:\mathbb{L}_p(\mathbb{R}^+,t^\gamma)\to\mathbb{L}_p(\mathbb{R}),\quad \mathbf{Z}_\beta\varphi(\xi):=e^{-\beta t}\varphi(e^{-\xi}),\ \ \xi\in\mathbb{R},$$
$$\mathbf{Z}^{-1}_\beta:\mathbb{L}_p(\mathbb{R})\to\mathbb{L}_p(\mathbb{R}^+,t^\gamma),\quad \mathbf{Z}^{-1}_\beta\psi(t):=t^{-\beta}\psi(-\ln t),\ \ t\in\mathbb{R}^+,$$

arrange an isometrical isomorphism between the corresponding $\mathbb{L}_p$-spaces. Moreover, the relations

$$\begin{gathered}\mathscr{M}_\beta=\mathscr{F}\mathbf{Z}_\beta,\quad \mathscr{M}^{-1}_\beta=\mathbf{Z}^{-1}_\beta\mathscr{F}^{-1},\quad \mathfrak{M}^0_a=\mathscr{M}^{-1}_\beta a\mathscr{M}_\beta=\mathbf{Z}^{-1}_\beta\mathscr{F}^{-1}a\mathscr{F}\mathbf{Z}_\beta=\mathbf{Z}^{-1}_\beta W^0_a\mathbf{Z}_\beta,\\ -1<\gamma<p-1,\ \ \beta:=\frac{1+\gamma}{p},\ \ 0<\beta<1,\end{gathered}\tag{4.2.1}$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and its inverse,

$$\mathscr{F}\varphi(\xi):=\int\limits_{-\infty}^{\infty}e^{i\xi x}\varphi(x)\,dx,\quad \mathscr{F}^{-1}\psi(x):=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\xi x}\psi(\xi)\,d\xi,\ \ x\in\mathbb{R},$$

show a close connection between Mellin $\mathfrak{M}^0_a$ and Fourier

$$W^0_a\varphi := \mathscr{F}^{-1}a\mathscr{F}\varphi, \ \ \varphi \in \mathbb{S}(\mathbb{R}),$$

convolution operators, as well as between the corresponding transforms. Here $\mathbb{S}(\mathbb{R})$ denotes the Schwartz class of infinitely smooth functions, decaying fast at infinity.

An $N \times N$ matrix function $a(\xi)$, $\xi \in \mathbb{R}$, is called a *Fourier* $\mathbb{L}_p$*-multiplier* if the operator $W^0_a : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R})$ is bounded. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

From (4.2.1) immediately follows the following

Proposition 4.2.1 (see [47]). *Let $1 < p < \infty$. The class of Mellin $\mathbb{L}_{p,\gamma}$-multipliers coincides with the Banach algebra $\mathfrak{M}_p(\mathbb{R})$ of Fourier $\mathbb{L}_p$-multipliers for arbitrary $-1 < \gamma < p-1$ and is independent of the parameter γ.*

Thus, a Mellin convolution operator $\mathfrak{M}^0_a$ in (4.2.1) is bounded in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ if and only if $a \in \mathfrak{M}_p(\mathbb{R})$.

It is known (see, e.g., [47]) that the Banach algebra $\mathfrak{M}_p(\mathbb{R})$ contains the algebra $\mathbf{V}_1(\mathbb{R})$ of all functions with bounded variation provided that

$$\beta := \frac{1+\gamma}{p}\,, \ \ 1 < p < \infty, \ \ -1 < \gamma < p-1. \tag{4.2.2}$$

As it was already mentioned, the primary aim of the present chapter is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in the Bessel potential spaces,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+). \tag{4.2.3}$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only one possible jump at infinity. We commence with the definition of the Besseel potential spaces and Bessel potentials, arranging isometrical isomorphisms between these spaces and enabling the lifting procedure, writing a Fredholm equivalent operator (equation) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ for the operator $\mathfrak{M}^0_a$ in (4.2.3).

For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space, known also as a fractional Sobolev space, is the subspace of the Schwartz space $\mathbb{S}'(\mathbb{R})$ of distributions having the finite norm

$$\big\|\varphi \mid \mathbb{H}^s_p(\mathbb{R})\big\| := \Bigg[\int\limits_{-\infty}^{\infty} \big|\mathscr{F}^{-1}\big(1+|\xi|^2\big)^{s/2}(\mathscr{F}\varphi)(t)\big|^p\,dt\Bigg]^{1/p} < \infty.$$

For an integer parameter $s = m = 1, 2, \dots$, the space $\mathbb{H}^s_p(\mathbb{R})$ coincides with the usual Sobolev space endowed with an equivalent norm

$$\big\|\varphi \mid \mathbb{W}^m_p(\mathbb{R})\big\| := \Bigg[\sum_{k=0}^{m}\int\limits_{-\infty}^{\infty}\Big|\frac{d^k\varphi(t)}{dt^k}\Big|^p\,dt\Bigg]^{1/p}.$$

If $s < 0$, one gets the space of distributions. Moreover, $\mathbb{H}^{-s}_{p'}(\mathbb{R})$ is the dual to the space $\mathbb{H}^s_p(\mathbb{R}^+)$, provided $p' := \frac{p}{p-1}$, $1 < p < \infty$. Note that $\mathbb{H}^s_2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_s = \int\limits_{\mathbb{R}} (\mathscr{F}\varphi)(\xi)\overline{(\mathscr{F}\psi)(\xi)}(1+\xi^2)^s\,d\xi, \ \ \varphi,\psi \in \mathbb{H}^s(\mathbb{R}).$$

By r_Σ we denote the operator restricting functions or distributions defined on $\mathbb{R}$ to the subset $\Sigma \subset \mathbb{R}$. Thus $\mathbb{H}^s_p(\mathbb{R}^+) = r_+(\mathbb{H}^s_p(\mathbb{R}))$, and the norm in $\mathbb{H}^s_p(\mathbb{R}^+)$ is defined by

$$\big\|f \mid \mathbb{H}^s_p(\mathbb{R}^+)\big\| = \inf_{\ell}\big\|\ell f \mid \mathbb{H}^s_p(\mathbb{R})\big\|,$$

where ℓf stands for any extension of f to a distribution in $\mathbb{H}^s_p(\mathbb{R})$.

Further, we denote by $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ the (closed) subspace of $\mathbb{H}^s_p(\mathbb{R})$ which consists of all distributions supported in the closure of $\mathbb{R}^+$.

Notice that $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ is always continuously embedded in $\mathbb{H}^s_p(\mathbb{R}^+)$, and if $s \in (\frac{1}{p}-1, \frac{1}{p})$, these two spaces coincide. Moreover, $\mathbb{H}^s_p(\mathbb{R}^+)$ may be viewed as the quotient-space $\mathbb{H}^s_p(\mathbb{R}^+) := \mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty, 0)$.

Let $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m \times m$ matrix function. The Fourier convolution operator (FCO) with the symbol a is defined by

$$W^0_a := \mathscr{F}^{-1} a \mathscr{F}.$$

If the operator

$$W^0_a : \mathbb{H}^s_p(\mathbb{R}) \to \mathbb{H}^{s-r}_p(\mathbb{R})$$

is bounded, we say that a is an $\mathbb{L}_p$-multiplier of order r and use "$\mathbb{L}_p$-multiplier" if the order is 0. The set of all $\mathbb{L}_p$-multipliers of order r (of order 0) is denoted by $\mathfrak{M}^r_p(\mathbb{R})$ (by $\mathfrak{M}_p(\mathbb{R})$, respectively).

For an $\mathbb{L}_p$-multiplier of order r, $a \in \mathfrak{M}^r_p(\mathbb{R})$, the Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ is defined by the equality

$$W_a = r_+ W^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+) \tag{4.2.4}$$

and the Hankel operator by the equality

$$H_a = r_+ \boldsymbol{V} W^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+), \quad \boldsymbol{V}\psi(t) := \psi(-t), \tag{4.2.5}$$

where $r_+ := r_{\mathbb{R}^+} : \mathbb{H}^s_p(\mathbb{R}) \to \mathbb{H}^s_p(\mathbb{R}^+)$ is the restriction operator to the semi-axes $\mathbb{R}^+$. Note that the generalized Hörmander's kernel of a FCO W_a depends on the difference of arguments $\mathscr{K}_a(t-\tau)$, while the Hörmander's kernel of a Hankel operator $r_+\boldsymbol{V}W^0_a$ depends of the sum of the arguments $\mathscr{K}_a(t+\tau)$.

We did not use in the definition of the class of multipliers $\mathfrak{M}^r_p(\mathbb{R})$ the parameter $s \in \mathbb{R}$. This is due to the fact that $\mathfrak{M}^r_p(\mathbb{R})$ is independent of s: if the operator W_a in (4.2.5) is bounded for some $s \in \mathbb{R}$, it is bounded for all other values of s. Another definition of the multiplier class $\mathfrak{M}^r_p(\mathbb{R})$ is written as follows: $a \in \mathfrak{M}^r_p(\mathbb{R})$ if and only if $\lambda^{-r} a \in \mathfrak{M}_p(\overline{\mathbb{R}}) = \mathfrak{M}^0_p(\overline{\mathbb{R}})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$. This assertion is one of the consequences of the following theorem.

Theorem 4.2.1. *Let $1 < p < \infty$. Then:*

(1) *For any $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$, the convolution operators (ΨDOs)*

$$\begin{gathered} \boldsymbol{\Lambda}^r_{\gamma} = W_{\lambda^r_{\gamma}} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ \boldsymbol{\Lambda}^r_{-\gamma} = r_+ W^0_{\lambda^r_{-\gamma}} \ell : \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+), \\ \lambda^r_{\pm\gamma}(\xi) := (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{gathered} \tag{4.2.6}$$

where $\ell : \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R})$ is an extension operator and r_+ is the restriction from the axes $\mathbb{R}$ to the semi-axes $\mathbb{R}^+$, arrange isomorphisms of the corresponding spaces. The final result is independent of the choice of an extension ℓ.

(2) *For arbitrary operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of order r, the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) & \xrightarrow{\ \mathbf{A}\ } & \mathbb{H}^{s-r}_p(\mathbb{R}^+) \\ \uparrow {\scriptstyle \boldsymbol{\Lambda}^{-s}_{\gamma}} & & \downarrow {\scriptstyle \boldsymbol{\Lambda}^{s-r}_{-\gamma}} \\ \mathbb{L}_p(\mathbb{R}^+) & \xrightarrow[\boldsymbol{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\boldsymbol{\Lambda}^{-s}_{\gamma}]{} & \mathbb{L}_p(\mathbb{R}^+) \end{array} . \tag{4.2.7}$$

Diagram (4.2.6) *provides an equivalent lifting of the operator $\mathbf{A}$ of order r to the operator $\boldsymbol{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\boldsymbol{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ of order* 0.

(3) *For any bounded convolution operator* $W_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ *of order* r *and for any pair of complex numbers* γ_1, γ_2 *such that* $\operatorname{Im}\gamma_j > 0$, $j = 1, 2$, *the lifted operator*

$$\mathbf{\Lambda}^{\mu}_{-\gamma_1} W_a \mathbf{\Lambda}^{\nu}_{\gamma_2} = W_{a_{\mu,\nu}} : \widetilde{\mathbb{H}}^{s+\nu}_p(\mathbb{R}^+) \to \mathbb{H}^{s-r-\mu}_p(\mathbb{R}^+), \quad a_{\mu,\nu}(\xi) := (\xi-\gamma_1)^{\mu} a(\xi)(\xi+\gamma_2)^{\nu} \tag{4.2.8}$$

is again a Fourier convolution.

In particular, the lifted operator W_{a_0} *in* $\mathbb{L}_p$*-spaces,* $\mathbf{\Lambda}^{s-r}_{-\gamma} W_a \mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ *has the symbol*

$$a_{s-r,-s}(\xi) = \lambda^{s-r}_{-\gamma}(\xi) a(\xi) \lambda^{-s}_{\gamma}(\xi) = \left(\frac{\xi-\gamma}{\xi+\gamma}\right)^{s-r} \frac{a(\xi)}{(\xi+i)^r}\,.$$

(4) *The Hilbert transform* $S_{\mathbb{R}^+} = i\mathbf{K}^1_1 = W_{-\,\mathrm{sign}}$ *is a Fourier convolution operator and*

$$\mathbf{\Lambda}^{s}_{-\gamma_1} \boldsymbol{K}^1_1 \mathbf{\Lambda}^{-s}_{\gamma_2} = W_{i\, g^s_{-\gamma_1,\gamma_2}\,\mathrm{sign}}, \tag{4.2.9}$$

where

$$g^s_{-\gamma_1,\gamma_2}(\xi) := \left(\frac{\xi-\gamma_1}{\xi+\gamma_2}\right)^{s}. \tag{4.2.10}$$

Proof. For the proof of items (1)–(3) we refer the reader to [47, Lemma 5.1] and [67, 72]. The item (4) is a consequence of the proved items (2) and (3) (see [47, 59]). □

Remark 4.2.1. The class of Fourier convolution operators is a subclass of pseudodifferential operators (ΨDOs). Moreover, for integer parameters $m = 1, 2, \dots$ the Bessel potentials $\mathbf{\Lambda}^m_{\pm} = W_{\lambda^m_{\pm\gamma}}$, which are Fourier convolutions of order m, are ordinary differential operators of the same order m:

$$\mathbf{\Lambda}^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \left(i\,\frac{d}{dt} \pm \gamma\right)^m = \sum_{k=0}^{m} \binom{m}{k} i^k (\pm\gamma)^{m-k}\, \frac{d^k}{dt^k}\,. \tag{4.2.11}$$

These potentials map both spaces (cf. (4.2.6))

$$\begin{aligned} \mathbf{\Lambda}^m_{\pm\gamma} &: \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ &: \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-m}_p(\mathbb{R}^+), \end{aligned} \tag{4.2.12}$$

but the mappings are not isomorphisms because the inverses $\mathbf{\Lambda}^{-m}_{\pm\gamma}$ are bounded only for one pair of spaces indicated in (4.2.6).

Remark 4.2.2. For any pair of multipliers $a \in \mathfrak{M}^r_p(\mathbb{R})$, $b \in \mathfrak{M}^s_p(\mathbb{R})$, the corresponding convolution operators on the half-axes W^0_a and W^0_b have the property $W^0_a W^0_b = W^0_b W^0_a = W^0_{ab}$.

For the corresponding Wiener–Hopf operators on the half-axes, a similar equality

$$W_a W_b = W_{ab} \tag{4.2.13}$$

holds if and only if either the function $a(\xi)$ has an analytic extension in the lower half-plane, or the function $b(\xi)$ has an analytic extension in the upper half-plane (see [47]).

Note that, actually, (4.2.8) is a consequence of (4.2.13).

4.3 Mellin convolutions with admissible meromorphic kernels

Now we consider kernels $\mathscr{K}(t)$ exposed in (4.1.11), which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at infinity, having poles at $c_0, c_1, \ldots \in \mathbb{C} \setminus \{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 4.3.1. We call a kernel $\mathscr{K}(t)$ in (4.1.11) admissible if and only if

(i) $\mathscr{K}(t)$ has only a finite number of poles $c_0, \ldots, c_\ell$ which belong to the positive semi-axes, i.e., $\arg c_0 = \cdots = \arg c_\ell = 0$;

(ii) The corresponding multiplicities are one: $m_0 = \cdots = m_\ell = 1$;

(iii) The remainder points $c_{\ell+1}, c_{\ell+2}, \dots$ do not condense to the positive semi-axes and their real parts are bounded uniformly

$$\lim_{j\to\infty} c_j \notin [0,\infty), \qquad \sup_{j=\ell+1,\ell+2,\dots} \operatorname{Re} c_j \leqslant K < \infty. \tag{4.3.1}$$

(iv) $\mathscr{K}(t)$ is a kernel of an operator, which is a composition of finite number of operators with admissible kernels.

Example 4.3.1. The function

$$\mathscr{K}(t) = \exp\Big(\frac{1}{t-c}\Big), \ \ \operatorname{Re} c < 0 \ \text{ or } \ \operatorname{Im} c \neq 0,$$

is an example of the admissible kernel which also satisfies the condition of the next Theorem 4.3.1. Other examples of operators with admissible kernels (which also satisfy the condition of the next Theorem 4.3.1) are operators which we encounter in (4.1.3), in (4.1.10) and in (4.2.4) and, in general, any finite sum in (4.1.11).

Example 4.3.2. The function

$$\mathscr{K}(t) = \frac{\ln t - c_1c_2}{t - c_1c_2}\,, \ \ \operatorname{Im} c_1 \neq 0, \ \ \operatorname{Im} c_2 \neq 0,$$

is another example of the admissible kernel and it represents the composition of operators $c_2\mathbf{K}^1_{c_1}\mathbf{K}^1_{c_2}$ (see (4.2.10)) with admissible kernels which also satisfies the condition of the next Theorem 4.3.1. More trivial examples of operators with admissible kernels (which also satisfy the condition of the next Theorem 4.3.1) are operators which we encounter in (4.1.3), in (4.1.10) and in (4.2.4) and, in general, any finite sum in (4.1.11).

Theorem 4.3.1. *Let conditions*

$$\beta := \frac{1+\gamma}{p}\,, \ \ 1 < p < \infty, \ \ -1 < \gamma < p-1, \tag{4.3.2}$$

hold, $\mathscr{K}(t)$ *in* (4.1.11) *be an admissible kernel and*

$$K_\beta := \sum_{j=0}^{\infty} 2^{m_j} |d_j|\, |c_j|^{\beta - m_j} < \infty. \tag{4.3.3}$$

Then the Mellin convolution $\mathfrak{M}^0_{a_\beta}$ *in* (4.1.9) *with the admissible meromorphic kernel* $\mathscr{K}(t)$ *in* (4.1.11) *is bounded in the Lebesgue space* $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ *and its norm has the estimate* $\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\| \leqslant MK_\beta$ *with some* $M > 0$.

We can drop the constant M *and replace* 2^{m_j} *by* $2^{\frac{m_j}{2}}$ *in estimate* (4.3.3) *provided* $\operatorname{Re} c_j < 0$ *for all* $j = 0, 1, \dots$.

Proof. The first $\ell + 1$ summands in the definition of the admissible kernel (4.1.11) correspond to the Cauchy operators

$$A_0\varphi(t) = \sum_{j=0}^{\ell} \frac{d_j}{\pi} \int_0^\infty \frac{\varphi(\tau)}{t - c_j\tau}\, d\tau, \ \ c_j > 0, \ \ j = 0, 1, \dots, \ell,$$

and their boundedness property in the weighted Lebesgue space

$$A_0 : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \tag{4.3.4}$$

under constraints (4.3.2) is well known (see [93] and also [83]). Therefore, we can ignore the first ℓ summands in the expansion of the kernel $\mathscr{K}(t)$ in (4.1.11). To the boundedness of the operator $\mathfrak{M}^0_{a^\ell_\beta}$ with the remainder kernel

$$\mathscr{K}^\ell(t) := \sum_{j=\ell+1}^{\infty} \frac{d_j}{(t-c_j)^{m_j}}\,, \quad c_j \neq 0, \quad j=0,1,\dots, \quad 0<\alpha_k := \arg c_k < 2\pi, \quad k=\ell+1,\ell+2,\dots$$

(see (4.1.11)), we apply estimate (4.1.8)

$$\left\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+,t^\gamma))\right\| \leqslant \int_0^\infty t^{\beta-1}|\mathscr{K}^\ell(t)|\,dt \leqslant \sum_{j=\ell+1}^{\infty}|d_j|\int_0^\infty \frac{t^{\beta-1}}{|t-c_j|^{m_j}}\,dt. \tag{4.3.5}$$

Now note that

$$\begin{aligned}|t-c_j|^{-m_j} &= \left(t^2+|c_j|^2-2\operatorname{Re}c_j t\right)^{-\frac{m_j}{2}} \\ &\leqslant \left(\frac{t^2+|c_j|^2}{2}\right)^{-\frac{m_j}{2}} \leqslant 2^{m_j}(t+|c_j|)^{-m_j} \quad \text{for all } \ t\geqslant 2K = 2\sup|\operatorname{Re}c_j|>0,\end{aligned}$$

due to constraints (4.3.1). On the other hand,

$$|t-c_j|^{-m_j} \leqslant M(t+|c_j|)^{-m_j} \quad \text{for all } \ 0\leqslant t\leqslant 2K$$

and a certain constant $M>0$. Therefore,

$$|t-c_j|^{-m_j} \leqslant M2^{m_j}(t+|c_j|)^{-m_j} \quad \text{for all } \ 0<t<\infty. \tag{4.3.6}$$

Next, we recall the formula from [84, Formula 3.194.4]

$$\int_0^\infty \frac{t^{\beta-1}}{(t+c)^m}\,dt = (-1)^{m-1}\binom{\beta-1}{m-1}\frac{\pi c^{\beta-m}}{\sin\pi\beta}\,, \quad -\pi<\arg c<\pi, \quad \operatorname{Re}\beta<1, \tag{4.3.7}$$

$$\binom{\beta-1}{m-1} := \frac{(\beta-1)\cdots(\beta-m+1)}{(m-1)}\,, \quad \binom{\beta-1}{0} := 1$$

to calculate the integrals. By inserting estimate (4.3.6) into (4.3.5) and applying (4.3.7), we get

$$\begin{aligned}\left\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+,t^\gamma))\right\| &\leqslant \sum_{j=\ell+1}^{\infty}|d_j|\int_0^\infty \frac{t^{\beta-1}}{|t-c_j|^{m_j}}\,dt \\ &\leqslant M_0\sum_{j=\ell+1}^{\infty}2^{m_j}|d_j|\int_0^\infty \frac{t^{\beta-1}}{(t+|c_j|)^{m_j}}\,dt \leqslant \frac{\pi M_0}{\sin\pi\beta}\sum_{j=\ell+1}^{\infty}2^{m_j}|d_j|\left|\binom{\beta-1}{m_j-1}\right|c_j^{\beta-m_j} \\ &\leqslant M\sum_{j=\ell+1}^{\infty}2^{m_j}|d_j|c_j^{\beta-m_j} = MK_\beta, \quad M := \frac{\pi M_0}{\sin\pi\beta}\,, \end{aligned} \tag{4.3.8}$$

since (see (4.3.7))

$$\left|\binom{\beta-1}{m_j-1}\right| \leqslant 1,$$

where K_β is from (4.3.3). The boundedness of (4.3.4) and estimate (4.3.8) imply the claimed estimate

$$\left\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+,t^\gamma))\right\| \leqslant MK_\beta.$$

If $\operatorname{Re}c_j<0$ for all $j=0,1,\dots$, we have

$$\frac{1}{|t-c_j|^{m_j}} = \left(t^2+|c|^2-2\operatorname{Re}c_j t\right)^{-\frac{m_j}{2}} \leqslant \left(t^2+|c|^2\right)^{-\frac{m_j}{2}} \leqslant 2^{\frac{m_j}{2}}\left(t+|c_j|\right)^{-m_j},$$

valid for all $t>0$ and a constant M does not emerge in the estimate. □

Let us find the symbol (the Mellin transform of the kernel) of operator (4.2.10) for $0<\arg c<2\pi$, $m=1,2,\dots$ (see (4.2.9), (4.2.10)). For this, we apply formula (4.3.7):

$$\begin{aligned}\mathscr{M}_\beta\mathscr{K}^m_c(\xi)&=\int\limits_0^\infty t^{\beta-i\xi-1}\mathscr{K}^m_c(t)\,dt=\frac{1}{\pi}\int\limits_0^\infty\frac{t^{\beta-i\xi-1}}{(t+(-c))^m}\,dt\\&=\binom{\beta-i\xi-1}{m-1}\frac{(-1)^{m-1}(-c)^{\beta-i\xi-m}}{\sin\pi(\beta-i\xi)}=\binom{\beta-i\xi-1}{m-1}\frac{(-1)^{m-1}e^{-i\pi(\beta-i\xi-m)}c^{\beta-i\xi-m}}{\sin\pi(\beta-i\xi)}\,,\end{aligned}$$

since if $-\pi<\arg(-c)<\pi$ and $0<\arg c<2\pi$, then $-c=e^{-\pi i}c$. In particular,

$$\mathscr{M}_\beta\mathscr{K}^1_c(\xi)=\frac{e^{-i\pi(\beta-i\xi-1)}c^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}\,,\quad 0<\arg c<2\pi, \tag{4.3.9a}$$

$$\mathscr{M}_\beta\mathscr{K}^1_{-d}(\xi)=\frac{d^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}\,,\quad -\pi<\arg d<\pi, \tag{4.3.9b}$$

$$\mathscr{M}_\beta\mathscr{K}^1_{-1}(\xi)=\frac{1}{\sin\pi(\beta-i\xi)}\,,\quad \xi\in\mathbb{R}. \tag{4.3.9c}$$

Now let us find the symbol of the Cauchy singular integral operator $K^1_1=-iS_{\mathbb{R}^+}$ (see (4.2.9), (4.2.10)). For this, we apply Plemeli formula and formula (4.3.7):

$$\begin{aligned}\mathscr{M}_\beta\mathscr{K}^1_1(t):&=\int\limits_0^\infty t^{\beta-i\xi-1}\mathscr{K}^1_1(t)\,dt=-\frac{1}{\pi}\int\limits_0^\infty\frac{t^{\beta-i\xi-1}\,dt}{t-1}\\&=\lim_{\varepsilon\to0}\frac{1}{2\pi}\int\limits_0^\infty\Big[\frac{t^{\beta-i\xi-1}}{t+e^{i(\pi-\varepsilon)}}+\frac{t^{\beta-i\xi-1}}{t+e^{-i(\pi-\varepsilon)}}\Big]\,dt\\&=\lim_{\varepsilon\to0}\frac{e^{i(\pi-\varepsilon)(\beta-i\xi-1)}+e^{-i(\pi-\varepsilon)(\beta-i\xi-1)}}{2\sin\pi(\beta-i\xi)}=\cot\pi(\beta-i\xi).\end{aligned}$$

For an admissible kernel with poles $\arg c_0=\arg c_\ell=0$ (and, therefore, $m_0=\dots=m_\ell=1$) and $0<\arg c_j<2\pi$, $j=\ell+1,\dots$, we get

$$\begin{aligned}\mathscr{M}_\beta\mathscr{K}(\xi)&=\cot\pi(\beta-i\xi)\sum_{j=0}^{\ell}d_jc_j^{\beta-i\xi-1}\\&\quad+\frac{1}{\sin\pi(\beta-i\xi)}\sum_{j=\ell+1}^{\infty}d_j\binom{\beta-i\xi-1}{m_j-1}(-1)^{m_j-1}e^{-i\pi(\beta-i\xi-m_j)}c_j^{\beta-i\xi-m_j}.\end{aligned}\tag{4.3.10}$$

Theorem 4.3.2. *If $\mathscr{K}$ is an admissible kernel, then the corresponding Mellin convolution operator with the kernel $\mathscr{K}$*

$$\mathbf{K}\varphi(t):=\int\limits_0^\infty\mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau}\,,\quad \mathbf{K}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+), \tag{4.3.11}$$

is bounded for all $1<p<\infty$ and $s\in\mathbb{R}$.

The condition on the parameter p can be relaxed to $1\leqslant p\leqslant\infty$, provided the admissible kernel $\mathscr{K}$ in (4.1.11) *has no poles on positive semi-axes: $\alpha_j=\arg c_j\neq0$ for all $j=0,1,\dots$.*

Proof. Due to representation (4.1.11), we have to prove the theorem only for a model kernel

$$\mathscr{K}^m_c(t):=\frac{1}{\pi(t-c)^m}\,,\quad c\neq0,\ \ 0<\arg c<2\pi,\ \ m=1,2,\dots . \tag{4.3.12}$$

The respective Mellin convolution operator $\mathbf{K}^m_c$ (see (4.2.10)) is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1\leqslant p\leqslant\infty$ for arbitrary $0<|\arg c|<\pi$ (cf. (4.1.2)).

To accomplish the boundedness result of $\mathbf{K}_c^m$ in $\mathbb{L}_p(\mathbb{R}^+)$, we need to consider the case $\arg c = 0$ (i.e., $c \in (0,\infty)$) and, therefore, $m = 1$ (see Definition 4.3.1). Then the operator $\mathbf{K}_c^1$ coincides with the "dilated" Cauchy singular integral operator with a constant multiplier

$$\mathbf{K}_c^1\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\varphi(\tau)}{t - c\,\tau}\,d\tau = -\frac{i}{c}\,(S_{\mathbb{R}^+}\varphi)\Big(\frac{t}{c}\Big), \tag{4.3.13}$$

where

$$S_{\mathbb{R}^+}\varphi(t) := \frac{1}{\pi i}\int_0^\infty \frac{\varphi(\tau)}{\tau - t}\,d\tau, \tag{4.3.14}$$

and is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 < p < \infty$ (cf., e.g., [47,83]).

Now let $0 \leqslant \arg c < 2\pi$ and $m = 1$. Then if $\varphi \in C_0^\infty(\mathbb{R}^+)$ is a smooth function with compact support and $k = 1, 2, \dots$, by integrating by parts we get

$$\begin{aligned}\frac{d^k}{dt^k}\,\mathbf{K}_c^1\varphi(t) &= \frac{1}{\pi}\int_0^\infty \frac{d^k}{dt^k}\,\frac{1}{t - c\,\tau}\varphi(\tau)\,d\tau\\ &= \frac{(-c)^{-k}}{\pi}\int_0^\infty \frac{d^k}{d\tau^k}\,\frac{1}{t - c\,\tau}\varphi(\tau)\,d\tau = \frac{c^{-k}}{\pi}\int_0^\infty \frac{1}{t - c\,\tau}\,\frac{d^k\varphi(\tau)}{d\tau^k}\,d\tau = c^{-k}\Big(\mathbf{K}_c^1\,\frac{d^k}{dt^k}\,\varphi\Big)(t). \end{aligned}\tag{4.3.15}$$

For $m = 2, 3, \dots$ and $0 < \arg c < 2\pi$ we get similarly

$$\begin{aligned}\frac{d}{dt}\,\mathbf{K}_c^m\varphi(t) &= \frac{1}{\pi}\int_0^\infty \frac{d}{dt}\,\frac{\tau^{m-1}}{(t - c\,\tau)^m}\,\varphi(\tau)\,d\tau = \sum_{j=0}^{m-1}\frac{(-c)^{-1-j}}{\pi}\int_0^\infty \frac{d}{dt}\,\frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\,\varphi(\tau)\,d\tau\\ &= -\sum_{j=0}^{m-1}\frac{(-c)^{-1-j}}{\pi}\int_0^\infty \frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\,\frac{d}{d\tau}\,\varphi(\tau)\,d\tau = -\sum_{j=0}^{m-1}(-c)^{-1-j}\Big(\mathbf{K}_c^{m-j}\,\frac{d}{dt}\,\varphi\Big)(t)\end{aligned}$$

and, recurrently,

$$\begin{gathered}\frac{d^k}{dt^k}\,\mathbf{K}_c^m\varphi(t) = (-1)^k\sum_{j=0}^{m-1}(-c)^{-k-j}\gamma_j^k\Big(\mathbf{K}_c^{m-j}\,\frac{d^k}{dt^k}\,\varphi\Big)(t), \quad k = 1, 2, \dots, \\ \gamma_j^1 = j + 1, \quad \gamma_0^k = 1, \quad \gamma_j^k := \sum_{r=0}^{j}\gamma_r^{k-1}, \quad j = 0, 1, \dots, m, \quad k = 1, 2, \dots .\end{gathered}\tag{4.3.16}$$

Recall now that for an integer $s = n$ the spaces $\mathbb{H}_p^n(\mathbb{R}^+)$, $\widetilde{\mathbb{H}}_p^n(\mathbb{R}^+)$ coincide with the Sobolev spaces $\mathbb{W}_p^n(\mathbb{R}^+)$, $\widetilde{\mathbb{W}}_p^n(\mathbb{R}^+)$, respectively (these spaces are isomorphic and the norms are equivalent), and $C_0^\infty(\mathbb{R}^+)$ is a dense subset in $\widetilde{\mathbb{W}}_p^n(\mathbb{R}^+) = \widetilde{\mathbb{H}}_p^n(\mathbb{R}^+)$. Then, using equalities (4.3.14), (4.3.16) and the boundedness of the operators $\mathbf{K}_c^{m-j}$ (see (4.3.12)–(4.3.14)), we proceed as follows:

$$\begin{aligned}\big\|\mathbf{K}_c^m\varphi \mid \mathbb{H}_p^n(\mathbb{R}^+)\big\| &= \sum_{k=0}^{n}\Big\|\frac{d^k}{dt^k}\,\mathbf{K}_c^m\varphi \mid \mathbb{L}_p(\mathbb{R}^+)\Big\|\\ &= \sum_{k=0}^{n}\sum_{j=0}^{m-1}|c|^{-k-j}\gamma_j^k\Big\|\mathbf{K}_c^{m-j}\,\frac{d^k}{dt^k}\,\varphi \mid \mathbb{L}_p(\mathbb{R}^+)\Big\| \leqslant M\sum_{k=0}^{n}\Big\|\frac{d^k}{dt^k}\,\varphi \mid \mathbb{L}_p(\mathbb{R}^+)\Big\| = M\big\|\varphi \mid \mathbb{H}_p^n(\mathbb{R}^+)\big\|,\end{aligned}$$

where $M > 0$ is a constant, and the boundedness of (4.3.11) follows for $s = 0, 1, 2, \dots$. The case of arbitrary $s > 0$ follows by the interpolation between the spaces $\mathbb{H}_p^m(\mathbb{R}^+)$ and $\mathbb{H}_p^0(\mathbb{R}^+) = \mathbb{L}_p(\mathbb{R}^+)$, also between the spaces $\widetilde{\mathbb{H}}_p^m(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}_p^0(\mathbb{R}^+) = \mathbb{L}_p(\mathbb{R}^+)$.

For $s<0$, the boundedness of (4.3.11) follows by duality: the adjoint operator to $\mathbf{K}^m_c$ is

$$\mathbf{K}^{m,*}_c\varphi(t) := \frac{1}{\pi}\int\limits_0^\infty \frac{t^{m-1}\varphi(\tau)\,d\tau}{(\tau-c\,t)^m} = \sum_{j=1}^m \omega_j \mathbf{K}^j_{c^{-1}}\varphi(t)$$

for some constant coefficients $\omega_1,\dots,\omega_m$. The operator $\mathbf{K}^{m,*}_c$ has the admissible kernel and, due to the proved part of the theorem, is bounded in the space setting $\mathbf{K}^{m,*}_c : \widetilde{\mathbb{H}}^{-s}_{p'}(\mathbb{R}^+) \to \mathbb{H}^{-s}_{p'}(\mathbb{R}^+)$, $p' := p/(p-1)$, since $-s>0$. The initial operator $\mathbf{K}^m_c : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$ is dual to $\mathbf{K}^{m,*}_c$ and, therefore, is bounded as well. □

Corollary 4.3.1. *Let $1<p<\infty$ and $s\in\mathbb{R}$. A Mellin convolution operator $\mathfrak{M}^0_a$ with an admissible kernel described in Definition* 4.3.1 (*also see Example* 4.3.2 *and Theorem* 4.3.1) *is bounded in the Bessel potential spaces*

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+).$$

The boundedness property

$$\mathfrak{M}^0_a : \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$$

does not hold in general for even a simplest Mellin convolution operator $\boldsymbol{K}_c$, except the case when the spaces $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+$ and $\mathbb{H}^s_p(\mathbb{R}^+)$ can be identified, i.e., except the case $\frac{1}{p}-1<s<\frac{1}{p}$. Indeed, to check this, consider a smooth function with a compact support $\varphi\in C^\infty_0(\mathbb{R}^+)$ which is constant on the unit interval: $\varphi(t)=1$ for $0<t<1$. Obviously, $\varphi\in\mathbb{H}^s_p(\mathbb{R}^+)$ and $\varphi\notin\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+$ for all $s>\frac{1}{p}$. Then

$$\boldsymbol{K}_c\varphi(t) = \frac{1}{\pi}\int\limits_0^\infty \frac{\varphi(\tau)}{t-c\,\tau}\,d\tau = \frac{1}{\pi}\int\limits_0^1 \frac{d\tau}{t-c\,\tau} + \frac{1}{\pi}\int\limits_1^\infty \frac{\varphi(\tau)}{t-c\,\tau}\,d\tau = c^{-1}\ln\tau + \varphi_0(t),$$

where $\varphi_0\in\mathbb{H}^s_p(\mathbb{R}^+)\cap C^\infty(\mathbb{R}^+)$, while the first summand ln τ does not belong to $\mathbb{H}^s(\mathbb{R}^+)$, since all functions in this space are continuous and uniformly bounded for $s>\frac{1}{p}$.

We can prove the following very partial result, which has important practical applications.

Theorem 4.3.3. *Let $1<p<\infty$, $c\in\mathbb{C}$ and $\mathbb{X}^s_p(\mathbb{R}^+)$ denote one of the spaces $\mathbb{H}^r_p(\mathbb{R}^+)$ or $\mathbb{W}^r_p(\mathbb{R}^+)$, while $\widetilde{\mathbb{X}}^s_p(\mathbb{R}^+)$ denote one of the spaces $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ or $\widetilde{\mathbb{W}^r_p}(\mathbb{R}^+)$.*

If $\frac{1}{p}-1<r<\frac{1}{p}+1$, the operator

$$\begin{aligned}\boldsymbol{A}_c := c\boldsymbol{K}_c - c^{-1}\boldsymbol{K}_{c^{-1}} &: \mathbb{X}^r_p(\mathbb{R}^+) \to \mathbb{X}^r_p(\mathbb{R}^+),\\ &: \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+) \to \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+)\end{aligned} \tag{4.3.17}$$

is bounded, while for $\frac{1}{p}-2<r<\frac{1}{p}$ the operator

$$\begin{aligned}\boldsymbol{A}^{\#}_c := \boldsymbol{K}_c - \boldsymbol{K}_{c^{-1}} &: \mathbb{X}^r_p(\mathbb{R}^+) \to \mathbb{X}^r_p(\mathbb{R}^+),\\ &: \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+) \to \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+)\end{aligned} \tag{4.3.18}$$

is bounded.

Proof. If $\frac{1}{p}-1<r<\frac{1}{p}$, the spaces $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ and $\mathbb{H}^r_p(\mathbb{R}^+)$ can be identified and the boundedness of (4.3.17), (4.3.18) follows from Theorem 4.3.2.

Now let $\frac{1}{p}<r<\frac{1}{p}+1$. Due to (1.6.4) and (4.2.12), the following diagrams

$$\begin{array}{ccc} \mathbb{H}^r_p(\mathbb{R}^+) & \xrightarrow{\ \boldsymbol{A}_c\ } & \mathbb{H}^r_p(\mathbb{R}^+) \\ \Big\uparrow{\scriptstyle \boldsymbol{\Lambda}^{-1}_{-1}} & & \Big\downarrow{\scriptstyle \boldsymbol{\Lambda}^{1}_{-1}} \\ \mathbb{H}^{r-1}_p(\mathbb{R}^+) & \xrightarrow[\boldsymbol{\Lambda}^1_{-1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{-1}]{} & \mathbb{H}^{r-1}_p(\mathbb{R}^+) \end{array}, \qquad \begin{array}{ccc} \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) & \xrightarrow{\ \boldsymbol{A}_c\ } & \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \\ \Big\uparrow{\scriptstyle \boldsymbol{\Lambda}^{-1}_{1}} & & \Big\downarrow{\scriptstyle \boldsymbol{\Lambda}^{1}_{1}} \\ \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) & \xrightarrow[\boldsymbol{\Lambda}^1_{1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{1}]{} & \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) \end{array} \tag{4.3.19}$$

are commutative. Diagrams (4.3.19) provide equivalent lifting of the operator $\boldsymbol{A}_c$ from the spaces $\mathbb{H}^r_p(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ to the operator $\boldsymbol{A}^+_c := \boldsymbol{\Lambda}^1_1\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_1$ in the space $\widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+)$ and the operator $\boldsymbol{A}^-_c := \boldsymbol{\Lambda}^1_{-1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{-1}$ in the space $\mathbb{H}^{r-1}_p(\mathbb{R}^+)$. On the other hand, $\boldsymbol{\Lambda}^1_{\pm1} = i\partial_t \pm I$ (see (4.2.11)) and it can be easily checked, using the integration by parts, that $\partial_t\boldsymbol{A}_c = -\boldsymbol{A}^\#_c\partial_t$. Then

$$\begin{aligned}\boldsymbol{A}^\pm_c &= \boldsymbol{\Lambda}^1_{\pm1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{\pm1} = (i\partial_t \pm I)\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{\pm1}\\ &= (\pm\boldsymbol{A}_c - \boldsymbol{A}^\#_c)\boldsymbol{\Lambda}^{-1}_1 + \boldsymbol{A}^\#_c(i\partial_t \pm I)\boldsymbol{\Lambda}^{-1}_{\pm1} = (\pm\boldsymbol{A}_c - \boldsymbol{A}^\#_c)\boldsymbol{\Lambda}^{-1}_1 + \boldsymbol{A}^\#_c.\end{aligned}$$

Since $\frac{1}{p} - 1 < r - 1 < \frac{1}{p}$ and the embeddings

$$\begin{aligned}\boldsymbol{\Lambda}^{-1}_{-1}\mathbb{H}^{r-1}_p(\mathbb{R}^+) &= \mathbb{H}^r_p(\mathbb{R}^+) \subset \mathbb{H}^{r-1}_p(\mathbb{R}^+),\\ \boldsymbol{\Lambda}^{-1}_1\widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) &= \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \subset \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+)\end{aligned}$$

are continuous, the operators

$$\begin{aligned}\boldsymbol{A}^-_c &= (-\boldsymbol{A}_c - \boldsymbol{A}^\#_c)\boldsymbol{\Lambda}^{-1}_1 + \boldsymbol{A}^\#_c : \mathbb{H}^{r-1}_p(\mathbb{R}^+) \to \mathbb{H}^{r-1}_p(\mathbb{R}^+),\\ \boldsymbol{A}^+_c &= (\boldsymbol{A}_c - \boldsymbol{A}^\#_c)\boldsymbol{\Lambda}^{-1}_1 + \boldsymbol{A}^\#_c : \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) \to \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+)\end{aligned}$$

are bounded. Then, according to the commutative diagrams (4.3.19), the operator $\boldsymbol{A}_c$ in (4.3.17) is bounded for $\mathbb{X}^r_p = \mathbb{H}^r_p$. For $\mathbb{X}^r_p = \mathbb{W}^r_p$, the boundedness is proved similarly or, alternatively, with the help of the interpolation theorems (see below Corollary 4.7.2 for similar arguments).

Now let $\frac{1}{p} - 2 < r < \frac{1}{p}$. Then

$$\frac{1}{p'} - 1 = -\frac{1}{p} < -r < \frac{1}{p'} + 1 = 2 - \frac{1}{p}\,, \quad p' := \frac{p}{p-1}\,. \tag{4.3.20}$$

The pair of the operator $\boldsymbol{K}_c$ and $-\overline{c}^{-1}\boldsymbol{K}_{\overline{c}^{-1}}$ are adjoint to each other. Therefore, the operator

$$\begin{aligned}\boldsymbol{A}_{\overline{c}} := \overline{c}\boldsymbol{K}_{\overline{c}} - \overline{c}^{-1}\boldsymbol{K}_{\overline{c}^{-1}} &: \mathbb{X}^r_p(\mathbb{R}^+) \to \mathbb{X}^r_p(\mathbb{R}^+),\\ &: \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+) \to \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+)\end{aligned} \tag{4.3.21}$$

is the adjoint to the operator $\boldsymbol{A}^\#_c$ in (4.3.18). Since the parameters $\{-r, p'\}$ satisfy the condition of the first part of the present theorem (see (4.3.20)), the operator $\boldsymbol{A}_{\overline{c}}$ in (4.3.21) is bounded and justifies the boundedness of the adjoint operator $\boldsymbol{A}^\#_c$ in (4.3.18). □

The next result is crucial in the present investigation. Note that the case $\arg c = 0$ is essentially different and will be considered in Theorem 4.5.1 below.

Theorem 4.3.4. *Let* $0 < \arg c < 2\pi$ *and* $0 < \arg(-c\,\gamma) < \pi$. *Then*

$$\boldsymbol{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi = c^{-s}\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\,\gamma}\varphi, \quad \varphi \in \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+), \tag{4.3.22}$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$.

Proof. First of all note that due to the mapping properties of the Bessel potential operators (see (4.2.6)) and the mapping properties of a Mellin convolution operator with an admissible kernel, both operators

$$\begin{aligned}\boldsymbol{\Lambda}^s_{-\gamma}\mathbf{K}^1_c &: \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^{r-s}_p(\mathbb{R}^+),\\ \mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\,\gamma} &: \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^{r-s}_p(\mathbb{R}^+)\end{aligned} \tag{4.3.23}$$

are correctly defined and bounded for all $s \in \mathbb{R}$, $1 < p < \infty$, since $-\pi < \arg(-\gamma) < 0$ and $0 < \arg(-c\,\gamma) < \pi$.

Second, let us consider the positive integer values $s = n = 1, 2, \ldots$. Then, with the help of formulae (4.2.11) and (4.3.14), it follows that

$$\begin{aligned}\mathbf{\Lambda}^n_{-\gamma}\mathbf{K}^1_c\varphi &= \Big(i\,\frac{d}{dt}-\gamma\Big)^n\mathbf{K}^1_c\varphi\\ &= \sum_{k=0}^n\binom{n}{k}i^k(-\gamma)^{n-k}\,\frac{d^k}{dt^k}\,\mathbf{K}^1_c\varphi = \sum_{k=0}^n\binom{n}{k}i^k(-\gamma)^{n-k}c^{-k}\Big(\mathbf{K}^1_c\,\frac{d^k}{dt^k}\,\varphi\Big)(t)\\ &= c^{-n}\mathbf{K}^1_c\bigg(\sum_{k=0}^n\binom{n}{k}i^k(-c\,\gamma)^{n-k}\,\frac{d^k}{dt^k}\,\varphi\bigg)(t) = c^{-n}\mathbf{K}^1_c\mathbf{\Lambda}^n_{-c\,\gamma}\varphi,\ \ \varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+),\end{aligned}$$

and we have proven formula (4.3.22) for positive integers $s=n=1,2,\dots$.

For negative $s=-1,-2,\dots$, formula (4.3.22) follows if we apply the inverse operator $\mathbf{\Lambda}^{-n}_{-\gamma}$ and $\mathbf{\Lambda}^{-n}_{-c\gamma}$ to the proved operator equality

$$\mathbf{\Lambda}^n_{-\gamma}\mathbf{K}^1_c = c^{-n}\mathbf{K}^1_c\mathbf{\Lambda}^n_{-c\,\gamma}$$

for positive $n=1,2,\dots$ from the left and from the right, respectively. We obtain

$$\mathbf{K}^1_c\mathbf{\Lambda}^{-n}_{-c\gamma} = c^{-n}\mathbf{\Lambda}^{-n}_{-\gamma}\mathbf{K}^1_c \ \text{ or } \ \mathbf{\Lambda}^{-n}_{-\gamma}\mathbf{K}^1_c = c^{n}\mathbf{K}^1_c\mathbf{\Lambda}^{-n}_{-c\gamma}$$

and (4.3.22) is proved also for a negative $s=-1,-2,\dots$.

In order to derive formula (4.3.22) for non-integer values of s, we can confine ourselves to the case $-2<s<-1$. Indeed, any non-integer value $s\in\mathbb{R}$ can be represented in the form $s=s_0+m$, where $-2<s_0<-1$ and m is an integer. Therefore, if for $s=s_0+m$ the operators in (4.3.23) are correctly defined and bounded, and if the relations in question are valid for $-2<s_0<-1$, then we can write

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = \mathbf{\Lambda}^{s_0+m}_{-\gamma}\mathbf{K}^1_c &= c^{-m}\mathbf{\Lambda}^{s_0}_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^m_{-c\,\gamma}\\ &= c^{-s_0-m}\mathbf{K}^1_c\mathbf{\Lambda}^{s_0}_{-c\,\gamma}\mathbf{\Lambda}^m_{-c\,\gamma} = c^{-s_0-m}\mathbf{K}^1_c\mathbf{\Lambda}^{s_0+m}_{-c\,\gamma} = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}.\end{aligned}$$

Thus let us assume that $-2<s<-1$ and consider the expression

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{1}{2\pi^2}\,r_+\int\limits_{-\infty}^{\infty}e^{-i\xi t}(\xi-\gamma)^s\int\limits_0^{\infty}e^{i\xi y}\int\limits_0^{\infty}\frac{\varphi(\tau)}{y-c\tau}\,d\tau\,dy\,d\xi, \tag{4.3.24}$$

where r_+ is the restriction to $\mathbb{R}^+$. It is clear that the integral in the right-hand side of (4.3.24) exists. Indeed, if $\varphi\in\mathbb{L}_2$, then $\mathbf{K}^1_c\varphi\in\mathbb{L}_2\cap C^\infty$ and $\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi\in\mathbb{H}^{-s}\cap C^\infty\subset\mathbb{L}_2\cap C^\infty$.

Now consider the function $e^{-izt}(z-\gamma)^s e^{izy}$, $z\in\mathbb{C}$. Since $\mathrm{Im}\gamma\neq0$, $s<-1$, for sufficiently small $\varepsilon>0$ this function is analytic in the strip between the lines $\mathbb{R}$ and $\mathbb{R}+i\varepsilon$ and vanishes at infinity for all finite $t\in\mathbb{R}$ and for all $y>0$. Therefore, the integration over the real line $\mathbb{R}$ in the first integral of (4.3.24) can be replaced by the integration over the line $\mathbb{R}+i\varepsilon$, i.e.,

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{1}{2\pi^2}\,r_+\int\limits_{-\infty}^{\infty}e^{-i\xi t+\varepsilon t}(\xi+i\varepsilon-\gamma)^s\int\limits_0^{\infty}e^{i\xi y-\varepsilon y}\int\limits_0^{\infty}\frac{\varphi(\tau)}{y-c\tau}\,d\tau\,dy\,dx. \tag{4.3.25}$$

Let us use the density of the set $C^\infty_0(\mathbb{R}^+)$ in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$. Thus for all finite $t\in\mathbb{R}$ and for all functions $\varphi\in C^\infty_0(\mathbb{R})$ with compact supports the integrand in the corresponding triple integral for (4.3.25) is absolutely integrable. Therefore, for such functions one can use Fubini–Tonelli theorem and change the order of integration in (4.3.25). Thereafter, one returns to the integration over the real line $\mathbb{R}$ and obtains

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{1}{2\pi^2}\,r_+\int\limits_0^{\infty}\varphi(\tau)\int\limits_0^{\infty}\frac{1}{y-c\tau}\int\limits_{-\infty}^{\infty}e^{i\xi(y-t)}(\xi-\gamma)^s\,d\xi\,dy\,d\tau. \tag{4.3.26}$$

In order to study the expression in the right-hand side of (4.3.26), one can use a well known formula

$$\int\limits_{-\infty}^{\infty}(\beta+ix)^{-\nu}e^{-ipx}\,dx = \begin{cases}0 & \text{for }\ p>0,\\ -\dfrac{2\pi(-p)^{\nu-1}e^{\beta\,p}}{\Gamma(\nu)} & \text{for }\ p<0,\end{cases}\qquad \mathrm{Re}\,\nu>0,\ \ \mathrm{Re}\,\beta>0$$

[84, Formula 3.382.6]. It can be rewritten in a more convenient form, viz.,

$$\int_{-\infty}^{\infty} e^{i\mu\,\xi}(\xi-\gamma)^s\,d\xi = \begin{cases} 0 & \text{if } \mu<0,\ \operatorname{Im}\gamma>0,\\ \dfrac{2\pi\,\mu^{-s-1}e^{-\frac{\pi}{2}si+\mu\,\gamma i}}{\Gamma(-s)} & \text{if } \mu>0,\ \operatorname{Im}\gamma>0. \end{cases} \tag{4.3.27}$$

Applying (4.3.27) to the last integral in (4.3.26), one obtains

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)}\,r_+\int_0^\infty \varphi(\tau)\,d\tau\int_t^\infty \frac{e^{i(y-t)\gamma}}{(y-t)^{1+s}(y-c\tau)}\,dy\\ &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)}\,r_+\int_0^\infty \varphi(\tau)\,d\tau\int_0^\infty \frac{y^{-s-1}e^{i\gamma\,y}}{y+t-c\tau}\,dy, \end{aligned} \tag{4.3.28}$$

where the integrals exist, since $-s-1>-1$ and $0<\arg\gamma<\pi$ (i.e., $\operatorname{Im}\gamma>0$).

Let us recall the formula

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu\,x}}{x+\beta}\,dx = \beta^{\nu-1}e^{\beta\,\mu}\Gamma(\nu)\Gamma(1-\nu,\beta\mu),\ \ \operatorname{Re}\nu>0,\ \ \operatorname{Re}\mu>0,\ \ \arg\beta|<\pi \tag{4.3.29}$$

(cf. [84, Formula 3.383.10]). Due to the conditions $0<\arg c<2\pi$, $t>0$, $\tau>0$, we have $|\arg(t-c\tau)|<\pi$ and, therefore, we can apply (4.3.29) to equality (4.3.28). Then (4.3.28) acquires the following final form:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{e^{-\frac{\pi}{2}si}}{\pi}\,r_+\int_0^\infty \frac{e^{-i\gamma(t-c\tau)}\Gamma(1+s,-i\gamma(t-c\tau))\varphi(\tau)}{(t-c\tau)^{1+s}}\,d\tau. \tag{4.3.30}$$

Consider now the reverse composition $\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t)$. Changing the order of integration in the corresponding expression (see (4.3.26) for a similar motivation), one obtains

$$\begin{aligned} \mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t) &:= \frac{1}{2\pi^2}\,r_+\int_0^\infty \frac{1}{t-c\,y}\int_{-\infty}^\infty e^{-i\xi\,y}(\xi-c\,\gamma)^s\int_0^\infty e^{i\xi\,\tau}\varphi(\tau)\,d\tau\,d\xi\,dy\\ &= \frac{1}{2\pi^2}r_+\int_0^\infty \varphi(\tau)\int_0^\infty \frac{1}{t-c\,y}\int_{-\infty}^\infty e^{i\xi(\tau-y)}(\xi-c\,\gamma)^s\,d\xi\,dy\,d\tau. \end{aligned} \tag{4.3.31}$$

In order to compute the expression in the right-hand side of (4.3.31), let us recall formula 3.382.7 from [84]:

$$\int_{-\infty}^{\infty} (\beta-ix)^{-\nu}e^{-ipx}\,dx = \begin{cases} 0 & \text{for } p<0,\\ \dfrac{2\pi\,p^{\nu-1}e^{-\beta\,p}}{\Gamma(\nu)} & \text{for } p>0, \end{cases} \qquad \operatorname{Re}\nu>0,\ \ \operatorname{Re}\beta>0,$$

and rewrite it in the form, more suitable for our consideration, viz.,

$$\int_{-\infty}^{\infty} e^{i\mu\,\xi}(\xi+\omega)^s\,d\xi = \begin{cases} 0 & \mu>0,\ \operatorname{Im}\omega>0,\\ \dfrac{2\pi\,(-\mu)^{-s-1}e^{\frac{\pi}{2}si-\mu\,\omega i}}{\Gamma(-s)} & \mu<0,\ \operatorname{Im}\omega>0, \end{cases} \tag{4.3.32}$$
$$\operatorname{Re}s<0,\ \ \mu\in\mathbb{R},\ \ \omega,\,s\in\mathbb{C}.$$

Using (4.3.32), we represent (4.3.31) in the form

$$\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t) = \frac{e^{\frac{\pi}{2}si}}{\pi\Gamma(-s)}\,r_+\int_0^\infty \varphi(\tau)\,d\tau\int_\tau^\infty \frac{e^{-ic\,\gamma(y-\tau)}}{(y-\tau)^{s+1}(t-c\,y)}\,dy$$

$$= -\frac{e^{\frac{\pi}{2}si}}{\pi c\Gamma(-s)}\, r_+\int_0^\infty \varphi(\tau)\,d\tau\int_0^\infty \frac{y^{-s-1}e^{-ic\gamma\, y}}{y+\tau-c^{-1}t}\,dy, \tag{4.3.33}$$

where the integrals exist, since $-s-1>-1$ and $-\pi<\arg(c\,\gamma)<0$ (i.e., $\operatorname{Im} c\,\gamma<0$).

Due to the conditions $0<\arg c<2\pi$, $t>0$, $\tau>0$, we have $|\arg(\tau-c^{-1}t)|<\pi$. Therefore, we can apply formula (4.3.29) to (4.3.33) and get the following representation:

$$\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t) = -\frac{c^{-1}e^{\frac{\pi}{2}si}}{\pi}\, r_+\int_0^\infty \frac{e^{-ic\gamma(c^{-1}t-\tau)}\Gamma(1+s,-ic\gamma(c^{-1}t-\tau))\varphi(\tau)}{(\tau-c^{-1}t)^{1+s}}\,d\tau$$

$$= \frac{c^s e^{-\frac{\pi}{2}si}}{\pi}\, r_+\int_0^\infty \frac{e^{-i\gamma(t-c\,\tau)}\Gamma(1+s,-i\gamma(t-c\,\tau))\varphi(\tau)}{(t-c\,\tau)^{1+s}}\,d\tau. \tag{4.3.34}$$

If we multiply (4.3.34) by c^{-s}, we get precisely the expression in (4.3.30) and, therefore, $\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t)=c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t)$, which proves the claimed equality (4.3.22) for $-2<s<-1$ and accomplishes the proof. □

Corollary 4.3.2. *Let $0<\arg c<2\pi$ and $0<\arg\gamma<\pi$. Then for arbitrary $\gamma_0\in\mathbb{C}$ such that $0<\arg\gamma_0<\pi$ and $-\pi<\arg(c\,\gamma_0)<0$, one has*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{4.3.35}$$

where

$$g^s_{-\gamma,-\gamma_0}(\xi) := \Big(\frac{\xi-\gamma}{\xi-\gamma_0}\Big)^s.$$

If, in addition, $1<p<\infty$ and $\frac{1}{p}-1<r<\frac{1}{p}$, then equality (4.3.35) can be supplemented as follows:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\big[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}}+\mathbf{T}\big]\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{4.3.36}$$

where $\mathbf{T}:\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)\to\mathbb{H}^r_p(\mathbb{R}^+)$ is a compact operator, and if c is a real negative number, then $c^{-s}:=|c|^{-s}e^{-\pi si}$.

Proof. It follows from equalities (4.2.13) and (4.3.22) that

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{\Lambda}^s_{-\gamma_0}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}$$

and (4.3.35) is proved. If $1<p<\infty$ and $\frac{1}{p}-1<r<\frac{1}{p}$, then the commutator

$$\mathbf{T} := W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c - \mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)\to\mathbb{H}^r_p(\mathbb{R}^+)$$

of Mellin and Fourier convolution operators is correctly defined and bounded. It is compact for $r=0$ and all $1<p<\infty$ (see [41, 52]). Due to Krasnoselsky's interpolation theorem (see [96] and also [133, Sections 1.10.1 and 1.17.4]), the operator $\mathbf{T}$ is compact in all $\mathbb{L}_r$-spaces for $\frac{1}{p}-1<r<\frac{1}{p}$. Therefore, equality (4.3.35) can be rewritten as

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\big[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}}+\mathbf{T}\big]\mathbf{\Lambda}^s_{-c\,\gamma_0},$$

and we are done. □

Remark 4.3.1. The assumption $\frac{1}{p}-1<r<\frac{1}{p}$ in (4.3.36) cannot be relaxed. Indeed, the operator $W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c=\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{K}^1_c:\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)\to\mathbb{H}^r_p(\mathbb{R}^+)$ is bounded for all $r\in\mathbb{R}$ (see (4.3.23)). But the operator $\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}}:\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)\to\mathbb{H}^r_p(\mathbb{R}^+)$ is bounded only for $\frac{1}{p}-1<r<\frac{1}{p}$, because the function $g^s_{-\gamma,-\gamma_0}(\xi)$ has an analytic extension into the lower half-plane but not into the upper one.

4.4 Quasi localization in Banach para-algebras

In the present section, we expose well known, but modified local principle from [38, 83, 127], which we apply intensively.

Let $\mathfrak{B}_1(\mathscr{C})$ and $\mathfrak{B}_2(\mathscr{C})$ be Banach spaces of functions on an (n-1)-dimensional ℓ-smooth hypersurface $\mathscr{C} \subset \mathscr{S} \subset \mathbb{R}^n$, $\ell \geqslant 1$, with the Lipshitz boundary $\Gamma = \partial\mathscr{C}$, and multiplication by uniformly bounded smooth $C^\ell(\overline{\mathscr{C}})$-functions are bounded operators in both spaces. If $\mathscr{C} = \mathbb{R}^{n-1}$, we consider one point compactification $\overline{\mathscr{C}} := \dot{\mathbb{R}}^{n-1}$ of $\mathscr{C} = \mathbb{R}^{n-1}$ and then $\Gamma = \emptyset$.

$\mathscr{C}$ can also be minded as a domain with the Lipschitz boundary in the Euclidean space $\mathbb{R}^{n-1}$.

Definition 4.4.1. A quadruple $\mathscr{L} = [\mathscr{L}_{jk}]_{2\times 2}$ of Banach spaces is called a **Banach para-algebra** if there exists a binary mapping (a multiplication)

$$\mathscr{L}_{jk} \times \mathscr{L}_{kr} \to \mathscr{L}_{jr}$$

for each choice of $j, k, r = 1, 2$, which is continuous, associative and bilinear.

Definition 4.4.2. Let $\mathfrak{A}$ be a Banach algebra. A set $\Delta \subset \mathfrak{A}$ is called a **localizing class** if:

i) $0 \notin \Delta$;

ii) for a pair of elements $a_1, a_2 \in \Delta$ an element $a \in \Delta$ exists such that $a_m a = a a_m = a$, $m = 1, 2$.

Definition 4.4.3. A system $\{\Delta_\omega\}_{\omega\in\Omega}$ of localizing classes in $\mathfrak{A}$ is said to be **covering** if from arbitrary collection $\{a_\omega\}_{\omega\in\Omega}$ of elements $a_\omega \in \Delta_\omega$ there can be selected a finite collection $\{a_{\omega_j}\}_{j=1}^N$ so that the sum $\sum\limits_{j=1}^N a_{\omega_j}$ is invertible in $\mathfrak{A}$.

In what follows, under the Banach algebra there are taken linear operator algebras on Banach (in particular-function) spaces and the quotient spaces

$$\mathscr{L}'_0(\mathfrak{B}_1(\mathscr{C}), \mathfrak{B}_2(\mathscr{C})) := \mathscr{L}(\mathfrak{B}_1(\mathscr{C}), \mathfrak{B}_2(\mathscr{C}))/\mathscr{K}(\mathfrak{B}_1(\mathscr{C}), \mathfrak{B}_2(\mathscr{C}))$$

of linear bounded operators with respect to the ideal of compact operators.

Definition 4.4.1 implies that the spaces $\mathscr{L}_{11}$ and $\mathscr{L}_{22}$ from a Banach para-algebra $\mathscr{L} = [\mathscr{L}_{jk}]_{2\times 2}$ are Banach algebras.

For a pair of Banach spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ the quadruple

$$\mathscr{L}_0(\mathfrak{B}_1, \mathfrak{B}_2) := [\mathscr{L}(\mathfrak{B}_k, \mathfrak{B}_j)]_{2\times 2}$$

represents a Banach para-algebra. Moreover, the quadruple of the quotient algebras $\mathscr{L}'_0(\mathfrak{B}_1, \mathfrak{B}_2) = [\mathscr{L}'_{jk}]_{2\times 2} = [\mathscr{L}(\mathfrak{B}_j, \mathfrak{B}_k)/\mathscr{K}(\mathfrak{B}_j, \mathfrak{B}_k)]_{2\times 2}$ represents a Banach para-algebra, as well. For simplicity, we dwell on these particular para-algebras.

Let $x \in \overline{\mathscr{C}}$ and consider the class of multiplication operators by functions

$$\Delta_x := \Big\{ vI : v \in C^\ell(\mathscr{C}),\ \ v(t) = 1 \text{ for } |t - x| < \varepsilon_1,\ \ v(x) \geqslant 0 \ \text{ and } \ v(t) = 0 \text{ for } |t - x| > \varepsilon_2 \Big\}, \quad (4.4.1)$$

where $\varepsilon_2 > \varepsilon_1 > 0$ are not fixed and vary from function to function. Δ_x is, obviously, a localizing class in the algebra of bounded linear operators $\mathscr{L}(\mathfrak{B}_1(\mathscr{C}), \mathfrak{B}_2(\mathscr{C}))$ and $\{\Delta_x\}_{x\in\overline{\mathscr{C}}}$ is a covering class. Indeed, for a system $\{v_x I\}_{x\in\overline{\mathscr{C}}}$ we consider the related covering

$$\overline{\mathscr{C}} = \bigcup_{x\in\overline{\mathscr{C}}} U_x, \quad U_x := \{y \in \overline{\mathscr{C}} \ : \ v_x(y) = 1\}.$$

The set $\overline{\mathscr{C}}$ is compact and there exists a finite covering system $\overline{\mathscr{C}} = \bigcup_{j=1}^N U_{x_j}$. The corresponding sum is strictly positive

$$\inf_{y\in\mathbb{R}^n} g(y) \geqslant 1 \quad \text{for} \quad g(y) := \sum_{j=1}^N v_{x_j}(y) \qquad (4.4.2)$$

and the multiplication operator $\sum\limits_{j=1}^{N} v_{x_j}I = gI$ has the inverse $g^{-1}I$. Thus, the system of localizing classes $\{\Delta_x\}_{x\in\overline{\mathscr{C}}}$ is covering.

Definition 4.4.4. A quotient class $[\boldsymbol{A}] \in \mathscr{L}'(\mathfrak{B}_1(\mathscr{C}), \mathfrak{B}_2(\mathscr{C}))$ is called **Δ_x-invertible from the left** (**Δ_x-invertible from the right**) if there exists a quotient class $[\mathbf{R}_x] \in \mathscr{L}'(\mathfrak{B}_2(\mathscr{C}), \mathfrak{B}_1(\mathscr{C}))$ and $v_x \in \Delta_x$ such that the operator equality $[\mathbf{R}_x\mathbf{A}v_xI_1] = [v_xI_1]$ ($[v_x\mathbf{A}\mathbf{R}_x] = [v_xI_2]$, respectively) holds, where I_1 and I_2 are the identity operators in the spaces $\mathfrak{B}_1(\mathscr{C})$ and $\mathfrak{B}_2(\mathscr{C})$.

$[\boldsymbol{A}]$ is called **Δ_x-invertible** if it is Δ_x-invertible from the left and from the right.

We can generalize Definition 4.4.4 for operators

$$\mathbf{A}_j \ : \ \mathfrak{B}_1(\mathscr{C}_j) \to \mathfrak{B}_2(\mathscr{C}_j), \quad j = 1, 2, \tag{4.4.3}$$

in the same pairs of function spaces $\mathfrak{B}_1(\mathscr{C}_1)$, $\mathfrak{B}_2(\mathscr{C}_1)$ and $\mathfrak{B}_1(\mathscr{C}_2)$, $\mathfrak{B}_2(\mathscr{C}_2)$ defined on different domains $\mathscr{C}_1, \mathscr{C}_2 \subset \mathbb{R}^n$. For this, we assume that for any pair of points $x_1 \in \overline{\mathscr{C}}_1$ and $x_2 \in \overline{\mathscr{C}}_2$ there exists a local diffeomorphism of neighbourhoods

$$\beta \ : \ \omega_1 \to \omega_2, \quad \beta(x_1) = x_2, \quad x_j \in \omega_j \subset \mathscr{C}_j, \quad j = 1, 2. \tag{4.4.4}$$

The operators

$$\beta_*\varphi(x) := \varphi(\beta(x)), \quad \beta_*^{-1}\psi(y) := \psi(\beta^{-1}(y))$$

are inverses to each-other and map the spaces

$$\beta_* \ : \ \mathfrak{B}_j(\omega_2) \to \mathfrak{B}_j(\omega_1), \quad \beta_*^{-1} \ : \ \mathfrak{B}_j(\omega_1) \to \mathfrak{B}_j(\omega_2).$$

Definition 4.4.5 (local quasi equivalence)**.** Let multiplication by uniformly bounded C^ℓ-functions on corresponding closed domains $\overline{\mathscr{C}_1}$ and $\overline{\mathscr{C}_1}$ are bounded operators in all respective spaces $\mathfrak{B}_2(\mathscr{C}_1)$ and $\mathfrak{B}_1(\mathscr{C}_2)$, $\mathfrak{B}_2(\mathscr{C}_2)$.

Two classes from the quotient spaces $[\boldsymbol{A}_1], [\boldsymbol{A}_2] \in \mathscr{L}'(\mathfrak{B}_1, \mathfrak{B}_2)$ (see (4.4.3)) are called **locally quasi equivalent from the left**, $[\mathbf{A}_1] \overset{\Delta_{x_1}-L}{\sim} \beta \overset{\Delta_{x_2}-L}{\sim} [\mathbf{A}_2]$ or $[\mathbf{A}_1] \overset{x_1-L}{\sim} \beta \overset{x_2-L}{\sim} [\mathbf{A}_2]$ (**locally quasi equivalent from the right**, $[\mathbf{A}_1] \overset{\Delta_{x_1}-R}{\sim} \beta \overset{\Delta_{x_2}-R}{\sim} [\mathbf{A}_2]$ or $[\mathbf{A}_1] \overset{x_1-R}{\sim} \beta \overset{x_2-R}{\sim} [\mathbf{A}_2]$) at $x_1 \in \overline{\mathscr{C}}_1$ and $x_2 \in \overline{\mathscr{C}}_2$, if (see (4.4.4))

$$\begin{gathered}\inf_{v_{x_1}\in\Delta_{x_1}} |||[v_{x_1}][\mathbf{A}_1 - \beta_*\mathbf{A}_2\beta_*^{-1}]||| = 0,\\ \left(\inf_{v_{x_1}\in\Delta_{x_1}} |||[\mathbf{A}_1 - \beta_*\mathbf{A}_2\beta_*^{-1}][v_{x_1}I]||| = 0\right),\end{gathered} \tag{4.4.5}$$

where the norm in the quotient space $\mathscr{L}'(\mathfrak{B}_1, \mathfrak{B}_2) = \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)/\mathscr{K}(\mathfrak{B}_1, \mathfrak{B}_2)$ coincides with the essential norm

$$\|[\boldsymbol{A}]\| := |||\boldsymbol{A}||| := \inf_{T\in\mathscr{K}(\mathfrak{B}_1,\mathfrak{B}_2)} \|A + T\|.$$

If two classes from the quotient spaces $[\boldsymbol{A}_1], [\boldsymbol{A}_2] \in \mathscr{L}'(\mathfrak{B}_1, \mathfrak{B}_2)$ are locally quasi equivalent both, from the left and from the right, they are called **locally quasi equivalent** at $x_1 \in \overline{\mathscr{C}}_1$ and $x_2 \in \overline{\mathscr{C}}_2$ $[\mathbf{A}_1] \overset{\Delta_{x_1}}{\sim} \beta \overset{\Delta_{x_2}}{\sim} [\mathbf{A}_2]$ or $[\mathbf{A}_1] \overset{x_1}{\sim} \beta \overset{x_2}{\sim} [\mathbf{A}_2]$.

If $\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}$ and $\beta(x) = x$ is the identity map, the equivalence at the point $x \in \mathscr{C}$ is denoted as follows: $[\mathbf{A}_1] \overset{\Delta_x-L}{\sim} [\mathbf{A}_2]$, $[\mathbf{A}_1] \overset{\Delta_x-R}{\sim} [\mathbf{A}_2]$, $[\mathbf{A}_1] \overset{\Delta_x}{\sim} [\mathbf{A}_2]$ or also $[\mathbf{A}_1] \overset{x-L}{\sim} [\mathbf{A}_2]$, $[\mathbf{A}_1] \overset{x-R}{\sim} [\mathbf{A}_2]$, $[\mathbf{A}_1] \overset{x}{\sim} [\mathbf{A}_2]$.

Definition 4.4.6. Let $\mathfrak{B}_1(\mathscr{C})$ and $\mathfrak{B}_2(\mathscr{C})$ be the same as in Definition 4.4.1. An operator $\mathbf{A} \ : \ \mathfrak{B}_1(\mathscr{C}) \to \mathfrak{B}_2(\mathscr{C})$ is called of **local type** if $v_1\boldsymbol{A}v_2I \ : \ \mathfrak{B}_1(\mathscr{C}) \to \mathfrak{B}_2(\mathscr{C})$ is compact for all $v_1, v_2 \in C^\ell(\mathscr{C})$, provided $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$ (see [127]); or, equivalently, if $v\boldsymbol{A} - \boldsymbol{A}vI \ : \ \mathfrak{B}_1(\mathscr{C}) \to \mathfrak{B}_2(\mathscr{C})$ is compact for all $v \in C^\ell(\mathscr{C})$ (see [125]).

Theorem 4.4.1 (Quasi Localization Principle). *Let* $\mathbf{A}$*,* $\mathfrak{B}_j(\mathscr{C}_k)$*,* $j,k=1,2$*, be the same as in Definition* 4.4.5 *and*

$$\mathbf{A}\ :\ \mathfrak{B}_1(\mathscr{C}_1)\to\mathfrak{B}_2(\mathscr{C}_1),\quad \boldsymbol{B}_y\ :\ \mathfrak{B}^1_y(\mathscr{C}^2_y)\to\mathfrak{B}^2_y(\mathscr{C}^2_y),\quad y=\beta_x(x),\quad x\in\mathscr{C}_1,$$

be operators of local type. The diffeomorphisms $\beta_x\ :\ \omega^1_x\to\omega^2_y$*,* $y=\beta_x(x)\in\mathscr{C}^2_y$ *of neighbourhoods of* $x\in\mathscr{C}_1$ *and of* $y\in\mathscr{C}^2_y$*, as well as the domains* $\mathscr{C}^2_y$ *and operators* $\boldsymbol{B}_y$*, depend on* $y=\beta_x(x)$ *and might be different for different* $x\in\overline{\mathscr{C}_1}$*.*

If the Quasi Equivalence $[\mathbf{A}]\overset{x}{\sim}\beta_x\overset{y=\beta_x(x)}{\sim}[\mathbf{B}_y]$ *holds at some point* $x\in\mathscr{C}_1$*, then the quotient class* $[\mathbf{A}]$ *is locally invertible at* $x\in\mathscr{C}_1$ *if and only if the quotient class* $[\mathbf{B}_y]$ *is locally invertible at* $y\in\mathscr{C}^2_y$*.*

If the Quasi Equivalence $[\mathbf{A}]\overset{x}{\sim}\beta_x\overset{y=\beta_x(x)}{\sim}[\mathbf{B}_y]$ *holds for all* $x\in\overline{\mathscr{C}_1}$ *and* $[\mathbf{B}_y]\in\mathscr{L}'(\mathfrak{B}_1(\mathscr{C}_2),\mathfrak{B}_2(\mathscr{C}_2))$ *are locally invertible at* $y\in\mathscr{C}^2_y$ *for all* $x\in\overline{\mathscr{C}}_1$*, then the quotient class* $[\boldsymbol{A}]$ *is globally invertible (i.e.,* $\mathbf{A}\ :\ \mathfrak{B}_1(\mathscr{C}_1)\to\mathfrak{B}_2(\mathscr{C}_1)$ *is a Fredholm operator).*

Proof. Let the left Quasi Equivalence $[\mathbf{A}]\overset{x-L}{\sim}\beta_x\overset{y=\beta_x(x)-R}{\sim}[\boldsymbol{B}_y]$ hold and $\boldsymbol{A}$ be Δ_x-invertible from the left. Then there exist $\boldsymbol{R}_x\in\mathscr{L}(\mathfrak{B}_2(\mathscr{C}_1),\mathfrak{B}_1)(\mathscr{C}_1)$, $v_1,v_2\in\Delta_x$ such that $\boldsymbol{R}_x\boldsymbol{A}v_1I=v_1I$ and

$$\begin{aligned}\boldsymbol{R}_x[\boldsymbol{A}-\beta_{x,*}\boldsymbol{B}_y\beta^{-1}_{x,*}]v_2I\big|\mathscr{L}(\mathfrak{B}_1(\mathscr{C}_1))\|&\leqslant\|R_xv_2I\big|\mathscr{L}(\mathfrak{B}_2(\mathscr{C}_1),\mathfrak{B}_1(\mathscr{C}_1))\|\\ &\times\|[\boldsymbol{A}-\beta_{x,*}\boldsymbol{B}_y\beta^{-1}_{x,*}]v_2I\big|\mathscr{L}(\mathfrak{B}_1(\mathscr{C}_1),\mathfrak{B}_2(\mathscr{C}_1))\|<1.\end{aligned}$$

Furthermore, let us pick up an element $v\in\Delta$ with the property $v_1v=v_2v=v$. Then $\boldsymbol{R}_x\beta_{x,*}\boldsymbol{B}_y\beta^{-1}_{x,*}vI=[I-\boldsymbol{D}_x]vI$, where

$$\boldsymbol{D}_x:=\boldsymbol{R}_x[\boldsymbol{A}-\beta_{x,*}\boldsymbol{B}_y\beta^{-1}_{x,*}]v_2I=[I-\boldsymbol{R}_x\beta_{x,*}\boldsymbol{B}_y\beta^{-1}_{x,*}]v_2I.$$

Since $\|\boldsymbol{D}_x\big|\mathscr{L}(\mathfrak{B}_1)\|<1$, the inverse $(I-\boldsymbol{D}_x)^{-1}$ to $I-\boldsymbol{D}_x$ exists and $\boldsymbol{R}^1_y\boldsymbol{B}_yvI=vI$ for $\boldsymbol{R}^1_y=\beta^{-1}_{x,*}(I-\boldsymbol{D}_x)^{-1}\boldsymbol{R}_x\beta_{x,*}$. Thus, $\boldsymbol{B}_y$ is Δ_y-invertible from the left.

Since Quasi Equivalence is symmetric, the left Quasi Equivalence and left local invertibility of $\boldsymbol{B}_y$ at $y=\beta_x(x)$ follows the left local invertibility of $\boldsymbol{A}$ at x.

The case of the right Quasi Equivalence $[\mathbf{A}]\overset{x-R}{\sim}\beta_x\overset{y=\beta_x(x)-R}{\sim}[\boldsymbol{B}_y]$ is similar.

Let the left Quasi Equivalence $[\mathbf{A}]\overset{x-L}{\sim}\beta_x\overset{y=\beta_x(x)-L}{\sim}[\mathbf{B}_y]$ hold for all $x\in\overline{\mathscr{C}_1}$ and $\boldsymbol{B}_y$ be Δ_y-invertible from the left for $y=\beta_x(x)$ and all $x\in\mathscr{C}_1$. By the first part of theorem $\boldsymbol{A}$ is then locally Δ_x invertible from the left for all $x\in\mathscr{C}_1$: there exist elements $\boldsymbol{R}_x\in\mathscr{L}(fB_2,fB_1)$ and $v_x\in\Delta_x$ such that $\boldsymbol{R}_x\boldsymbol{A}_xv_xI=v_xI$. Since the system $\{\Delta_x\}_{x\in\overline{\mathscr{C}_1}}$ is covering, there exists a finite collection of elements $v_{x_1},\dots,v_{x_N}$ such that the sum $v_0I=\sum\limits_{m=1}^{N}v_{x_m}I$ is invertible. By taking $\boldsymbol{R}=\sum\limits_{m=1}^{N}\boldsymbol{R}_{x_m}v_{x_m}I$ and recalling that $\boldsymbol{A}$ is of local type, which provides the commutativity $\boldsymbol{A}v_{y_m}I=v_{y_m}\boldsymbol{A}+\boldsymbol{T}_m$, where $\boldsymbol{T}_m\in\mathscr{L}(\mathfrak{B}_1(\mathscr{C}_1),\mathfrak{B}_2(\mathscr{C}_1))$ are all compact operators, $m=1,\dots,N$ (see Definition 4.4.6), we get

$$\boldsymbol{R}\boldsymbol{A}=\sum_{m=1}^{N}\boldsymbol{R}_{x_m}v_{x_m}\boldsymbol{A}=\sum_{m=1}^{N}\boldsymbol{R}_{x_m}[\boldsymbol{A}v_{x_m}I+\boldsymbol{T}_m]=\sum_{m=1}^{N}v_{x_m}I+\boldsymbol{T}=v_0I+\boldsymbol{T}\,,$$

where v_0I is invertible and

$$\boldsymbol{T}=\sum_{m=1}^{N}\boldsymbol{R}_{x_m}\boldsymbol{T}_m\in\mathscr{L}(\mathfrak{B}_1(\mathscr{C}_1),\mathfrak{B}_1(\mathscr{C}_1))$$

is a compact operator. Hence, the operator $\boldsymbol{A}$ has a left regularizer and the quotient class [A] is invertible from the left and the inverse reads $[\boldsymbol{A}]^{-1}=[v_0^{-1}I]\boldsymbol{R}$.

The case of the right Quasi Equivalence $[\mathbf{A}]\overset{x-R}{\sim}\beta_x\overset{y=\beta_x(x)-R}{\sim}[\mathbf{B}_y]$ is similar. □

4.5 Algebra generated by Mellin and Fourier convolution operators

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denote one point compactification of the real axes $\mathbb{R}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the two point compactification of $\mathbb{R}$. By $C(\dot{\mathbb{R}})$ (by $C(\overline{\mathbb{R}})$, respectively) we denote the space of continuous functions $g(x)$ on $\mathbb{R}$ which have the equal limits at infinity $g(-\infty) = g(+\infty)$ (limits at infinity can be different $g(-\infty) \neq g(+\infty)$). By $PC(\dot{\mathbb{R}})$ it is denoted the space of piecewise-continuous functions on $\dot{\mathbb{R}}$ having the limits $a(t \pm 0)$ at all points $t \in \dot{\mathbb{R}}$, including infinity.

Unlike the operators W_a^0 and $\mathfrak{M}_a^0$ (see Section 3.1) possessing the property

$$W_a^0 W_b^0 = W_{ab}^0, \ \ \mathfrak{M}_a^0 \mathfrak{M}_b^0 = \mathfrak{M}_{ab}^0 \ \text{ for all } \ a, b \in \mathfrak{M}_p(\mathbb{R}), \tag{4.5.1}$$

the composition of the convolution operators on the semi-axes W_a and W_b cannot be computed by the rules similar to (4.5.1). Nevertheless, the following propositions hold.

Proposition 4.5.1 ([47, § 2])**.** *Let $1 < p < \infty$ and $a, b \in \mathfrak{M}_p(\overline{\mathbb{R}}^+) \cap PC(\dot{\mathbb{R}})$ be scalar $\mathbb{L}_p$-multipliers, piecewise-continuous on $\mathbb{R}$ including infinity. Then the commutant $[W_a, W_b] := W_a W_b - W_b W_a$ of the operators W_a and W_b is a compact operator in the Lebesgue space $[W_a, W_b] : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$.*

Moreover, if, in addition, the symbols $a(\xi)$ and $b(\xi)$ of the operators W_a and W_b have no common discontinuity points, i.e., if

$$\big[a(\xi + 0) - a(\xi + 0)\big]\big[b(\xi + 0) - b(\xi + 0)\big] = 0 \ \textit{ for all } \ \xi \in \dot{\mathbb{R}},$$

then $\mathbf{T} = W_a W_b - W_{ab}$ is a compact operator in $\mathbb{L}_p(\mathbb{R}^+)$.

Note that the algebra of $N \times N$ matrix multipliers $\mathfrak{M}_2(\mathbb{R})$ coincides with the algebra of $N \times N$ matrix functions essentially bounded on $\mathbb{R}$. For $p \neq 2$, the algebra $\mathfrak{M}_p(\mathbb{R})$ is rather complicated. There are multipliers $g \in \mathfrak{M}_p(\mathbb{R})$ which are elliptic, i.e., $\operatorname{ess\,inf} |g(x)| > 0$, but $1/g \notin \mathfrak{M}_p(\mathbb{R})$. In connection with this, let us consider the subalgebra $PC\mathfrak{M}_p(\mathbb{R})$ which is the closure of the algebra of piecewise-constant functions on $\mathbb{R}$ in the norm of multipliers $\mathfrak{M}_p(\mathbb{R})$

$$\big\| a \mid \mathfrak{M}_p(\mathbb{R}) \big\| := \big\| W_a^0 \mid \mathbb{L}_p(\mathbb{R}) \big\|.$$

Note that any function $g \in PC\mathfrak{M}_p(\mathbb{R}) \subset PC(\mathbb{R})$ has limits $g(x \pm 0)$ for all $x \in \overline{\mathbb{R}}$, including infinity. Let

$$C\mathfrak{M}_p(\overline{\mathbb{R}}) := C(\overline{\mathbb{R}}) \cap PC\mathfrak{M}_p^0(\mathbb{R}), \quad C\mathfrak{M}_p^0(\dot{\mathbb{R}}) := C(\dot{\mathbb{R}}) \cap PC\mathfrak{M}_p(\mathbb{R}),$$

where the functions $g \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ (the functions $h \in C(\dot{\mathbb{R}})$) might have jump only at infinity $g(-\infty) \neq g(+\infty)$ (are continuous at infinity $h(-\infty) = h(+\infty)$).

$PC\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra and contains all functions of bounded variation as a subset for all $1 < p < \infty$ (Stechkin's theorem, see [47, Section 2]). Therefore, $\coth \pi(i\beta + \xi) \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ for all $p \in (1, \infty)$.

Proposition 4.5.2 ([47, § 2])**.** *If $g \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ is an $N \times N$ matrix multiplier, then its inverse $g^{-1} \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ if and only if it is elliptic, i.e., $\det g(x \pm 0) \neq 0$ for all $x \in \overline{\mathbb{R}}$. If this is the case, the corresponding Mellin convolution operator $\mathfrak{M}_g^0 : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$ is invertible and $(\mathfrak{M}_g^0)^{-1} = \mathfrak{M}_{g^{-1}}^0$.*

Moreover, any $N \times N$ matrix multiplier $b \in C\mathfrak{M}_p^0(\dot{\mathbb{R}})$ can be approximated by polynomials

$$r_n(\xi) := \sum_{j=-m}^{m} c_m \Big(\frac{\xi - i}{\xi + i}\Big)^m, \ \ r_m \in C\mathfrak{M}_p^0(\overline{\mathbb{R}}),$$

with constant $N \times N$ matrix coefficients, whereas any $N \times N$ matrix multiplier $g \in C\mathfrak{M}_p^0(\overline{\mathbb{R}})$ having a jump discontinuity at infinity can be approximated by $N \times N$ matrix functions $d \coth \pi(i\beta + \xi) + r_m(\xi)$, $0 < \beta < 1$.

The Hilbert transform on the semi-axis

$$S_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int_0^\infty \frac{\varphi(y)}{y-x}\,dy \tag{4.5.2}$$

is the Fourier convolution $S_{\mathbb{R}^+} = W_{-\,\mathrm{sign}}$ on the semi-axis $\mathbb{R}^+$ with the discontinuous symbol $-\,\mathbf{sign}\,\xi$ (see [47, Lemma 1.35]), and it is also the Mellin convolution

$$S_{\mathbb{R}^+} = \mathfrak{M}^0_{s_\beta} = \mathbf{Z}_\beta W^0_{s_\beta}\mathbf{Z}^{-1}_\beta, \tag{4.5.3}$$

$$s_\beta(\xi) := \coth\pi(i\beta+\xi) = \frac{e^{\pi(i\beta+\xi)} + \boldsymbol{e}^{-\pi(i\beta+\xi)}}{e^{\pi(i\beta+\xi)}\boldsymbol{e}^{-\pi(i\beta+\xi)}} = -i\cot\pi(\beta i\xi),\quad \xi\in\mathbb{R}$$

(cf. (4.1.1) and (4.1.7)). Indeed, to verify (4.5.3), rewrite $S_{\mathbb{R}^+}$ in the form

$$S_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int_0^\infty \frac{\varphi(y)}{1-\frac{x}{y}}\,\frac{dy}{y} = \int_0^\infty K\Big(\frac{x}{y}\Big)\varphi(y)\,\frac{dy}{y}\,,$$

where $K(t) := (1/\pi i)(1-t)^{-1}$. Further, using the formula

$$\int_0^\infty \frac{t^{z-1}}{1-t}\,dt = \pi\cot\pi z,\quad \operatorname{Re} z<1$$

(cf. [84, formula 3.241.3]), one shows that the Mellin transform $\mathscr{M}_\beta K(\xi)$ coincides with the function $s_\beta(\xi)$ from (4.5.3).

Next Theorem 4.5.1 is an enhancement of Theorem 4.2.4.

Theorem 4.5.1. *Let $1<p<\infty$ and $s\in\mathbb{R}$. For arbitrary $\gamma_j\in\mathbb{C}$, $\operatorname{Im}\gamma_j>0$ $(j=1,2)$, the Hilbert transform*

$$\boldsymbol{K}^1_1 = -iS^+_{\mathbb{R}} = -iW_{-\,\mathbf{sign}} = W_{i\,\mathbf{sign}} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+) \tag{4.5.4}$$

(*see* (4.3.13), (4.3.14) *and* (4.5.2)*; the case $c=1$, $\arg c=0$, Theorem* 4.3.4). $\boldsymbol{K}^1_1$ *is a Fourier convolution operator and*

$$\mathbf{\Lambda}^s_{-\gamma_1}\mathbf{K}^1_1\mathbf{\Lambda}^{-s}_{\gamma_2} = W_{i\,\mathbf{sign}\,g^s_{-\gamma_1,\gamma_2}} : \mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+), \tag{4.5.5}$$

where $g^s_{-\gamma_1,\gamma_2}(\xi)$ is defined in (4.2.10).

Proof. Formula (4.5.5) follows from (4.2.8) and (4.5.4). □

We need certain results concerning the compactness of Mellin and Fourier convolutions in $\mathbb{L}_p$-spaces. These results are scattered in literature. For the convenience of the reader, we reformulate them here as Propositions 4.5.3–4.5.7. For more details, the reader can consult [30, 47, 52].

Proposition 4.5.3 ([52, Proposition 1.6])**.** *Let $1<p<\infty$, $a\in C(\dot{\mathbb{R}}^+)$, $b\in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ and $a(0)=a(\infty)=b(\infty)=0$. Then the operators $a\mathfrak{M}^0_b, \mathfrak{M}^0_b\,aI : \mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ are compact.*

Proposition 4.5.4 ([47, Lemma 7.1], [52, Proposition 1.2])**.** *Let $1<p<\infty$, $a\in C(\dot{\mathbb{R}}^+)$, $b\in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ and $a(\infty)=b(\infty)=0$. Then the operators $aW_b, W_b\,aI : \mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ are compact.*

Proposition 4.5.5 ([52, Lemma 2.5, Lemma 2.6], [30])**.** *Assume that $1<p<\infty$. Then*

(1) *if $g\in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ and $g(\infty)=0$, the Hankel operator $H_g : \mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ is compact;*

(2) *if the functions $a\in C(\dot{\mathbb{R}})$, $b\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, $c\in C(\overline{\mathbb{R}}^+)$ satisfy one of the conditions*

   (i) *$c(0)=b(+\infty)=0$ and $a(\xi)=0$ for all $\xi>0$;*

(ii) $c(0)=b(-\infty)=0$ *and* $a(\xi)=0$ *for all* $\xi<0$;

(iii) $c(0)=b(\pm\infty)=a(0)=0$,

then the operators $cW_a\mathfrak{M}^0_b$, $c\mathfrak{M}^0_b W_a$, $W_a\mathfrak{M}^0_b\, cI$, $\mathfrak{M}^0_b W_a\, cI:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ *are all compact.*

Proof. Let us comment on item (1) which is actually well known. The kernel $k(x+y)$ of the operator H_a is approximated by the Laguerre polynomials $k_m(x+y)=e^{-x-y}p_m(x+y)$, $m=1,2,\dots$, where $p_m(x+y)$ are polynomials of order m so that the corresponding Hankel operators converge in norm $\|H_a-H_{a_m}|\,|\mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\|\to 0$, where $a_m=\mathscr{F}k_m$ are the Fourier transforms of the Laguerre polynomials (see, e.g. [82]). Since

$$|k_m(x+y)|=\left|e^{-x-y}p_m(x+y)\right|\leqslant C_m e^{-x}e^{-y}x^m y^m,\quad m=1,2,\dots,$$

for some constant C_m, the condition on the kernel

$$\int\limits_0^\infty\Bigg[\int\limits_0^\infty |k_m(x+y)|^{p'}\,dy\Bigg]^{p/p'}dx<\infty,\quad p':=\frac{p}{p-1}\,,$$

holds and ensures the compactness of the operator $H_{a_m}:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$. Then the limit operator $H_a=\lim\limits_{m\to\infty}H_{a_m}$ is compact as well.

Items (i) and (ii) are proved in [52].

The item (iii) follows from (i) and (ii) and the representation $cW_a\mathfrak{M}^0_b=cW_{\chi_-a}\mathfrak{M}^0_b+cW_{\chi_+a}\mathfrak{M}^0_b$, where $\chi_\pm$ are the characteristic functions of the semi-axes $\mathbb{R}^\pm$. □

Proposition 4.5.6 ([47, Lemma 7.1], [52, Proposition 1.2]). *Let* $1<p<\infty$, $a\in C(\dot{\overline{\mathbb{R}}}{}^+)$, $b\in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ *and* $a(\infty)=b(\infty)=0$. *Then the operators* $aW_b, W_b\, aI:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 4.5.7 ([47, Lemma 7.4], [52, Lemma 1.2]). *Let* $1<p<\infty$ *and let* a *and* b *satisfy at least one of the conditions*

(i) $a\in C(\overline{\mathbb{R}}{}^+)$, $b\in\mathfrak{M}^0_p(\mathbb{R})\cap PC(\overline{\mathbb{R}})$;

(ii) $a\in PC(\overline{\mathbb{R}}{}^+)$, $b\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$.

Then the commutants $[aI,W_b]$ *and* $[aI,\mathfrak{M}^0_b]$ *are compact operators in the space* $\mathbb{L}_p(\mathbb{R}^+)$.

Remark 4.5.1. Note that if both, a symbol b and a function a, have jumps at finite points, the commutants $[aI,W_b]$ and $[aI,\mathfrak{M}^0_b]$ are not compact. Only jumps of a symbol at infinity does not matter.

Proposition 4.5.8 ([52]). *The Banach algebra generated by the Cauchy singular integral operator* $S_{\mathbb{R}^+}$ *and the identity operator* I *on the semi-axis* $\mathbb{R}^+$ *contains Fourier convolution operators with symbols having discontinuity of the jump type only at zero and at infinity and Mellin convolution operators with continuous symbols on* $\dot{\mathbb{R}}$ (*including infinity*).

Moreover, the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ *generated by the Cauchy singular integral operators with "shifts"*

$$S^c_{\mathbb{R}^+}\varphi(x):=\frac{1}{\pi i}\int\limits_0^\infty\frac{e^{-ic(x-y)}\varphi(y)}{y-x}\,dy=W_{-\,\mathbf{sign}(\xi-c)}\varphi(x)\ \ \textit{for all}\ \ c\in\mathbb{R}$$

and by the identity operator I *on the semi-axis* $\mathbb{R}^+$ *over the field of* $N\times N$ *complex valued matrices coincides with the Banach algebra generated by Fourier convolution operators with piecewise-constant* $N\times N$ *matrix symbols containing all Fourier convolution* W_a *and Hankel* H_b *operators with* $N\times N$ *matrix symbols* (*multipliers*) $a,b\in PC\mathfrak{M}_p(\overline{\mathbb{R}})$.

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators, i.e., by the operators

$$\mathbf{A} := \sum_{j=1}^{m} \mathfrak{M}^0_{a_j} W_{b_j} \tag{4.5.6}$$

and there compositions, in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$. Here $\mathfrak{M}^0_{a_j}$ are Mellin convolution operators with continuous $N \times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\dot{\mathbb{R}})$, W_{b_j} are Fourier convolution operators with $N \times N$ matrix symbols $b_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$. The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous $N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at infinity.

This and more general algebras were studied in [52] and also in earlier works [41, 51, 132].

Remark 4.5.2. If in (4.5.6) we admit more general symbols $a_j \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ which have different limits at infinity $a_j(-\infty) \neq a_j(+\infty)$, this will not be a generalization.

Indeed, if $a_j \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ has different limits at infinity $a_j(-\infty) \neq a_j(+\infty)$, we can represent

$$a_j(\xi) = a^0_j(\xi) + a_j(-\infty)\,\frac{1 - \coth \pi(\frac{i}{p} + \xi)}{2} + a_j(+\infty)\,\frac{1 + \coth \pi(\frac{i}{p} + \xi)}{2}\,, \quad a^0_j(\pm\infty) = 0$$

and the corresponding Mellin operator is written as follows:

$$\begin{aligned} \mathfrak{M}^0_{a_j} &= \mathfrak{M}^0_{a^0_j} + \frac{a_j(-\infty)}{2}\,[I - S_{\mathbb{R}^+}] + \frac{a_j(-\infty)}{2}\,[I + S_{\mathbb{R}^+}] \\ &= \mathfrak{M}^0_{a^0_j} + \frac{a_j(-\infty)}{2}\,[I - W_{-\,\mathbf{sign}}] + \frac{a_j(-\infty)}{2}\,[I + W_{-\,\mathbf{sign}}] \end{aligned}$$

(see (4.5.4) and (4.3.14)). Therefore, the discontinuity at infinity of symbols of Mellin convolution operators is taken over in Fourier convolution operators and we can even assume in (4.5.6) that $a^0_j(\pm\infty) = 0$ for all $j = 1, \ldots, m$.

In order to keep the exposition self-contained and to improve formulations from [52], the results concerning the Banach algebra generated by operators (4.5.6) are presented here with some modification and the proofs.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a with piecewise-constant symbols $a(\xi)$ in the space $\mathbb{L}_p(\mathbb{R}^+)$ (cf. Proposition 4.5.7). Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N = 1$, the following is true.

Corollary 4.5.1. *The quotient algebra* $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ *is commutative in the scalar case* $N = 1$.

To expose the symbol of operator (4.5.6), consider the infinite clockwise oriented “rectangle” $\mathfrak{R} := \Gamma_1 \cup \Gamma^-_2 \cup \Gamma^+_2 \cup \Gamma_3$, where (cf. Fig. 4.1)

$$\Gamma_1 := \{\infty\} \times \overline{\mathbb{R}}, \;\; \Gamma^{\pm}_2 := \overline{\mathbb{R}}^+ \times \{\pm\infty\}, \;\; \Gamma_3 := \{0\} \times \overline{\mathbb{R}}.$$

The symbol $\mathscr{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (4.5.6) is a function on the set $\mathfrak{R}$, viz.,

$$\mathscr{A}_p(\omega) := \begin{cases} \displaystyle\sum_{j=1}^{m} a_j(\xi)(b_j)_p(\infty, \xi), & \omega = (\infty, \xi) \in \overline{\Gamma_1}, \\ \displaystyle\sum_{j=1}^{m} a_j(\infty) b_j(\eta), & \omega = (\eta, +\infty) \in \Gamma^+_2, \\ \displaystyle\sum_{j=1}^{m} a_j(\infty) b_j(-\eta), & \omega = (\eta, -\infty) \in \Gamma^-_2, \\ \displaystyle\sum_{j=1}^{m} a_j(\xi)(b_j)_p(0, \xi), & \omega = (0, \xi) \in \overline{\Gamma_3}. \end{cases} \tag{4.5.7}$$

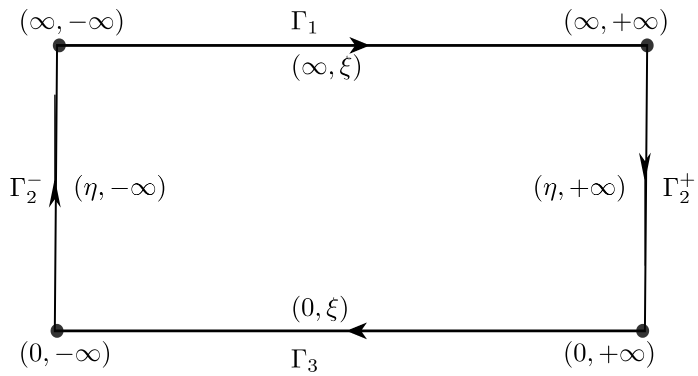

Figure 4.1. The domain $\mathfrak{R}$ of definition of the symbol $\mathscr{A}_p^s(\omega)$.

The symbol $\mathscr{A}_p(\omega)$, when $\omega=(\infty,\xi)$ ranges through the infinite interval Γ_1 (cf. Fig. 4.1), it fills the gap between the values

$$\sum_{j=1}^m a_j(\infty)b_j(-\infty) \ \text{ and } \ \sum_{j=1}^m a_j(\infty)b_j(+\infty)$$

and, when $\omega=(0,\xi)$ ranges through the infinite interval Γ_3 (cf. Fig. 4.1), it fills the gap between the values

$$\sum_{j=1}^m a_j(\xi)b_j(0-0) \quad \text{and} \quad \sum_{j=1}^m a_j(\xi)b_j(0+0).$$

The connecting function $g_p(\infty,\xi)$ in (4.5.7) for a piecewise continuous function $g\in PC(\overline{\mathbb{R}})$ is defined as follows:

$$\begin{aligned} g_p(x,\xi) &:= \frac{g(x+0)+g(x-0)}{2}+\frac{g(x+0)-g(x-0)}{2i}\cot\pi\Big(\frac{1}{p}-i\xi\Big)\\ &= e^{i\pi\,\frac{g_x^+ + g_x^-}{2}}\,\frac{\cos\pi(\frac{1}{p}-\frac{g_x^+ + g_x^-}{2}-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\,,\quad \xi\in\mathbb{R}, \qquad (4.5.8)\\ g_x^\pm &:= \frac{1}{\pi i}\ln g(x\pm 0),\quad \operatorname{Re} g_x^\pm=\frac{1}{\pi}\arg g(x\pm 0),\quad x\in\dot{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}. \end{aligned}$$

The function $g_p(\infty,\xi)$ fills up the discontinuity (the jump) of $g(\xi)$ at ∞ between $g(-\infty)$ and $g(+\infty)$ with an oriented arc of the circle such that from every point of the arc the oriented interval $[g(-\infty),g(+\infty)]$ is seen under the angle π/p. Moreover, the oriented arc lies above the oriented interval if $\frac{1}{2}<\frac{1}{p}<1$ (i.e., if $1<p<2$) and the oriented arc is under the oriented interval if $0<\frac{1}{p}<\frac{1}{2}$ (i.e., if $2<p<\infty$). For $p=2$, the oriented arc coincides with the oriented interval (cf. Fig. 4.2 on page 117)).

A similar geometric interpretation is valid for the function $g_p(t,\xi)$, which connects the point $g(t-0)$ with $g(t+0)$ at the point t where $g(\xi)$ has a jump discontinuity.

To make the symbol $\mathscr{A}_p(\omega)$ continuous, we endow the rectangle $\mathfrak{R}$ with a special topology. Thus, let us define the distance on the curves Γ_1, $\Gamma_2^\pm$, Γ_3 and on $\overline{\mathbb{R}}$ by

$$\rho(x,y):=\Big|\arg\frac{x-i}{x+i}-\arg\frac{y-i}{y+i}\Big| \ \text{ for arbitrary } \ x,y\in\overline{\mathbb{R}}.$$

In this topology, the length $|\mathfrak{R}|$ of $\mathfrak{R}$ is 6π, and the symbol $\mathscr{A}_p(\omega)$ is continuous everywhere on $\mathfrak{R}$. The image of the function $\det\mathscr{A}_p(\omega)$, $\omega\in\mathfrak{R}$ ($\det\mathscr{B}_p(\omega)$) is a closed curve in the complex plane. It follows from the continuity of the symbol at the angular points of the rectangle $\mathfrak{R}$ where the one-sided limits

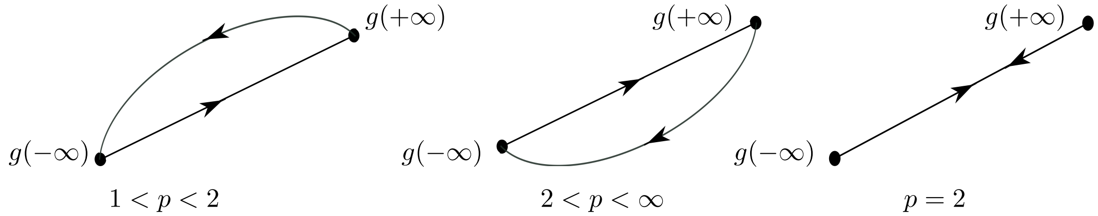

Figure 4.2. Arc condition.

coincide. Thus

$$\mathscr{A}_p(\pm\infty, \infty) = \sum_{j=1}^{m} a_j(\infty) b_j(\mp\infty),$$
$$\mathscr{A}_p(\pm\infty, 0) = \sum_{j=1}^{m} [a_j(\infty) b_j(0 \mp 0).$$

Hence, if the symbol of the corresponding operator is elliptic, i.e., if

$$\inf_{\omega \in \mathfrak{R}} |\det \mathscr{A}_p(\omega)| > 0, \tag{4.5.9}$$

the increment of the argument $(1/2\pi)\arg \mathscr{A}_p(\omega)$, when ω ranges through $\mathfrak{R}$ in the positive direction, is an integer, is called the winding number or the index and it is denoted by $\mathbf{ind}\det \mathscr{A}_p$.

Theorem 4.5.2. *Let $1 < p < \infty$ and let $\mathbf{A}$ be defined by* (4.5.6)*. The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, the index of the operator has the value*

$$\mathbf{Ind}\,\mathbf{A} = -\,\mathbf{ind}\det \mathscr{A}_p. \tag{4.5.10}$$

The operator $\mathbf{A}$ is locally invertible at $0 \in \mathbb{R}^+$ if and only if its symbol $\mathscr{A}_p^s(\omega)$, defined in (4.5.7)*, is elliptic on Γ_1, i.e.,*

$$\inf_{\omega \in \Gamma_1} |\det \mathscr{A}_p^s(\omega)| = \inf_{\xi \in \mathbb{R}} \left|\det \mathscr{A}_p^s(\xi, \infty)\right| > 0.$$

Proof. Note that our study is based on a localization technique. For more details concerning this approach, we refer the reader to [47, 49, 83, 127].

Let us apply the Gohberg–Krupnik local principle to the operator $\mathbf{A}$ in (4.5.8), "freezing" the symbol of $\mathbf{A}$ at a point $x \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. For $x \in \mathbb{R}$ and $\ell \in \mathbb{N}$, $\ell \geq 1$, let $C_x^\ell(\overline{\mathbb{R}})$ denote the set of all ℓ-times differentiable non-negative functions which are supported in a neighborhood of $x \in \mathbb{R}$ and are identically one everywhere in a smaller neighborhood of x. For $x \in \{-\infty\} \cup \{+\infty\} \cup \{\infty\}$, the functions from the corresponding classes $C_{+\infty}^\ell(\overline{\mathbb{R}})$ and $C_{-\infty}^\ell(\overline{\mathbb{R}})$ vanish on semi-infinite intervals $[-\infty, c)$ and $(-c, \infty]$, respectively, for certain $c > 0$, and are identically one in smaller neighborhoods. It is easily seen that the system of localizing classes $\{C_x^\ell(\overline{\mathbb{R}})\}_{x \in \overline{\mathbb{R}}}$ is covering in the algebras $C(\overline{\mathbb{R}})$, $\mathfrak{M}_p(\overline{\mathbb{R}})$, respectively (cf. [38, 47, 49, 83]).

Let us now consider a system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x) \in \mathfrak{R} \times \overline{\mathbb{R}^+}}$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. These localizing classes depend on two variables, viz. on $\omega \in \mathfrak{R}$ and $x \in \overline{\mathbb{R}^+}$. In particular, the class $\mathfrak{L}_{\omega,x}$ contains the operator $\Lambda_{\omega,x}$,

$$\Lambda_{\omega,x} := \begin{cases} \left[h_0 \mathfrak{M}_{v_\xi}^0 W_{g_\infty}\right] = \left[h_0 \mathfrak{M}_{v_\xi}^0\right] & \text{if } \omega = (\xi, \infty) \in \Gamma_1, \ \ x = 0, \\ \left[h_x \mathfrak{M}_{v_{\pm\infty}}^0 W_{g_\infty}\right] = \left[h_x \mathfrak{M}_{v_{\pm\infty}}^0 W_{g_{\mp\infty}}\right] & \text{if } \omega = (\pm\infty, \infty) \in \Gamma_2^{\pm} \cap \Gamma_1, \ \ x \in \mathbb{R}^+, \\ \left[h_\infty \mathfrak{M}_{v_{\pm\infty}}^0 W_{g_\eta}\right] = \left[h_\infty \mathfrak{M}_{v_{\pm\infty}}^0 W_{g_{\mp\,\eta}}\right] & \text{if } \omega = (\pm\infty, \eta) \in \Gamma_2^{\pm}, \ \ x = \infty, \\ \left[h_\infty \mathfrak{M}_{v_\xi}^0 W_{g_0}\right] = \left[\mathfrak{M}_{v_\xi}^0 W_{g_0}\right] & \text{if } \omega = (\xi, 0) \in \overline{\Gamma}_3, \ \ x = \infty, \end{cases} \tag{4.5.11}$$

where $h_x \in C^1_x(\overline{\mathbb{R}}^+)$, $v_\xi \in C^1_\xi(\overline{\mathbb{R}}^+)$, $g_\eta \in C^1_\eta(\overline{\mathbb{R}}^+)$, and $[\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denotes the coset containing the operator $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$.

To verify the equalities in (4.5.11), one has to show that the difference between the operators in the square brackets is compact.

Consider the first equality in (4.5.11). The operator

$$h_0 W_{g_\infty} - h_0 I = h_0 W_{(g_\infty - 1)} = h_0 W_{g_0}$$

is compact, since both functions h_0 and $1 - g_\infty = g_0$ have compact supports, so Proposition 4.5.3 applies.

To check the second equality in (4.5.11), let us note that $h_x(0) = 0$, $v_{\pm\infty}(\mp\infty) = 0$ and $g_{\pm\infty}(\xi) = 0$ for all $\mp\xi > 0$. From the fourth part of Proposition 4.5.5 we conclude that for any $x \in \mathbb{R}^+$ the operator $h_x \mathfrak{M}^0_{v_{\pm\infty}} W_{g_{\pm\infty}}$ is compact. This leads to the claimed equality, since

$$\big[h_x \mathfrak{M}^0_{v_{\pm\infty}} W_{g_\infty}\big] = \big[h_x \mathfrak{M}^0_{v_{\pm\infty}} \{W_{g_{-\infty}} + W_{g_{+\infty}}\}\big] = \big[h_x \mathfrak{M}^0_{v_{\pm\infty}} W_{g_{\mp\infty}}\big].$$

The third identity in (4.5.11) can be verified analogously. Concerning the fourth identity in (4.5.11): one can replace h_∞ by 1 because the difference $h_\infty W_{g_0} - W_{g_0} = (1 - h_\infty) W_{g_0} = h_0 W_{g_0}$ is compact due to Proposition 4.5.3.

Now consider other properties of the system $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$. Propositions 4.5.3–4.5.6 imply that

$$\big[h_x \mathfrak{M}^0_{v_\xi} W_{g_\infty}\big] = 0 \ \text{ for all } \ (\xi,\eta,x) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}^+ \setminus \mathfrak{R} \times \overline{\mathbb{R}}^+.$$

Therefore, the system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ is covering: for a given system $\{\Lambda_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ of localizing operators one can select a finite number of points $(\omega_1, x_1) = (\xi_1, \eta_1, x_1), \dots, (\omega_s, x_s) = (\xi_s, \eta_s, x_s) \in \mathfrak{R}$ and add appropriately chosen terms $[h_{x_{s+j}} \mathfrak{M}^0_{v_{\xi_{s+j}}} W_{g_{s+j}}] = 0$ with $(\xi_{s+j}, \eta_{s+j}, x_{s+j})) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}^+ \setminus (\mathfrak{R} \times \overline{\mathbb{R}}^+)$, $j = 1, 2, \dots, r$, so that the equality

$$\sum_{j=1}^{r} \sum_{k=1}^{s} \big[c_{x_j} \mathfrak{M}^0_{a_{\xi_j}} W_{b_{\eta_k}}\big] = \big[c \mathfrak{M}^0_a W_b\big] \tag{4.5.12}$$

holds and the functions $c \in C(\overline{\mathbb{R}}^+)$, $a \in C\mathfrak{M}_p(\overline{\mathbb{R}})$, $b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ are all elliptic. This implies the invertibility of the coset $[c\mathfrak{M}^0_a W_b]$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the inverse coset is $[c\mathfrak{M}^0_a W_b]^{-1} = [c^{-1} \mathfrak{M}^0_{a^{-1}} W_{b^{-1}}]$.

Note that the choice of a finite number of terms in (4.5.12) is possible due to the Borel–Lebesgue lemma and the compactness of the sets $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ (two point and one point compactification of $\mathbb{R}$ and of $\mathbb{R}^+$, respectively).

Moreover, localization in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ leads to the following local representatives of the cosets containing Mellin and Fourier convolution operators with symbols $a, b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$:

$$[\mathfrak{M}^0_a] \overset{\mathfrak{M}^0_{v_{\xi_0}}}{\sim} [\mathfrak{M}^0_{a(\xi_0)}] = [a(\xi_0) I] \ \text{ if } \ \xi_0 \in \overline{\mathbb{R}}, \tag{4.5.13a}$$

$$[\mathfrak{M}^0_a] \overset{v_{x_0} I}{\sim} [\mathfrak{M}^0_{a^\infty}] \ \text{ if } \ x_0 \in \overline{\mathbb{R}^+}, \ \ x_0 \neq 0, \tag{4.5.13b}$$

$$[\mathfrak{M}^0_a] \overset{v_\infty I}{\sim} [\mathfrak{M}^0_a] \ \text{ if } \ x_0 = \infty, \tag{4.5.13c}$$

$$[\mathfrak{M}^0_a] \overset{v_0 I}{\sim} [\mathfrak{M}^0_a] \ \text{ if } \ x_0 = 0, \tag{4.5.13d}$$

$$[W_b] \overset{W_{v_{\eta_0}}}{\sim} [W_{b(\eta_0)}] = [b(\eta_0) I] \ \text{ if } \ \eta_0 \in \mathbb{R} \setminus \{0\}, \tag{4.5.13e}$$

$$[W_b] \overset{W_{v_0}}{\sim} [W_{b^0}] = [\mathfrak{M}^0_{b_p(0,\,\cdot\,)}] \ \text{ if } \ \eta = 0, \tag{4.5.13f}$$

$$[W_b] \overset{W_{v_\infty}}{\sim} [W_{b^\infty(\infty,\,\cdot\,)}] = [\mathfrak{M}^0_{b_p(\infty,\,\cdot\,)}] \ \text{ if } \ \eta_0 = \infty, \tag{4.5.13g}$$

$$[W_b] \overset{v_{x_0} I}{\sim} [W_{b^\infty}] = [\mathfrak{M}^0_{b_p(\infty,\,\cdot\,)}] \ \text{ if } \ x_0 \in \mathbb{R}^+, \tag{4.5.13h}$$

$$[W_b] \stackrel{v}{\approx}{}^{I} [W_b] \text{ if } x_0 = \infty, \tag{4.5.13i}$$

where

$$g^\infty(\xi) := \frac{g(+\infty)+g(-\infty)}{2} + \frac{g(+\infty)-g(-\infty)}{2}\,\mathbf{sign}\,\xi = g(-\infty)\chi_-(\xi) + g(+\infty)\chi_+(\xi),$$
$$g^0(\xi) := \frac{g(0+0)+g(0-0)}{2} + \frac{g(0+0)+g(0-0)}{2}\,\mathbf{sign}\,\xi = g(0-0)\chi_-(\xi) + g(0+0)\chi_+(\xi),$$

and $\chi_\pm(\xi) := (1/2)(1 \pm \mathbf{sign}\,\xi)$. Note that in the equivalency relations (4.5.13e)–(4.5.13g) we used the identities (cf. (4.5.2) and (4.5.8))

$$W_{g^\infty} = \frac{g(-\infty)-g(+\infty)}{2} - \frac{g(-\infty)-g(+\infty)}{2}\,S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(\infty,\,\cdot\,)},$$
$$W_{g^0} = \frac{g(0+0)+g(0-0)}{2} - \frac{g(0+0)-g(0-0)}{2}\,S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(0,\,\cdot\,)}\,,$$

which means that the Fourier convolution operators with homogeneous of order 0 symbols $g^\infty(\xi)$ and $g^0(\xi)$ are, simultaneously, Mellin convolutions with the symbols $g_p(\infty,\xi)$, $g_p(0,\xi)$.

Using the equivalence relations (4.5.13a)–(4.5.13h) and the compactness of the corresponding operators, cf. Propositions 4.5.3–4.5.5, one easily finds the following local representatives of the operator (coset) $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}\mathbb{L}_p(\mathbb{R}^+)$ (see (4.5.8) for the operator $\mathbf{A}$):

$$\begin{aligned}
[\mathbf{A}] &\stackrel{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\xi_0)} W_{(b_j)^\infty}\Big] \\
&= \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\,\cdot\,)}\Big] \stackrel{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\xi_0)}\Big] \\
&= \big[\mathscr{A}_p(\xi_0,\infty)I\big] \text{ if } \omega = (\xi_0,\infty) \in \Gamma_1, \ \ x_0 = 0,
\end{aligned} \tag{4.5.14a}$$

$$\begin{aligned}
[\mathbf{A}] &\stackrel{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\pm\infty)} W_{(b_j)^\infty}\Big] = \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\pm\infty)(b_j)_p(\infty,\,\cdot\,)}\Big] \\
&= \big[\mathfrak{M}^0_{\mathscr{A}_p(\pm\infty,\,\cdot\,)}\big] \stackrel{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \big[\mathscr{A}_p(\pm\infty,\infty)I\big] \\
&\qquad \text{if } \omega = (\pm\infty,\infty) \in \overline{\Gamma_2^\pm} \cap \overline{\Gamma_1}, \ \ 0 < x_0 < \infty,
\end{aligned} \tag{4.5.14b}$$

$$\begin{aligned}
[\mathbf{A}] &\stackrel{\Lambda_{(\pm\infty,\mp\eta_0),\infty}}{\sim} \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(\mp\eta_0)}\Big] = \Big[\sum_{j=1}^m a_j(\pm\infty) b_j(\mp\eta_0) I\Big] \\
&= \big[\mathscr{A}_p(\pm\infty,\mp\eta_0)I\big] \text{ if } \eta_0 > 0, \ \omega = (\pm\infty,\mp\eta_0) \in \Gamma_2^\pm, \ x_0 = \infty,
\end{aligned} \tag{4.5.14c}$$

$$\begin{aligned}
[\mathbf{A}] &\stackrel{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j} W_{b_j^0}\Big] \\
&= \Big[\sum_{j=1}^m a_j(\xi_0) \mathfrak{M}_{(b_j)_p(0,\,\cdot\,)}\Big] \stackrel{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^m a_j(\xi_0)(b_j)_p(0,\xi_0)\Big] \\
&= \big[\mathscr{A}_p(\xi_0,0)I\big] \text{ if } \omega = (\xi_0,0) \in \overline{\Gamma}_3, \ \ x_0 = \infty,
\end{aligned} \tag{4.5.14d}$$

$$\begin{aligned}
[\mathbf{A}] &\stackrel{\Lambda_{(\pm\infty,\eta),\infty}}{\sim} \Big[\sum_{j=1}^m \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(0)}\Big] = \Big[\sum_{j=1}^m a_j(\pm\infty) b_j(0) I\Big] = \\
&= \big[\mathscr{A}_p(\pm\infty,0)I\big] \text{ if } \omega = (\pm\infty,0) \in \overline{\Gamma}_3, \ \ x_0 = \infty.
\end{aligned} \tag{4.5.14e}$$

It is remarkable that the local representatives (4.5.14a)–(4.5.14e) are just the quotient classes of multiplication operators by constant $N \times N$ matrices $[\mathscr{A}_p(\xi_0,\eta_0)I]$. If $\det \mathscr{A}_p(\xi_0,\eta_0) = 0$, these representatives are not invertible, both locally and globally. On the other hand, they are globally invertible if $\det \mathscr{A}_p(\xi_0,\eta_0) \neq 0$. Thus, the conditions of the local invertibility for all points $\omega_0 = (\xi_0,\eta_0) \in \mathfrak{R}$

and the global invertibility of the operators under consideration coincide with the ellipticity condition for the symbol $\inf\limits_{(\xi_0,\eta_0)\in\mathfrak{R}} \det \mathscr{A}_p(\xi_0,\eta_0) \neq 0$.

The index $\mathbf{Ind}\,\mathbf{A}$ is a continuous integer-valued multiplicative function $\mathbf{Ind}\,\mathbf{AB} = \mathbf{Ind}\,\mathbf{A} + \mathbf{Ind}\,\mathbf{B}$ defined on the group of Fredholm operators of $\mathfrak{A}_p(\mathbb{R}^+)$. On the other hand, the index function $\mathbf{ind}\det\mathscr{A}_p$ defined on L_p-symbols $\mathscr{A}_p$ possesses the same property $\mathbf{ind}\det\mathscr{A}_p\mathscr{B}_p = \mathbf{ind}\det\mathscr{A}_p + \mathbf{ind}\det\mathscr{B}_p$, see explanations after (4.5.9). Moreover, the set of operators (4.5.8) is dense in the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ and the corresponding set of their symbols is dense in the algebra $C(\mathfrak{R})$ of all continuous functions on $\mathfrak{R}$. For $p = 2$, these algebras even coincide. Therefore, there is an algebraic homeomorphism between the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the algebra of their symbols which is a dense subalgebra of $C(\mathfrak{R})$. Hence, two various index functions can only be connected by the relation $\mathbf{Ind}\,\mathbf{A} = M_0\,\mathbf{ind}\det\mathscr{A}_p$ with an integer constant M_0 independent of $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Since for any Fourier convolution operator $\mathbf{A} = W_a$ the index formula is $\mathbf{Ind}\,\mathbf{A} = -\,\mathbf{ind}\det\mathscr{A}_p$ [41, 42, 47], the constant $M_0 = -1$, and the index formula (4.5.10) is proved.

Concerning the concluding assertion of the theorem: $\mathbf{A}$ is, after lifting to $\mathbb{L}_p$-space, locally equivalent at 0 to the Mellin convolution operator $\mathfrak{M}^0_{\mathscr{A}^s_p(\infty,\xi)}$, which commutes with the dilation

$$\mathfrak{M}^0_a V_\lambda = V_\lambda \mathfrak{M}^0_a, \quad V_\lambda\varphi(t) := \varphi(\lambda t) \ \text{ for all } \ \lambda > 0$$

and, therefore, is locally invertible at 0 if and only if it is globally invertible (see [45, 47, 127]) and this is the case if and only if $\inf\limits_{\xi\in\mathbb{R}} |\mathscr{A}^s_p(\infty,\xi)| > 0$. □

Remark 4.5.3. Let us emphasize that formula (4.5.10) does not contradict the invertibility of "pure Mellin convolution" operators $\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ with an elliptic matrix symbol $a \in C\mathfrak{M}^0_p(\mathbb{R})$, $\inf\limits_{\xi\in\mathbb{R}} |a(\xi)| > 0$, stated in Proposition 4.1.1, even if $\mathbf{ind}\, a \neq 0$.

In fact, computing the symbol of $\mathfrak{M}^0_a$ by formula (4.5.7), one obtains

$$(\mathfrak{M}^0_a)_p(\omega) := \begin{cases} a(\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ a(+\infty), & \omega = (+\infty,\eta) \in \Gamma^+_2, \\ a(-\infty), & \omega = (-\infty,\eta) \in \Gamma^-_2, \\ a(\xi), & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases}$$

Noting that on the sets Γ_1 and Γ_3 the variable ω runs in opposite direction, the increment of the argument $[\arg\det(\mathfrak{M}^0_a)_p(\omega)]_{\mathfrak{R}} = 0$ is zero, implying $\mathbf{Ind}\,\mathfrak{M}^0_a = 0$.

In contrast to the above, the pure Fourier convolution operators $W_b : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ with elliptic matrix symbol $b \in C\mathfrak{M}^0_p(\mathbb{R})$, $\inf\limits_{\xi\in\mathbb{R}} |b_p(\xi,\eta)| > 0$ can possess non-zero indices. Since

$$b_p(\omega) := \begin{cases} b_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ b(-\eta), & \omega = (+\infty,\eta) \in \Gamma^+_2, \\ b(\eta), & \omega = (-\infty,\eta) \in \Gamma^-_2, \\ b(0), & \omega = (\xi,0) \in \overline{\Gamma_3}, \end{cases}$$

one arrives at the well-known formula

$$\mathbf{Ind}\, W_b = -\,\mathbf{ind}\, b_p.$$

Moreover, in the case where the symbol $b(-\infty) = b(+\infty)$ is continuous, one has $b_p(\xi,\eta) = b(\xi)$. Thus the ellipticity of the corresponding operator leads to the formula

$$\mathbf{ind}\, b_p = \mathbf{ind}\det b.$$

If $\mathscr{A}_p(\omega)$ is the symbol of an operator $\mathbf{A}$ of (4.5.6), the set $\mathscr{R}(\mathscr{A}_p) := \{\mathscr{A}_p(\omega) \in \mathbb{C} : \ \omega \in \mathfrak{R}\}$ coincides with the essential spectrum of $\mathbf{A}$. Recall that the essential spectrum $\sigma_{ess}(\mathbf{A})$ of a bounded operator $\mathbf{A}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathbf{A} - \lambda I$ is not Fredholm in $\mathbb{L}_p(\mathbb{R}^+)$ or, equivalently, the coset $[\mathbf{A}-\lambda I]$ is not invertible in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Then, due to Banach theorem, the essential norm $|||\mathbf{A}|||$ of the operator $\mathbf{A}$ can be estimated as follows

$$\sup_{\omega\in\omega}|\mathscr{A}_p(\omega)|\leqslant |||\mathbf{A}|||:=\inf_{\mathbf{T}\in\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))}\big\|(\mathbf{A}+\mathbf{T})\mid\mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\big\|. \tag{4.5.15}$$

Inequality (4.5.15) enables one to extend continuously the symbol map (4.5.7)

$$[\mathbf{A}]\to\mathscr{A}_p(\omega),\quad [\mathbf{A}]\in\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+)) \tag{4.5.16}$$

on the whole Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$. Now, applying Theorem 4.5.2 and standard methods, cf. [52, Theorem 3.2], one can derive the following result.

Corollary 4.5.2. *Let $1<p<\infty$ and $\mathbf{A}\in\mathfrak{A}_p(\mathbb{R}^+)$. The operator $\mathbf{A}:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\mathbf{Ind}\,\mathbf{A}=-\,\mathbf{ind}\,\mathscr{A}_p.$$

Corollary 4.5.3. *The set of maximal ideals of the commutative Banach quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$, generated by scalar $N=1$ operators in* (4.5.6)*, is homeomorphic to $\mathfrak{R}$, and the symbol map in* (4.5.7), (4.5.16) *is a Gelfand homeomorphism of the corresponding Banach algebras.*

Proof. The proof is based on Theorem 4.5.2 and Corollary 4.5.2 and is similar to [52, Theorem 3.1]. The details of the proof is left to the reader. ☐

Remark 4.5.4. All the above results are valid in a more general setting, viz., for the Banach algebra $\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$ generated in the weighted Lebesgue space of N-vector-functions $\mathbb{L}^N_p(\mathbb{R}^+,x^\alpha)$ by the operators

$$\mathbf{A}:=\sum_{j=1}^m\Big[d^1_j\mathfrak{M}^0_{a^1_j}W_{b^1_j}+d^2_j\mathfrak{M}^0_{a^2_j}H_{c^1_j}+d^3_jW^0_{b^2_j}H_{c^2_j}\Big] \tag{4.5.17}$$

when coefficients $d^1_j,d^2_j,d^3_j\in PC^{N\times N}(\overline{\mathbb{R}})$ are piecewise-continuous $N\times N$ matrix functions, symbols of Mellin convolution operators $\mathfrak{M}^0_{a^1_j}$, $\mathfrak{M}^0_{a^2_j}$, Winer–Hopf (Fourier convolution) operators $W_{b^1_j}$, $W_{b^2_j}$ and Hankel operators $H_{c^1_j}$, $H_{c^2_j}$ are $N\times N$ piecewise-continuous matrix $\mathbb{L}_p$-multipliers $a^k_j,b^k_j,c^k_j\in PC^{N\times N}\mathfrak{M}_p(\mathbb{R})$.

The spectral set $\Sigma(\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+))$ of such Banach algebra (viz., the set where the symbols are defined, e.g., $\mathfrak{R}$ for the Banach algebra $\mathfrak{A}^{N\times N}_p(\mathbb{R}^+)$ investigated above) is more sophisticated and described in the papers [45,46,52,132]. Let $\mathfrak{C}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ be the sub-algebra of $\mathfrak{P}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)=\mathfrak{P}\mathfrak{A}^{1\times1}_{p,\alpha}(\mathbb{R}^+)$ generated by the scalar operators (4.5.17) with continuous coefficients $c_j,h_j\in C(\overline{\mathbb{R}})$ and the scalar piecewise-continuous $\mathbb{L}_p$-multipliers $a_j,b_j,d_j,g_j\in PC\mathfrak{M}_p(\mathbb{R})$. The quotient-algebra $\mathfrak{C}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ with respect to the ideal of all compact operators is a commutative algebra and the spectral set $\Sigma(\mathfrak{P}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+))$ is homeomorphic to the set of maximal ideals.

We will not elaborate more on further details concerning the Banach algebra $\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$, since the result exposed above are sufficient for the purpose of this and subsequent papers dealing with the BVPs in domains with corners at the boundary.

4.6 Mellin convolution operators in the Bessel potential spaces. The boundedness and lifting

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in the Bessel potential spaces,

$$\mathfrak{M}^0_a:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+). \tag{4.6.1}$$

The symbols of these operators are $N\times N$ matrix functions $a\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$ continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Theorem 4.6.1. *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

If conditions of Theorem 4.3.4 hold, the Mellin convolution operator between the Bessel potential spaces

$$\mathbf{K}^1_c : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{4.6.2}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{4.6.3}$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and the function $g^s_{-c\gamma,\gamma}$ is defined in (4.2.10).

If conditions of Corollary 4.3.2 hold, the Mellin convolution operator between the Bessel potential spaces (4.6.2) *is lifted to the equivalent operator*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}} = c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}} + \mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{4.6.4}$$

where $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof. The equivalent operator after lifting is

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$$

(see Theorem 4.2.1). To proceed we need two formulae

$$\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = W_{g^s_{-c\gamma,\gamma}}, \quad W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} = W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}}. \tag{4.6.5}$$

The first one holds because $0 < \arg\gamma < \pi$ (see (4.2.8)) and the second one holds because $g^s_{-\gamma,-\gamma_0}(\xi)$ has a smooth, uniformly bounded analytic extension in the complex lower half-plane (see (4.2.13)).

If conditions of Theorem 4.3.4 hold, we apply formula (4.3.23), the first formula in (4.6.5) and derive the equality in (4.6.3):

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}}.$$

If conditions of Corollary 4.3.2 hold, we apply formulae (4.3.35), (4.3.36), both formulae in (4.6.5) and derive the equality in (4.6.4):

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma}\\ &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}} = c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} + \mathbf{T}.\end{aligned}$$

□

Remark 4.6.1. The case of operator $\mathbf{K}^1_1$ is not covered by the foregoing Theorem 4.6.1, where $\arg c \neq 0$. This case is essentially different as underlined in Theorem 4.5.1 because $\mathbf{K}^1_1$ is a Hilbert transform $\mathbf{K}^1_1 = -\pi i S_{\mathbb{R}^+} = \pi i W_{\mathbf{sign}}$ and $\mathbf{K}^1_1$ between the Bessel potential spaces (4.6.2) is lifted to the equivalent Fourier convolution operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_1\mathbf{\Lambda}^{-s}_{\gamma} = W_{\pi i g^s_{-\gamma,\gamma}\,\mathbf{sign}} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{4.6.6}$$

as it follows from Theorem 4.5.1.

Theorem 4.6.2. *Let $c_j, d_j \in \mathbb{C}$, $0 < \arg c_j < 2\pi$, $0 < \arg\gamma < \pi$, $-\pi < \arg(c_j\gamma) < 0$ for $j = 1, \dots, m$ and $0 < \arg(c_j\gamma) < \pi$ for $j = m+1, \dots, n$.*

The Mellin convolution operator between the Bessel potential spaces

$$\boldsymbol{A} = \sum_{j=1}^n d_j\mathbf{K}^1_{c_j} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{4.6.7}$$

is lifted to the equivalent operator

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\boldsymbol{A}\mathbf{\Lambda}^{-s}_{\gamma} &= \sum_{j=0}^m d_j c_j^{-s}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,-\gamma}} + \sum_{j=m+1}^n d_j c_j^{-s}W_{g^s_{-\gamma,-\gamma_j}}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma_j,\gamma}} &\text{(4.6.8a)}\\ &= \sum_{j=0}^m d_j c_j^{-s}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,\gamma}} + \sum_{j=m+1}^n d_j c_j^{-s}\mathbf{K}^1_{c_j}W_{g^s_{-\gamma,-\gamma_j}g^s_{-c_j\gamma_j,\gamma}} + \mathbf{T} &\text{(4.6.8b)}\end{aligned}$$

in the space $\mathbb{L}_p(\mathbb{R}^+)$, where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and γ_j are such that $0 < \arg\gamma_j < \pi$, $-\pi < \arg(c_j\,\gamma_j) < 0$ for $j = m+1, \dots, n$. $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof. The proof is a direct consequence of Theorem 4.6.1. □

Theorem 4.6.3. *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

If conditions of Theorem 4.3.4 *hold, the Mellin convolution operator between the Bessel potential spaces*

$$\mathbf{K}^2_c : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{4.6.9}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_\gamma = c^{-s}[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma,\gamma}} \tag{4.6.10}$$

in the space $\mathbb{L}_p(\mathbb{R}^+)$, where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and the function $g^s_{-c\gamma,\gamma}$ is defined in (4.2.10) *and the last summand in* (4.6.10)

$$\mathbf{T} := s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_\gamma : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+) \tag{4.6.11}$$

is a compact operator.

If conditions of Corollary 4.3.2 *hold, the Mellin convolution operator between the Bessel potential spaces* (4.6.9) *is lifted to the equivalent operator in the space $\mathbb{L}_p(\mathbb{R}^+)$*

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_\gamma &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c]W_{g^s_{-c\gamma_0,\gamma}} + s\,\gamma\,c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma_0,\gamma}}\\ &= c^{-s}[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c]W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\,\gamma_0,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi-c\gamma_0)^{-1}g^s_{-c\gamma_0,-\gamma_0}g^s_{-\gamma,\gamma}} + \mathbf{T},\end{aligned} \tag{4.6.12}$$

where $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof. Let conditions of Theorem 4.3.4 hold (that means $\operatorname{Im}\gamma > 0$ and $\operatorname{Im} c\,\gamma < 0$). Then

$$\frac{1}{(t-c)^2} = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\Big[\frac{1}{t-c-\varepsilon i} - \frac{1}{t-c+\varepsilon i}\Big]$$

and we have

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_\gamma &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\,\mathbf{\Lambda}^s_{-\gamma}\big[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\big]\mathbf{\Lambda}^{-s}_\gamma\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\Big[(c+\varepsilon i)^{-s}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - (c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\Big]\mathbf{\Lambda}^{-s}_\gamma\\ &= \lim_{\varepsilon\to 0}\Big\{\frac{(c+\varepsilon i)^{-s} - (c-\varepsilon i)^{-s}}{2\varepsilon i}\,\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma}\\ &\qquad - (c-\varepsilon i)^{-s}\frac{1}{2\varepsilon i}\big[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\big]\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\\ &\qquad\quad - (c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\,\frac{1}{2\varepsilon i}\big[\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - \mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\big]\Big\}\mathbf{\Lambda}^{-s}_\gamma\\ &= -sc^{-s-1}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_\gamma + c^{-s}\mathbf{K}^2_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_\gamma\\ &\qquad + c^{-s}\mathbf{K}^1_c\lim_{\varepsilon\to 0}\mathscr{F}^{-1}\,\frac{(\xi - c\,\gamma - \varepsilon\gamma i)^s - (\xi - c\,\gamma + \varepsilon\gamma i)^s}{2\varepsilon i}\,\mathscr{F}\mathbf{\Lambda}^{-s}_\gamma\\ &= c^{-s}\big[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\big]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^{s-1}_{-c\,\gamma}\mathbf{\Lambda}^{-s}_\gamma\\ &= c^{-s}\big[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\big]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma,\gamma}}\end{aligned}$$

Formula (4.6.10) is proved.

Formula (4.6.12) is derived from (4.6.10) as in Theorem 4.6.1. □

Remark 4.6.2. The case of operators $\mathbf{K}^n_c$, $n = 3, 4, \ldots$, can be treated similarly as in Corollary 4.6.3: with the help of perturbation the operator $\mathbf{K}^n_c$ can be represented in the form

$$\mathbf{K}^n_c\varphi = \lim_{\varepsilon\to 0}\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi,\ \ \forall\,\varphi \in \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+),$$

$$\mathbf{K}_{c_{1,\varepsilon},\dots,c_{n,\varepsilon}}\varphi(t):=\int\limits_0^\infty \mathscr{K}_{c_{1,\varepsilon},\dots,c_{n,\varepsilon}}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau}=\sum_{j=1}^n d_j(\varepsilon)\mathbf{K}^1_{c_{j,\varepsilon}}\varphi(t),$$

$$\mathscr{K}_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}(t):=\frac{1}{(t-c_{1,\varepsilon})\cdots(t-c_{n,\varepsilon})}=\sum_{j=1}^n\frac{d_j(\varepsilon)}{t-c_{j,\varepsilon}}\,, \tag{4.6.13}$$

$$c_{j,\varepsilon}=c(1+\varepsilon e^{i\omega_j}),\ \ \omega_j\in(-\pi,\pi),\ \ \arg c_{j,\varepsilon},\arg c_{j,\varepsilon}\,\gamma_j\neq 0,\ \ j=1,\dots,m.$$

The points $\omega_1,\dots,\omega_n\in(-\pi,\pi]$ are pairwise different, i.e., $\omega_j\neq\omega_k$ for $j\neq k$ (we remind that $\arg c\neq 0$ because $n=3,4,\dots$). By equating the numerators in formula (4.6.2) we find the coefficients $d_1(\varepsilon),\dots,d_{n-1}(\varepsilon)$.

Since the operators $\mathbf{K}^3_c,\mathbf{K}^4_c,\dots$ encounter in applications rather rarely, we have confined ourselves with the exact formulae only for the operators $\mathbf{K}^1_c$ and $\mathbf{K}^2_c$.

4.7 Mellin convolution operators in the Bessel potential spaces. Fredholm properties

Let us write the symbol of a model operator

$$\mathbf{A}:=d_0I+W_{a_0}+\sum_{j=1}^n W_{a_j}\mathbf{K}^1_{c_j}W_{b_j} \tag{4.7.1}$$

acting in the Bessel potential spaces $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+)$, compiled of the identity I, of Fourier $W_{a_0},\dots,W_{a_n}$, $W_{b_1},\dots,W_{b_n}$ and Mellin $\mathbf{K}^1_{c_1},\dots,\mathbf{K}^1_{c_n}$ convolution operators.

We assume that $a_0,\dots,a_n,b_1,\dots,b_n\in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\})$, $c_1,\dots,c_n\in\mathbb{C}$ and, if $s\leqslant\frac{1}{p}-1$ or $s\geqslant\frac{1}{p}$, the functions $a_1(\xi),\dots,a_n(\xi)$ have bounded analytic extensions in the lower half-plane $\operatorname{Im}\xi<0$, while the functions $b_1(\xi),\dots,b_n(\xi)$ have bounded analytic extensions in the upper half-plane $\operatorname{Im}\xi>0$ to ensure the proper mapping properties of the operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+)$. For $\frac{1}{p}-1<s<\frac{1}{p}$, such constraints are not necessary.

Now we describe the symbol $\mathscr{A}^s_p(\omega)$ of the operator $\mathbf{A}$. For this, we lift the operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+)$ to the $\mathbb{L}_p$-setting and apply equality (4.2.13) to the operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma}:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+), \tag{4.7.2}$$

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma}&=d_0\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{\gamma}+\mathbf{\Lambda}^s_{-\gamma}W_{a_0}\mathbf{\Lambda}^{-s}_{\gamma}+\sum_{j=1}^n W_{a_j}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{c_j}\mathbf{\Lambda}^{-s}_{\gamma}W_{b_j}\\&=d_0W_{(\frac{\xi-\gamma}{\xi+\gamma})^s}+W_{a_0(\xi)(\frac{\xi-\gamma}{\xi+\gamma})^s}+\sum_{j=1}^n W_{a_j}\mathbf{K}^1_{c_j}W_{(\frac{\xi-c\gamma}{\xi+\gamma})^s}W_{b_j}\end{aligned} \tag{4.7.3}$$

(see Theorem 4.2.1, diagram (4.2.7)), if conditions of Theorem 4.3.4 hold (see (4.6.4)) and to the operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma}=d_0W_{(\frac{\xi-\gamma}{\xi+\gamma})^s}+W_{a_0(\xi)(\frac{\xi-\gamma}{\xi+\gamma})^s}+\sum_{j=1}^n W_{a_j}\mathbf{K}^1_{c_j}W_{(\frac{\xi-\gamma}{\xi-\gamma_0})^s(\frac{\xi-c\gamma_0}{\xi+\gamma})^s}W_{b_j}+\mathbf{T}, \tag{4.7.4}$$

where $\mathbf{T}:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ is a compact operator, if conditions of Corollary 4.3.2 hold (see (4.6.5)).

We declare the symbol of the lifted operator (4.7.2)–(4.7.4) in the space $\mathbb{L}_p(\mathbb{R}^+)$ as the symbol of the operator $\mathbf{A}$ in the Bessel potential space. This symbol, written according formulae (4.5.7) and (4.5.8), has the form

$$\mathscr{A}^s_p(\omega):=d_0\mathscr{I}^s_p(\omega)+\mathscr{W}^s_{a_0,p}(\omega)+\sum_{j=1}^n\mathscr{W}^0_{a_j,p}(\omega)\mathscr{K}^{1,s}_{c_j,p}(\omega)\mathscr{W}^0_{b_j,p}(\omega), \tag{4.7.5}$$

where $\mathscr{I}^s_p(\omega)$, $\mathscr{W}^s_{a_0,p}(\omega)$, $\mathscr{W}^0_{a_j,p}(\omega)$, $\mathscr{K}^{1,s}_{c_j,p}(\omega)$ and $\mathscr{W}^0_{b_j,p}(\omega)$ are the symbols of the operators $W_{(\frac{\xi-\gamma}{\xi-\gamma})^s}$ in $\mathbb{L}_p$ (of I in $\mathbb{H}^s_p$), of $W_{a_0(\xi)(\frac{\xi-\gamma}{\xi-\gamma})^s}$ in $\mathbb{L}_p$ (of W_{a_0} in $\mathbb{H}^s_p$), of W_{a_j} in $\mathbb{L}_p$ (and in $\mathbb{H}^s_p$), of $\mathbf{K}^1_{c_j}W_{(\frac{\xi-c\gamma}{\xi-\gamma})^s}$ in $\mathbb{L}_p$ (of $\mathbf{K}^1_{c_j}$ in $\mathbb{H}^s_p$), of W_{b_j} in $\mathbb{L}_p$ (and in $\mathbb{H}^s_p$). Now it suffices to expose the symbols $\mathscr{I}^s_p(\omega)$, $\mathscr{W}^s_{a_0,p}(\omega)$, $\mathscr{W}^0_{a_j,p}(\omega)$ and $\mathscr{K}^{1,s}_{c_j,p}(\omega)$ of the operators I, W_{a_0}, $W_{a_j}(j=1,2,\dots,n)$ and $\mathbf{K}^1_c$ separately (the symbol $\mathscr{W}^0_{b_j,p}(\omega)$ of $W_{b_j}(j=1,2,\dots,n)$ is written analogously):

$$\mathscr{I}^s_p(\omega) := \begin{cases} g^s_{-\gamma,\gamma,p}(\infty,\xi), & \omega=(\xi,\infty)\in\overline{\Gamma}_1, \\ \Big(\dfrac{\eta-\gamma}{\eta+\gamma}\Big)^{\mp s}, & \omega=(+\infty,\eta)\in\Gamma^\pm_2, \\ e^{\pi si}, & \omega=(\xi,0)\in\overline{\Gamma}_3, \end{cases} \tag{4.7.6a}$$

$$\mathscr{W}^s_{a,p}(\omega) := \begin{cases} a^s_p(\infty,\xi), & \omega=(\xi,\infty)\in\overline{\Gamma}_1, \\ a(\mp\eta)\Big(\dfrac{\eta-\gamma}{\eta+\gamma}\Big)^{\mp s}, & \omega=(+\infty,\eta)\in\Gamma^\pm_2, \\ e^{\pi si}a_p(0,\xi), & \omega=(\xi,0)\in\overline{\Gamma}_3, \end{cases} \tag{4.7.6b}$$

$$\mathscr{W}^0_{a,p}(\omega) := \begin{cases} a_p(\infty,\xi), & \omega=(\xi,\infty)\in\overline{\Gamma}_1, \\ a(\mp\eta), & \omega=(+\infty,\eta)\in\Gamma^\pm_2, \\ a_p(0,\xi), & \omega=(\xi,0)\in\overline{\Gamma}_3, \end{cases} \tag{4.7.6c}$$

$$\mathscr{K}^{1,s}_{c,p}(\omega) := \begin{cases} \dfrac{c^{-s}e^{-i\pi(\frac1p-i\xi-1)}c^{\frac1p-i\xi-1}}{\sin\pi(\frac1p-i\xi)}, & \omega=(\xi,\infty)\in\overline{\Gamma}_1,,\ \ \omega=(\xi,0)\in\overline{\Gamma}_3 \\ 0, & \omega=(\pm\infty,\eta)\in\Gamma^\pm_2 \ \text{ for } \ \arg c\neq 0, \end{cases} \tag{4.7.6d}$$

$$\mathscr{K}^{1,s}_{1,p}(\omega) := \begin{cases} -i\cot\pi\Big(\dfrac1p-i\xi\Big), & \omega=(\xi,\infty)\in\overline{\Gamma}_1, \\ \pm1, & \omega=(\pm\infty,\eta)\in\Gamma^\pm_2, \\ i\cot\pi\Big(\dfrac1p-i\xi\Big), & \omega=(\xi,0)\in\overline{\Gamma}_3, \end{cases} \tag{4.7.6e}$$

$$a^s_p(\infty,\xi) := \frac{e^{2\pi si}a(\infty)+a(-\infty)}{2} - \frac{e^{2\pi si}a(\infty)-a(-\infty)}{2i}\cot\pi\Big(\frac1p-i\xi\Big),$$

$$a_p(x,\xi) := \frac{a(x+0)+a(x-0)}{2} - \frac{a(x+0)-a(x-0)}{2i}\cot\pi\Big(\frac1p-i\xi\Big),\quad x=0,\infty,$$

$$g^s_{-\gamma,\gamma,p}(\infty,\xi) := \frac{e^{2\pi si}+1}{2} - \frac{e^{2\pi si}-1}{2i}\cot\pi\Big(\frac1p-i\xi\Big) = e^{\pi si}\frac{\sin\pi(\frac1p-s-i\xi)}{\sin\pi(\frac1p-i\xi)},\quad \xi\in\mathbb{R},\quad \eta\in\mathbb{R}^+,$$

where

$$0<\arg c<2\pi,\quad -\pi<\arg(c\gamma)<0,\quad 0<\arg\gamma<\pi,$$

and $c^s=|c|^se^{is\arg c}$, $(-c)^\delta=|c|^\delta e^{i\delta(\arg c\mp\pi)}$ for $c,\delta\in\mathbb{C}$; the sign "$-$" is chosen for $\pi<\arg c<2\pi$ and the sign "$+$" is chosen for $0<\arg c<\pi$.

Note that we got the equal symbol $\mathscr{K}^{1,s}_{c,p}(\omega)$ of the operator $\mathbf{K}^1_{c_j}$ in cases (4.7.3) and (4.7.4), since the functions

$$g^s_{-\gamma,-\gamma_0}(\xi)g^s_{-c\gamma_0,\gamma}(\xi) := \Big(\frac{\xi-\gamma}{\xi-\gamma_0}\Big)^s\Big(\frac{\xi-c\gamma_0}{\xi+\gamma}\Big)^s \ \text{ and } \ g^s_{-c\gamma,\gamma}(\xi) := \Big(\frac{\xi-c\gamma}{\xi+\gamma}\Big)^s$$

have equal limits at infinity

$$g^s_{-c\gamma,\gamma}(\pm\infty) = g^s_{-\gamma,-\gamma_0}(\pm\infty)g^s_{-c\gamma_0,\gamma}(\pm\infty)=1 \ \text{ and } \ g^s_{-c\gamma,\gamma}(0)=g^s_{-\gamma,-\gamma_0}(0)g^s_{-c\gamma_0,\gamma}(0)=(-c)^s.$$

If $a(-\infty)=1$ and $a(+\infty)=e^{2\pi\alpha i}$, then $a^-_\infty=0$, $a^+_\infty=2\alpha$ and the symbol $a^s_p(\infty,\xi)$ acquires the form

$$a^s_p(\infty,\xi) = e^{\pi(s+\alpha)i}\,\frac{\sin\pi(\frac1p-s+\alpha-i\xi)}{\cos\pi(\frac1p-i\xi)}\,. \tag{4.7.6f}$$

Note that the Mellin convolution operator

$$\mathbf{K}^1_{-1}\varphi(t) := \int\limits_0^\infty \frac{\varphi(\tau)\,d\tau}{t+\tau} = \mathfrak{M}^0_{\mathscr{M}_{\frac{1}{p}}\mathscr{K}^1_{-1}}, \quad \mathscr{M}_{\frac{1}{p}}\mathscr{K}^1_{-1}(\xi) = \frac{\pi d^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}$$

(see (4.3.9b)), which we encounter in applications, has a rather simple symbol in the Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$ (see (4.7.6c), where $c=-1=e^{i\pi}$):

$$\mathscr{K}^{1,s}_{-1,p}(\omega) := \begin{cases} \dfrac{e^{-\pi si}}{\sin\pi(\beta-i\xi)}\,, & \omega=(\xi,\infty)\in\overline{\Gamma_1}\cup\overline{\Gamma_3},\\ 0, & \omega=(\pm\infty,\eta)\in\Gamma^\pm_2,. \end{cases}$$

Theorem 4.7.1. *Let $1<p<\infty$, $s\in\mathbb{R}$. The operator*

$$\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+), \tag{4.7.7}$$

defined in (4.7.1)*, is Fredholm if and only if its symbol $\mathscr{A}^s_p(\omega)$, defined in* (4.7.5) *and* (4.7.6a)–(4.7.6f)*, is elliptic.*

If $\mathbf{A}$ is Fredholm, the index of the operator has the value

$$\mathbf{Ind}\,\mathbf{A} = -\,\mathbf{ind}\det\mathscr{A}^s_p.$$

Proof. Let $c_j,d_j\in\mathbb{C}$, $0<\arg c_j<2\pi$. Lifting the operator $\mathbf{A}$ to the space $\mathbb{L}_p(\mathbb{R}^+)$ we get

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma = d_0\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_\gamma + \mathbf{\Lambda}^s_{-\gamma}W_{a_0}\mathbf{\Lambda}^{-s}_\gamma + \sum_{j=1}^n W_{a_j}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{c_j}\mathbf{\Lambda}^{-s}_\gamma W_{b_j}, \tag{4.7.8}$$

where $c^{-s}=|c|^{-s}e^{-is\arg c}$ and γ_j are such that $0<\arg\gamma_j<\pi$, $-\pi<\arg(c_j\,\gamma_j)<0$ for $j=m+1,\dots,n$.

To derive (4.7.8), we have applied the following property of convolution operators $\mathbf{\Lambda}^s_{-\gamma}W_{a_j}=W_{a_j}\mathbf{\Lambda}^s_{-\gamma}$ and $W_{b_j}\mathbf{\Lambda}^s_\gamma=\mathbf{\Lambda}^s_\gamma W_{b_j}$, $\mathbf{\Lambda}^{\mp s}_{\pm\gamma}=W_{\lambda^{\mp s}_{\pm\gamma}}$, which are based on the analytic extension properties of the symbols $\lambda^s_{-\gamma}$, $a_1(\xi),\dots,a_n(\xi)$ in the lower half-plane $\operatorname{Im}\xi<0$ and of symbols $\lambda^{-s}_\gamma, b_1(\xi),\dots,b_n(\xi)$ in the upper half-plane $\operatorname{Im}\xi>0$ (see (4.2.6)).

The model operators I, $\mathbf{K}^1_c$ and W_a lifted to the space $\mathbb{L}_p(\mathbb{R}^+)$ acquire the form

$$\begin{gathered}\mathbf{\Lambda}^s_\gamma I\mathbf{\Lambda}^{-s}_\gamma = W_{g^s_{-\gamma,\gamma}}, \quad \mathbf{\Lambda}^s_\gamma W_{a_k}\mathbf{\Lambda}^{-s}_\gamma = W_{a_k g^s_{-\gamma,\gamma}},\\ \mathbf{\Lambda}^s_\gamma\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_\gamma = \begin{cases} c^{-s}\mathbf{K}^1_c W_{g^s_{-c\,\gamma,\gamma}} & \text{for } -\pi<\arg(c\,\gamma)<0,\\ c^{-s}\mathbf{K}^1_c W_{g^s_{-\,\gamma,-\gamma_0}g^s_{-c\,\gamma_0,\gamma}}+\mathbf{T} & \text{for } 0<\arg(c\,\gamma)<\pi,\ \ -\pi<\arg(c\,\gamma_0)|<0,\end{cases}\end{gathered} \tag{4.7.9}$$

where $\mathbf{T}$ is a compact operator. Here, as above, $0<\arg c<2\pi$, $0<\arg\gamma<\pi$, $0<\arg\gamma_0<\pi$ and either $-\pi<\arg(c\,\gamma)<0$ or, if $-\pi<\arg(c\,\gamma)<0$, then $-\pi<\arg(c\,\gamma_0)|<0$. Here $c^{-s}=|c|^{-s}e^{-is\arg c}$.

Therefore, the operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma$ in (4.7.8) is rewritten as follows:

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma &= d_0W_{g^s_{-\gamma,\gamma}} + W_{a_0g^s_{\gamma,\gamma}} + \sum_{j=1}^m c_j^{-s}W_{a_j}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,\gamma}}W_{b_j}\\ &\quad + \sum_{j=m+1}^n c_j^{-s}W_{a_j}\mathbf{K}^1_{c_j}W_{g^s_{-\gamma,-\gamma_j}g^s_{-c_j\gamma_j,\gamma}}W_{b_j} + \mathbf{T}:\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+),\end{aligned} \tag{4.7.10}$$

where $\mathbf{T}$ is a compact operator and we ignore it when writing the symbol of $\mathbf{A}$.

We declare the symbol of the lifted operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma$ (see (4.7.10)) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ as the symbol of the initial operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+)$ in (4.7.1).

The function $g^s_{-\gamma,\gamma} \in C(\mathbb{R})$ is continuous on $\mathbb{R}$, but has different limits at infinity

$$g^s_{-\gamma,\gamma}(-\infty) = 1, \quad g^s_{-\gamma,\gamma}(+\infty) = e^{2\pi si}, \;\; g^s_{-\gamma,\gamma}(0) = e^{\pi si}, \tag{4.7.11a}$$

while the functions $g^s_{-\gamma,-\gamma_0}, g^s_{-c\gamma,\gamma}, g^s_{-c\gamma_0,\gamma} \in C(\mathbb{R})$ are continuous on $\mathbb{R}$ including infinity

$$\begin{gathered} g^s_{-c\,\gamma,\gamma}(\pm\infty) = g^s_{-\gamma,-\gamma_0}(\pm\infty) = g^s_{-c\,\gamma_0,\gamma}(\pm\infty) = 1, \\ g^s_{-\gamma,-\gamma_0}(0) g^s_{-c\,\gamma_0,\gamma}(0) = \left(\frac{-\gamma}{-\gamma_0}\right)^s \left(\frac{-c\gamma_0}{\gamma}\right)^s = (-c)^s, \\ g^s_{-c\,\gamma,\gamma}(0) = (-c)^s \;\text{ if }\; 0 < \arg c < 2\pi. \end{gathered} \tag{4.7.11b}$$

In the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$, the symbols of the first two operators in (4.7.10) are written according to formulae (4.5.7)–(4.5.8) by taking into account equalities (4.7.11a) and (4.7.11a). The symbols of these operators have, respectively, forms (4.7.6a) and (4.7.6c).

The symbols of operators $W_{a_1}, \ldots, W_{a_n}$ and $W_{b_1}, \ldots, W_{b_n}$ are written with the help of formulae (4.5.7)–(4.5.8) and have form (4.7.6b).

The lifted Mellin convolution operators

$$\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$$

are of mixed type and comprise both the Mellin convolution operators $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathscr{K}^1_{c_j,p}(\xi)}$, where the symbol $\mathscr{K}^1_{c_j,p}(\xi) := \mathscr{M}_{\frac{1}{p}} \mathscr{K}^1_{c_j}(\xi)$ is defined in (4.3.9b) and (4.3.9c), and the Fourier convolution operators $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\,\gamma,-\gamma_0} g^s_{-c_j\,\gamma_0,\gamma}}$. The symbol of the operators $\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma$ from (4.7) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ is found according formulae (4.5.7)–(4.5.8), has form (4.7.6d) and is declared the symbol of $\mathbf{K}^1_{c_j} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$. The symbols of Fourier convolution factors $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\,\gamma,-\gamma_0} g^s_{-c_j\,\gamma_0,\gamma}}$, which contribute the symbol of $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathscr{K}^1_{c_j,p}}$, are written again according formulae (4.5.7)–(4.5.8) by taking into account equalities (4.7.11a) and (4.7.11b).

Theorem 4.6.2 applied to the lifted operator gives the result formulated in Theorem 4.7.1. □

Corollary 4.7.1. *Let $1 < p < \infty$, $s \in \mathbb{R}$. The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+),$$

defined in (4.5.15)*, is locally invertible at $0 \in \mathbb{R}^+$ if and only if its symbol $\mathscr{A}^s_p(\omega)$, defined in* (4.7.5) *and* (4.7.6a)*–*(4.7.6f)*, is elliptic on Γ_1, i.e.,*

$$\inf_{\omega\in\Gamma_1} \left|\det \mathscr{A}^s_p(\omega)\right| = \inf_{\xi\in\mathbb{R}} \left|\det \mathscr{A}^s_p(\xi,\infty)\right| > 0.$$

Proof. For the definition of the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p(\Omega) = \mathbb{B}^s_{p,p}(\Omega)$, $\widetilde{\mathbb{W}}^s_p(\Omega) = \widetilde{\mathbb{B}}^s_{p,p}(\Omega)$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$, including the half-axes $\mathbb{R}^+$, we refer to the monograph [133]. □

Corollary 4.7.2. *Let $1 < p < \infty$, $s \in \mathbb{R}$. If the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$, defined in* (4.5.15)*, is Fredholm* (*is invertible*) *for all $a \in (s_0, s_1)$ and $p \in (p_0, p_1)$, where $-\infty < s_0 < s_1 < \infty$, $1 < p_0 < p_1 < \infty$, then*

$$\mathbf{A} : \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+) \to \mathbb{W}^s_p(\mathbb{R}^+), \;\; s \in (s_0, s_1), \;\; p \in (p_0, p_1), \tag{4.7.12}$$

is Fredholm (*is invertible, respectively*) *and has the equal index*

$$\mathbf{Ind}\,\mathbf{A} = -\,\mathbf{ind}\det \mathscr{A}^s_p \tag{4.7.13}$$

in the Sobolev–Slobodeckij (*Besov*) *spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$.*

Proof. First of all recall that the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$ emerge as the result of interpolation with the real interpolation method between the Bessel potential spaces

$$\begin{aligned}&\big(\mathbb{H}^{s_0}_{p_0}(\Omega), \mathbb{H}^{s_1}_{p_1}(\Omega)\big)_{\theta,p} = \mathbb{W}^s_p(\Omega), \quad s := s_0(1-\theta) + s_1\theta,\\ &\big(\widetilde{\mathbb{H}}^{s_0}_{p_0}(\Omega), \widetilde{\mathbb{H}}^{s_1}_{p_1}(\Omega)\big)_{\theta,p} = \widetilde{\mathbb{W}}^s_p(\Omega), \quad p := \frac{1}{p_0}\,(1-\theta) + \frac{1}{p_1}\,\theta, \quad 0<\theta<1.\end{aligned} \tag{4.7.14}$$

If $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm (or is invertible) for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$, it has a regularizer $\mathbf{R}$ (has the inverse $\mathbf{A}^{-1} = \mathbf{R}$, respectively), which is bounded in the setting

$$\mathbf{R} : \mathbb{W}^s_p(\mathbb{R}^+) \to \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+),$$

due to the interpolation (4.7.14) and

$$\mathbf{RA} = I + \mathbf{T}_1, \quad \mathbf{AR} = I + \mathbf{T}_2,$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and in $\mathbb{H}^s_p(\mathbb{R}^+)$, respectively ($\mathbf{T}_1 = \mathbf{T}_2 = 0$ if $\mathbf{A}$ is invertible).

Due to the Krasnoselskij interpolation theorem (see [133]), $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$ and in $\mathbb{W}^s_p(\mathbb{R}^+)$, respectively, for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$ and, therefore, $\mathbf{A}$ in (4.7.12) is Fredholm (is invertible, respectively).

The index formula (4.7.13) follows from the embedding properties of the Sobolev–Slobodeckij and the Bessel potential spaces by standard well-known arguments. □

Chapter 5

BVPs for the Laplace–Beltrami equations on surfaces with Lipschitz boundary

The objective of the present chapter is to investigate the general Mixed type boundary value problems for the Laplace–Beltrami equation on a surface with the Lipschitz boundary $\mathscr{C}$ in a non-classical setting, when solutions are sought in the Bessel potential spaces $\mathbb{H}^s_p(\mathscr{C})$, $\frac{1}{p} < s < 1 + \frac{1}{p}$, $1 < p < \infty$. Fredholm criteria and the unique solvability criteria are found. By the localization the problem is reduced to the investigation of Model Dirichlet, Neumann and mixed boundary value problems for the Laplace equation in a planar angular domain $\Omega_\alpha \subset \mathbb{R}^2$ of magnitude α. The model mixed BVP is investigated in earlier paper [69] and here we study Model Dirichlet and Neumann boundary value problems in a non-classical setting. The problems are investigated by the potential method and by reducing to locally equivalent 2×2 systems of Mellin convolution equations with meromorphic kernels on the semi-infinite axes $\mathbb{R}^+$ in the Bessel potential spaces. Such equations were studied recently by R. Duduchava in [59] and V. Didenko and R. Duduchava in [37].

5.1 Introduction and formulation of the problems

Many problems in mathematical physics, e.g., cracks in elastic media, electromagnetic scattering by surfaces, etc., are reformulated in the form of a boundary value problem for an elliptic partial differential equation in domains and surfaces with angular points at the boundary. In the recent paper [15], investigation of such BVPs with the help of localization are reduced to the investigation of a family of model problems in plane with finite number of angular points on the boundary of magnitude $\alpha_j \in (0, 2\pi)$, $j = 1, \dots, m$, which, in its turn, are reduced to the investigation of the associated model BVPs in angles with vertex at 0 and the same magnitude.

Consider a hypersurface $\mathscr{C} \subset \mathbb{R}^3$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$, which is a smooth subsurface of a closed hypersurface $\mathscr{S}$ in the Euclidean space $\mathbb{R}^3$. Let $\mathscr{M}_\Gamma$ denote the angular points (the knots) of Γ. Let $\boldsymbol{\nu} := (\nu_1, \nu_2, \nu_3)^\top$ be the normal vector field on the surface $\mathscr{C}$.

On $\mathscr{C}$ we consider the mixed BVP

$$\begin{cases} \boldsymbol{\Delta}_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on } \Gamma_D, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = h(s), & \text{on } \Gamma_N, \end{cases} \tag{5.1.1}$$

where $\boldsymbol{\Delta}_{\mathscr{C}}$ is the Laplace–Beltrami operator

$$\boldsymbol{\Delta}_{\mathscr{C}} := \mathscr{D}_1^2 + \mathscr{D}_2^2 + \mathscr{D}_3^2$$

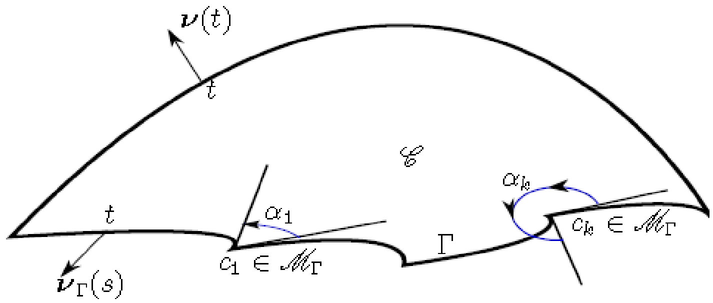

Figure 5.1.

and $\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}}$, $j = 1,2,3$, are Günter's tangent derivatives on the surface. Note that for the flat case $\mathscr{C} \subset \mathbb{R}^2$ Günter's tangent derivatives coincide with the coordinate derivatives $\mathscr{D}_j := \partial_j$ and the Laplace–Beltrami operator with the Laplace operator $\boldsymbol{\Delta}_{\mathscr{C}} = \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$.

$\boldsymbol{\nu}_\Gamma := (\nu_{\Gamma,1}, \nu_{\Gamma,2}, \nu_{\Gamma,3})^\top$ is the normal vector field to the boundary Γ tangent to $\mathscr{S}$ and $\partial_{\boldsymbol{\nu}_\Gamma} = \nu_{\Gamma,1}\mathscr{D}_1 + \nu_{\Gamma,2}\mathscr{D}_2 + \nu_{\Gamma,3}\mathscr{D}_3$ is the normal derivative.

Problem (5.1.1) is considered in the non-classical setting

$$u\in\mathbb{H}^s_p(\mathscr{C}),\ \ f\in\widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathscr{C}),\ \ g\in\mathbb{H}^{s-\frac{1}{p}}_p(\Gamma_D),\ \ h\in\mathbb{H}^{s-1-\frac{1}{p}}_p(\Gamma_N),\ \ \Gamma=\Gamma_D\cup\Gamma_N,\ \ 1<p<\infty,\ s>\frac{1}{p}\,. \tag{5.1.2}$$

Note that the upper constraint in $\frac{1}{p} < s < 1 + \frac{1}{p}$ is necessary to ensure an invariant definition of the Bessel potential and Besov spaces on non-smooth boundary Γ, while the lower constraint ensures the existence of the Dirichlet trace u^+ and, together with the Green formulae, also the existence of the Neumann trace $(\partial_{\boldsymbol{\nu}} u)^+$ of a solution on the boundary. These constraints cannot be relaxed.

For the definitions of the Bessel potential $\mathbb{H}^s_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$, $\mathbb{H}^r_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ and Sobolev–Slobodečkii $\widetilde{\mathbb{W}}^r_p(\mathbb{R}^+)$, etc., spaces for $r \in \mathbb{R}$, $1 < p < \infty$, we refer to the classical source [133] and also the papers [54, 68, 69].

Here we define only the space $\widetilde{\mathbb{H}}^{-1}_{p,0}(\mathscr{C})$ mentioned above. Let $\widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C})$ be a subspace of $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ consisting of functions supported just on the boundary Γ, i.e.:

$$\widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C}) := \Big\{ f \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}) : \ \ \langle f, \varphi\rangle = 0 \ \text{ for all } \ \varphi \in C^1_0(\mathscr{C}) \Big\}. \tag{5.1.3}$$

$\widetilde{\mathbb{H}}^{-1}_0(\mathscr{C})$ is a subspace of $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ orthogonal to $\widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C})$. $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ is decomposed into the direct sum of the subspaces:

$$\widetilde{\mathbb{H}}^{-1}(\mathscr{C}) = \widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C}) \oplus \widetilde{\mathbb{H}}^{-1}_0(\mathscr{C}). \tag{5.1.4}$$

The space $\widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C})$ is non-empty (see [89, § 5.1] and excluding it from $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ it is necessary to make BVPs uniquely solvable (cf. [89] and the next Theorem 5.1.1).

Let

$$\widetilde{\mathbb{H}}^{-r}_{p,0}(\mathscr{C}) = \widetilde{\mathbb{H}}^{-r}_p(\mathscr{C}) \cap \widetilde{\mathbb{H}}^{-1}_0(\mathscr{C}), \quad r > \frac{1}{p}\,. \tag{5.1.5}$$

Theorem 5.1.1 (see Theorem 5.2.1 below)**.** *The BVP* (5.1.1) *has a unique solution in the classical weak setting*

$$u \in \mathbb{H}^1(\mathscr{C}),\ \ f \in \widetilde{\mathbb{H}}^{-1}_0(\mathscr{C}),\ \ g \in \mathbb{H}^{\frac{1}{2}}(\Gamma),\ \ h \in \mathbb{H}^{-\frac{1}{2}}(\Gamma). \tag{5.1.6}$$

A natural question arises: why we investigate the BVP (5.1.1) in the non-classical setting, when in the classical setting the solvability result is easily obtainable. Besides that this is an interesting mathematical problem in many cases, for example, in approximation methods, it is important to know a maximal smoothness of a solution. From the solvability results in non-classical setting it is possible to conclude smoothness property of a solution.

To formulate the appropriate main theorems of the present chapter we need the following definition.

Definition 5.1.1. The BVP (5.1.1) in setting (5.1.2) (the BVP (5.7.1), the BVP (5.8.1)) is Fredholm if the homogeneous problem $f = g = 0$ ($f = h = 0$, respectively) has a finite number of solutions and the BVP has a solution if and only if the data f, g, h satisfy a finite number of orthogonality conditions.

Let $\mathscr{M}_\Gamma$ denote the set of knots of the boundary $\Gamma := \partial\mathscr{C}$, where the smoothness of the curve Γ is violated, the angle between the left and right tangent half-lines at $c_j \in \mathscr{M}_\Gamma$ (inside the surface $\mathscr{C}$) is $\alpha_j \neq 0, \pi, 2\pi$ or, c_j is a smoothness point of Γ, but Dirichlet and Neumann boundary conditions collide there. The set $\mathscr{M}_\Gamma$ consists of three subsets: $\mathscr{M}_\Gamma = \mathscr{M}_D \cup \mathscr{M}_N \cup \mathscr{M}_{DN}$; the first subset $\mathscr{M}_D$ consists of all knots c_j where the Dirichlet conditions collide and $\alpha_j \neq \pi$; $\mathscr{M}_N$ consists of all knots c_j where Neumenn conditions collide and $\alpha_j \neq \pi$; $\mathscr{M}_{DN}$ consists of all knots c_j where Dirichlet and Neumenn conditions collide and here α_j can be smoothness point $0 < \alpha_j < 2\pi$.

Next, we formulate the main theorem of the present chapter which was proved in [61].

Theorem 5.1.2. *The BVP on a surface* (5.1.1) *in the non-classical setting* (5.1.2) *is Fredholm if and only if the following holds:*

(i) *If at $c_j \in \mathscr{M}_D$ collide the Dirichlet conditions, then either $\alpha = \pi$ or $\alpha \neq \pi$ and*

$$e^{i2\pi(s-\frac{1}{p})}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha_j-\pi)\Big(\frac{1}{p}-s-1-i\xi\Big) \neq 0 \ \ \text{for all} \ \ \xi\in\mathbb{R}. \tag{5.1.7}$$

(ii) *If at $c_j \in \mathscr{M}_N$ collide the Neumann conditions, then either $\alpha = \pi$ or $\alpha \neq \pi$ and*

$$e^{i2\pi(s-\frac{1}{p})}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha_j-\pi)\Big(\frac{1}{p}-s-i\xi\Big) \neq 0 \ \ \text{for all} \ \ \xi\in\mathbb{R}. \tag{5.1.8}$$

(iii) *If at $c_j \in \mathscr{M}_N$ collide the Dirichlet and Neumann conditions, then either $\alpha = \pi$ or $\alpha \neq \pi$ and*

$$e^{i2\pi(s-1/p)}\sin^2\pi(s-i\xi) - \cos^2[\pi/p+\alpha s-i(\pi-\alpha)\xi] \neq 0 \quad \text{for all} \quad \xi\in\mathbb{R}. \ \text{for all} \ \xi\in\mathbb{R}. \tag{5.1.9}$$

If conditions (5.1.7), (5.1.8) *and* (5.1.9) *hold* (*i.e., the BVP* (5.1.1), (5.1.2) *is Fredholm*), *the subset* $(\frac{1}{p},\infty)\times(1,\infty)$ *of the Euclidean plane $\mathbb{R}^2$, where the pairs (s,p) range, decomposes into an infinite union $\mathscr{R}_0 \cup \mathscr{R}_1 \cup \cdots$ of non-intersecting connected subsets of regular pairs, for which the BVP* (5.1.1) *is Fredholm in setting* (5.1.2).

If the connected subset $\mathscr{R}_0$ contains the point $(1,2)$ (i.e., $s = 1$, $p = 2$), then the BVP (5.1.1) *is uniquely solvable in setting* (5.1.2) *for all pairs $(s,p) \in \mathscr{R}_0$.*

The formulated Theorem 5.1.2 is proved at the end of Section 5.8. Theorem is proved based on a local principle, which reduces the proof to the investigation of the model problems, Dirichlet, Neumann and Mixed BVPs on a model domain, an angle of magnitude α (see Section 5.3). We will investigate model Dirichlet, Neumann and Mixed BVPs in Sections 5.6–5.8.

We can formulate more transparent criteria of solvability of BVP (5.1.1), where $\mathscr{C}$ is a hypersurface with a smooth boundary $\Gamma = \partial\mathscr{C}$ and, consequently, the set of knots $\mathscr{M}_\Gamma$ consists of only points where the Dirichlet and Neumann boundary conditions collide $\mathscr{M}_\Gamma = \mathscr{M}_{DN}$.

Corollary 5.1.1 (cf. [68])**.** *Let $\mathscr{C}$ be a hypersurface with a smooth boundary $\mathscr{M}_\Gamma = \mathscr{M}_{DN}$ and $1 < p < \infty$, $1/p < s < 1/p+1$. The BVP* (5.1.1) *is Fredholm in the non-classical setting* (*see* (5.1.6)) *if and only if*

$$\cos^2\pi\, s - \Big|\sin 2\pi\Big(s-\frac{1}{p}\Big)\Big| \neq 0. \tag{5.1.10}$$

In other words, the isotherm curves on Fig. 5.2 does not cross the point $(s-k, 1/p)$, where $k = 0, 1, \dots$ is an integer such that $\frac{1}{2} < s-k \leqslant \frac{3}{2}$.

In particular, the BVP (5.1.1) *has a unique solution u in the non-classical setting* (5.1.6) *if the point $(s, 1/p)$ belongs to the open curved quadrangle ABCD on Fig. 5.2.*

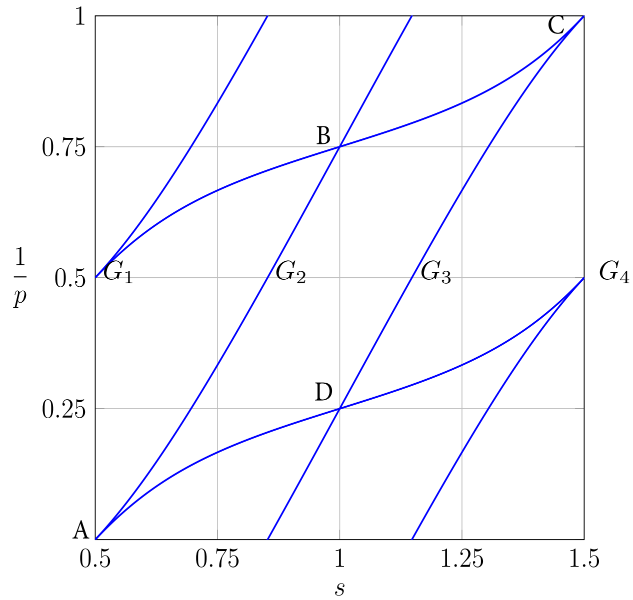

Figure 5.2. The symbol (5.1.10) plot.

Investigations of the boundary integral equations run into difficulties due to the absence of results on Mellin convolution equations in the Bessel potential space setting $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $f \in \mathbb{H}^s_p(\mathbb{R}^+)$. In the recent paper [19], L. Castro and D. Kapanadze reduce BVPs (5.6.1) and (5.7.1) in the $\mathbb{H}^{1+\varepsilon}(\Omega_\alpha)$ space settings to equivalent Wiener–Hopf ± Hankel operators, by manipulating with the even and odd extensions and the reflection operators. The obtained equations were investigated in $\mathbb{L}_2(\mathbb{R}^+)$ and in the special potential space defined by Mellin transforms.

In [97], P. A. Krutitskii investigated boundary value problems for the Helmholtz equation in a planar 2D domain Ω outer to a finite number of domains and cuts, with Dirichlet, Neumann, mixed and impedance conditions on the boundary and faces of cuts. Unique solvability was proved in classical strong setting $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ by reducing the problems to boundary Fredholm integral equations. Singularities at the tips of cuts were described as well.

In the present chapter, we apply the potential method and reduce investigation of BVPs (5.6.1) and (5.7.1) to the investigation of simpler equivalent systems.

The chapter is organized as follows. In Section 5.2, we apply Lax–Milgram Lemma (see Lemma 1.1.4) and prove the solvability of Dirichlet, Neumann and Mixed boundary value problems for the the Laplace–Beltrami equation on a hypersurface $\mathscr{C}$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ in the classical $W^1(\mathscr{C})$ setting. In Section 5.3, we expose quasi localisation method for the boundary value problem for a second order elliptic partial differential equation on a hypersurface with the Lipschitz bounbdary (cf. (5.3.1)) and prove Theorem 5.3.1 on Quasi Localization. In Section 5.4, we recall auxiliary materials on potential operators and representation of solutions to BVPs in model domain, then on Mellin convolution operators in the Bessel potential spaces (see Section 5.5). We prove criteria of Fredholm property and unique solvability in non-classical setting of model Dirichlet problem (5.6.1) (Section 5.6), of model Neumann BVP (5.7.1) (Section 5.7) and of the model Mixed BVP (5.8.1) (Section 5.8). At the end of Section 5.8, we prove the main theorem of the section, Theorem 5.1.2, based on Quasi Localization (cf. Section 5.3) and investigation of model BVPs in Sections 5.6–5.8.

5.2 Solvability of BVPs for the anisotropic Laplace-Beltrami equation on a hypersurface in the classical setting

The exposition in the present section follows the paper [70].

We will use the notation from Section 5.1 and consider, along with the Mixed BVP (slightly more general than in (5.1.1)),

$$\begin{cases} \operatorname{div}_{\mathscr{C}} \mathscr{A} \nabla_{\mathscr{S}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on } \Gamma_D, \\ \langle \boldsymbol{\nu}_\Gamma(s), (\mathscr{A} \nabla_{\mathscr{C}} u)^+(s) \rangle = h(s), & \text{on } \Gamma_N, \end{cases} \tag{5.2.1}$$

the particular cases – the Dirichlet BVP (when Γ_N is the empty set $\Gamma_N = \emptyset$)

$$\begin{cases} \operatorname{div}_{\mathscr{C}} \mathscr{A} \nabla_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on} \quad \Gamma \end{cases} \tag{5.2.2}$$

and the Neumann BVP (when Γ_D is the empty set $\Gamma_D = \emptyset$)

$$\begin{cases} \operatorname{div}_{\mathscr{C}} \mathscr{A} \nabla_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ \langle \boldsymbol{\nu}_\Gamma(s), (\mathscr{A} \nabla_{\mathscr{C}} u)^+(s) \rangle = h(s), & \text{on} \quad \Gamma. \end{cases} \tag{5.2.3}$$

Here $\operatorname{div}_{\mathscr{C}} \mathscr{A} \nabla_{\mathscr{C}}$ is the "anisotropic" Laplace-Beltrami operator and $\mathscr{A}$ is a positive definite 3×3 matrix function

$$(\mathscr{A} \varphi, \varphi)_{\mathscr{C}} \geqslant C > 0 \quad \text{for all} \quad \|\varphi | L_2(cC)\| = 1 \tag{5.2.4}$$

and $\langle \boldsymbol{\nu}_\Gamma(s), \mathscr{A} (\nabla_{\mathscr{C}} u)^+(s) \rangle$ denotes the "Neumann" operator, the scalar product of 3-vectors $\boldsymbol{\nu}_\Gamma(s)$ and $\mathscr{A} (\nabla_{\mathscr{C}} u)$ in $\mathbb{R}^3$. For $\mathscr{A}(t) \equiv 1$, the operator $\operatorname{div}_{\mathscr{C}} \mathscr{A} \nabla_{\mathscr{C}}$ becomes the Laplace–Beltrami operator $\operatorname{div}_{\mathscr{C}} \nabla_{\mathscr{C}} = \boldsymbol{\Delta}_{\mathscr{C}}$ and the "Neumann" operator becomes the normal derivative $\langle \boldsymbol{\nu}_\Gamma(s), (\nabla_{\mathscr{C}} u)^+(s) \rangle = \langle \boldsymbol{\nu}_\Gamma, \nabla_{\mathscr{C}} u \rangle^+(s)$.

The BVPs (5.2.1)–(5.2.3) are investigated in the following classical weak settings in 3-vector spaces

$$u \in \mathbb{H}^1(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}_0^{-1}(\mathscr{C}), \quad g \in \mathbb{H}^{\frac{1}{2}}(\Gamma_D), \quad h \in \mathbb{H}^{-\frac{1}{2}}(\Gamma_N), \quad \Gamma = \Gamma_D \cup \Gamma_N \tag{5.2.5}$$

for the mixed BVP (also cf. (5.1.2)), and in the weak settings

$$f \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \quad g \in \mathbb{H}^{1/2}(\Gamma), \quad h \in \mathbb{H}^{-1/2}(\Gamma) \tag{5.2.6}$$

for the Dirichlet and Neumann BVPs.

The main objective of the present section is to prove the following

Theorem 5.2.1. *Let $\mathscr{C} \subset \mathbb{R}^3$ be a hypersurface with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$.*

The Mixed BVP (5.2.1) *in the classical setting* (5.2.5) *has a unique solution.*

The Dirichlet BVP (5.2.2) *in the classical setting* (5.2.6) *has a unique solution.*

For the solvability of the Neumann problem (5.2.3) *in the classical setting* (5.2.5) *the following necessary and sufficient compatibility condition has to hold:*

$$(f, 1)_{\mathscr{C}} - (h, 1)_\Gamma = 0. \tag{5.2.7}$$

Note that if f and h are regular integrable functions, the compatibility condition (5.2.7) acquires the form

$$\int_{\mathscr{C}} f(y)\, d\sigma - \oint_\Gamma h(s) ds = 0. \tag{5.2.8}$$

The formulated theorem will be proved at the end of the present section. Prior to this, we will expose auxiliary material for this proof.

Remark 5.2.1. Theorem 5.2.1 was proved in [56] for Dirichlet and Neumann BVPs with the help of potential method and in [70] for Mixed, Dirichlet and Neumann BVPs using Lax-Milgram Lemma in case of smooth boundary Γ.

Moreover, for the Dirichlet (5.2.2) and Neumann (5.2.3) BVPs and non-classical setting

$$u \in \mathbb{H}^s_p(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathscr{C}), \quad g \in \mathbb{H}^{s-\frac{1}{p}}_p(\Gamma), \quad h \in \mathbb{H}^{s-1-\frac{1}{p}}_p(\Gamma), \quad 1 < p < \infty, \ s > \frac{1}{p} \tag{5.2.9}$$

in case of a hypersurface with the smooth boundary, the unique solvability holds as well (see [56]).

For the mixed BVP (5.2.3) in the non-classical setting

$$u\in\mathbb{H}^s_p(\mathscr{C}),\ \ f\in\widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathscr{C}),\ \ g\in\mathbb{H}^{s-\frac{1}{p}}_p(\Gamma_D),\ \ h\in\mathbb{H}^{s-1-\frac{1}{p}}_p(\Gamma_N),\ \ \Gamma=\Gamma_D\cup\Gamma_N,\ \ 1<p<\infty,\ s>\frac{1}{p},\quad (5.2.10)$$

even for a hypersurface with the smooth boundary and for Dirichlet (5.2.2) and Neumann (5.2.3) BVPs for a hypersurface with the Lipshitz boundary and the non-classical setting (5.2.9), the solvability conditions change dramatically (see Theorem 5.1.2 above and Sections 5.6–5.8 below).

Mixed BVPs for the Laplace equation in domains were investigated by Lax-Milgram Lemma by many authors (see, e.g., the recent lecture notes online [103]).

Let $\mathscr{M}$ be a non-trivial, $\operatorname{mes}\mathscr{M}\neq\emptyset$, smooth closed or open hypersurface, $s\in\mathbb{R}$ and $1<p<\infty$. If $\mathscr{M}$ is definitely closed, we use $\mathscr{S}$, while in case $\mathscr{M}$ is definitely open, we use $\mathscr{C}$.

By $\mathbb{X}^s_p(\mathscr{M})$ we denote one of the spaces: $\mathbb{H}^s_p(\mathscr{M})$, Sobolev-Slobodecki $\mathbb{W}^s_p(\mathscr{M})$ (if $\mathscr{M}$ is closed or open) and by $\widetilde{\mathbb{X}}^s_p(\mathscr{C})$ denote one of the spaces: $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ and $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$. Consider the space

$$\mathbb{X}^s_{p,\#}(\mathscr{M}):=\{\varphi\in\mathbb{X}^s_2(\mathscr{M})\ :\ (\varphi,1)=0\}. \tag{5.2.11}$$

It is obvious that $\mathbb{X}^s_{p,\#}(\mathscr{M})$ does not contain constants: if $c_0=\text{const}\in\mathbb{X}^s_{p,\#}(\mathscr{M})$, then

$$0=(c_0,1)=c_0(1,1)=c_0\operatorname{mes}\mathscr{M}$$

and $c_0=0$. Moreover, $\mathbb{X}^s_p(\mathscr{M})$ decomposes into the direct sum

$$\mathbb{X}^s_p(\mathscr{M})=\mathbb{X}^s_{p,\#}(\mathscr{M})+\{\text{const}\} \tag{5.2.12}$$

and the dual (adjoint) space is

$$(\mathbb{X}^s_{p,\#}(\mathscr{M}))^*=\mathbb{X}^{-s}_{p',\#}(\mathscr{M}),\quad p':=\frac{p}{p-1}. \tag{5.2.13}$$

Indeed, decomposition (5.2.12) follows from the representation

$$\varphi=\varphi_0+\varphi_{\text{aver}},\quad \varphi_0\in\mathbb{X}^s_{p,\#}(\mathscr{M}),\quad \varphi_{\text{aver}}:=\frac{1}{\operatorname{mes}\mathscr{M}}(\varphi,1)$$

of arbitrary function $\varphi\in\mathbb{X}^s_p(\mathscr{M})$), because the average of the difference of a function and its average is zero: $(\varphi_0)_{\text{aver}}=(\varphi-\varphi_{\text{aver}})_{\text{aver}}=0$.

Description (5.2.13) of the dual space follows from the fact that the dual space to $\mathbb{X}^s_p(\mathscr{M})$ is $\mathbb{X}^{-s}_{p'}(\mathscr{M})$ (see [133]) and, therefore, due to the decomposition (5.2.12) and Hahn–Banach theorem, the dual space to $\mathbb{X}^s_{p,\#}(\mathscr{M})$ should be embedded into $\mathbb{X}^{-s}_{p'}(\mathscr{M})$. The only functional from $\mathbb{X}^{-s}_{p'}(\mathscr{M})$ that vanishes on the entire space $\mathbb{X}^s_{p,\#}(\mathscr{M})$ is constant $1\in\mathbb{X}^{-s}_{p'}(\mathscr{M})$ (see definition (5.2.11)). After detaching this functional the remainder coincides, due to (5.2.12), with the space $\mathbb{X}^{-s}_{p',\#}(\mathscr{M})$, which is the dual to $\mathbb{X}^s_{p,\#}(\mathscr{M})$.

Lemma 5.2.1. *The equivalent norm in the space $\mathbb{W}^m_{p,\#}(\mathscr{M})$ is defined as follows:*

$$\big\|\varphi\big|\mathbb{W}^m_{p,\#}(\mathscr{M})\big\|_0:=\sum_{1\leqslant|\alpha|\leqslant m}\big\|\mathscr{D}^\alpha\varphi\big|\mathbb{L}_p(\mathscr{M})\big\|. \tag{5.2.14}$$

In particular, in the space $\mathbb{W}^1_{p,\#}(\mathscr{M})$ the equivalent norm is

$$\big\|\varphi\big|\mathbb{W}^1_{p,\#}(\mathscr{M})\big\|_0:=\big\|\nabla_{\mathscr{S}}\varphi\big|\mathbb{L}_p(\mathscr{M})\big\|. \tag{5.2.15}$$

Proof. By $\mathbb{W}^{m,0}_{p,\#}(\mathscr{M})$ denote the same subspace $\mathbb{W}^m_{p,\#}(\mathscr{M})$ of $\mathbb{W}^m_p(\mathscr{M})$, but equipped with the standard norm of the subspace

$$\big\|\varphi\big|\mathbb{W}^{m,0}_{p,\#}(\mathscr{M})\big\|=\big\|\varphi\big|\mathbb{W}^m_p(\mathscr{M})\big\|:=\sum_{0\leqslant|\alpha|\leqslant m}\big\|\mathscr{D}^\alpha\varphi\big|\mathbb{L}_p(\mathscr{M})\big\|. \tag{5.2.16}$$

Then the embedding $\mathbb{W}^{m,0}_{p,\#}(\mathscr{M}) \subset \mathbb{W}^{m}_{p,\#}(\mathscr{M})$ is continuous

$$\big\|\varphi\big|\mathbb{W}^{m}_{p,\#}(\mathscr{M})\big\|_0 \leqslant \big\|\varphi\big|\mathbb{W}^{m,0}_{p,\#}(\mathscr{M})\big\|. \tag{5.2.17}$$

On the other hand, this embedding is bijective due to representations (5.2.12). Then the Banach inverse mapping theorem states that these norms are equivalent: along with inequality (5.2.17) also the inverse inequality

$$\big\|\varphi\big|\mathbb{W}^{m,0}_{p,\#}(\mathscr{M})\big\|_0 \leqslant C\big\|\varphi\big|\mathbb{W}^{m}_{p,\#}(\mathscr{M})\big\|$$

holds with some constant C, which is independent of a function φ. □

Let $\Gamma_0 \subset \Gamma$ be a non-trivial subset of the boundary $\Gamma = \partial\mathscr{C}$ of the surface $\mathscr{C}$ and $\widetilde{\mathbb{X}}^s_p(\Gamma_0, \mathscr{C})$, $s > 1/p$, $1 < p < \infty$, denote the subspace of $\mathbb{X}^s_p(\mathscr{C})$ which consists of functions with vanishing trace on Γ_0.

Lemma 5.2.2. *Along with the standard norm of subspace of $\mathbb{W}^m_p(\mathscr{C})$ $(m = 1, 2, \dots)$, the equivalent norm in the space $\widetilde{\mathbb{W}}^m_p(\Gamma_0, \mathscr{C})$ is defined as follows:*

$$\big\|\varphi\big|\widetilde{\mathbb{W}}^m_p(\Gamma_0, \mathscr{C})\big\|_0 := \sum_{1\leqslant|\alpha|\leqslant m} \big\|\mathscr{D}^\alpha\varphi\big|\mathbb{L}_p(\mathscr{C})\big\|. \tag{5.2.18}$$

In particular, in the space $\widetilde{\mathbb{W}}^1_p(\Gamma_0, \mathscr{C})$ the equivalent norm is

$$\big\|\varphi\big|\widetilde{\mathbb{W}}^1_p(\Gamma_0, \mathscr{C})\big\|_0 := \big\|\nabla_{\mathscr{S}}\varphi\big|\mathbb{L}_p(\mathscr{M})\big\|. \tag{5.2.19}$$

Proof. By $\widetilde{\mathbb{W}}^{m,0}_p(\Gamma_0, \mathscr{C})$ denote the same subspace $\widetilde{\mathbb{W}}^m_p(\Gamma_0, \mathscr{C})$ of $\mathbb{W}^m_p(\mathscr{C})$, but equipped with the standard norm of the subspace

$$\big\|\varphi\big|\widetilde{\mathbb{W}}^{m,0}_p(\Gamma_0, \mathscr{C})\big\| = \big\|\varphi\big|\mathbb{W}^m_p(\mathscr{C})\big\| := \sum_{0\leqslant|\alpha|\leqslant m} \big\|\mathscr{D}^\alpha\varphi\big|\mathbb{L}_p(\mathscr{C})\big\|. \tag{5.2.20}$$

Then the embedding $\widetilde{\mathbb{W}}^{m,0}_p(\Gamma_0, \mathscr{C}) \subset \widetilde{\mathbb{W}}^m_p(\Gamma_0, \mathscr{C})$ is continuous

$$\big\|\varphi\big|\widetilde{\mathbb{W}}^m_p(\Gamma_0, \mathscr{C})\big\|_0 \leqslant \big\|\varphi\big|\widetilde{\mathbb{W}}^{m,0}_p(\Gamma_0, \mathscr{C})\big\|. \tag{5.2.21}$$

On the other hand, this embedding is bijective due to the definition. Then the Banach inverse mapping theorem states that these norms are equivalent: along with inequality (5.2.21) also the inverse inequality

$$\big\|\varphi\big|\widetilde{\mathbb{W}}^{m,0}_p(\Gamma_0, \mathscr{C})\big\|_0 \leqslant C\big\|\varphi\big|\widetilde{\mathbb{W}}^m_p(\Gamma_0, \mathscr{C})\big\|$$

holds with some constant C, which is independent of a function φ. □

Theorem 5.2.2. *Let $\mathscr{S}$ be ℓ-smooth $\ell = 1, 2, \dots$, $1 < p < \infty$, $|s| \leqslant \ell$ and $\mathscr{A}(t)$ be a positive definite 3×3 matrix function. Let $\mathbb{X}^s_p(\mathscr{S})$ be the same vector-space as above.*

Let the matrix-function $\mathscr{H} \in [C^{\ell-1}(\mathbb{R}^n)]^{3\times3}$ have one of the following properties:

i. $\mathscr{H}$ has a non-negative definite real part $(\mathrm{Re}\,\mathscr{H}\varphi, \varphi)_{\mathscr{C}} \geqslant 0$ and $\mathrm{mes\,supp\,Re}\,\mathscr{H} \neq 0$;

ii. $\mathrm{mes\,supp\,Im}\,\mathscr{H} \neq 0$ and the complex part $\mathrm{Im}\,\mathscr{H}$ is either positive definite or negative definite:

$$(\varepsilon\mathrm{Im}\,\mathscr{H}\varphi, \varphi)_{\mathscr{C}} \geqslant C > 0 \quad \text{for all} \quad \|\varphi|L_2(\mathscr{C})\| = 1, \tag{5.2.22}$$

where $\varepsilon = 1$ or $\varepsilon = -1$.

Then the perturbed operator

$$\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}-\mathscr H I\ :\ \mathbb X^{s+1}_p(\mathscr S)\to\mathbb X^{s-1}_p(\mathscr S) \tag{5.2.23}$$

is invertible.

Moreover, the operator $\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}$ *is also invertible between the spaces with detached constants* (*see* (5.2.11))

$$\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}\ :\ \mathbb X^{s+1}_{p,\#}(\mathscr S)\to\mathbb X^{s-1}_{p,\#}(\mathscr S) \tag{5.2.24}$$

and, therefore, $\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}$ *has the fundamental solution in setting* (5.2.24).

The invertibility is also interpreted as the existence of the fundamental solution to the operators $\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}-\mathscr H I$ *and* $\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}$ *in the appropriate spaces.*

In particular, the perturbed Laplace–Beltrami operator $\boldsymbol\Delta_{\mathscr S}-\mathscr H I$ (*the particular case* $\mathscr A(t)\equiv 1$) *is invertible in setting* (5.2.23) *(has the fundamental solution), while the Laplace–Beltrami operator* $\boldsymbol\Delta_{\mathscr S}$ *is invertible in setting* (5.2.24) (*has the fundamental solution*).

Proof. First of all note that the operators in (5.2.22) and (5.2.23) are bounded. For the operator in (5.2.22) this is trivial, while for the operator in (5.2.23) we need to check that the image of the operator belongs to the subspace $\mathbb X^{s-1}_p(\mathscr S)$, i.e., is orthogonal to the identity 1 (see (5.2.11)). Indeed, by applying formula (1.3.37) we get

$$\big(\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}\varphi,1\big)_{\mathscr S}=\big(\mathscr A\nabla_{\mathscr S}\varphi,\nabla_{\mathscr S}1\big)_{\mathscr S}=0.$$

This operator in the setting

$$\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}\ :\ \mathbb X^{1}_{2,\#}(\mathscr S)\to\mathbb X^{-1}_{2,\#}(\mathscr S) \tag{5.2.25}$$

is coercive:

$$-\big(\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}\varphi,\varphi\big)_{\mathscr S}=\big(\mathscr A\,\nabla_{\mathscr S}\varphi,\nabla_{\mathscr S}\varphi\big)_{\mathscr S}\geqslant C\|\varphi|\mathbb X^1_{2\#}(\mathscr S)\|^2. \tag{5.2.26}$$

Then, due to Lemma 1.1.5, this operator is invertible in setting (5.2.25).

Moreover, this operator $\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}$ is elliptic and even has negative definite symbol $\xi^t op\mathscr A\xi$, $\xi\in\mathbb R^m$ (ellipticity follows from the invertibility in setting (5.2.26), as well). As an elliptic operator on the closed hypersurface the operator in (5.2.24) is Fredholm for all $s\in\mathbb R$ and $1<p<\infty$ (it has a parametrix if $\mathscr S$ is infinitely smooth, see [88, 126, 130] and the proof of Theorem 1.7.1 for a similar arguments). Since all operators in the homotopy

$$\mathbf B_\lambda=(1-\lambda)\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}-\lambda\,\Lambda^2(\mathscr X,D),\quad 0\leqslant\lambda\leqslant 1,$$

where

$$\Lambda_2(\mathscr X,D)\ :\ \mathbb X^{s+1}_{p,\#}(\mathscr S)\to\mathbb X^{s-1}_{p,\#}(\mathscr S)$$

is the Bessel potential operator with positive definite symbol and arranges the isometrical isomorphism of the spaces (see (1.7.17)), have positive definite symbol, they are Fredholm operators in the setting

$$\mathbf B_\lambda\ :\ \mathbb X^{s+1}_{p,\#}(\mathscr S)\to\mathbb X^{s-1}_{p,\#}(\mathscr S)$$

for all $0\leqslant\lambda\leqslant 1$, $1<p<\infty$, $|s|\leqslant\ell$. Then

$$\mathbf{Ind}\,\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}=\mathbf{Ind}\;\mathbf B_0=\mathbf{Ind}\;\mathbf B_\lambda=\mathbf{Ind}\;\mathbf B_1=\mathbf{Ind}\;\lambda^2(\mathscr X,D)=0$$

and Theorem 1.1.1 can be applied, which states that the operator in (5.2.24) is invertible for all $1<p<\infty$, $|s|\leqslant\ell$.

Since the operator in (5.2.23) is the perturbation by lower order operator $\mathscr H I$ (i.e., by a compact operator) of the invertible operator in (5.2.23), the operator in (5.2.23) is Fredholm and has trivial index $\mathbf{Ind}\,[\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}-\mathscr H]=0$. Then to prove that the operator in (5.2.23) is invertible we just need to check that it has trivial kernel, i.e., if $(\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}-\mathscr H)\varphi=0$, then $\varphi=0$. Due to equality (1.3.37),

$$\begin{aligned}\big(-(\mathrm{div}_{\mathscr S}\mathscr A\,\nabla_{\mathscr S}-\mathscr H)\varphi,\varphi\big)_{\mathscr S}&\\ =\big(\mathscr A\nabla_{\mathscr S}\varphi,\nabla_{\mathscr S}\varphi\big)_{\mathscr S}&+\big(\mathrm{Re}\,\mathscr H\varphi,\varphi\big)_{\mathscr S}+i\big(\mathrm{Im}\,\mathscr H\varphi,\varphi\big)_{\mathscr S},\ \ \forall\,\varphi\in\mathbb W^1_2(\mathscr S).\end{aligned} \tag{5.2.27}$$

If φ is a solution to the homogeneous equation $(\mathrm{div}_{\mathscr{S}}\mathscr{A}\,\nabla_{\mathscr{S}}-\mathscr{H})\varphi=0$, equality (5.2.24) takes the form

$$(\mathscr{A}\nabla_{\mathscr{S}}\varphi,\nabla_{\mathscr{S}}\varphi)_{\mathscr{S}}+(\mathrm{Re}\,\mathscr{H}\varphi,\varphi)_{\mathscr{S}}=0,\quad (\mathrm{Im}\,\mathscr{H}\varphi,\varphi)_{\mathscr{S}}=0. \tag{5.2.28}$$

Now let $\mathrm{Re}\,\mathscr{H}(t)\geqslant 0$ for all $t\in\mathscr{S}$ and $\mathrm{mes\,supp}\,\mathrm{Re}\,\mathscr{H}\neq 0$ (case (i)). Then from the first equality in (5.2.27) it follows

$$C\|\nabla_{\mathscr{S}}\varphi\|\leqslant(\mathscr{A}\nabla_{\mathscr{S}}\varphi,\nabla_{\mathscr{S}}\varphi)_{\mathscr{S}}=0,\quad (\mathrm{Re}\,\mathscr{H}\varphi,\varphi)_{\mathscr{S}}=0$$

(the inequality is due to the positive definiteness of $\mathscr{A}$). From the first inequality we get $\nabla_{\mathscr{S}}\varphi\equiv 0$ and, consequently, $\varphi=C=\mathrm{const}$ (this is easy to ascertain by analysing the definition of Günter's derivatives; see, e.g., [56]). By inserting this in the second equality in (5.2.27) we get

$$0=(\mathrm{Re}\,\mathscr{H}\varphi,\varphi)_{\mathscr{S}}=C\int_{\mathscr{S}}\mathscr{H}(t)d\sigma,$$

and the conclusion $\varphi(t)=C=0$ is immediate, because $\mathrm{mes\,supp}\,\mathrm{Re}\,\mathscr{H}\neq 0$.

In the case (ii), from the second equality in (5.2.28) we have

$$C\|\varphi\|\leqslant\varepsilon(\mathrm{Im}\,\mathscr{H}\,\varphi,\nabla_{\mathscr{S}}\varphi)_{\mathscr{S}}=0$$

(the inequality is due to the positive definiteness of $\varepsilon\mathrm{Im}\,\mathscr{H}$) and, again, $\varphi=0$. □

Corollary 5.2.1 (cf. [56])**.** *For the operator* $\mathrm{div}_{\mathscr{C}}\mathscr{A}\nabla_{\mathscr{C}}$ *on the open hypersurface* $\mathscr{C}$ *with the boundary* $\partial\mathscr{C}:=\Gamma$ *the following Green formulae are valid*

$$\begin{aligned}&(\mathrm{div}_{\mathscr{C}}\mathscr{A}\nabla_{\mathscr{C}}\varphi,\psi)_{\mathscr{C}}=(\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_{\mathscr{C}}\varphi)^+\rangle,\psi^+)_\Gamma-(\mathscr{A}\nabla_{\mathscr{C}}\varphi,\nabla_{\mathscr{C}}\psi)_{\mathscr{C}}\,,\\&(\mathrm{div}_{\mathscr{C}}\mathscr{A}\nabla_{\mathscr{C}}\varphi,\psi)_{\mathscr{C}}-(\varphi,\mathrm{div}_{\mathscr{C}}\mathscr{A}\nabla_{\mathscr{C}}\psi)_{\mathscr{C}}\\&\qquad=(\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_{\mathscr{C}}\varphi)^+\rangle,\psi^+)_\Gamma-(\varphi^+,\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_{\mathscr{C}}\psi)^+\rangle)_\Gamma,\end{aligned} \tag{5.2.29}$$

where $(\varphi,\psi)_{\mathscr{C}}$ *denotes the scalar product of functions. The normal boundary derivative* $\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\nabla_{\mathscr{C}}\varphi)^+\rangle$ *we have encountered already in the mixed BVP* (5.2.1).

Note that a function $\varphi\in\mathbb{W}^s_p(\mathscr{C})$ (and $\varphi\in\mathbb{H}^s_p(\mathscr{C})$) has the trace $\varphi^+\in\mathbb{W}^{s-\frac{1}{p}}_p(\Gamma)$ on the boundary, provided $1<p<\infty$ and $s>\frac{1}{p}$ (see [133] for details). Therefore, if we look for a solution to Dirichlet BVP (5.2.2) in the space $\mathbb{W}^1(\mathscr{C})$, the trace u^+ on Γ_D exists and belongs to the space $\mathbb{H}^{1/2}(\Gamma_D)$.

Concerning the existence of the Neumann trace $\langle\boldsymbol{\nu}_\Gamma,\mathscr{A}\,\nabla_{\mathscr{C}}u\rangle^+$ in (5.2.1) and (5.2.3) for a solution $u\in\mathbb{W}^1(\mathscr{C})$, it is not guaranteed by the general trace theorem. But in this case, the first Green formula (5.2.29) ensures the existence of the Neumann trace. Indeed, by setting $\varphi=u$ and inserting the data $(\mathrm{div}_{\mathscr{C}}A\nabla_{\mathscr{C}})u(t)=f(t)$ from (5.2.1) into the first Green formula (5.2.29) we obtain

$$(\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_{\mathscr{C}}u)^+\rangle,\psi^+)_\Gamma-(\mathscr{A}\,\nabla_{\mathscr{C}}u,\nabla_{\mathscr{C}}\psi)_{\mathscr{C}}=(\mathrm{div}_{\mathscr{C}}(\mathscr{A}\,\nabla_{\mathscr{C}}u),\psi)_{\mathscr{C}}=(f,\psi)_{\mathscr{C}}$$

and, finally, we get

$$(\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_{\mathscr{C}}u)^+\rangle,\psi^+)_\Gamma=(\mathscr{A}\,\nabla_{\mathscr{C}}u,\nabla_{\mathscr{C}}\psi)_{\mathscr{C}}+(f,\psi)_{\mathscr{C}} \tag{5.2.30}$$

for arbitrary $\psi\in\mathbb{W}^1(\mathscr{C})$. Since $\psi^+\in\mathbb{H}^{1/2}(\Gamma)$, the scalar product $(\mathscr{A}\nabla_{\mathscr{C}}u,\nabla_{\mathscr{C}}\psi)_{\mathscr{C}}$ in the right-hand side of equality (5.2.30) is correctly defined and defines correct duality in the left-hand side of the equality. Since $\psi^+\in\mathbb{H}^{1/2}(\Gamma)$ is arbitrary, by the duality argument this gives that $\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_{\mathscr{C}}u)^+\rangle$ should be in the dual space, i.e., in $\mathbb{H}^{-1/2}(\Gamma)$.

To prove the above formulated Theorem 5.2.1, we need more properties of trace operator (called retraction) and their inverses, called co-retractions (see [133, § 2.7]).

To keep the exposition simpler we recall a very particular case of Lemma 4.8 from [53], which we apply in the present investigation.

A differential operator

$$\boldsymbol{B}(t,\mathscr{D})=\sum_{|\alpha|\leqslant m}a_\alpha(t)\mathscr{D}^\alpha,\quad t\in\mathscr{C},$$

on a hypersurface $\mathscr{C}\subset\mathbb{R}^3$ with the Lipschitz boundary $\Gamma=\partial\mathscr{C}$ is called **normal** if its symbol (the characteristic polynomial)

$$\mathscr{B}(t,\xi)=\sum_{|\alpha|\leqslant m}a_\alpha(t)(-i\xi)^\alpha,\quad t\in\mathscr{C},\quad \xi\in\mathbb{R}^3,$$

does not vanish on normal vactor field on the boundary $\inf_{s\in\Gamma}|\det\mathscr{B}(s,\boldsymbol{\nu}(s))|>0$.

Lemma 5.2.3 (see Lemma 4.8 in [53])**.** *Let $s\notin\mathbb{N}$, $1<p<\infty$, and $\mathscr{C}$ be a hypersurface with the Lipschitz boundary $\Gamma=\partial\mathscr{C}$. Further, let $s>0$, $\boldsymbol{B}(D)$ be a normal differential operator of the first order defined in the vicinity of the boundary Γ and $\boldsymbol{A}(D)$ be a normal differential operator of the second order defined on the surface $\mathscr{C}$. Then there exists a continuous linear operator*

$$\mathscr{R}:\mathbb{W}^s_p(\Gamma)\otimes\mathbb{W}^{s-1}_p(\Gamma)\longrightarrow\mathbb{H}^{s+\frac1p}_p(\mathscr{C})\tag{5.2.31}$$

such that

$$(\mathscr{R}\Phi)^+=\varphi_0,\quad (\boldsymbol{B}(D)\mathscr{R}\Phi)^+=\varphi_1,\quad \boldsymbol{A}(D)\mathscr{B}\Phi\in\widetilde{\mathbb{H}}^{s-2+\frac1p}_p(\mathscr{C})\tag{5.2.32}$$

for arbitrary pair of functions $\Phi=(\varphi_0,\varphi_1)^\top$, where $\varphi_0\in\mathbb{W}^s_p(\Gamma)$ and $\varphi_1\in\mathbb{W}^{s-1}_p(\Gamma)$.

Proof of Theorem 5.2.1. We commence by the reduction of the BVP (5.2.1) to an equivalent one with the homogeneous Dirichlet condition. For this, we extend the boundary data $g\in\mathbb{W}^{1/2}(\Gamma_D)$ up to some function $\widetilde{g}\in\mathbb{W}^{1/2}(\Gamma)$ on the entire boundary Γ and apply Lemma 5.2.3: there exists a function $G\in\mathbb{W}^1(\mathscr{C})$ such that $G^+(t)=g(t)$ for $t\in\Gamma_D$ (actually $G^+=\widetilde{g}$ almost everywhere on the boundary) and $\operatorname{div}_\mathscr{C}(\mathscr{A}\,\nabla_\mathscr{C}G)\in\widetilde{\mathbb{W}}^{-1}(\overline{\mathscr{C}})$.

For a new unknown function $v:=u-G$ we have the following equivalent reformulation of the BVP (5.2.1):

$$\begin{cases}\operatorname{div}_\mathscr{C}(\mathscr{A}\nabla_\mathscr{C}v)(t)=f_0(t), & t\in\mathscr{C},\\ v^+(s)=0, & \text{on}\quad\Gamma_D,\\ \langle\boldsymbol{\nu}_\Gamma(s),(\mathscr{A}\,\nabla_\mathscr{C}v)^+(s)\rangle=h_0(s), & \text{on}\quad\Gamma_N,\end{cases}\tag{5.2.33}$$

where

$$\begin{gathered}f_0:=f+\operatorname{div}_\mathscr{C}(\mathscr{A}\nabla_\mathscr{C}G)\in\widetilde{\mathbb{W}}^{-1}(\mathscr{C}),\quad h_0:=h+\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_\mathscr{C}G)^+\rangle\in\mathbb{W}^{-1/2}(\Gamma),\\ v^+\in\widetilde{\mathbb{W}}^1(\Gamma_D,\mathscr{C})\mathbb{W}^{1/2}(\Gamma_N).\end{gathered}\tag{5.2.34}$$

To justify the last inclusion $v^+\in\widetilde{\mathbb{W}}^{1/2}(\Gamma_N)$ note that, due to our construction, the trace of a solution vanishes on Γ_D: $v^+\,|_{\Gamma_D}=0$.

By inserting the data from the reformulated boundary value problem (5.2.33) into the first Green identity (5.2.29), where $\varphi=\psi=v$, we get

$$\begin{aligned}(\mathscr{A}\,\nabla_\mathscr{C}v,\nabla_\mathscr{C}v)_\mathscr{C}&=(\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_\mathscr{C}v)^+\rangle,v^+)_{\Gamma_D}+(\langle\boldsymbol{\nu}_\Gamma,(\mathscr{A}\,\nabla_\mathscr{C}v)^+\rangle,v^+)_{\Gamma_N}-(\operatorname{div}_\mathscr{C}(\mathscr{A}\,\nabla_\mathscr{C}v),v)_\mathscr{C}\\ &=(h_0,v^+)_{\Gamma_N}-(f_0,v)_\mathscr{C},\quad v\in\widetilde{\mathbb{W}}^1(\Gamma_D,\mathscr{C}).\end{aligned}\tag{5.2.35}$$

In the left-hand side of equality (5.2.35) we have a symmetric bilinear form, which is positive definite

$$(\mathscr{A}\nabla_\mathscr{C}v,\nabla_\mathscr{C}v)_\mathscr{C}\geqslant C\|\nabla_\mathscr{C}v\|^2=\|v|\widetilde{\mathbb{W}}^1(\Gamma_D,\mathscr{C})\|^2\quad\forall\,v\in\widetilde{\mathbb{W}}^1(\Gamma_D,\mathscr{C}),$$

because $\mathscr{A}(s)$ is strictly positive definite matrix and we have applied Lemma 5.2.2 on equivalent norms in the space $\widetilde{\mathbb{W}}^1(\Gamma_D,\mathscr{C})$.

$(h_0, v^+)_{\Gamma_N}$ and $(f_0, v)_{\mathscr{C}}$ in equality (5.2.35) are correctly defined continuous functionals, because $h_0 \in \mathbb{W}^{-1/2}(\Gamma)$, $f_0 \in \widetilde{\mathbb{W}}^{-1}(\mathscr{C})$, while their counterparts in the functional belong to the dual spaces $v^+ \in \widetilde{\mathbb{W}}^{1/2}(\Gamma_D)$ and $v \in \widetilde{\mathbb{W}}^1(\Gamma_N, \mathscr{C}) \subset \mathbb{W}^1(\mathscr{C})$.

Application of the Lax-Milgram Lemma 1.1.4 accomplishes the proof of the unique solvability of the mixed BVP (5.2.1) in setting (5.2.5).

Now we prove the unique solvability of the Dirichlet BVP (5.2.2) in setting (5.2.6). We commence, as above, with an equivalent reformulation: due to Lemma 5.2.3, we can pick up a function $G \in \mathbb{W}^1(\mathscr{C})$ such that $G^+ = g$ and $\operatorname{div}_{\mathscr{C}}(\mathscr{A}\,\nabla_{\mathscr{C}} G) \in \widetilde{\mathbb{W}}^{-1}(\mathscr{C})$.

For a new unknown function $v := u - G$ we have the following equivalent reformulation of the BVP (5.2.2):

$$\begin{cases} \operatorname{div}_{\mathscr{C}}(A\,\nabla_{\mathscr{C}} v)(t) = f_0(t), & t \in \mathscr{C}, \\ v^+(s) = 0, & \text{on} \quad \Gamma, \end{cases} \tag{5.2.36}$$

where

$$f_0 := f + \operatorname{div}_{\mathscr{C}}(\mathscr{A}\,\nabla_{\mathscr{C}} G) \in \widetilde{\mathbb{W}}^{-1}(\mathscr{C}), \quad v \in \widetilde{\mathbb{W}}^1(\mathscr{C}).$$

By inserting the data from the reformulated boundary value problem (5.2.36) into the first Green identity (5.2.29), where $\varphi = \psi = v$, we get

$$(A\nabla_{\mathscr{C}} v, \nabla_{\mathscr{C}} v)_{\mathscr{C}} = (\langle \boldsymbol{\nu}_\Gamma, (A\,\nabla_{\mathscr{C}} v)^+\rangle, v^+)_\Gamma - (\operatorname{div}_{\mathscr{C}}(A\,\nabla_{\mathscr{C}} v), v)_{\mathscr{C}} = -(f_0, v)_{\mathscr{C}}, \quad v \in \widetilde{\mathbb{W}}^1(\mathscr{C}).$$

What we get is similar to identity (5.2.35) which we derived in the foregoing case: the positive definite form in the left-hand side and a single correctly defined functional in the right-hand side. Again, applying the Lax–Milgram Lemma, the unique solvability of the Dirichlet BVP (5.2.2) can be proved in setting (5.2.6).

In conclusion, we prove the unique solvability of the Dirichlet BVP (5.2.3) in setting (5.2.6). Let us insert the data from the boundary value problem (5.2.3) into the first Green identity (5.2.29), where $\varphi = \psi = u$. We get

$$\begin{aligned} (\mathscr{A}\,\nabla_{\mathscr{C}} u, \nabla_{\mathscr{C}} u)_{\mathscr{C}} &= (\langle \boldsymbol{\nu}_\Gamma, \mathscr{A}\,\nabla_{\mathscr{C}} u\rangle^+, u^+)_\Gamma - ((\operatorname{div}_{\mathscr{C}} \mathscr{A}\,\nabla_{\mathscr{C}})u, u)_{\mathscr{C}} \\ &= (h, u)_\Gamma - (f, u)_{\mathscr{C}}, \quad u \in \mathbb{W}^1_{2,\#}(\mathscr{C}). \end{aligned} \tag{5.2.37}$$

We have to look for a solution in the subspace $\mathbb{W}^1_{2,\#}(\mathscr{C})$ (see (5.2.11)) because the constants are trivial solutions of the homogeneous BVP (5.2.1) with $f = h = 0$. Since identity (5.2.37) has to be valid for constant $u(t) = 1$ and the left-hand side vanishes for such solution, we get the necessary condition of solvability $(h, 1)_\Gamma - (f, 1)_{\mathscr{C}} = 0$, which is the compatibility condition (5.2.7).

In the left-hand side of equality (5.2.37) we have a symmetric bilinear form, which is positive definite

$$(\mathscr{A}\,\nabla_{\mathscr{C}} v, \nabla_{\mathscr{C}} v)_{\mathscr{C}} \geqslant C\|\nabla_{\mathscr{C}} v\|^2 = \|v|\mathbb{W}^1_{2,\#}(\mathscr{C})\|^2 \quad \forall\, v \in \mathbb{W}^1_{2,\#}(\mathscr{C}),$$

because $\mathscr{A}(s)$ is strictly positive definite matrix and we have applied Lemma 5.2.1 on equivalent norms in the space $\mathbb{W}^12, \#(\mathscr{C})$.

Further, both functionals in the right-hand side of (5.2.37) are bounded on the subspace $\mathbb{W}^1_{2,\#}(\mathscr{C})$. Application of the Lax-Milgram Lemma 1.1.4 accomplishes the proof of the unique solvability of the Neumann BVP (5.2.3) in setting (5.2.6), provided the compatibility condition (5.2.6) holds. □

5.3 Quasi localization of boundary value problems

The exposition of Quasi Localization of BVPs follows from the paper [15]. Similar localization is also applied in [17].

In recent years, there is a substantial interest to investigate the boundary value problems in domains with Lipschitz boundary. Let $\mathscr{C} \subset \mathbb{R}^3$ be a 2-dimensional hypersurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$; $\mathscr{C}$ (cf. Fig. 5.1 on page 130). The boundary $\Gamma := \partial\mathscr{C}$ is decomposed into two closed parts $\Gamma = \Gamma_1 \cup \Gamma_2$, each consisting of finite number of smooth arcs, having in common only endpoints.

Look for a vector-function $u(x) = (u_1(x), u_2(x), u_3(x))^\top$ on $\mathscr{C}$ which solves the mixed boundary value problem

$$\begin{cases} \boldsymbol{A}(\mathscr{D})u = f & \text{in} \quad \mathscr{C}, \\ [\boldsymbol{B}_1(\mathscr{D})u]^+ = g_i & \text{on} \quad \Gamma_1, \\ [\boldsymbol{B}_2(\mathscr{D})u]^+ = g_2 & \text{on} \quad \Gamma_2 \end{cases} \tag{5.3.1}$$

for a second order elliptic operator $\boldsymbol{A}(\mathscr{D})$ with constant scalar or 3×3 matrix coefficients in the domains and 0 or 1-st order normal boundary operators $\boldsymbol{B}_k(\mathscr{D})$ of order $0 \leqslant r_k \leqslant 1$, $k = 1, 2$, with constant scalar or 3×3 matrix coefficients:

$$\boldsymbol{A}(\mathscr{D}) = \sum_{|\alpha| \leqslant 2} a_\alpha \mathscr{D}^\alpha, \quad \boldsymbol{B}_k(\mathscr{D}) = b_0^k + \sum_{j=1}^{3} b_j^k \mathscr{D}_j, \quad k = 1, 2. \tag{5.3.2}$$

Here $\mathscr{D} = (\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3)^\top$ denotes Günter's gradient.

We denote by $t_0, t_1, \dots, t_n \neq \omega_0$ the knots on the boundary Γ where either Γ has an angle (of magnitude $\alpha_j \neq \pi$, measured from the inner domain), or Γ is smooth at t_j (i.e., $\alpha_j = \pi$), but t_j is an endpoint of both Γ_1 and Γ_2 (at such points two different boundary conditions collide, see (5.3.1)). Let $t_0, t_1, \dots, t_{n_0}$ be the knots where the different boundary conditions $\boldsymbol{B}_1(\mathscr{D})$ and $\boldsymbol{B}_2(\mathscr{D})$ do not collide, while at the rest knots $t_{n_0+1}, \dots, t_n$ different boundary conditions collide (in the left and right neighbourhoods of t_j boundary condition is prescribed with different boundary operators $\boldsymbol{B}_1(\mathscr{D})$ and $\boldsymbol{B}_2(\mathscr{D})$).

The BVP (5.3.1) is considered in the non-classical setting

$$u \in \mathbb{H}^s_p(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathscr{C}), \quad g_j \in \mathbb{H}^{s-r_j-1/p}_p(\Gamma_j), \quad 1 < p < \infty, \quad s > \frac{1}{p} \quad j = 1, 2\,. \tag{5.3.3}$$

Our objective is to find a criterion of unique solvability of particular BVPs of type (5.3.1) when, for example, operators in the domains (surfaces) are Laplace–Beltrami, Lamé or Hepmholz operators. In the present section, we will reduce investigation of the BVP (5.3.1) to the investigation of local representatives – model BVPs in model domains of the type described below.

To formulate the model problems, let us introduce the operators

$$\boldsymbol{A}(\nabla) = \sum_{|\alpha| \leqslant 2} a_\alpha \partial^\alpha, \quad \boldsymbol{B}_k(\nabla) = b_0^k + \sum_{j=1}^{3} b_j^k \partial_j, \quad k = 1, 2, \tag{5.3.4}$$

defined now on the Euclidean space $\mathbb{R}^2$ and its subdomains $\mathbb{R}^2_+$ and the model domains Ω_{α_k} (see Fig. 5.3 below). $\nabla = (\partial_1, \partial_2)^\top$ denotes the classical gradient.

I model problem. A local representative of the BVP (5.3.1) at an inner point $t \in \mathscr{C}_j$ is problem in the entire Euclidean space $\mathbb{R}^2$:

$$\boldsymbol{A}(\nabla)u = f \quad \text{in} \quad \mathbb{R}^2 \tag{5.3.5}$$

in the non-classical setting

$$u \in \mathbb{H}^s_p(\mathbb{R}^2), \ \ f \in \mathbb{H}^{s-2}_p(\mathbb{R}^2). \tag{5.3.6}$$

The fundamental solution is the inverse to the model differential equation and the invertibility is granted. In this case we do not need even ellipticity of the operator.

II model problem. A local representative of the BVP (5.3.1) at a boundary point $t \in \Gamma \cap \partial\mathscr{C}_j$, different from knots $t \neq t_0, \dots, t_n$ and where in the neighbourhood the boundary condition is prescribed by the operator $\boldsymbol{B}_\ell(\mathscr{D})$, is a model problem in a half-plane $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}^+$

$$\begin{cases} \boldsymbol{A}(\nabla)u = f & \text{in} \quad \mathbb{R}^2_+, \\ (\boldsymbol{B}_\ell(\nabla)u)^+ = g_\ell & \text{on} \quad \mathbb{R} := \partial\mathbb{R}^2_+ \end{cases} \tag{5.3.7}$$

in the non-classical setting

$$u \in \mathbb{H}^s_p(\mathbb{R}^2_+), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathbb{R}^2_+), \quad g_\ell \in \mathbb{H}^{s-r_k-\frac{1}{p}}_{p,0}(\mathbb{R}^+). \tag{5.3.8}$$

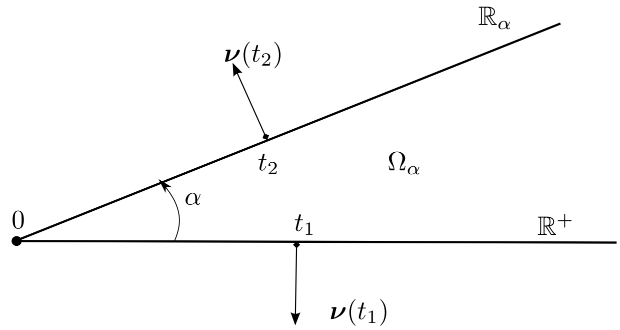

Figure 5.3.

Only the ellipticity of the symbol ensures the unique solvability of BVP (5.3.7), (5.3.8) and we drop the details again.

III model problem. Assume that at a knot $t = t_k \in \Gamma \cap \partial\mathscr{C}_j$, $0 \leqslant k \leqslant n_0$, the boundary condition in the neighbourhood is prescribed by $\boldsymbol{B}_\ell(\mathscr{D})$. A local representative of the BVP (5.3.1) at such vertex is the model problem in an angular domain Ω_{α_k} (cf. Fig. 5.3):

$$\begin{cases} \boldsymbol{A}(\nabla)u = f & \text{in} \quad \Omega_{\alpha_k}, \\ (\boldsymbol{B}_\ell(\nabla)u)^+ = g & \text{on} \quad \Gamma_{\alpha_k} := \partial\Omega_{\alpha_k} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_k} \end{cases} \tag{5.3.9}$$

in the non-classical setting

$$u \in \mathbb{H}^s_p(\Omega_{\alpha_k}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_k}), \quad g \in \mathbb{H}^{s-r_\ell-\frac{1}{p}}_p(\Gamma_{\alpha_k}). \tag{5.3.10}$$

Here Ω_{α_k} is the angle of magnitude α_k between the half-axes $\mathbb{R}^+$ and the beam $\mathbb{R}_{\alpha_k}$ inclined to $\mathbb{R}^+$ by the angle $\alpha = \alpha_k$. Ω_{α_k} is **oriented counterclockwise** (cf. Fig. 5.3):

$$\Gamma_{\alpha_j} := \partial\Omega_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}, \quad \mathbb{R}^+ = [0,\infty), \quad \mathbb{R}_{\alpha_j} := \left\{ e^{i\alpha_j t} = (t\cos\alpha_j, t\sin\alpha_j) : \ \ t \in \mathbb{R}^+ \right\}. \tag{5.3.11}$$

The unit normal vector field on the boundary Γ_α of the model domain and the corresponding normal derivative are given by the following formulae:

$$\begin{gathered} \boldsymbol{\nu}(t) = \begin{cases} (0,-1)^\top & \text{for} \ \ t \in \mathbb{R}^+, \\ (-\sin\alpha_j, \cos\alpha_j) & \text{for} \ \ t \in \mathbb{R}_\alpha, \end{cases} \\ \partial_{\boldsymbol{\nu}}\varphi(t) := \begin{cases} -\partial_{x_2}\varphi(x_1,x_2)\big|_{(x_1,x_2)=(t,0)} & \text{for} \ \ t \in \mathbb{R}^+, \\ (-\sin\alpha\partial_{x_1} + \cos\alpha\partial_{x_2})\varphi(x_1,x_2)\big|_{(x_1,x_2)=(t\sin\alpha, t\cos\alpha)} & \text{for} \ \ t \in \mathbb{R}_\alpha. \end{cases} \end{gathered} \tag{5.3.12}$$

IV model problem. Assume that at a boundary knot $t = t_k \in \Gamma \cap \partial\mathscr{C}_j$, $n_0 + 1 \leqslant k \leqslant n$, the boundary condition in the left neighbourhood is prescribed by $\boldsymbol{B}_1(\mathscr{D})$ and in the right neighbourhood-by $\boldsymbol{B}_2(\mathscr{D})$. A local representative of the BVP (5.3.1) at such vertex is the mixed type model problem in an angular domain Ω_{α_k} (see Fig. 5.3):

$$\begin{cases} \boldsymbol{A}(\nabla)u = f & \text{in} \quad \Omega_{\alpha_k}, \\ (\boldsymbol{B}_1(\nabla)u)^+ = g^1 & \text{on} \quad \mathbb{R}_{\alpha_k}, \\ (\boldsymbol{B}_2(\nabla)u)^+ = g^2 & \text{on} \quad \mathbb{R}^+ \end{cases} \tag{5.3.13}$$

in the non-classical setting

$$u \in \mathbb{H}^s_p(\Omega_{\alpha_k}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_k}), \quad g^1 \in \mathbb{H}^{s-r_1-\frac{1}{p}}_p(\mathbb{R}_{\alpha_k}), \quad g^2 \in \mathbb{H}^{s-r_2-\frac{1}{p}}_p(\mathbb{R}^+), \tag{5.3.14}$$

Remark 5.3.1. Further model case is when the boundary condition in the left neighbourhood is prescribed by $\boldsymbol{B}_2(\mathscr{D})$ and in the right neighbourhood-by $\boldsymbol{B}_1(\mathscr{D})$:

$$\begin{cases} \boldsymbol{A}_j(\nabla)u = f & \text{in} \quad \Omega_{\alpha_k}, \\ (\boldsymbol{B}_2(\nabla)u)^+ = g^2 & \text{on} \quad \mathbb{R}_{\alpha_k}, \\ (\boldsymbol{B}_1(\mathscr{D})u)^+ = g^1 & \text{on} \quad \mathbb{R}^+. \end{cases} \tag{5.3.15}$$

Theorem 5.3.1 (Quasi Localization Principle)**.** *The initial mixed boundary value problem* (5.3.1) *in the non-classical setting* (5.3.3) *is Fredholm if and only if:*

- *III model BVPs* (5.3.9) *in the non-classical setting* (5.3.10) (*or the alternative BVP* (5.3.11)–(5.3.10)) *are Fredholm for all knots* $t_0, \dots, t_{n_0}$*;*
- *IV model BVPs* (5.3.13) *in the non-classical setting* (5.3.14) *are Fredholm for all knots* $t_{n_0+1}, \dots, t_n$.

Proof. The unique solvability (the Fredholmness) of the BVP (5.3.1) can be reformulated as the invertibility (Fredholmness) of the operator between Banach spaces, direct product of Bessel potential spaces:

$$\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D}) \ : \ \mathfrak{B}_1 \to \mathfrak{B}_2, \tag{5.3.16}$$

$$\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D}) := \begin{bmatrix} \boldsymbol{A}(\mathscr{D}) \\ \mathbf{Tr}_{\Gamma_1}\boldsymbol{B}_1(\mathscr{D}) \\ \mathbf{Tr}_{\Gamma_2}\boldsymbol{B}_2(\mathscr{D}) \end{bmatrix}, \quad \mathfrak{B}_1 := \begin{bmatrix} \mathbb{H}^s_p(\mathscr{C}) \\ \mathbb{H}^s_p(\mathscr{C}) \\ \mathbb{H}^s_p(\mathscr{C}) \end{bmatrix}, \quad \mathfrak{B}_2 := \begin{bmatrix} \widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathscr{C}) \\ \mathbb{H}^{s-r_1-1/p}_p(\Gamma_1) \\ \mathbb{H}^{s-r_2-1/p}_p(\Gamma_2) \end{bmatrix},$$

where $\mathbf{Tr}_{\Gamma_{\alpha_k}}$ is the trace operator from the hypersurface $\mathscr{C}$ to the part of the boundaery Γ_{α_k}, $k = 1, 2$.

The unique solvability (the Fredholmness) of the model BVPs (5.3.5)–(5.3.15) can be reformulated as the invertibility (Fredholmness) of the following operators between Banach spaces:

$$\boldsymbol{M}_I(\nabla) \ : \ \mathfrak{B}_1 \to \mathfrak{B}_2, \tag{5.3.17}$$
$$\boldsymbol{M}_I(\nabla) = \boldsymbol{A}(\nabla), \quad \mathfrak{B}_1 := \mathbb{H}^s_p(\mathbb{R}^2), \quad \mathfrak{B}_2 := \mathbb{H}^{s-2}_p(\mathbb{R}^2),$$

for the I Model BVP (5.3.5)–(5.3.6);

$$\boldsymbol{M}_{II}(\nabla) \ : \ \mathfrak{B}_1 \to \mathfrak{B}_2, \tag{5.3.18}$$

$$\boldsymbol{M}_{II}(\nabla) := \begin{bmatrix} \boldsymbol{A}(\nabla) \\ \mathbf{Tr}_{\mathbb{R}}\boldsymbol{B}_\ell(\nabla) \end{bmatrix}, \quad \mathfrak{B}_1 := \begin{bmatrix} \mathbb{H}^s_p(\mathbb{R}^2_+) \\ \mathbb{H}^s_p(\mathbb{R}^2) \end{bmatrix}, \quad \mathfrak{B}_2 := \begin{bmatrix} \widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathbb{R}^2) \\ \mathbb{H}^{s-r_\ell-1/p}_p(\mathbb{R}) \end{bmatrix},$$

for the II Model BVP (5.3.7)–(5.3.8);

$$\boldsymbol{M}_{III}(\nabla) \ : \ \mathfrak{B}_1 \to \mathfrak{B}_2, \tag{5.3.19}$$

$$\boldsymbol{M}_{III}(\nabla) := \begin{bmatrix} \boldsymbol{A}(\nabla) \\ \mathbf{Tr}_{\Gamma_{\alpha_k}}\boldsymbol{B}_\ell(\nabla) \end{bmatrix}, \quad \mathfrak{B}_1 := \begin{bmatrix} \mathbb{H}^s_p(\Omega_{\alpha_k}) \\ \mathbb{H}^s_p(\Omega_{\alpha_k}) \end{bmatrix}, \quad \mathfrak{B}_2 := \begin{bmatrix} \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_k}) \\ \mathbb{H}^{s-r_1-1/p}_p(\Gamma_{\alpha_k}) \end{bmatrix},$$

for the III Model BVP (5.3.9)–(5.3.10);

$$\boldsymbol{M}_{IV}(\nabla) \ : \ \mathfrak{B}_1 \to \mathfrak{B}_2, \tag{5.3.20}$$

$$\boldsymbol{M}_{IV}(\nabla) := \begin{bmatrix} \boldsymbol{A}(\nabla) \\ \mathbf{Tr}_{\mathbb{R}^+}\boldsymbol{B}_1(\nabla) \\ \mathbf{Tr}_{\mathbb{R}_{\alpha_k}}\boldsymbol{B}_2(\nabla) \end{bmatrix}, \quad \mathfrak{B}_1 := \begin{bmatrix} \mathbb{H}^s_p(\Omega_{\alpha_k}) \\ \mathbb{H}^s_p(\Omega_{\alpha_k}) \\ \mathbb{H}^s_p(\Omega_{\alpha_k}) \end{bmatrix}, \quad \mathfrak{B}_2 := \begin{bmatrix} \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_k}) \\ \mathbb{H}^{s-r_1-1/p}_p(\Gamma_{\alpha_k}) \\ \mathbb{H}^{s-r_2-1/p}_p(\Gamma_{\alpha_k}) \end{bmatrix},$$

for the IV Model BVP (5.3.13)–(5.3.14),

For the localizing class Δ_c at a point $c \in \overline{\mathscr{C}} = \mathscr{C} \cup \Gamma$ we take operators of multiplication by smooth functions $v_c I$ which is 1 at some neighbourhood $U_c \subset \overline{\mathscr{C}}$ and has support at some larger neighbourhood. Similar localizing class Δ^0 is chosen for the operators $\boldsymbol{M}_I, \boldsymbol{M}_{II}, \boldsymbol{M}_{III}, \boldsymbol{M}_{IV}$ of the model BVPs, consisting of operators of multiplication by smooth functions $v^0 I$ which is 1 at some neighbourhood U^0 of the respective point $0 \in \mathbb{R}^2$ (for the I model BVP), $0 \in \mathbb{R}^2_+$ (for the II model BVP) and $0 \in \Omega_{\alpha_k}$ (for the III and IV model BVPs), and has support at some larger neighbourhood.

Instead of initial operator $\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D})$ and model operators $\boldsymbol{M}_I(\nabla)$, $\boldsymbol{M}_{II}(\nabla)$, $\boldsymbol{M}_{III}(\nabla)$ and $\boldsymbol{M}_{IV}(\nabla)$ we consider the quotient classes $[\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D})]$, $[\boldsymbol{M}_I(\nabla)]$, $[\boldsymbol{M}_{II}(\nabla)]$, $[\boldsymbol{M}_{III}(\nabla)]$ and $[\boldsymbol{M}_{IV}(\nabla)]$ in the respective quotient spaces $\widehat{\mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)} := \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)/\boldsymbol{T}(\mathfrak{B}_1, \mathfrak{B}_2)$ of linear operators $\mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ with respect to the compact operators $\boldsymbol{T}(\mathfrak{B}_1, \mathfrak{B}_2)$. This approach has two advantages. First, the Fredholmness criteria for operators turns into the invertibility of the corresponding quotient class (see Corollary 1.1.3 on page 11). Second, quotient classes $[v_c I]$ and $[v^0 I]$ of operators from the localizing classes commute with the corresponding quotient classes of operators:

$$[\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D})][v_c I] = [v_c I][\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D})], \quad [\boldsymbol{M}_I(\nabla)][v^0 I] = [v^0 I][\boldsymbol{M}_I(\nabla)], \quad [\boldsymbol{M}_{II}(\nabla)][v^0 I] = [v^0 I][\boldsymbol{M}_{II}(\nabla)],$$
$$[\boldsymbol{M}_{III}(\nabla)][v^0 I] = [v^0 I][\boldsymbol{M}_{III}(\nabla)], \quad [\boldsymbol{M}_{IV}(\nabla)][v^0 I] = [v^0 I][\boldsymbol{M}_{IV}(\nabla)].$$

Next, note that if $\beta_c : U_c \to U^0$ is a diffeomorphism of the neighbourhood U_c of a point $c \in \overline{\mathscr{C}}$ and of the point 0 in the model domain of the corresponding model BVPs I–IV, the corresponding quotient classes are locally quasi equivalent:

$$[\boldsymbol{M}_{\boldsymbol{A}}(\mathscr{D})] \overset{c}{\sim} \beta_c \overset{0=\beta_c(c)}{\sim} [\boldsymbol{M}_0(\nabla)],$$

where $\boldsymbol{M}_0(\nabla)$ is one of the model operators $\boldsymbol{M}_I(\nabla)$, $\boldsymbol{M}_{II}(\nabla)$, $\boldsymbol{M}_{III}(\nabla)$ and $\boldsymbol{M}_{IV}(\nabla)$, chosen depending on the point $c \in \overline{\mathscr{C}}$ according to the algorithm described above.

Now note that the quotient classes of operators $\boldsymbol{M}_I(\nabla)$ in (5.3.17) and $\boldsymbol{M}_{II}(\nabla)$ in (5.3.18) are invertible. Indeed, the operator $\boldsymbol{M}_I(\nabla)$ in (5.3.17) is invertible itself and the inverse is given by the Newton's potential

$$\boldsymbol{N}_{\boldsymbol{M}_I(\nabla)}\varphi(x) := \int\limits_{\mathbb{R}^2} \mathscr{K}_{\boldsymbol{M}_I(\nabla)}(x-y)\varphi(y)\,dy, \quad x \in \mathbb{R}^2,$$

where $\mathscr{K}_{\boldsymbol{M}_I(\nabla)}(x)$ is the fundamental solution to $\boldsymbol{M}_I(\nabla)$.

The operator $\boldsymbol{M}_{II}(\nabla)$ in (5.3.18) is invertible itself if $\boldsymbol{B}_\ell(\nabla) = const$ and is Fredholm (has one dimensional kernel and cokernel) if $\boldsymbol{B}_\ell(\nabla) = a_0 + a_1\partial_1 + a_2\partial_2$. The inverse (the regularizer) is written by analogy of Poisson integrals for the Laplace equation $\Delta u(x) = f(x)$ with the Dirichlet $u^+ = g$ and the Neumann $(-\partial_{x_2} u)^+ = h$ boundary conditions.

There remains to note that the proof follows now from Theorem 4.4.1 on Quasi Localization. □

5.4 Potential operators

It is well known that the Laplace operator Δ has the fundamental solution $\mathscr{K}_\Delta$

$$\mathscr{K}_\Delta(x) := \frac{1}{2\pi}\ln|x|, \quad \Delta\mathscr{K}_\Delta(x) = \delta(x), \quad x \in \mathbb{R}^2,$$

which is used to define the standard double layer $\boldsymbol{W}_\Delta$, the single layer $\boldsymbol{V}_\Delta$ and the Newton $\boldsymbol{N}_\Delta$ potentials on the angle Ω_α:

$$\begin{gathered}\boldsymbol{V}_\Delta\varphi(x) := \frac{1}{2\pi}\int\limits_{\Gamma_\alpha} \ln|x-\tau|\varphi(\tau)\,d\sigma, \quad \boldsymbol{W}_\Delta\varphi(x) := \frac{1}{2\pi}\int\limits_{\Gamma_\alpha} \partial_{\boldsymbol{\nu}(\tau)}\ln|x-\tau|\varphi(\tau)\,d\sigma,\\ \boldsymbol{N}_\Delta\varphi(x) := \frac{1}{2\pi}\int\limits_{\Omega_\alpha} \ln|x-y|\varphi(y)\,dy, \quad x \in \Omega_\alpha.\end{gathered} \tag{5.4.1}$$

For the standard properties of these potentials we refer to [53].

Let us recall the Plemelji formulae

$$\begin{gathered}(\boldsymbol{W}_\Delta\varphi)^\pm(t)=\pm\frac{1}{2}\,\varphi(t)+\boldsymbol{W}_{\Delta,0}\varphi(t),\quad (\partial_{\boldsymbol{\nu}_\Delta(t)}\boldsymbol{V}_\Delta\psi)^\pm)(t)=\mp\frac{1}{2}\,\psi(t)+\boldsymbol{W}^*_{\Delta,0}\psi(t),\\ (\partial_{\boldsymbol{\nu}_\Delta(t)}\boldsymbol{W}_\Delta\psi)^\pm(t)=\boldsymbol{V}_{\Delta,+1}\psi(t),\quad (\boldsymbol{V}_\Delta\varphi)^\pm(t)=\boldsymbol{V}_{\Delta,-1}\varphi(t),\ \ t\in\Gamma_\alpha:=\partial\Omega_\alpha,\end{gathered}\tag{5.4.2}$$

where the pseudodifferential operators (ΨDO)

$$\begin{gathered}\boldsymbol{V}_{\Delta,-1}\varphi(t):=\frac{1}{2\pi}\int\limits_{\Gamma_\alpha}\ln|t-\tau|\varphi(\tau)\,d\sigma,\quad \boldsymbol{V}_{\Delta,+1}\varphi(t:=\frac{1}{2\pi}\int\limits_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(t)}\partial_{\boldsymbol{\nu}(\tau)}\ln|t-\tau|\varphi(\tau)\,d\sigma,\\ \boldsymbol{W}_{\Delta,0}\varphi(t):=\frac{1}{2\pi}\int\limits_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(\tau)}\ln|t-\tau|\varphi(\tau)\,d\sigma,\quad \boldsymbol{W}^*_{\Delta,0}\varphi(t):=\frac{1}{2\pi}\int\limits_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(t)}\ln|t-\tau|\varphi(\tau)\,d\sigma,\\ t\in\Gamma_\alpha,\end{gathered}\tag{5.4.3}$$

of orders -1, 0, 0 and $+1$ are associated with the layer potentials of the Helmholtz equation. The operator $\boldsymbol{V}_{\Delta,-1}$ has weakly singular kernel and the integral exists in the Lebesgue sense, while the operators $\boldsymbol{W}_{\Delta,0}$ and $\boldsymbol{W}^*_{\Delta,0}$ have singular kernel of order -1 and the integrals exist in the Cauchy Mean Value sense. $\boldsymbol{V}_{\Delta,+1}$ is a hypersingular integral operator and it is interpreted in [68, § 1]. The standard mapping property is listed below (see [53, 65, 89] for details):

$$\begin{aligned}\boldsymbol{V}_{\Delta,-1}&:\mathbb{H}^s_p(\Gamma_\alpha)\to\mathbb{H}^{s+1}_p(\Gamma_\alpha),\\ \boldsymbol{W}_{\Delta,0}&:\mathbb{H}^s_p(\Gamma_\alpha)\to\mathbb{H}^s_p(\Gamma_\alpha),\\ \boldsymbol{W}^*_{\Delta,0}&:\mathbb{H}^s_p(\Gamma_\alpha)\to\mathbb{H}^s_p(\Gamma_\alpha),\\ \boldsymbol{V}_{\Delta,+1}&:\mathbb{H}^s_p(\Gamma_\alpha)\to\mathbb{H}^{s-1}_p(\Gamma_\alpha),\ \ s\in\mathbb{R},\ \ 1<p<\infty.\end{aligned}\tag{5.4.4}$$

Next, we need to find explicit forms of pseudodifferential operators (PsDOs) $W_{\Delta,0}$ and $W^*_{\Delta,0}$ for the use in Chapters 4–5.

Let us consider the following Mellin convolutions operators, where the first one is known as the Cauchy singular integral operator (see [37, 47, 59]):

$$\mathbf{S}_{\mathbb{R}^+}\phi(t):=\frac{1}{\pi i}\int\limits_0^\infty\frac{\phi(\tau)\,d\tau}{\tau-t},\quad \mathbf{K}_c\phi(t):=\frac{1}{\pi}\int\limits_0^\infty\frac{\phi(\tau)\,d\tau}{t-c\,\tau},\ \ 0<\arg c<2\pi,\ \ \phi\in\mathbb{L}_p(\mathbb{R}^+).\tag{5.4.5}$$

The pull back operator $\boldsymbol{J}_\alpha:\mathbb{H}^s_p(\mathbb{R}_\alpha)\to\mathbb{H}^s_p(\mathbb{R}^+)$ and its inverse $\boldsymbol{J}^{-1}:\mathbb{H}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}_\alpha)$ are defined as follows:

$$\begin{gathered}\boldsymbol{J}_\alpha\varphi(t)=\varphi(t\,\cos\,\alpha,t\,\sin\,\alpha),\ \ t\in\mathbb{R}^+,\\ \boldsymbol{J}^{-1}_\alpha\psi(x_1,x_2)=\psi\Big(\sqrt{x_1^2+x_2^2}\Big),\ \ (x_1,x_2)^\top\in\mathbb{R}_\alpha.\end{gathered}\tag{5.4.6}$$

Note that the tangent vector $\ell(x)$ to the boundary of the model domain Γ_α and the corresponding tangent derivative are given by the formulae (we remind that $\mathbb{R}_\alpha$ is oriented from ∞ to 0):

$$\begin{gathered}\ell(t)=\begin{cases}(1,0)^\top & \text{for}\ \ x\in\mathbb{R}^+,\\ -(\cos\,\alpha,\sin\,\alpha) & \text{for}\ \ x\in\mathbb{R}_\alpha,\end{cases}\\ \partial_\ell\varphi(x):=\begin{cases}\partial_t\varphi(t,0) & \text{for}\ \ x=(t,0)\in\mathbb{R}^+,\\ -(\cos\,\alpha\,\partial_{x_1}+\sin\,\alpha\,\partial_{x_2})\varphi(t\cos\,\alpha\,,t\,\sin\,\alpha) & \text{for}\ \ x=(t\cos\,\alpha\,,t\,\sin\,\alpha)^\top\in\mathbb{R}_\alpha.\end{cases}\end{gathered}\tag{5.4.7}$$

Theorem 5.4.1 (cf. [61, 69]). *For the singular integral operator on the boundary Γ_α of the model domain $\boldsymbol{W}_{\Delta,0}$ and its dual (conjugate) $\boldsymbol{W}^*_{\Delta,0}$ the following explicit representations hold:*

$$r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\boldsymbol{J}^{-1}_\alpha\varphi(t)=-\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi(t)=-\frac{1}{4i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi(t),\tag{5.4.8a}$$

$$r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}_\alpha}\boldsymbol{J}^{-1}_\alpha\varphi(t)=-\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}^+}\varphi(t)=\frac{1}{4i}\left[\boldsymbol{K}_{e^{i\alpha}}-\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi(t),\ \ t\in\mathbb{R}^+,\tag{5.4.8b}$$

$$r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi=r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi=r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}^+}\varphi=r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi=0,\tag{5.4.8c}$$

where $r_{\mathbb{R}^+}$ and $r_{\mathbb{R}_\alpha}$ are the restriction operators to the spaces on the corresponding subsets $\mathbb{R}^+$ and $\mathbb{R}_\alpha$.

For the pseudodifferential operators $\boldsymbol{V}_{\Delta,-1}$ and $\boldsymbol{V}_{\Delta,+1}$ the following explicit representations hold:

$$r_{\mathbb{R}^+}\partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}_\alpha}\boldsymbol{J}_\alpha^{-1}\varphi(t) = \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}^+}\varphi(t) = -\frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\varphi(t), \tag{5.4.9a}$$

$$r_{\mathbb{R}^+}\boldsymbol{V}_{\Delta,+1} r_{\mathbb{R}_\alpha}\boldsymbol{J}_\alpha^{-1}\varphi(t) = \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\partial_\ell \boldsymbol{V}_{\Delta,+1} r_{\mathbb{R}^+}\partial_\tau\varphi(t) = \frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\partial_\tau\varphi(t), \tag{5.4.9b}$$

$$r_{\mathbb{R}^+}\partial_t \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}^+}\varphi(t)\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha} = \partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}_\alpha}\boldsymbol{J}_\alpha^{-1}\varphi(t) = \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\varphi(t), \tag{5.4.9c}$$

$$r_{\mathbb{R}^+}\boldsymbol{V}_{\Delta,+1} r_{\mathbb{R}^+}\varphi(t) = \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{V}_{\Delta,+1} r_{\mathbb{R}_\alpha}\boldsymbol{J}_\alpha^{-1}\varphi(t) = -\varphi(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi(t), \quad t \in \mathbb{R}^+, \tag{5.4.9d}$$

where the operator ∂_ℓ is defined from (5.4.7).

Proof. Using the parametrizations $x = (x_1, x_2)^\top = (t, 0)^\top$ of $\mathbb{R}^+$ and $y = (y_1, y_2)^\top = (\tau\cos\alpha, \tau\sin\alpha)^\top$ of $\mathbb{R}_\alpha$ (cf. (5.3.11)), recalling the form of the normal derivative $\partial_{\boldsymbol{\nu}}(y)$ on $\Gamma_\alpha = \mathbb{R}^+ \cup \mathbb{R}_\alpha$ (cf. (5.3.12)) and taking into account that $\mathbb{R}_\alpha$ is oriented from ∞ to 0, we get

$$\begin{aligned} r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}\varphi(x) &= \frac{1}{2\pi}\int_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(y)}\ln|x-y|\varphi(y)d\sigma = -\frac{1}{2\pi}\int_0^\infty \Big[\partial_{y_2}\ln|(t,0)-(y_1,y_2)|\big|_{(y_1,y_2)=(\tau,0)}\varphi(\tau,0) \\ &+ [-\sin\alpha\,\partial_{y_1} + \cos\alpha\,\partial_{y_2}]\ln|(t,0)-(y_1,y_2)|\varphi(y_1,y_2)\big|_{(y_1,y_2)=(\tau\cos\alpha,\tau\sin\alpha)}\Big]d\tau. \end{aligned}$$

Here we apply the equality

$$\ln|x-y| = \ln|(x_1,x_2)-(y_1,y_2)=|\frac{1}{2}\ln\left[(x_1-y_1)^2+(x_2-y_2)^2\right] \tag{5.4.10}$$

and continue as follows:

$$\begin{aligned} r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}\varphi(x) &= -\frac{1}{2\pi}\int_0^\infty \Big[\frac{y_2\varphi_+(\tau)}{(t-y_1)^2+y_2^2}\Big|_{(y_1,y_2)=(\tau,0)} \\ &\quad + \frac{[(t-y_1)\sin\alpha + y_2\cos\alpha]\varphi_\alpha(\tau)}{(t-y_1)^2+y_2^2}\Big|_{(y_1,y_2)=(\tau\cos\alpha,\tau\sin\alpha)}\Big]d\tau \\ &= -\frac{1}{2\pi}\int_0^\infty \frac{t\sin\alpha\,\varphi_\alpha(\tau)\,d\tau}{(t-\tau\cos\alpha)^2+\tau^2\sin^2\alpha} = -\frac{1}{2\pi}\int_0^\infty \frac{t\sin\alpha\varphi_\alpha(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha} \\ &= -\frac{1}{4i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e^{-i\alpha}}\right]\varphi_\alpha(t) = -\frac{1}{4i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi_\alpha(t), \\ &\varphi_+(\tau) := \varphi(\tau,0), \quad \varphi_\alpha(\tau) := \varphi(\tau\cos\alpha,\tau\sin\alpha) = \boldsymbol{J}_\alpha\varphi(t), \quad \tau\in\mathbb{R}^+. \end{aligned} \tag{5.4.11}$$

The obtained equality proves the first equalities in (5.4.8a) and (5.4.8c), because the integral on $\mathbb{R}^+$ in the third line of (5.4.11) turned to 0. Similarly,

$$\begin{aligned} r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0}\varphi(x) &= \frac{1}{2\pi}\int_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(x)}\ln|x-y|\varphi(y)d\sigma = -\frac{1}{2\pi}\int_0^\infty \Big[\partial_{x_2}\ln|(x_1,x_2)-(\tau,0)|\big|_{(x_1,x_2)=(t,0)}\varphi(\tau,0) \\ &+ \partial_{x_2}\ln|(x_1,x_2)-(\tau\cos\alpha,\tau\sin\alpha)|\big|_{(x_1,x_2)=(t,0)}\varphi(\tau\cos\alpha,\tau\sin\alpha)\Big]d\tau \\ = -\frac{1}{2\pi}\int_0^\infty &\Big[\frac{x_2\varphi(\tau,0)}{(x_1-\tau)^2+x_2^2}\Big|_{(x_1,x_2)=(t,0)} + \frac{(x_2-\tau\sin\alpha)\varphi_\alpha(\tau)}{(x_1-\tau\cos\alpha)^2+(x_2-\tau\sin\alpha)^2}\Big|_{(x_1,x_2)=(t,0)}\Big]d\tau \\ &= \frac{1}{2\pi}\int_0^\infty \frac{\tau\sin\alpha\,\varphi_\alpha(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha} = \frac{1}{4i}\left[\boldsymbol{K}_{e^{i\alpha}} - \boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi_\alpha(t). \end{aligned} \tag{5.4.12}$$

The obtained equality proves the first equality in (5.4.8b) and the third equality in (5.4.8c) (because the integral on $\mathbb{R}^+$ in the third line of (5.4.12) turned to 0).

If $x=(x_1,x_2)^\top=(t\cos\alpha, t\sin\alpha)^\top\in\mathbb{R}_\alpha$ and $y=(y_1,y_2)^\top=(\tau,0)^\top\in\mathbb{R}^+$ (cf. (5.3.11)), as in the foregoing case, we get the following:

$$\begin{aligned}
r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}\varphi(x)&=\frac{1}{2\pi}\int\limits_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(y)}\ln|x-y|\varphi(y)d\sigma\\
&=-\frac{1}{2\pi}\int\limits_0^\infty\Big[\partial_{y_2}\ln|(t\cos\alpha,t\sin\alpha)-(y_1,y_2)|\big|_{(y_1,y_2)=(\tau,0)}\varphi_+(\tau)\\
&\quad+[-\sin\alpha\partial_{y_1}+\cos\alpha\partial_{y_2}]\ln|(t\cos\alpha,t\sin\alpha)-(y_1,y_2)|\big|_{(y_1,y_2)=(\tau\cos\alpha,\tau\sin\alpha)}\varphi_\alpha(\tau)\Big]d\tau\\
&=-\frac{1}{2\pi}\int\limits_0^\infty\Big[\frac{-(t\sin\alpha-y_2)\varphi_+(\tau)}{(t\cos\alpha-y_1)^2+(t\sin\alpha-y_2)^2}\Big|_{(y_1,y_2)=(\tau,0)}\\
&\quad+\frac{[\sin\alpha(t\cos\alpha-y_1)-\cos\alpha(t\sin\alpha-y_2)]\varphi_\alpha(\tau)}{(t\cos\alpha-y_1)^2+(t\sin\alpha-y_2)^2}\Big|_{(y_1,y_2)=(\tau\cos\alpha,\tau\sin\alpha)}\Big]d\tau\\
&=\frac{1}{2\pi}\int\limits_0^\infty\frac{t\sin\alpha\varphi_+(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha}=\frac{1}{4i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi_+(t),
\end{aligned}\tag{5.4.13}$$

The obtained equality proves the second equalities in (5.4.8a) and (5.4.8c), because the integral on $\mathbb{R}_\alpha$ in the fourth line of (5.4.13) turned to 0. Similarly,

$$\begin{aligned}
r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0}\varphi(x)&=\frac{1}{2\pi}\int\limits_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(x)}\ln|x-y|\varphi(y)d\sigma\\
&=-\frac{1}{2\pi}\int\limits_0^\infty\Big[[-\sin\alpha\partial_{x_1}+\cos\alpha\partial_{x_2}]\ln|(x_1,x_2)-(\tau,0)|\big|_{(x_1,x_2)=(t\cos\alpha,t\sin\alpha)}\varphi_+(\tau)\\
&\quad+[-\sin\alpha\partial_{x_1}+\cos\alpha\partial_{x_2}]\ln|(x_1,x_2)-(\tau\cos\alpha,\tau\sin\alpha)|\big|_{(x_1,x_2)=(t\cos\alpha,t\sin\alpha)}\varphi_\alpha(\tau)\Big]d\tau\\
&=-\frac{1}{2\pi}\int\limits_0^\infty\Big[\frac{\{-\sin\alpha(x_1-\tau)+\cos\alpha\, x_2\}\varphi_+(\tau)}{(x_1-\tau)^2+x_2^2}\Big|_{(x_1,x_2)=(t\cos\alpha,t\sin\alpha)}\\
&\quad+\frac{[-\sin\alpha(x_1-\tau\cos\alpha)+\cos\alpha(x_2-\tau\sin\alpha)]\varphi_\alpha(\tau)}{(x_1-\tau\cos\alpha)^2+(x_2-t\sin\alpha)^2}\Big|_{(x_1,x_2)=(t\cos\alpha,t\sin\alpha)}\Big]d\tau\\
&=-\frac{1}{2\pi}\int\limits_0^\infty\frac{\tau\sin\alpha\varphi_+(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha}=-\frac{1}{4i}\left[\boldsymbol{K}_{e^{i\alpha}}-\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi_+(t).
\end{aligned}\tag{5.4.14}$$

The obtained equality proves the second equality in (5.4.8b) and the fourth equality in (5.4.8c), because the integral on $\mathbb{R}_+$ in the fifth line of (5.4.14) turned to 0.

Prior to calculating the operator $\boldsymbol{V}_{\Delta,+1}$ from (5.4.3), consider its kernel $\frac{1}{2\pi}\partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\ln|x-y|$.

Using equalities (5.4.10), (5.4.7) and, as above, the parametrizations of $\mathbb{R}_\alpha$ and $\mathbb{R}^+$ we calculate the kernel for $x=(t,0)^\top\in\mathbb{R}^+$, $y=(\tau\cos\alpha\,\tau\sin\alpha)^\top\in\mathbb{R}_\alpha$, as follows:

$$\begin{aligned}
\partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\mathscr{K}_\Delta(x-y) &= -\partial_{x_2}(-\sin\alpha\,\partial_{y_1}+\cos\alpha\,\partial_{y_2})\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \partial_{y_2}\{-\sin\alpha\,\partial_{y_1}+\cos\alpha\,\partial_{y_2}\}\,\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= [\cos\alpha\Delta\mathscr{K}_\Delta(x-y)-\partial_{y_1}\{\cos\alpha\,\partial_{y_1}+\sin\alpha\,\partial_{y_2}\}\,\mathscr{K}_\Delta(x-y)]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\delta(x-y)+\partial_{y_1}\partial_{\ell(y)}\mathscr{K}_\Delta(x-y)\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\,\delta(0)+\frac{1}{4\pi}\partial_{\ell(y)}\partial_{y_1}\ln\left[(x_1-y_1)^2+(x_2-y_2)^2\right]\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\,\delta(0)-\frac{1}{2\pi}\partial_{\ell(y)}\frac{x_1-y_1}{(x_1-y_1)^2+(x_2-y_2)^2}\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}},
\end{aligned}\tag{5.4.15}$$

since, $\delta(x-y)=0$ for $x\in\mathbb{R}^+$ and $y\in\mathbb{R}^\alpha$ with Dirac's delta function $\delta(x)$.

In the case $x=(t\cos\alpha,\,t\sin\alpha)^\top\in\mathbb{R}_\alpha$, $y=(\tau,0)^\top\in\mathbb{R}^+$ we calculate similarly:

$$\begin{aligned}
\partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\mathscr{K}_\Delta(x-y) &= -(-\sin\alpha\,\partial_{x_1}+\cos\alpha\,\partial_{x_2})\partial_{y_2}\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau,0)}}\\
&= \{-\sin\alpha\,\partial_{y_1}+\cos\alpha\,\partial_{y_2}\}\,\partial_{y_2}\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau,0)}}\\
&= [\cos\alpha\Delta\mathscr{K}_\Delta(x-y)-\partial_{y_1}\{\cos\alpha\,\partial_{y_1}+\sin\alpha\,\partial_{y_2}\}\,\mathscr{K}_\Delta(x-y)]\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau,0)}}\\
&= \left[\cos\alpha\delta(x-y)+\partial_{\ell(y)}\{\cos\alpha\,\partial_{y_1}+\sin\alpha\,\partial_{y_2}\}\,\mathscr{K}_\Delta(x-y)\right]\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau,0)}}\\
&= \left[\cos\alpha\,\delta(0)-\frac{1}{2\pi}\partial_{\ell(y)}\frac{\cos\alpha(x_1-y_1)+\sin\alpha(x_2-y_2)}{(x_1-y_1)^2+(x_2-y_2)^2}\right]\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau,0)}}.
\end{aligned}\tag{5.4.16}$$

For the case $x=(t,0)^\top\in\mathbb{R}^+$, $y=(\tau,0)^\top\in\mathbb{R}^+$ we get

$$\begin{aligned}
\partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\mathscr{K}_\Delta(x-y) = \partial_{x_2}\partial_{y_2}\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau,0)}} &= -\partial_{y_2}^2\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau,0)}}\\
&= \left[-\Delta\mathscr{K}_\Delta(x-y)+\partial_{y_1}^2\mathscr{K}_\Delta(x-y)\right]\Big|_{\substack{x=(t,0)\\ y=(\tau,0)}}\\
&= -\delta(t-\tau)+\partial_\tau^2\mathscr{K}_\Delta(t-\tau) = -\delta(t-\tau)+\frac{1}{2\pi}\partial_\tau\frac{1}{\tau-t}.
\end{aligned}\tag{5.4.17}$$

For the case $x=(t\cos\alpha\,t,\,\sin\alpha)^\top\in\mathbb{R}_\alpha$, $y=(\tau\cos\alpha,\,\tau\sin\alpha)^\top\in\mathbb{R}_\alpha$ we get

$$\begin{aligned}
\partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\mathscr{K}_\Delta(x-y) &= (-\sin\alpha\,\partial_{x_1}+\cos\alpha\,\partial_{x_2})(-\sin\alpha\,\partial_{y_1}+\cos\alpha\,\partial_{y_2})\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= -\{-\sin\alpha\,\partial_{y_1}+\cos\alpha\,\partial_{y_2}\}^2\,\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= -\left[\Delta-(\cos\alpha\,\partial_{y_1}+\sin\alpha\,\partial_{y_2})^2\right]\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= -\left[\delta(x-y)-\partial_{\ell(y)}^2\mathscr{K}_\Delta(x-y)\right]_{\substack{x=(t\cos\alpha,t\sin\alpha)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}
\end{aligned}$$

$$= -\left[\delta(x-y) + \frac{1}{2\pi}\partial_{\ell(y)}\frac{-\cos\,\alpha(x_1-y_1) - \sin\,\alpha(x_2-y_2)}{(x_1-y_1)^2+(x_2-y_2)^2}\right]\Bigg|_{\substack{x=(t\cos\,\alpha,t\sin\,\alpha)\\ y=(\tau\cos\,\alpha,\tau\sin\,\alpha)}}$$

$$= -\delta(t-\tau) + \frac{1}{2\pi}\partial_\tau\frac{1}{\tau-t}. \tag{5.4.18}$$

Now we calculate the operator $r_{\mathbb{R}^+}\boldsymbol{V}_{\Delta,+1}$ from (5.4.3), using the derived representations of the kernel (5.4.15), (5.4.17) and integration by parts:

$$\begin{aligned}
r_{\mathbb{R}^+}\boldsymbol{V}_{\Delta,+1}\varphi(t) &= -\varphi_+(t) - \frac{1}{2\pi}\int\limits_{\mathbb{R}^+}\frac{\partial_\tau\varphi_+(\tau)d\tau}{\tau-t} + \frac{1}{2\pi}r_{\mathbb{R}^+}\int\limits_{\mathbb{R}_\alpha}\frac{(x_1-y_1)\partial_{\ell(y)}\varphi(y)d\sigma}{(x_1-y_1)^2+(x_2-y_2)^2}\Bigg|_{\substack{x=(t,0)\\ y=(\tau\cos\,\alpha,\tau\sin\,\alpha)}}\\
&= -\varphi_+(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi_+(t) - \frac{1}{2\pi}\int\limits_0^\infty\frac{t-\tau\cos\,\alpha}{t^2+\tau^2-2t\tau\,\cos\alpha}(\boldsymbol{J}_\alpha\partial_\ell\varphi)(\tau)d\tau\\
&= -\varphi_+(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi_+(t) - \frac{1}{4\pi}\int\limits_0^\infty\left[\frac{1}{t-e^{i\alpha}\tau} + \frac{1}{t-e^{-i\alpha}\tau}\right](\boldsymbol{J}_\alpha\partial_\ell\varphi)(\tau)d\tau,\\
&= -\varphi_+(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi_+(t) + \frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\partial_\tau\varphi_\alpha(t),
\end{aligned}$$

since $(\boldsymbol{J}_\alpha\partial_\ell\varphi)(\tau) = -(\partial_\tau\varphi_\alpha)(\tau)$, where, we remind, $\varphi_+(t) := \varphi(t,0)$, $\varphi_\alpha(t) := \boldsymbol{J}_\alpha\varphi(t)$. Thus, the first formula in (5.4.9b) and the first formula in (5.4.9d) are proved.

Next, we calculate the operator $\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{V}_{\Delta,+1}$ using the derived representations of the kernel (5.4.16), (5.4.18) and integration by parts:

$$\begin{aligned}
\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{V}_{\Delta,+1}\varphi(t) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}^+}\frac{[\cos\,\alpha(x_1-y_1)+\sin\,\alpha(x_2-y_2)]\partial_{\ell(y)}\varphi(y)d\sigma}{(x_1-y_1)^2+(x_2-y_2)^2}\Bigg|_{\substack{x=(t\cos\,\alpha,t\sin\,\alpha)\\ y=(\tau,0))}}\\
&\quad - \varphi_\alpha(t) - \frac{1}{2\pi}\int\limits_{\mathbb{R}^+}\frac{\partial_\tau\varphi_\alpha(\tau)d\tau}{\tau-t} = -\varphi_\alpha(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi_\alpha(t)\\
&\quad + \frac{1}{2\pi}\int\limits_0^\infty\frac{[\cos\,\alpha(t\cos\,\alpha-\tau)+t\sin^2\,\alpha]\partial_\tau\varphi_+(\tau)d\tau}{t^2+\tau^2-2t\tau\,\cos\alpha}\\
&= \frac{1}{2\pi}\int\limits_0^\infty\frac{t-\tau\cos\,\alpha)\partial_\tau\varphi_+(\tau)d\tau}{t^2+\tau^2-2t\tau\,\cos\alpha} - \varphi_\alpha(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi_\alpha(t)\\
&= \frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\partial_\tau\varphi_+(t) - \varphi_\alpha(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\partial_\tau\varphi_\alpha(t).
\end{aligned}$$

Thus, the second formula in (5.4.9b) and the second formula in (5.4.9d) are proved.

Now we look at the singular integral operator $r_{\mathbb{R}^+}\partial_\ell\boldsymbol{V}_{\Delta,-1}$:

$$\begin{aligned}
r_{\mathbb{R}^+}\partial_\ell\boldsymbol{V}_{\Delta,-1}\varphi(t) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}^+}\partial_{\ell(x)}\ln|x-y|\,\varphi(y)d\sigma\Bigg|_{\substack{x=(t,0))\\ y=(\tau,0)}}\\
&\quad + \frac{1}{2\pi}\int\limits_{\mathbb{R}_\alpha}\partial_{\ell(x)}\ln|x-y|\,\varphi(y)d\sigma\Bigg|_{\substack{x=(t,0)\\ y=(\tau\cos\,\alpha,\tau\sin\,\alpha)}}\\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}^+}\frac{(t-\tau)\varphi_+(\tau)\,d\tau}{(t-\tau)^2} - \frac{1}{2\pi}\int\limits_0^\infty\frac{(t-\tau\cos\,\alpha)\varphi_\alpha(\tau)\,d\tau}{(t-\tau\cos\,\alpha)^2+\tau^2\sin^2\alpha}\\
&= -\frac{1}{2\pi}\int\limits_0^\infty\frac{\varphi_+(\tau)\,d\tau}{\tau-t} - \frac{1}{4\pi}\int\limits_0^\infty\left[\frac{e^{i\alpha}}{t-e^{i\alpha}\tau} + \frac{e^{-i\alpha}}{t-e^{-i\alpha}\tau}\right]\varphi_\alpha(\tau)d\tau
\end{aligned}$$

$$= \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\varphi_+(t) - \frac{1}{4}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}} + e^{-i\alpha}\boldsymbol{K}_{e^{-i\alpha}}\right]\varphi_\alpha(t), \quad t \in \mathbb{R}^+, \quad t \in \mathbb{R}^+.$$

Thus, the first formula in (5.4.9a) and the first formula in (5.4.9c) are proved.

In conclusion, we look at the singular integral operator $\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\partial_\ell \boldsymbol{V}_{\Delta,-1}$:

$$\begin{aligned}
\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\partial_\ell \boldsymbol{V}_{\Delta,-1}\varphi(t) &= \frac{1}{2\pi}\int_{\mathbb{R}^+}\partial_{\ell(x)}\ln|x-y|\,\varphi(y)d\sigma\Big|_{\substack{x=(t\cos\,\alpha,t\sin\,\alpha))\\ y=(\tau,0)}} \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}_\alpha}\partial_{\ell(x)}\ln|x-y|\,\varphi(y)d\sigma\Big|_{\substack{x=(t\cos\,\alpha,t\sin\,\alpha))\\ y=(\tau\cos\,\alpha,\tau\sin\,\alpha))}} \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}^+}\frac{\cos\,\alpha(x_1-\tau)+x_2\sin\,\alpha}{(x_1-\tau)^2+x_2^2}\varphi_+(\tau)\,d\tau\Big|_{x=(t\cos\,\alpha,t\sin\,\alpha)} \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}^+}\frac{\cos\,\alpha(x_1-\tau\cos\,\alpha)+(x_2-\tau\sin\,\alpha)\sin\,\alpha}{(x_1-\tau\cos\,\alpha)^2+(x_2-\tau\sin\,\alpha)^2}\varphi_\alpha(\tau)\,d\tau\Big|_{x=(t\cos\,\alpha,t\sin\,\alpha)} \\
&= -\frac{1}{2\pi}\int_0^\infty\frac{\cos\,\alpha(t\cos\,\alpha-\tau)+t\,\sin^2\alpha}{(t\,\cos\alpha-\tau)^2+t\sin^2\alpha}\varphi_+(\tau)d\tau - \frac{1}{2\pi}\int_{\mathbb{R}^+}\frac{\varphi_\alpha(\tau)\,d\tau}{\tau-t} \\
&= -\frac{1}{2\pi}\int_0^\infty\frac{t-\tau\,\cos\alpha}{t^2+\tau^2-2t\tau\,\cos\alpha}\varphi_\alpha(\tau)d\tau + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\varphi_\alpha(t) \\
&= -\frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\varphi_+(t) + \frac{1}{2i}\mathbf{S}_{\mathbb{R}^+}\varphi_\alpha(t), \quad t \in \mathbb{R}^+.
\end{aligned}$$

Thus, the second formula in (5.4.9a) and the second formula in (5.4.9c) are proved. □

5.5 Mellin convolution equations in Bessel potential spaces

Let us recall from [37] the results on the Fredholm properties of operators

$$\mathbf{A} := d_0 I + \sum_{j=1}^n d_j \mathbf{K}^1_{c_j} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+), \tag{5.5.1}$$

where $\mathbf{K}^1_{c_1}, \dots, \mathbf{K}^1_{c_n}$ are admissible Mellin convolution operators and $d_0, \dots, d_n$ are $m \times m$ constant matrix coefficients. $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and $\mathbb{H}^s_p(\mathbb{R}^+)$ are the spaces of m-vector functions.

To this end, consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Fig. 4.1 on page 116)

$$\Gamma_1 := \{+\infty\} \times \overline{\mathbb{R}}, \;\; \Gamma_2^\pm := \overline{\mathbb{R}}^+ \times \{\pm\infty\}, \;\; \Gamma_3 := \{0\} \times \overline{\mathbb{R}}.$$

According to [37, formulae (52)–(53d)], the symbol $\mathscr{A}^s_p(\omega)$ of the operator $\mathbf{A}$ is

$$\mathscr{A}^s_p(\omega) := d_0 \mathscr{I}^s_p(\omega) + \sum_{j=1}^n d_j \mathscr{K}^{1,s}_{c_j,p}(\omega), \tag{5.5.2}$$

where

$$\mathscr{I}^s_p(\omega) := \begin{cases} g^s_{-\gamma,\gamma,p}(\infty,\xi), & \omega = (\infty,\xi) \in \overline{\Gamma}_1, \\ \left(\dfrac{\pm\eta-\gamma}{\pm\eta+\gamma}\right)^s, & \omega = (\eta,\pm\infty) \in \Gamma_2^\pm, \\ e^{\pi s i}, & \omega = (0,\xi) \in \overline{\Gamma}_3, \;\; \xi,\eta \in \mathbb{R}, \end{cases} \tag{5.5.3a}$$

$$g^s_{-\gamma,\gamma,p}(\infty,\xi):=\frac{e^{2\pi si}+1}{2}+\frac{e^{2\pi si}-1}{2i}\,\cot\pi\Big(\frac{1}{p}-i\xi\Big)=e^{\pi si}\,\frac{\sin\pi(\frac{1}{p}+s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\,,\ \ \xi\in\mathbb{R}, \tag{5.5.3b}$$

$$\mathscr{K}^{1,s}_{c,p}(\omega):=\begin{cases}-\dfrac{e^{-i\pi(\frac{1}{p}-i\xi)}c^{\frac{1}{p}-i\xi-s-1}}{\sin\pi(\frac{1}{p}-i\xi)}\,, & \omega=(\infty,\xi)\in\overline{\Gamma}_1,\\ 0, & \omega=\eta,\pm\infty)\in\Gamma_2^{\pm},\\ -\dfrac{e^{-i\pi(\frac{1}{p}+s-i\xi)}c^{\frac{1}{p}-i\xi-s-1}}{\sin\pi(\frac{1}{p}-i\xi)}, & \omega=(0,\xi)\in\overline{\Gamma}_3,\end{cases} \tag{5.5.3c}$$

$$0<\arg c<2\pi,\ \ c^{-s}=|c|^{-s}e^{i(2\pi-\arg c)s},\ \ c^{\gamma}=|c|^{\gamma}e^{i\gamma\arg c}.$$

The function $\det\mathscr{A}^s_p(\omega)$ is continuous on the rectangle $\mathfrak{R}$. The statement is easy to verify, analyzing the symbols in (5.5.2), (5.5.3a)–(5.5.3b) and taking into account that

$$\mathscr{I}^s_p(-\infty,-\infty)=1,\quad \mathscr{I}^s_p(0,-\infty)=\mathscr{I}^s_p(0,+\infty)=e^{\pi si},\quad \mathscr{I}^s_p(+\infty,+\infty)=e^{2\pi si},$$
$$\mathscr{K}^{1,s}_{-1,p}(-\infty,-\infty)=\mathscr{K}^{1,s}_{-1,p}(0,-\infty)=\mathscr{K}^{1,s}_{-1,p}(0,+\infty)=\mathscr{K}^{1,s}_{-1,p}(+\infty,+\infty)=0,$$
$$g^s_{-\gamma,\gamma,p}(\infty,-\infty)=1,\quad g^s_{-\gamma,\gamma,p}(\infty,+\infty)=e^{2\pi si}.$$

Therefore, the image of the function $\det\mathscr{A}^s_p(\omega)$ is a closed curve in the complex plane and, if the symbol is elliptic

$$\inf_{\omega\in\mathfrak{R}}\big|\det\mathscr{A}^s_p(\omega)\big|>0,$$

the increment of the argument $\frac{1}{2\pi}\arg\mathscr{A}^s_p(\omega)$, when ω ranges through $\mathfrak{R}$ in the direction of orientation, is an integer. It is called the winding number or the index of the curve $\Gamma:=\{z\in\mathbb{C}:z=\det\mathscr{A}_p(\omega),\ \omega\in\mathfrak{R}\}$ and is denoted by $\mathbf{ind}\det\mathscr{A}^s_p$.

Propositions 5.5.1–5.5.3, exposed below, are well known and will be applied in the next section in the proof of the main theorems.

Proposition 5.5.1 ([59] and [37, Theorem 5.4])**.** *Let $1<p<\infty$, $s\in\mathbb{R}$. The operator*

$$\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+) \tag{5.5.4}$$

defined in (5.5.1) *is Fredholm if and only if its symbol $\mathscr{A}^s_p(\omega)$ defined in* (5.5.2), (5.5.3a)–(5.5.3b) *is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\mathbf{Ind}\,\mathbf{A}=-\,\mathbf{ind}\det\mathscr{A}^s_p.$$

The operator $\mathbf{A}$ in (5.5.4) *is locally invertible at* 0 *if and only if it is globally invertible.*

The operator $\mathbf{A}$ in (5.5.4) *is locally invertible at* 0 *if and only if its symbol $\mathscr{A}^s_p(\omega)$ is elliptic on the set Γ_1 only, $\inf_{\omega\in\Gamma_1}\big|\det\mathscr{A}^s_p(\omega)\big|>0$.*

Proposition 5.5.2 ([59, Corollary 6.3])**.** *Let $1<p<\infty$, $s\in\mathbb{R}$ and let $\mathbf{A}$ be defined by* (5.5.1)*. If the operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\to\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm* (*is invertible*) *for all $s\in(s_0,s_1)$ and $p\in(p_0,p_1)$, where $-\infty<s_0<s_1<\infty$, $1<p_0<p_1<\infty$, then $\boldsymbol{A}$ is Fredholm* (*is invertible, respectively*) *in the Sobolev–Slobodečkii space setting*

$$\mathbf{A}:\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)\to\mathbb{W}^s_p(\mathbb{R}^+)\ \ \textit{for all}\ \ s\in(s_0,s_1),\ \ p\in(p_0,p_1)$$

and has the same index

$$\mathbf{Ind}\,\mathbf{A}=-\,\mathbf{ind}\det\mathscr{A}^s_p.$$

Proposition 5.5.3 ([40,65])**.** *Let two pairs of parameter-dependent Banach spaces $\mathfrak{H}^s_1$ and $\mathfrak{H}^s_2$, $s_1<s<s_2$, have intersections $\mathfrak{H}^{s'}_j\cap\mathfrak{H}^{s''}_j$ dense in $\mathfrak{H}^{s'}_j$ and in $\mathfrak{H}^{s''}_j$ for all $j=1,2$, $s',s''\in(s_1,s_2)$.*

If a linear bounded operator $A:\mathfrak{H}^s_1\to\mathfrak{H}^s_2$ is Fredholm for all $s\in(s_1,s_2)$, it has the same kernel and co-kernel for all values of this parameter $s\in(s_1,s_2)$.

In particular, if $A:\mathfrak{H}^s_1\to\mathfrak{H}^s_2$ is Fredholm for all $s\in(s_1,s_2)$ and is invertible for only one value $s_0\in(s_1,s_2)$, it is invertible for all values of this parameter $s\in(s_1,s_2)$.

5.6 Model Dirichlet BVP

In the present section, we investigate model Dirichlet Boundary value problem, associated with the BVP (5.2.2) and described in general in the foregoing Section 5.5. We derive an equivalent boundary integral equation in the model domain (5.6.10) and investigate it.

Results for the model Dirirchlet BVP (5.6.1) (Fredholm criteria, the unique solvability) were obtained in [60, 61].

Let us commence with the formulation of the model Dirichlet BVP associated with the BVP (5.2.2) in the non-classical setting (5.1.2) at a knot $c_j \in \mathscr{M}_D$ (where Dirichlet conditions collide):

$$\begin{cases} \Delta u(t) = f(t), & t \in \Omega_{\alpha_j}, \\ u^+(s) = g(s) & \text{on } \Gamma_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}, \end{cases} \tag{5.6.1}$$

$$u \in \mathbb{H}^s_p(\Omega_{\alpha_j}), \;\; f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_j}), \;\; g \in \mathbb{H}^{s-\frac{1}{p}}_p(\Gamma_{\alpha_j}), \;\; 1 < p < \infty, \;\; \frac{1}{p} < s < 1 + \frac{1}{p}.$$

Here Ω_j is the model domain, associated with this problem (cf. Fig. 5.3 on page 141 and formula (5.3.11)) and $\Gamma_{\alpha_j} := \partial\Omega_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}$ is the boundary. We assume that $\alpha \neq \pi$, because for $\alpha = \pi$ we have the case $\Gamma_\pi = \mathbb{R}$ and BVP (5.6.1) is trivially solvable.

As a particular case of Theorem 5.1.1 we get the following

Corollary 5.6.1. *The boundary value problem* (5.6.1) *has a unique solution in the classical weak setting* $p = 2$, $s = 1$.

Let $C^s_0(\Gamma_\alpha)$ denote the set of Hölder continuous functions with exponent s and compact supports. It is well known that $C^s_0(\Gamma_\alpha)$ is a dense subset of $\mathbb{H}^s_p(\Gamma_\alpha)$ for $0 < s < 1 + \frac{1}{p}$.

The next proposition is a standard consequence of the Green formulae and can easily be found e.g. in [53, 65, 89].

Proposition 5.6.1 (Representation of a solution to BVP)**.** *Any solution* $u \in \mathbb{H}^s_p(\Omega_{\alpha_j})$ *to the BVP* (5.6.1) (*and also to the BVPs* (5.7.1) *and* (5.8.1) *in the forthcoming sections*) *is represented as follows:*

$$u(x) = \boldsymbol{N}_\Delta f(x) + \boldsymbol{W}_\Delta u^+(x) - \boldsymbol{V}_\Delta[\partial_{\boldsymbol{\nu}} u]^+(x), \;\; x \in \Omega_{\alpha_j}, \tag{5.6.2}$$

where u^+ *and* $[\partial_{\boldsymbol{\nu}} u]^+$ *are the Dirichlet and the Neumann traces of the solution* u *on the boundary* Γ_{α_j}.

Lemma 5.6.1. *Let* $1 < p < \infty$, $-1 - \frac{1}{p} < s < 1 + \frac{1}{p}$, $g_0 \in C^s_0(\Gamma_\alpha)$, $g_0(0) = 1$, *is a fixed function. Let us consider the linear functional*

$$F_0(\varphi) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\Gamma_{\alpha,\varepsilon}} \psi(\tau)\, d\sigma, \;\; \psi \in \mathbb{H}^s_p(\Gamma_\alpha),$$

where $\Gamma_{\alpha,\varepsilon}$ *is the intersection of* Γ_α *with the circle of radius* ε *centered at the vertex* $0 \in \Gamma_\alpha$.

Then for arbitrary $\varphi \in \mathbb{H}^s_p(\Gamma_\alpha)$ *and* $\psi \in \mathbb{W}^s_p(\Gamma_\alpha)$ *the following representations hold:*

$$\begin{gathered} \varphi = F_0(\varphi)g_0 + \varphi_+ + \varphi_\alpha, \quad \varphi_+ \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+), \quad \varphi_\alpha \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}_\alpha), \\ \psi = F_0(\psi)g_0 + \psi_+ + \psi_\alpha, \quad \psi_+ \in \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+), \quad \psi_\alpha \in \widetilde{\mathbb{W}}^s_p(\mathbb{R}_\alpha), \\ F_0(\varphi_+) = F_0(\varphi_\alpha) = F_0(\psi_+) = F_0(\psi_\alpha) = 0. \end{gathered} \tag{5.6.3}$$

Proof. It is easy to check that for $\varphi \in C^s_0(\Gamma_\alpha)$ there holds $F_0(\varphi) = \varphi(0)$ and, since $g_0(0) = 1$, we get $\varphi_+(0) = 0$, $\varphi_\alpha(0) = 0$. The inclusions $\varphi_+ \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $\varphi_\alpha \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}_\alpha)$ follow automatically. Since the subset $C^s_0(\Gamma_\alpha)$ is dense in $\mathbb{H}^s_p(\Gamma_\alpha)$ (also in $\mathbb{W}^s_p(\Gamma_\alpha)$) and F_0 is a linear bounded functional in $\mathbb{H}^s_p(\Gamma_\alpha)$ (also in $\mathbb{W}^s_p(\Gamma_\alpha)$), the both representations in (5.6.3) remain valid for arbitrary function $\varphi \in \mathbb{H}^s_p(\Gamma_\alpha)$ (for arbitrary function $\psi \in \mathbb{W}^s_p(\Gamma_\alpha)$). □

We remind that the Dirichlet trace $u^+ = g \in \mathbb{W}_p^{s-\frac{1}{p}}(\Gamma_\alpha)$ is a known function and let $(\partial_\nu u)^+ = \psi \in \mathbb{W}_p^{s-1-\frac{1}{p}}(\Gamma_\alpha)$ denote the unknown Neuman's trace. Then the representation formula (5.6.2) for a solution to the Dirichlet BVP (5.6.1) has the form

$$u = \boldsymbol{N}_\Delta f + \boldsymbol{W}_\Delta g - \boldsymbol{V}_\Delta \psi. \tag{5.6.4}$$

By applying the Plemelji Formulae (5.4.2) to (5.6.4) we get

$$(\partial_\nu u)^+ = \psi = (\partial_\nu \boldsymbol{N}_\Delta f)^+ + \boldsymbol{V}_{\Delta,+1} g + \frac{1}{2}\,\psi - \boldsymbol{W}^*_{\Delta,0}\psi$$

and rewrite the obtained equality as follows:

$$\frac{1}{2}\,\psi + \boldsymbol{W}^*_{\Delta,0}\psi = G, \;\; G := (\partial_\nu \boldsymbol{N}_\Delta f)^+ + \boldsymbol{V}_{\Delta,+1} g, \;\; \psi, G \in \mathbb{W}_p^{s-1-\frac{1}{p}}(\Gamma_\alpha). \tag{5.6.5}$$

Since $I = r_{\mathbb{R}^+} + r_{\mathbb{R}_\alpha}$, applying equalities (5.4.8c) we rewrite equation (5.6.5) as follows:

$$\frac{1}{2}\,\psi + r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha}\psi + r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+}\psi = G, \;\; G, \psi \in \mathbb{W}_p^{s-1-\frac{1}{p}}(\Gamma_\alpha). \tag{5.6.6}$$

Now we recall representation (5.4.8b), restrict equation (5.6.6) to $\mathbb{R}^+$ by applying $r_{\mathbb{R}^+}$, which gives us the first equation in (5.6.7) below. Then restrict equation (5.6.6) to $\mathbb{R}_\alpha$ and apply the pull back operator $\boldsymbol{J}_\alpha$ and its inverse (see (5.4.6)) and get the second equation in (5.6.7). Thus, we get the system of two equations on the half-axes with two unknown functions:

$$\begin{cases} \frac{1}{2}\,\psi_1 + (r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha}\boldsymbol{J}_\alpha^{-1})\psi_2 + F_0(\psi) g_2 = G_1, \\ \frac{1}{2}\,\psi_2 + (\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+})\psi_1 + F_0(\psi) g_1 = G_2, \end{cases} \tag{5.6.7}$$

$$\begin{gathered} g_1 := r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha} g_0, \quad g_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+} g_0, \\ \psi_1 := r_{\mathbb{R}^+}\psi, \quad \psi_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\psi, \quad G_1 := r_{\mathbb{R}^+} G, \quad G_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha} G, \\ \psi_1, \psi_2, \in \widetilde{\mathbb{W}}_p^{s-1-\frac{1}{p}}(\mathbb{R}^+), \;\; g_1, g_2, G_1, G_2 \in \mathbb{W}_p^{s-1-\frac{1}{p}}(\mathbb{R}^+). \end{gathered} \tag{5.6.8}$$

Since one-dimensional operator $F_0(\,\cdot\,)$ does not influence Fredholm property of system (5.6.7), the system

$$\begin{cases} \frac{1}{2}\,\psi_1 + (r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha}\boldsymbol{J}_\alpha^{-1})\psi_2 = G_1, \\ \frac{1}{2}\,\psi_2 + (\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+})\psi_1 = G_2, \end{cases} \qquad \psi_1, \psi_2 \in \widetilde{\mathbb{W}}_p^{s-1-\frac{1}{p}}(\mathbb{R}^+), \;\; G_1, G_2 \in \mathbb{W}_p^{s-1-\frac{1}{p}}(\mathbb{R}^+), \tag{5.6.9}$$

is Fredholm-equivalent to system (5.6.7)–(5.6.8).

Due to formula (5.4.8b), system (5.6.9) of boundary integral equations coincides with the following system of integral equations of Mellin type:

$$\begin{gathered} \begin{cases} \psi_1 - \frac{1}{2i}\left[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}\right]\psi_2 = G_1, \\ \psi_2 + \frac{1}{2i}\left[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}\right]\psi_1 = G_2, \end{cases} \\ \psi_1, \psi_2 \in \widetilde{\mathbb{W}}_p^{s-1-\frac{1}{p}}(\mathbb{R}^+), \;\; G_1, G_2 \in \mathbb{W}_p^{s-1-\frac{1}{p}}(\mathbb{R}^+). \end{gathered} \tag{5.6.10}$$

Theorem 5.6.1. *Let $1 < p < \infty$, $\frac{1}{p} < s < 1 + \frac{1}{p}$.*

The model Dirichlet boundary value problem in the non-classical setting (5.6.1) *is Freholm if and only if the system of boundary integral equation* (5.6.10) *is Fredholm.*

Now we can prove the main theorem of the present section.

Theorem 5.6.2. *Let $1 < p < \infty$, $-1-\frac{1}{p} < s < 1+\frac{1}{p}$. The Model Dirichlet BVP in the non-classical setting* (5.6.1) *is Fredholm* (*and the system of boundary integral equations* (5.6.10) *is Fredholm*) *if and only if either $\alpha = \pi$, or $(p,s) = (2,1)$ or $\alpha \neq \pi$, $(p,s) \neq (2,1)$, $\alpha \neq \pi$ and the following holds:*

$$e^{i2\pi(s-\frac{1}{p})}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha-\pi)\Big(\frac{1}{p}-s-1-i\xi\Big) \neq 0, \quad \forall\,\xi\in\mathbb{R}. \tag{5.6.11}$$

If condition (5.6.11) *holds, the semi-strip $(\frac{1}{p},\infty)\times(0,1)$ of the Euclidean plane $\mathbb{R}^2$, where the pair $(s,\frac{1}{p})$ ranges, decomposes into an infinite union $\mathscr{R}_0\cup\mathscr{R}_1\cup\cdots$ of non-intersecting connected subsets of regular pairs, for which the BVP* (5.6.1) *is Fredholm.*

If the point $(1,\frac{1}{2})$ (i.e., $s=1$, $p=2$) belongs to the connected subset $\mathscr{R}_0$, then the BVP (5.6.1) *is uniquely solvable for all pairs $(s,\frac{1}{p})\in\mathscr{R}_0$.*

The same unique solvability holds for the system of integral equations (5.6.7).

Proof. The unique solvability of the BVP (5.6.1) in the cases $(p,s)=(2,1)$ and $\alpha=\pi$ are already proved in Corollary 5.6.1 on page 151 and the Model case II on page 140, respectively. Thus, we assume that $(p,s)\neq(2,1)$, $\alpha\neq\pi$.

Let us investigate the Fredholm properties of system (5.6.10). An equivalent task is to study the Fredholm property of the corresponding operator

$$\boldsymbol{D}_\alpha = I - \frac{1}{2i}\,d\big[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}\big] : \widetilde{\mathbb{W}}{}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+) \to \mathbb{W}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+). \tag{5.6.12a}$$

For this, it suffices, due to Proposition 5.5.2, to prove the same theorem for the operator

$$\boldsymbol{D}_\alpha = I - \frac{1}{2i}\,d\big[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}\big] : \widetilde{\mathbb{H}}{}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+) \to \mathbb{H}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+). \tag{5.6.12b}$$

Here d is the 2×2 constant matrix

$$d := \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}. \tag{5.6.13}$$

The symbol of the operator $\boldsymbol{D}_\alpha$ in (5.6.12b) on the set $\overline{\Gamma}_1$, according to the formulae (5.5.3a)–(5.5.3c), reads:

$$\mathscr{D}^{s-1-\frac{1}{p}}_{\alpha,p}(\infty,\xi) = \begin{bmatrix} e^{i\pi(s-\frac{1}{p})}\,\dfrac{\sin\pi(s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)} & -e^{-i\pi s}\,\dfrac{\sin(\alpha-\pi)(\frac{1}{p}-s-1-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)},\\ e^{-i\pi s}\,\dfrac{\sin(\alpha-\pi)(\frac{1}{p}-s-1-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)} & e^{i\pi(s-\frac{1}{p})}\,\dfrac{\sin\pi(s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\end{bmatrix}, \tag{5.6.14}$$

because

$$\mathscr{I}^{s-1-\frac{1}{p}}_p(\infty,\xi) = e^{2\pi(s-1-\frac{1}{p})i}\,\frac{\sin\pi(\frac{1}{p}+s-1-\frac{1}{p}-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)} = -e^{i\pi(s-\frac{1}{p})}\,\frac{\sin\pi(s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\,; \tag{5.6.15}$$

$$\frac{1}{2i}\Big[\mathscr{K}^{1,s-1-\frac{1}{p}}_{e^{i\alpha},p}(\infty,\xi) - \mathscr{K}^{1,s-1-\frac{1}{p}}_{e^{i(2\pi-\alpha)},p}(\infty,\xi)\Big] = -e^{-i\pi(\frac{1}{p}-i\xi)}\,\frac{e^{i\alpha(\frac{1}{p}-s-1-i\xi)} - e^{i(2\pi-\alpha)(\frac{1}{p}-s-1-i\xi)}}{2i\sin\pi(\frac{1}{p}-i\xi)}$$

$$= e^{-i\pi s}\,\frac{e^{i(\alpha-\pi)(\frac{1}{p}-s-1-i\xi)} - e^{-i(\alpha-\pi)(\frac{1}{p}-s-1-i\xi)}}{2i\sin\pi(\frac{1}{p}-i\xi)} = e^{-i\pi s}\,\frac{\sin(\alpha-\pi)(\frac{1}{p}-s-1-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\,.$$

Since $\det\mathscr{D}^{s-1-\frac{1}{p}}_p(\infty,\xi)$ coincides with the function in (5.6.11), due to Proposition 5.5.1, the operator in (5.6.12b) is locally Fredholm and, therefore, globally Fredholm if condition (5.6.11) holds.

The determinant of the symbol

$$\det\mathscr{D}^{s-1-\frac{1}{p}}_p(\infty,\xi) = e^{i2\pi(s-\frac{1}{p})}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha-\pi)\Big(\frac{1}{p}-s-1-i\xi\Big)$$

is a periodic function with respect to the parameters s and $\frac{1}{p}$ and vanishes on curves which divide the strip $(1,\infty)\times(0,1)\subset\mathbb{R}^2$ into connected subsets $\mathscr{R}_0,\mathscr{R}_1,\ldots$. Due to Corollary 5.6.1, the BVP (5.6.1) is uniquely solvable for $s=1$ and $p=2$. Then, due to Proposition 5.5.3, the BVP (5.6.1) is uniquely solvable for all pairs $(s,\frac{1}{p})\in\mathscr{R}_0$, provided $(1,\frac{1}{2})\in\mathscr{R}_0$. $\square$

5.7 Model Neumann BVP

In the present section, we investigate model Neumann Boundary value problem, associated with the BVP (5.1.1) and described in general in the foregoing Section 5.5. We derive an equivalent boundary integral equation for the model domain (5.7.5) and investigate it.

Results for the model Neumann BVP (5.7.1) (Fredholm criteria, the unique solability) was obtained in [60, 61].

Let us commence with the formulation of the model Neumann BVP associated with the BVP (5.1.1) in the non-classical setting (5.1.2) at a knot $c_j \in \mathscr{M}_D$ (where Neumann conditions collide):

$$\begin{cases}\Delta u(t)=f(t), & t\in\Omega_{\alpha_j},\\ (\partial_{\boldsymbol{\nu}} u)^+(s)=h(s) & \text{on } \Gamma_{\alpha_j}=\mathbb{R}^+\cup\mathbb{R}_{\alpha_j},\end{cases}\tag{5.7.1}$$

$$u\in\mathbb{H}^s_p(\Omega_{\alpha_j}),\quad f\in\widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_j}),\quad h\in\mathbb{H}^{s-1-\frac{1}{p}}_p(\Gamma_{\alpha_j}),\quad 1<p<\infty,\quad \frac{1}{p}s<1+\frac{1}{p}.$$

Here the model domain Ω_{α_j} and the boundary Γ_{α_j} are the same as in Section 5.6 (see Fig. 5.3 on page 141 and formula (5.3.11)). The unit normal vector field $\boldsymbol{\nu}(t)$ and the normal derivative $\partial_{\boldsymbol{\nu}}$ are defined above in (5.3.12). We assume, as above, that $\alpha\neq\pi$, because for $\alpha=\pi$ we have the case $\Gamma_\pi=\mathbb{R}$ and BVP (5.7.1) is trivially solvable.

As a particular case of Theorem 5.1.1 we get the following

Corollary 5.7.1. *The boundary value problems* (5.7.1) *has a unique solution in the classical weak setting* $p=2$, $s=1$.

If the Neuman trace $(\partial_\nu u)^+=h\in\mathbb{W}^{s-1-\frac{1}{p}}_p(\Gamma_\alpha)$ is known and $u^+=\varphi\in\mathbb{W}^{s-\frac{1}{p}}_p(\Gamma_\alpha)$ denotes the unknown Dirichlet trace, the representation formula (5.6.2) for a solution to BVP (5.7.1) takes the form

$$u=\boldsymbol{N}_\Delta f+\boldsymbol{W}_\Delta\varphi-\boldsymbol{V}_\Delta h.\tag{5.7.2}$$

By applying the Plemelji Formulae (5.4.2) to (5.7.2) we get

$$u^+=\varphi=(\boldsymbol{N}_\Delta f)^+ +\frac{1}{2}\,\varphi+\boldsymbol{W}_{\Delta,0}\varphi-\boldsymbol{V}_{\Delta,-1}h,\quad \varphi\in\Gamma_\alpha.$$

Since $I=r_{\mathbb{R}^+}+r_{\mathbb{R}_\alpha}$, rewrite the obtained equation as follows:

$$\frac{1}{2}\,\varphi-r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi-r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi=H,,\quad H:=(\partial_\nu\boldsymbol{N}_\Delta f)^+-\boldsymbol{V}_{\Delta,-1}h,\quad \varphi,H\in\mathbb{W}^{s-\frac{1}{p}}_p(\Gamma_\alpha).\tag{5.7.3}$$

By using representation (5.6.3), similarly to (5.6.6)–(5.6.8), equation (5.7.3) is rewritten as an equivalent system of boundary integral equations on the semi-axes $\mathbb{R}^+$:

$$\begin{cases}\dfrac{1}{2}\,\varphi_1-r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi_2-F_0(\varphi)h_2=H_1,\\ \dfrac{1}{2}\,\varphi_2-\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi_1-F_0(\varphi)h_1=H_2,\end{cases}\tag{5.7.4}$$

$$h_1:=r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}g_0,\quad h_2:=\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}g_0,$$
$$\varphi_1:=r_{\mathbb{R}^+}\varphi,\quad \varphi_2:=\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\varphi,\quad H_1:=r_{\mathbb{R}^+}H,\quad H_2:=\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}H,$$
$$\varphi_1,\varphi_2\in\widetilde{\mathbb{W}}^{s-\frac{1}{p}}_p(\mathbb{R}^+),\quad h_1,h_2,H_1,H_2\in\mathbb{W}^{s-\frac{1}{p}}_p(\mathbb{R}^+).$$

Due to formula (5.4.8a), system (5.7.4) of boundary integral equations coincides with the following system of integral equations of Mellin type:

$$\begin{cases}\psi_1-\dfrac{1}{2i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\psi_2=G_1,\\ \psi_2+\dfrac{1}{2i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\psi_1=G_2,\end{cases}\tag{5.7.5}$$

$$\psi_1,\psi_2\in\widetilde{\mathbb{W}}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+),\quad G_1,G_2\in\mathbb{W}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+).$$

Theorem 5.7.1. *Let* $1<p<\infty$, $\frac{1}{p}<s<1+\frac{1}{p}$.

The model Neumann boundary value problem in the non-classical setting (5.7.1) *is Freholm if and only if the system of boundary integral equation* (5.7.5) *is Fredholm.*

Now we can prove the main theorem of the present section.

Theorem 5.7.2. *Let* $1<p<\infty$, $-1-\frac{1}{p}<s<1+\frac{1}{p}$. *The Model Neumann BVP in the non-classical setting* (5.7.1) *is Fredholm* (*and the system of boundary integral equations* (5.7.5) *is Fredholm*) *if and only if either* $\alpha=\pi$ *or* $(p,s)=(2,1)$ *or* $\alpha\neq\pi$, $(p,s)\neq(2,1)$ *and the following holds:*

$$e^{i2\pi(s-\frac{1}{p})}\sin^2\pi(s-i\xi)+e^{-i2\pi s}\sin^2(\alpha-\pi)\Big(\frac{1}{p}-s-i\xi\Big)\neq 0,\ \ \forall\,\xi\in\mathbb{R}. \tag{5.7.6}$$

If condition (5.7.6) *holds, the subset* $(\frac{1}{p},\infty)\times(1,\infty)$ *of the Euclidean plane* $\mathbb{R}^2$, *where the pairs* (s,p) *range, decomposes into an infinite union* $\mathscr{R}_0\cup\mathscr{R}_1\cup\cdots$ *of non-intersecting connected subsets of regular pairs, for which the BVP* (5.7.1) *is Fredholm.*

If point $(1,2)$ (*i.e.,* $s=1$, $p=2$) *belongs to the connected subset* $\mathscr{R}_0$, *then the BVP* (5.7.1) *is uniquely solvable for all pairs* $(s,p)\in\mathscr{R}_0$.

The same unique solvability holds for the system of integral equations (5.7.4).

Proof. The unique solvability of the BVP (5.7.1) in the cases $(p,s)=(2,1)$ and $\alpha=\pi$ are already proved in Corollary 5.7.1 on page 154 and the Model case II on page 140, resepctively. Thus, we assume that $(p,s)\neq(2,1)$, $\alpha\neq\pi$.

Let us investigate the Fredholm properties of system (5.7.5). An equivalent task is to study the Fredholm property of the corresponding operator

$$\boldsymbol{N}_\alpha=I-\frac{1}{2i}\,d\big[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\big]:\widetilde{\mathbb{W}}{}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+)\to\mathbb{W}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+). \tag{5.7.7a}$$

For this, it suffices, due to Proposition 5.5.2, to prove the same theorem for the operator

$$\boldsymbol{N}_\alpha=I-\frac{1}{2i}\,d\big[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\big]:\widetilde{\mathbb{H}}{}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+)\to\mathbb{H}^{s-1-\frac{1}{p}}_p(\mathbb{R}^+). \tag{5.7.7b}$$

Here the 2×2 matrix d is defined in (5.6.13).

The symbol of the operator $\boldsymbol{D}_\alpha$ in (5.7.7b) on the set $\overline{\Gamma}_1$, according to formulae (5.5.3a)–(5.5.3c), reads:

$$\mathscr{D}^{s-1-\frac{1}{p}}_{\alpha,p}(\infty,\xi)=\begin{bmatrix} -e^{i\pi(s-\frac{1}{p})}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)} & -e^{-i\pi s}\dfrac{\sin(\alpha-\pi)(\frac{1}{p}-s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\\ e^{-i\pi s}\dfrac{\sin(\alpha-\pi)(\frac{1}{p}-s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)} & -e^{i\pi(s-\frac{1}{p})}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\end{bmatrix}, \tag{5.7.8}$$

because

$$\begin{aligned}\frac{1}{2i}\Big[e^{i\alpha}\mathscr{K}^{1,s-1-\frac{1}{p}}_{e^{i\alpha},p}(\infty,\xi)-e^{-i\alpha}\mathscr{K}^{1,s-1-\frac{1}{p}}_{e^{i(2\pi-\alpha)},p}(\infty,\xi)\Big]&=-e^{-i\pi(\frac{1}{p}-i\xi)}\,\frac{e^{i\alpha(\frac{1}{p}-s-i\xi)}-e^{i(2\pi-\alpha)(\frac{1}{p}-s-i\xi)}}{2i\sin\pi(\frac{1}{p}-i\xi)}\\ =e^{-i\pi s}\,\frac{e^{i(\alpha-\pi)(\frac{1}{p}-s-i\xi)}-e^{-i(\alpha-\pi)(\frac{1}{p}-s-i\xi)}}{2i\sin\pi(\frac{1}{p}-i\xi)}&=e^{-i\pi s}\,\frac{\sin(\alpha-\pi)(\frac{1}{p}-s-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)},\end{aligned}$$

and for $\mathscr{I}^{s-1-\frac{1}{p}}_p(\infty,\xi)$ cf. (5.6.15).

Since $\det\mathscr{N}^{s-1-\frac{1}{p}}_p(\infty,\xi)$ coincides with the function in (5.7.6), due to Proposition 5.5.1, the operator in (5.7.7a) is locally Fredholm and, therefore, globally Fredholm if condition (5.7.6) holds.

The determinant of the symbol

$$\det\mathscr{N}^{s-1-\frac{1}{p}}_p(\infty,\xi)=e^{i2\pi(s-\frac{1}{p})}\sin^2\pi(s-i\xi)+e^{-i2\pi s}\sin^2(\alpha-\pi)(\frac{1}{p}-s-i\xi)$$

is a periodic function with respect to the parameters s and $\frac{1}{p}$ and vanishes on curves which divide the strip $(1,\infty)\times(0,1)\subset\mathbb{R}^2$ into connected subsets $\mathscr{R}_0,\mathscr{R}_1,\dots$. Due to Corollary 5.7.1, the BVP (5.7.1) is uniquely solvable for $s=1$ and $p=2$. Then, due to Proposition 5.5.3, the BVP (5.7.1) is uniquely solvable for all pairs $(s,\frac{1}{p})\in\mathscr{R}_0$, provided $(1,\frac{1}{2})\in\mathscr{R}_0$. □

5.8 Model Mixed BVP and proof of Theorem 5.1.2

In the present section, we investigate model Mixed (Dirichlet–Neumann) Boundary value problem, associated with the BVP (5.1.1) and described in general in the foregoing Section 5.5. We derive an equivalent boundary integral equation for the model domain (5.8.7) and investigate it.

Moreover, at the end of this section we prove the main theorem of the present Chapter 5, Theorem 5.1.2.

The results for the model mixed BVP (5.8.1) (Fredholm criteria, the unique solability) were obtained in [68,69]. Similar results for the BVPs with mixed impedance conditions are proved in [18].

Let us commence with the formulation of the model Mixed BVP associated with the BVP (5.1.1) in the non-classical setting (5.1.2) at a knot $c_j\in\mathscr{M}_D$ (where Neumann and Dirichlet conditions collide):

$$\begin{cases}\Delta u(t)=f(t), & t\in\Omega_{\alpha_j},\\ u^+(s)=g(s) & \text{on }\ \mathbb{R}^+,\\ (\partial_{\boldsymbol{\nu}}u)^+(s)=h(s) & \text{on }\ \mathbb{R}_{\alpha_j},\end{cases}\tag{5.8.1}$$

$$u\in\mathbb{H}^s_p(\Omega_{\alpha_j}),\ \ f\in\widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_j}),\ \ g\in\mathbb{H}^{s-\frac{1}{p}}_p(\mathbb{R}^+),\ \ h\in\mathbb{H}^{s-1-\frac{1}{p}}_p(\mathbb{R}_{\alpha_j}),\ \ 1<p<\infty,\ \ \frac{1}{p}<s<1+\frac{1}{p},$$

at a knot $c_j\in\mathscr{M}_{DN}$ (where the Dirichlet and Neumann conditions collide). We assume, as above, that $\alpha\neq\pi$, bacause for $\alpha=\pi$ we have the case $\Gamma_\pi=\mathbb{R}$ and the unique solvability of BVP (5.8.1) is well known.

As a particular case of Theorem 5.1.1 we get the following

Corollary 5.8.1. *The boundary value problems* (5.8.1) *has a unique solution in the classical weak setting* $p=2,\ s=1$.

Let $g_0\in\mathbb{H}^{s-1/p}_p(\Gamma_\alpha)$ and $h_0\in\mathbb{H}^{s-1-1/p}_p(\Gamma_\alpha)$ be some fixed extensions of the boundary conditions $g\in\mathbb{H}^{s-1/p}_p(\mathbb{R}^+)$ and $h\in\mathbb{H}^{s-1-1/p}_p(\mathbb{R}_\alpha)$ in BVP (5.8.1), initially defined on the parts of the boundary $\Gamma_\alpha=\mathbb{R}^+\cup\mathbb{R}^\alpha$. Since the difference between such two extensions belong to the spaces $\widetilde{\mathbb{H}}^{s-1/p}_p(\mathbb{R}_\alpha)$ and $\widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}^+)$, respectively, we seek two unknown functions $\varphi\in\widetilde{\mathbb{H}}^{s-1/p}_p(\mathbb{R}_\alpha)$ and $\psi\in\widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}^+)$, for which the boundary conditions in (5.8.1) hold on the entire boundary. It is usual to consider $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}^s_p(\mathbb{R}_\alpha)$ as subsets of $\mathbb{H}^s(\Gamma_\alpha)$ by extending functions from $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}^s_p(\mathbb{R}_\alpha)$ by 0 to $\mathbb{R}_\alpha$ and to $\mathbb{R}^+$, respectively. Then if $u(x)$ is a solution to the BVP (5.8.1), the following holds:

$$\begin{aligned}u^+(t)=g_0(t)+\varphi(t)&=\begin{cases}g(t) & \text{if}\quad t\in\mathbb{R}^+,\\ g_0(t)+\varphi(t) & \text{if}\quad t\in\mathbb{R}_\alpha,\end{cases}\\ (\partial_{\boldsymbol{\nu}}u)^+(t)=h_0(t)+\psi(t)&=\begin{cases}h_0(t)+\psi(t) & \text{if}\quad t\in\mathbb{R}^+,\\ h(t) & \text{if}\quad t\in\mathbb{R}_\alpha.\end{cases}\end{aligned}\tag{5.8.2}$$

By introducing the data of the boundary value problem (5.8.1) into the representation formula (5.6.2) of a solution, we get

$$u(x)=\boldsymbol{W}u^+(x)-\boldsymbol{V}[\partial_{\boldsymbol{\nu}}u]^+(x)=\boldsymbol{W}[g_0+\varphi](x)-\boldsymbol{V}_{\mathbb{R}^+}[h_0+\psi](x),\quad x\in\Omega_\alpha.\tag{5.8.3}$$

The known and unknown functions in (5.8.3) belong to the following spaces (cf. (5.8.1))

$$g_0\in\mathbb{H}^{s-1/p}_p(\Gamma_\alpha),\quad h_0\in\mathbb{H}^{s-1-1/p}_p(\Gamma_\alpha),\quad \varphi\in\widetilde{\mathbb{H}}^{s-1/p}_p(\mathbb{R}_\alpha),\quad \psi\in\widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}^+).\tag{5.8.4}$$

Inserting the boundary conditions from (5.8.1) into (5.8.2) and applying the Plemelji formulae (5.4.2) we get

$$\begin{cases} g_0(t)+\varphi(t)=u^+(t)=\dfrac{1}{2}(g_0(t)+\varphi(t))+\boldsymbol{W}_0[g_0+\varphi](t)-\boldsymbol{V}_{-1}[h_0+\psi](t), \\ h_0(t)+\psi(t)=(\partial_{\boldsymbol{\nu}}u)^+(t)=\boldsymbol{V}_{+1}[g_0+\varphi](t)+\dfrac{1}{2}(h_0(t)+\psi(t))-\boldsymbol{W}_0^*[h_0+\psi](t), \quad t\in\Gamma_\alpha. \end{cases}$$

The obtained system is rewritten in the form

$$\begin{cases} \dfrac{1}{2}\varphi-\boldsymbol{W}_0\varphi+\boldsymbol{V}_{-1}\psi=G_0, \\ \dfrac{1}{2}\psi+\boldsymbol{W}_0^*\psi-\boldsymbol{V}_{+1}\varphi=H_0 \quad \text{on} \quad \Gamma_\alpha, \end{cases} \tag{5.8.5}$$

where

$$G_0:=-\frac{1}{2}g_0+\boldsymbol{W}_0g_0-\boldsymbol{V}_{-1}h_0\in\mathbb{H}_p^{s-1/p}(\Gamma_\alpha),$$
$$H_0:=-\frac{1}{2}h_0+\boldsymbol{V}_{+1}g_0-\boldsymbol{W}_0^*h_0\in\mathbb{H}_p^{s-1-1/p}(\Gamma_\alpha).$$

Since supp $\varphi\subset\mathbb{R}_\alpha$ and supp $\psi\subset\mathbb{R}^+$, we restrict the first equation in system (5.8.5) to $\mathbb{R}_\alpha$ while the second equation to $\mathbb{R}^+$. Let r_+ and r_α be the corresponding restriction operators: $r_+\varphi=r_\alpha\psi=0$. By restricting system (5.8.5) properly and by applying equalities (5.4.8c), we get the equations

$$\begin{cases} \frac{1}{2}\varphi+r_\alpha\boldsymbol{V}_{-1}\psi=r_\alpha G_0 \quad \text{on} \quad \mathbb{R}_\alpha, \\ \frac{1}{2}\psi-r_+\boldsymbol{V}_{+1}\varphi=r_+H_0 \quad \text{on} \quad \mathbb{R}^+. \end{cases} \tag{5.8.6}$$

By applying the operator $\boldsymbol{J}_\alpha\partial_\ell=-\partial_tJ_\alpha$ (composition of the pull back $\boldsymbol{J}_\alpha$ from (5.4.6) and the tangent derivative ∂_ℓ from (5.4.7)) to the first equation and using formulae (5.4.9a)–(5.4.9b), we rewrite system (5.8.6) in the following form:

$$\begin{cases} \varphi_0-\dfrac{1}{2\pi}\left[\mathbf{K}^1_{e^{i\alpha}}+\mathbf{K}^1_{e^{-i\alpha}}\right]\psi=G_1, \\ \psi-\dfrac{1}{2\pi}\left[\mathbf{K}^1_{e^{i\alpha}}+\mathbf{K}^1_{e^{-i\alpha}}\right]\varphi_0=H_1 \quad \text{on} \quad \mathbb{R}^+, \end{cases} \tag{5.8.7}$$

where

$$\varphi_0(t):=\boldsymbol{J}_\alpha\partial_\ell\varphi(t)=-\partial_t\varphi(t\cos\alpha,t\sin\alpha)^\top, \quad H_1:=2r_+H_0, \quad G_1:=2\boldsymbol{J}_\alpha\partial_\ell G_1, \tag{5.8.8}$$
$$\varphi_0,\psi,G_1,H_1\in\mathbb{H}_p^{s-1-1/p}(\mathbb{R}^+).$$

Theorem 5.8.1. *Let* $1<p<\infty$ *and* $1/p<s<1+1/p$. *A solution* $u\in\mathbb{H}_p^s(\Omega_\alpha)$ *to the mixed BVP* (5.8.1) *is represented by formula* (5.8.3), *where the unknown functions* $\psi,\varphi_0\in\widetilde{\mathbb{H}}_p^{s-1-1/p}(\mathbb{R}^+)$ *are solutions to system* (5.8.7) *and* $\varphi\in\widetilde{\mathbb{H}}_p^{s-1-1/p}(\mathbb{R}_\alpha)$ *is recovered from* $\varphi_0(t)=\partial_t\varphi_+(t)$ *(see* (5.8.8)*) by the formula*

$$\varphi(x)=\varphi(t\cos\alpha,t\sin\alpha):=-\int_0^t\varphi_0(\tau)\,d\tau, \quad t\in\mathbb{R}^+, \tag{5.8.9}$$

where $x:=(t\cos\alpha,t\sin\alpha)^\top\in\mathbb{R}_\alpha$.

Vice versa: if the functions $\psi,\varphi_0\in\widetilde{\mathbb{H}}_p^{s-1-1/p}(\mathbb{R}^+)$ *are solutions to system* (5.8.7) *and* $\varphi\in\widetilde{\mathbb{H}}_p^{s-1/p}(\mathbb{R}_\alpha)$ *is recovered by formula* (5.8.9), *the function* $u\in\mathbb{H}^1(\Omega_\alpha)$, *represented by formula* (5.8.3), *is a solution to the model mixed BVP* (5.8.1).

Proof. Every step in deriving equation system (5.8.7) from the model mixed BVP (5.8.1) is reversible and the one-to-one correspondence of solutions to the equation system (5.8.7) and solution to the model mixed BVP (5.8.1), established by representation formula (5.8.3), can easily be checked. □

Now we can prove the main theorem of the present section.

Theorem 5.8.2. *Let $1 < p < \infty$, $\frac{1}{p} < s < 1 + \frac{1}{p}$. The Model Mixed BVP in the non-classical setting* (5.8.1) *is Fredholm (and the system of boundary integral equations* (5.8.7) *is Fredholm) if and only if either $(p, s) = (2, 1)$ or $(p, s) \neq (2, 1)$ and the following holds:*

$$e^{i2\pi(s-1/p)} \sin^2 \pi(s - i\xi) - \cos^2[\pi/p + \alpha s - i(\pi - \alpha)\xi] \neq 0 \quad \textit{for all} \quad \xi \in \mathbb{R}. \tag{5.8.10}$$

If condition (5.8.10) *holds, the subset $(\frac{1}{p}, \infty) \times (1, \infty)$ of the Euclidean plane $\mathbb{R}^2$, where the pairs (s, p) range, decomposes into an infinite union $\mathscr{R}_0 \cup \mathscr{R}_1 \cup \cdots$ of non-intersecting connected subsets of regular pairs, for which the BVP* (5.8.1) *is Fredholm.*

If point $(1, 2)$ (i.e., $s = 1$, $p = 2$) belongs to the connected subset $\mathscr{R}_0$, then BVP (5.8.1) *is uniquely solvable for all pairs $(s, p) \in \mathscr{R}_0$.*

The same unique solvability holds for the system of integral equations (5.8.7).

Proof. The unique solvability of the BVP (5.7.1) in the case $(p, s) = (2, 1)$ is already proved in Corollary 5.8.1 on page 156. Thus, we assume that $(p, s) \neq (2, 1)$.

Let us rewrite system (5.8.7) in the matrix form

$$\boldsymbol{B}_\alpha \Phi = \mathbf{G}, \tag{5.8.11}$$

where

$$\Phi := \begin{pmatrix} \varphi_0 \\ \psi \end{pmatrix}, \quad \mathbf{G}_\alpha := \begin{pmatrix} G_1 \\ H_1 \end{pmatrix} \in \mathbb{H}^{s-1-1/p}_p(\mathbb{R}^+),$$
$$\boldsymbol{B}_\alpha = \begin{bmatrix} I & -\boldsymbol{A}_\alpha \\ -\boldsymbol{A}_\alpha & I \end{bmatrix}, \quad \boldsymbol{A}_\alpha = \frac{1}{2\pi}\left[\mathbf{K}^1_{e^{i\alpha}} + \mathbf{K}^1_{e^{-i\alpha}}\right].$$

and investigate the operator $\boldsymbol{B}_\alpha : \widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}^+) \to \mathbb{H}^{s-1-1/p}_p(\mathbb{R}^+)$ with the help of Theorem 1.4.2. For this, we have to write the symbol $\mathscr{A}^{s-1-1/p}_{\alpha,p}(\omega)$ of $\boldsymbol{A}_\alpha$ of the operator $\boldsymbol{B}_\alpha$ in (5.7.7b) on the set $\overline{\Gamma}_1$. To this end, note that for $c_1 = c_+ = e^{i\alpha}$ and $c_2 = c_- = e^{-i\alpha}$ we can choose $\gamma = e^{i\theta}$, $0 < \theta < \pi$, such that $\arg(c_\pm\gamma) = \pm\alpha + \beta < 0$ provided $\dfrac{\pi}{2} < \alpha < \pi$. If $0 < \alpha < \dfrac{\pi}{2}$, there is needed a couple $\gamma = e^{i\theta}$, $\gamma_0 = e^{i\theta_0}$, $0 < \theta < \pi$, $0 < \theta_0 < \pi$, such that $\arg(c_+\gamma) = \alpha + \beta < 0$ and $\arg(c_+\gamma_0) = \alpha + \beta_0 < 0$. Since these values exist but there choice does not influence the symbol of operators, we drop further details about them.

The symbol of the operator $\boldsymbol{D}_\alpha$ in (5.7.7b) on the set $\overline{\Gamma}_1$, according to formulae (5.5.3a)–(5.5.3c), reads:

$$\begin{aligned}\mathscr{B}^{s-1-1/p}_{\alpha,p}(\infty,\xi) &= \begin{bmatrix} \mathscr{I}^{s-1-1/p}_p(\infty,\xi) & \mathscr{A}^{s-1-1/p}_{\alpha,p}(\infty,\xi) \\ -\mathscr{A}^{s-1-1/p}_{\alpha,p}(\infty,\xi) & \mathscr{I}^{s-1-1/p}_p(\infty,\xi)\end{bmatrix}, \\ &= \begin{bmatrix} -e^{i\pi(s-1/p)}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(1/p-i\xi)} & \dfrac{\cos[\pi/p+\alpha s - i(\pi-\alpha)\xi]}{\sin\pi(1/p-i\xi)} \\ -\dfrac{\cos[\pi/p+\alpha s - i(\pi-\alpha)\xi]}{\sin\pi(1/p-i\xi)} & -e^{i\pi(s-1/p)}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(1/p-i\xi)}\end{bmatrix}, \\ &\text{for} \quad \omega = (\infty,\xi) \in \overline{\Gamma_1}, \quad 1/p < s < 1 + 1/p, \quad 1 < p < \infty,\end{aligned} \tag{5.8.12}$$

because (cf. (5.5.3c))

$$\begin{aligned}\mathscr{A}^{s-1-1/p}_{\alpha,p}((\infty,\xi)) &= \frac{1}{2i}\left[\mathscr{K}^{1,s-1-1/p}_{e^{i\alpha},p}(\infty,\xi) + \mathscr{K}^{1,s-1-1/p}_{e^{i(2\pi-\alpha)},p}(\infty,\xi)\right] \\ &= -\frac{e^{-\pi(1/p-i\xi)i}e^{-i\alpha(i\xi+s)} + e^{\pi(1/p-i\xi)i}e^{i\alpha(i\xi+s)}}{2\sin\pi(1/p-i\xi)} \\ &= -\frac{\cos[\pi/p+\alpha s - i(\pi-\alpha)\xi]}{\sin\pi(1/p-i\xi)}\end{aligned}$$

and (cf. (5.6.15))

$$\mathscr{I}_p^{s-1-1/p}(\infty,\xi) = -e^{i\pi(s-1/p)}\,\frac{\sin\pi(s-i\xi)}{\sin^2\pi(1/p-i\xi)}.$$

Since

$$\det\,\mathscr{A}_{\alpha,p}^{s-1-1/p}((\infty,\xi) = \frac{e^{i2\pi(s-1/p)}\sin^2\pi(s-i\xi) - \cos^2[\pi/p+\alpha s - i(\pi-\alpha)\xi]}{\sin^2\pi(1/p-i\xi)},$$

due to Proposition 5.5.1 and condition (5.8.10), the operator in (5.7.7a) is locally Fredholm and, therefore, globally Fredholm provided condition (5.8.10) holds.

The determinant of the symbol $\det\mathscr{A}_{\alpha,p}^{s-1-1/p}((\infty,\xi)$ in (5.8.10) vanishes on curves which divide the strip $(1,\infty)\times(0,1)\subset\mathbb{R}^2$ into connected subsets $\mathscr{R}_0,\mathscr{R}_1,\dots$. Due to Corollary 5.8.1, the BVP (5.8.1) is uniquely solvable for $s=1$ and $p=2$. Then, due to Proposition 5.5.3, the BVP (5.8.1) is uniquely solvable for all pairs $(s,1/p)\in\mathscr{R}_0$, provided $(1,\frac{1}{2})\in\mathscr{R}_0$. □

Next, we are going to prove Theorem 5.1.2. For this we need one auxiliary result, formulated in Corollary 5.8.2 and which is a direct consequence of Theorem 5.3.1.

Corollary 5.8.2 (Quasi Localization Principle)**.** *The initial mixed boundary value problem* (5.1.1) *in the non-classical setting is Fredholm if and only if the boundary value problems* (5.6.1)*,* (5.7.1) *and* (5.8.1) *are Fredholm in the non-classical setting for all knots* $c_j\in\mathscr{M}_\Gamma$*.*

Proof of Theorem 5.1.2. Due to the Quasi Localization Principle, Corollary 5.8.2, the BVP (5.1.1) is Fredholm if local representatives (the corresponding BVPs (5.6.1), (5.7.1) and (5.8.1)) at the knots $c_j\in\mathscr{M}=\mathscr{M}_D\cup\mathscr{M}_N\cup\mathscr{M}_{DN}$ are all Fredholm. Due to Theorem 5.6.2, Theorem 5.7.2 and Theorem 5.8.2, conditions (5.1.7), (5.1.8) and (5.1.9) are necessary and sufficient for the corresponding Dirichlet, Neumann and Mixed BVPs to be Fredholm in appropreate non-classical settings.

The determinants of the symbols in (5.1.7), (5.1.8) and (5.1.9) are periodic functions with respect to the parameters s and $1/p$ and vanish on curves which divide the strip $(1,\infty)\times(0,1)\subset\mathbb{R}^2$ into connected subsets $\mathscr{R}_0,\mathscr{R}_1,\dots$. Due to Theorem 2.1.1, the BVP (5.1.1) is uniquely solvable for $s=1$ and $p=2$. Then, due to Proposition 5.5.3, the BVP (5.1.1) is uniquely solvable for all pairs $(s,1/p)\in\mathscr{R}_0$, provided $(1,\frac{1}{2})\in\mathscr{R}_0$. □

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