Memoirs on Differential Equations and Mathematical Physics

Volume 77, 2019, 1–12

M. A. Belozerova, G. A. Gerzhanovskaya

ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS WITH SLOWLY VARYING DERIVATIVES OF ESSENTIALLY NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Abstract. Differential equations of the second order with nonlinearities of rather general type that are in some sense near to the power ones are considered. For some class of solutions with derivatives of the first order that are slowly varying functions as the argument tends to the critical point, the conditions of the existence and asymptotic representations are found.

2010 Mathematics Subject Classification. 34C41, 34E10.

Key words and phrases. Regular variation, regularly varying function, slowly varying function, nonlinear differential equation, regularly varying solution.

რეზიუმე. განხილულია საკმარისად ზოგადი ტიპის მეორე რიგის დიფერენციალური განტოლებები, რომლებიც გარკვეული აზრით ახლოს არიან ხარისხოვან განტოლებებთან. ამონახსნთა გარკვეული კლასისთვის, რომელთა პირველი წარმოებულები ნელად ცვალებადი ფუნქციებია, როცა არგუმენტი კრიტიკული წერტილისკენ მიისწრაფის, დადგენილია ამოხსნადობის პირობები და ასიმპტოტური წარმოდგენები.

Let us consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp\big(R(|\ln|yy'||)\big), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\, \to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ are continuous functions, $R :]0; +\infty[\, \to]0; +\infty[$ is a continuously differentiable function, that is, regularly varying at infinity of order μ, $0 < \mu < 1$, and has a monotone derivative. Here, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0; Y_i[$ [1], or the interval $]Y_i; y_i^0]$ $(i = 0, 1)$. Moreover, it is supposed that every function φ_i $(i = 0, 1)$ is regularly varying of order σ_i [4, Chapter 1, § 1.1, p. 9] as the argument tends to Y_i and $\sigma_0 + \sigma_1 \ne 1$.

The solution y of equation (1) defined on the interval $[t_0, \omega[\, \subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the conditions

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0 \tag{2}$$

are satisfied.

Let the function $\varphi : \Delta_Y \to]0, +\infty[$ be regularly varying of order σ as $z \to Y$ $(z \in \Delta_Y, Y \in \{0, \infty\}$, Δ_Y is a one-sided neighborhood of $Y)$. We say that the function φ satisfies the condition S if for any slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ such that

$$\lim_{\substack{z\to Y\\ z\in\Delta_Y}} \frac{zL'(z)}{L(z)} = 0,$$

the equality

$$\Theta(zL(z)) = \Theta(z)(1 + o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y)$$

takes place, where $\Theta(z) = \varphi(z)|z|^{-\sigma}$.

Some classes of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) were investigated earlier (see, e.g., [3]). The sufficiently important class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equations like (1) has been considered only for the cases $R(z) \equiv 0$ and $\varphi_0(z)|z|^{-\sigma_0}$ satisfies the condition S. Later, it has turned out to extend the results on more general cases (see, e.g., [1]). But the functions that do not satisfy the condition S, but contain in the left-hand side the derivative of an unknown function as in a general case of equation (1), have not been considered before. Notice that the derivative of every $P_\omega(Y_0, Y_1, \pm\infty)$-solution is a slowly varying function as $t \uparrow \omega$. It makes a lot of difficulties when conducting investigations.

We need the following auxiliary notation

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t - \omega & \text{as } \ \omega < +\infty, \end{cases} \qquad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i = 0, 1)$$

and in case $\lim\limits_{t\uparrow\omega} |\pi_\omega(t)| \operatorname{sign} y_0^0 = Y_0$,

$$N(t) = \alpha_0 p(t)|\pi_\omega(t)|^{\sigma_0+1}\Theta_0\big(|\pi_\omega(t)| \operatorname{sign} y_0^0\big) \ \text{ as } \ t \in [b, \omega[\,,$$

$$I_0(t) = \alpha_0 \int_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\, d\tau,$$

$$A_\omega^0 = \begin{cases} b & \text{as } \displaystyle\int_b^\omega p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\, d\tau = +\infty, \\ \omega & \text{as } \displaystyle\int_b^\omega p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\, d\tau < +\infty. \end{cases}$$

Here, we choose $b \in [a, \omega[$ in such a way that $|\pi_\omega(t)| \operatorname{sign} y_0^0 \in \Delta_0$ as $t \in [b, \omega[\,$.

[1] If $Y_i = +\infty$ $(Y_i = -\infty)$, we respectively suppose that $y_i^0 > 0$ $(y_i^0 < 0)$.

Theorem 1. *The conditions*

$$Y_0 = \begin{cases} \pm\infty & as \ \ \omega = +\infty, \\ 0 & as \ \ \omega < +\infty, \end{cases} \quad \pi_\omega(t) y_0^0 y_1^0 > 0 \ \ as \ \ t \in [a;\omega[\tag{3}$$

are necessary for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1)*. If the function φ_0 satisfies the condition S and*

$$\lim_{t \uparrow \omega} \frac{R'(|\ln|\pi_\omega(t)||) I_0(t)}{\pi_\omega(t) I_0'(t)} = 0, \tag{4}$$

then the conditions

$$\begin{gathered} y_1^0 I_0(t)(1 - \sigma_0 - \sigma_1) > 0 \ \ as \ \ t \in [a, \omega[\,, \\ \lim_{t \uparrow \omega} y_1^0 |I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = 0 \end{gathered} \tag{5}$$

together with conditions (3) *are necessary and sufficient for the existence of the above-mentioned solutions of equation* (1)*. Moreover, for each $P_\omega(Y_0, Y_1, \pm\infty)$-solution of equation* (1) *the asymptotic representations*

$$\begin{gathered} \frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t)) \exp(R(|\ln|y(t)y'(t)||))} = (1 - \sigma_0 - \sigma_1) I_0(t)[1 + o(1)], \\ \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)} [1 + o(1)] \end{gathered} \tag{6}$$

take place as $t \uparrow \omega$.

If condition (4) is not valid, there takes place the next theorem with another condition (7). Note that if the limit of the left-hand side of equality (4) is equal to infinity, then condition (7) takes place in most cases.

Theorem 2. *Let the function p in equation* (1) *be continuously differentiable in its domain. If the function φ_0 satisfies the condition S and*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) N'(t)}{R'(|\ln|\pi_\omega(t)||) N(t)} = 0, \tag{7}$$

then the conditions

$$\begin{gathered} \alpha_0 y_1^0 (1 - \sigma_0 - \sigma_1) \ln|\pi_\omega(t)| > 0 \ \ as \ \ t \in [a, \omega[\,, \\ \lim_{t \uparrow \omega} y_1^0 \exp\Big(\frac{1}{1 - \sigma_0 - \sigma_1} R\big(|\ln|\pi_\omega(t)||\big)\Big) = Y_1 \end{gathered} \tag{8}$$

together with conditions (3) *are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation* (1)*. Moreover, for every such solution the asymptotic representations*

$$\begin{gathered} \frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t)) \exp(R(|\ln|y(t)y'(t)||))} = \frac{(1 - \sigma_0 - \sigma_1) N(t)}{R'(|\ln|\pi_\omega(t)||)} [1 + o(1)], \\ \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)} [1 + o(1)] \end{gathered} \tag{9}$$

take place as $t \uparrow \omega$.

Proof of Theorem 1. The necessity. Let the function $y : [t_0, \omega[\to \Delta_{Y_0}$ be a $P_\omega(Y_0, Y_1, \pm\infty)$-solution of equation (1). By virtue of (2), the equality

$$\frac{y''(t) y(t)}{(y'(t))^2} = \Big(\frac{y'(t)}{y(t)}\Big)' \cdot \Big(\frac{y'(t)}{y(t)}\Big)^{-2} + 1$$

implies that

$$\Big(\frac{y'(t)}{y(t)}\Big)' \cdot \Big(\frac{y'(t)}{y(t)}\Big)^{-2} = -1 + o(1) \ \text{ as } \ t \uparrow \omega.$$

From this, in view of (2), we have the following asymptotic representations:

$$y(t)=\pi_\omega(t)y'(t)[1+0(1)],\quad y''(t)=o\Big(\frac{y'(t)}{\pi_\omega(t)}\Big)\ \text{ as }\ t\uparrow\omega. \tag{10}$$

From the first formula we get the first one of representations (6) and condition (3). It also follows from (10) that there exists a slowly varying continuously differentiable function $L:\Delta_{Y_0}\to]0,+\infty[$ such that $y(t)=\pi_\omega(t)L(\pi_\omega(t))$. By the condition S, we obtain $\Theta_0(y(t))=\Theta_0(|\pi_\omega(t)|\operatorname{sign}y_0^0)[1+o(1)]$ as $t\uparrow\omega$.

Moreover, from the first formula of (10), using the properties of logarithmic functions and the function R, we find that the asymptotic representations

$$R\big(|\ln|y(t)y'(t)||\big)=R\big(|\ln|\pi_\omega(t)||\big)[1+o(1)],\quad R'\big(|\ln|y(t)y'(t)||\big)=R'\big(|\ln|\pi_\omega(t)||\big)[1+o(1)] \tag{11}$$

take place as $t\uparrow\omega$.

Let us rewrite (1) in the form

$$\frac{y''(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}}=I_0'(t)\exp\big(R(|\ln|y(t)y'(t)||)\big)[1+o(1)]\ \text{ as }\ t\uparrow\omega. \tag{12}$$

Suppose now that condition (4) holds and denote

$$\lim_{t\uparrow\omega}I_0(t)=J_0.$$

Let us show that the function $\exp(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||))$ is slowly varying as $z\to J_0$. Here, I_0^{-1} is the function, inverse to I_0. By conditions (4), (11) and (10), we have

$$\begin{aligned}&\lim_{z\to J_0}\frac{z\big(\exp(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||))\big)'}{\exp\big(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||)\big)}\\
&=\lim_{z\to J_0}\frac{z\exp(R\big(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||\big))R'\big(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||\big)}{I_0'(I_0^{-1}(z))\exp\big(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||)\big)}\Big(\frac{y'(I_0^{-1}(z))}{y(I_0^{-1}(z))}+\frac{y''(I_0^{-1}(z))}{y'(I_0^{-1}(z))}\Big)\\
&=\lim_{z\to J_0}\frac{zR'\big(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||\big)}{I_0'(I_0^{-1}(z))}\,\frac{y'(I_0^{-1}(z))}{y(I_0^{-1}(z))}\Big(1+\frac{y''(I_0^{-1}(z))y(I_0^{-1}(z))}{(y'(I_0^{-1}(z)))^2}\Big)=0.\end{aligned}$$

Therefore, using (12), we get

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))}=(1-\sigma_0-\sigma_1)I_0(t)[1+o(1)]\ \text{ as }\ t\uparrow\omega. \tag{13}$$

Thus representation (6) is valid. Taking into account the sign of the function $y'(t)$, we obtain the first and the second of conditions (5). Using the second of relations (10), by (13) and (12), we have

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)\varphi_1(y'(t))}{|y'(t)|^{1-\sigma_0}}=0.$$

The third of conditions (5) follows from the latter relation, and thus the necessity is proved.

The sufficiency. Suppose that the function φ_1 satisfies the condition S and conditions (3)–(5) of the theorem hold. We denote $g(v_0,v_1)=\exp(R(|\ln|v_0v_1||))L_1(v_1)$, where $L_1:\Delta_{Y_1}\to]0,+\infty[$ is a slowly varying function as $z\to Y_1$ ($z\in\Delta_{Y_1}$) such that

$$L_1(z)=\Theta_1(z)[1+o(1)]\ \text{ as }\ z\to Y_1\ \ (z\in\Delta_{Y_1}),\qquad \lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_1}}}\frac{zL_1'(z)}{L_1(z)}=0. \tag{14}$$

According to the properties of the function R and (14), we get

$$\lim_{\substack{v_i\to Y_i\\ v_i\in\Delta_{Y_i}}}\frac{v_i\frac{\partial g}{\partial v_i}(v_0,v_1)}{g(v_0,v_1)}=0\ \text{ uniformly by }\ v_j\in\Delta_{Y_j},\ \ j\neq i,\ \ i,j=0,1. \tag{15}$$

So, we can take $\widetilde{\Delta}_{Y_i} \subset \Delta_{Y_i}$ $(i = 0,1)$ in a form such that

$$\left| \frac{v_i \frac{\partial g}{\partial v_i}(v_0, v_1)}{g(v_0, v_1)} \right| < \zeta \ \ (i = 0,1) \ \text{ as } \ (v_0, v_1) \in \widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}. \tag{16}$$

Here, $0 < \zeta < \frac{|1-\sigma_0-\sigma_1|}{8}$, ζ is sufficiently small and

$$\widetilde{\Delta}_{Y_i} = \begin{cases} [\widetilde{y}_i^{\,0}, Y_i[\,, & \text{if } \Delta_{Y_i} = [y_i^0, Y_i[\,, \ \ y_i^0 \le \widetilde{y}_i^{\,0} < Y_i, \\ ]Y_i, \widetilde{y}_i^{\,0}], & \text{if } \Delta_{Y_i} =]Y_i, y_i^0], \ \ Y_i > \widetilde{y}_i^{\,0} \ge y_i^0, \end{cases} \quad i = 0,1.$$

Consider now the function

$$F(s_0, s_1) = \begin{pmatrix} \dfrac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)} \\ \dfrac{s_1}{s_0} \end{pmatrix}$$

on the set $\widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}$. Using (15), we have

$$\lim_{\substack{s_1 \to Y_1 \\ s_1 \in \widetilde{\Delta}_{Y_1}}} \frac{s_1 \big(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}\big)'_{s_1}}{\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}} = 1 - \sigma_0 - \sigma_1 \ \text{ uniformly by } \ s_0 \in \widetilde{\Delta}_{Y_0}, \tag{17}$$

$$\lim_{\substack{s_0 \to Y_0 \\ s_0 \in \widetilde{\Delta}_{Y_0}}} \frac{s_0 \big(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}\big)'_{s_0}}{\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}} = -R'(|\ln|s_0 s_1||)\,\mathrm{sign}(s_0) = 0 \ \text{ uniformly by } \ s_1 \in \widetilde{\Delta}_{Y_1}.$$

Therefore, we get

$$\lim_{\substack{s_1 \to Y_1 \\ s_1 \in \widetilde{\Delta}_{Y_1}}} \frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)} = \Upsilon \ \text{ uniformly by } \ s_0 \in \widetilde{\Delta}_{Y_0},$$

$$\Upsilon = \begin{cases} +\infty, & \text{if } Y_1 = \infty, \ 1 - \sigma_0 - \sigma_1 > 0, \ \text{ or } \ Y_1 = 0, \ 1 - \sigma_0 - \sigma_1 < 0, \\ 0, & \text{if } Y_1 = \infty, \ 1 - \sigma_0 - \sigma_1 < 0, \ \text{ or } \ Y_0 = 0, \ 1 - \sigma_0 - \sigma_1 > 0. \end{cases}$$

Let us show that F establishes the one-to-one correspondence between the set $\widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}$ and the set

$$F(\widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}) = \begin{cases} \Big[\dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})}; \Upsilon \Big) \times \Delta_0 & \text{as } \ \dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})} < \Upsilon, \\ \Big(\Upsilon; \dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})} \Big] \times \Delta_0 & \text{as } \ \dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})} > \Upsilon. \end{cases} \tag{18}$$

Here,

$$\Delta_0 = \begin{cases} \Big[\dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}}; Y_0^0 \Big) & \text{as } \ \lambda_0 < 0, \ \dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}} < Y_0^0, \\ \Big(Y_0^0; \dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}} \Big] & \text{as } \ \lambda_0 < 0, \ \dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}} > Y_0^0, \end{cases} \tag{19}$$

$$Y_0^0 = \begin{cases} 0 & \text{as } \ Y_0 = 0, \\ -\infty & \text{as } \ Y_0 = 0, \ \ \omega < +\infty, \\ +\infty & \text{as } \ Y_0 = 0, \ \ \omega = +\infty. \end{cases}$$

Let us consider the behavior of the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}$ on the straight lines

$$s_0 = k s_1, \quad k \in \mathbb{R} \setminus \{0\}. \tag{20}$$

On every such a line we have

$$\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)}=\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\,.$$

Moreover, we get

$$\left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\right)'_{s_1}=\frac{|s_1|^{1-\sigma_0-\sigma_1}}{s_1g(ks_1,s_1}\left(1-\sigma_0-\sigma_1-\frac{s_1L_1'(s_1)}{L(s_1)}-2ks_1R'(|\ln|ks_1^2||)\operatorname{sign}(\ln|ks_1^2|)\right).$$

Taking into account (16), from the latter equality we obtain

$$\operatorname{sign}\left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\right)'_{s_1}=\operatorname{sign}(y_1^0(1-\sigma_0-\sigma_1)).$$

Therefore, the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}$ is strongly monotone on every line of type (20). Suppose that the correspondence F is not of one-to-one type. Then

$$\exists\,(p_0,p_1),(q_0,q_1)\in\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1},\ \ p_0,p_1)\neq(q_0,q_1):\quad F(p_0,p_1)=F(q_0,q_1).$$

Taking into account the definitions of the sets $\widetilde{\Delta}_{Y_0},\widetilde{\Delta}_{Y_1}$, the latter equality implies that

$$\frac{|p_1|^{1-\sigma_0-\sigma_1}}{g(p_0,p_1)}=\frac{|q_1|^{1-\sigma_0-\sigma_1}}{g(q_0,q_1)}\,,\quad\frac{p_0}{p_1}=\frac{q_0}{q_1}=c\in\mathbb{R}\setminus\{0\}.\tag{21}$$

Thus, the points (p_0,p_1) and (q_0,q_1) lie on a line of type (20). But in this case equalities (21) fail to take place, because the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_1,cs_1)}$ is strongly monotone on the line. Therefore there exists the inverse function $F^{-1}:F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})\to\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1}$. Taking into account the character of the function F, we have

$$F^{-1}(w_0,w_1)=\begin{pmatrix}F_1^{-1}(w_0,w_1)\\F_0^{-1}(w_0,w_1)\end{pmatrix}=\begin{pmatrix}F_1^{-1}(w_0,w_1)\\\dfrac{1}{w_0}F_1^{-1}(w_0,w_1)\end{pmatrix}.$$

Since by (16) the Jakobian of the function F is different from zero as $(s_0,s_1)\in\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1}$, the function F^{-1} is continuously differentiable on $F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})$.

Taking

$$\begin{cases}\dfrac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y')\exp(R(|\ln|y(t)y'(t)||))}=(1-\sigma_0-\sigma_1)I_0(t)\operatorname{sign}(y')[1+z_1(x)],\\\dfrac{y'(t)}{y(t)}=\dfrac{1}{\pi_\omega(t)}\,[1+z_2(x)],\end{cases}\tag{22}$$

where

$$x=\beta\ln|\pi_\omega(t)|,\quad\beta=\begin{cases}1&\text{as }\ \omega=+\infty,\\-1&\text{as }\ \omega<\infty,\end{cases}$$

we can reduce equation (1) to the system

$$\begin{cases}z_1'=\beta G_0(x)[1+z_1]\Bigg(\left(1-\sigma_0-\sigma_1-\dfrac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\right)\\\qquad\times\dfrac{K_1(x,z_1,z_2)}{[1+z_1]|1+z_2|^{\sigma_0}}-K_2(x,z_1,z_2)\,\dfrac{R'(|\ln|\pi_\omega(t)||)}{G_0(x)}\left(1+\dfrac{K_1(x,z_1,z_2)G_0(x)}{[1+z_1][1+z_2]^{\sigma_0-1}}\right)-1\Bigg),\\z_2'=\beta[1+z_2]\left(\dfrac{G_0(x)K_1(x,z_1,z_2)}{(1-\sigma_0-\sigma_1)[1+z_1]\,[1+z_2]^{\sigma_0}}-z_2\right),\end{cases}\tag{23}$$

where

$$\begin{aligned}
\Psi_0(x,z_1,z_2) &= F_0^{-1}\Big((1-\sigma_0-\sigma_1)I_0(t(X))[1+z_1(x)],\,\frac{1}{\pi_\omega(t(x))}\,[1+z_2(x)]\Big),\\
\Psi_1(x,z_1,z_2) &= F_1^{-1}\Big((1-\sigma_0-\sigma_1)I_0(t(x))[1+z_1(x)],\,\frac{1}{\pi_\omega(t(x))}\,[1+z_2(x)]\Big),\\
G_0(x) &= \frac{\pi_\omega(t(x))I_0'(t(x))}{I_0(t(x))}\,,\\
K_1(x,z_1,z_2) &= \frac{\Theta_0(\Psi_0(t(x),z_1,z_2))}{(1-\sigma_0-\sigma_1)\Theta_0(|\pi_\omega(t(x))|\operatorname{sign} y_0^0)}\,,\\
K_2(x,z_1,z_2) &= \frac{R'\big(|\ln|\Psi_0(t(x),z_1,z_2)\Psi_1(t(x),z_1,z_2)||\big)}{R'(|\ln|\pi_\omega(t(x))||)}\,.
\end{aligned}$$

By (3), it is clear that

$$\lim_{t\uparrow\omega}\frac{1}{\pi_\omega(t)}=Y_1.$$

Moreover, it follows from the first and the second of conditions (5) that

$$\lim_{t\uparrow\omega}(1-\sigma_0-\sigma_1)I_0(t)=\Upsilon.$$

Therefore, we can choose $t_0\in[a,\omega[$ in a form such that

$$\begin{pmatrix}(1-\sigma_0-\sigma_1)I_0(t)[1+z_1(x)]\\ \dfrac{1}{\pi_\omega(t)}\,[1+z_2(x)]\end{pmatrix}\in F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})\ \text{ as }\ t\in[t_0,\omega[,\quad |z_i|\le\frac{1}{2}\,,\quad i=1,2.$$

Then we consider the system of differential equations (23) on the set

$$\Omega=[x_0,+\infty[\times D,\ \text{ where }\ x_0=\beta\ln|\pi_\omega(t_0)|,$$
$$D=\Big\{(z_1,z_2):\ |z_i|\le\frac{1}{2}\,,\ i=1,2\Big\}.$$

Rewrite the system in the form

$$\begin{cases} z_1'=G_0(x)(A_{11}z_1+A_{12}z_2+R_1(x,z_1,z_2)+R_2(z_2)),\\ z_2'=A_{21}z_1+A_{22}z_2+R_3(x,z_1,z_2)+R_4(z_2),\end{cases}\tag{24}$$

where

$$A_{11}=A_{22}=-\beta,\quad A_{12}=-\beta\sigma_0,\quad A_{21}=0,$$

$$\begin{aligned}
R_1(x,z_1,z_2) &= -\beta[1+z_1]\Big(K_2(x,z_1,z_2)\,\frac{R'(|\ln|\pi_\omega(t(x))||)}{G_0(x)}\Big(1+\frac{K_1(x,z_1,z_2)G_0(x)}{(1+z_1)(1+z_2)^{\sigma_0-1}}\Big)\\
&\qquad+\frac{K_1(x,z_1,z_2)}{(1+z_1)|1+z_2|^{\sigma_0}}\,\frac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\Big)\\
&\quad+\beta\,\frac{|K_1(x,z_1,z_2)|(1-\sigma_0-\sigma_1)-1}{|1+z_2|^{\sigma_0}}\,,\\
R_2(z_2) &= \beta\big(|1+z_2|^{-\sigma_0}+\sigma_0z_2\big),\\
R_3(x,z_1,z_2) &= \beta\,\frac{[1+z_2]G_0(x)K_1(x,z_1,z_2)}{(1-\sigma_0-\sigma_1)[1+z_1]\,[1+z_2]^{\sigma_0}}\,,\quad R_4(z_2)=-\beta z_2^2.
\end{aligned}$$

For $(w_0,w_1)\in F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})$, we have the equality

$$\frac{|F_1^{-1}(w_0,w_1)|^{1-\sigma_0-\sigma_1}}{g(F_0^{-1}(w_0,w_1),F_1^{-1}(w_0,w_1))}=w_1.$$

Since (16), (3) and the second of conditions (4) are filfilled, it follows from this equality that

$$\lim_{x\to\infty} \Psi_i(t(x), z_1, z_2) = Y_i \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2};\frac{1}{2}\Big]$$

as $i = 0, 1$. Therefore, by (14), we have

$$\lim_{x\to\infty} \frac{\Psi_1(t(x), z_1, z_2)L_1'(\Psi_1(t(x), z_1, z_2))}{L_1(\Psi_1(t(x), z_1, z_2))} = 0 \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2}\,;\frac{1}{2}\Big]. \tag{25}$$

Moreover, it follows from the properties of the function F and conditions (3)–(5) that the function $\Psi_1(t, z_1, z_2)$ is slowly varying as $t \uparrow \omega$ uniformly by $(z_1, z_2) \in [-\frac{1}{2}\,; \frac{1}{2}] \times [-\frac{1}{2}\,; \frac{1}{2}]$. Since

$$\Psi_0(t, z_1, z_2) = \frac{\pi_\omega(t)\Psi_1(t, z_1, z_2)}{1 + z_2}\,,$$

and the function φ_0 together with the logarithmic function satisfy the condition S, we have

$$\lim_{x\to\infty} K_1(x, z_1, z_2) = \frac{1}{1 - \sigma_0 - \sigma_1} \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2}\,;\frac{1}{2}\Big], \tag{26}$$

$$\lim_{x\to\infty} K_2(x, z_1, z_2) = 1 \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2}\,;\frac{1}{2}\Big]. \tag{27}$$

Since the function R is regularly varying at infinity of order μ, $0 < \mu < 1$, we obtain

$$\lim_{t\uparrow\omega} R'\big(|\ln|\pi_\omega(t)||\big) = 0. \tag{28}$$

Third of conditions (5) implies

$$\lim_{x\to\infty} G_0(x) = 0. \tag{29}$$

By (4) and (25)–(29), we get the limit relations

$$\lim_{|z_1|+|z_2|\to 0} \frac{R_i(z_2)}{|z_1| + |z_2|} = 0 \ \text{ uniformly by } x: \ x \in \,]x_0, +\infty[$$

as $i = 2, 4$ and

$$\lim_{x\to+\infty} R_i(x, z_1, z_2) = 0 \ \text{ uniformly by } z_1, z_2: \ (z_1, z_2) \in D$$

as $i = 1, 3$.

By the definition of the function G_0 it is clear that $\int\limits_{x_0}^{\infty} G_0(x)\,dx = \infty$.

So, for the system of differential equations (24) all conditions of Theorem 2.8 from [2] are fulfilled. According to this theorem, system (24) has at least one solution $\{z_i\}_{i=1}^2 : [x_1, +\infty[\, \to \mathbb{R}^2$ $(x_1 \geq x_0)$ tending to zero as $x \to +\infty$. By (22) and (23), this solution corresponds to such solution y of equation (1) that admits asymptotic representations (6) as $t \uparrow \omega$. By our representations and (1), it is clear that the obtained solution is indeed the $P_\omega(Y_0, Y_1, \pm\infty)$-solution. □

Proof of Theorem 2. The necessity. Let the function $y : [t_0, \omega[\, \to R$ be a $P_\omega(Y_0, Y_1, \pm\infty)$-solution of equation (1). We obtain (10) and (12) just as in the proof of Theorem 1. The second of representations (9) follows from these relations. Let us rewrite (12) by using the first of asymptotic representations (10) in the form

$$\frac{y''(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}} = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))y'(t)[1 + o(1)]}{y(t)} \ \text{ as } \ t \uparrow \omega. \tag{30}$$

Suppose that conditions (7) are valid. By the properties of the function R, there exists a twice continuously differentiable function $\widetilde{R} :]0; +\infty[\, \to\,]0; +\infty[$ such that

$$\widetilde{R}(z) = R(z)[1 + o(1)], \quad \widetilde{R}'(z) = R'(z)[1 + o(1)], \quad \lim_{z\to+\infty} \frac{\widetilde{R}''(z)R(z)}{(\widetilde{R}'(z))^2} = \frac{\mu}{\mu - 1}\,. \tag{31}$$

By conditions (2), (11), (31), (7) and the first of asymptotic representations (10), from the equality

$$\left(\frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\right)' = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))y'(t)}{y(t)}$$
$$\times\left(\frac{y(t)}{\pi_\omega(t)y'(t)}\left(\frac{N'(t)\pi_\omega(t)}{N(t)\widetilde{R}'(|\ln|\pi_\omega(t)||)} - \frac{\widetilde{R}(|\ln|\pi_\omega(t)||)}{(\widetilde{R}'(|\ln|\pi_\omega(t)||))^2}\frac{\widetilde{R}''(|\ln|\pi_\omega(t)||)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\right) + 1 + \frac{y(t)y''(t)}{(y'(t))^2}\right)$$

we have the following representation

$$\left(\frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\right)' = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\,[1+o(1)]$$

as $i = 1, 3$. So, using the properties of the function φ_1 and (30), we get

$$\frac{y'(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}} = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\,(1-\sigma_0-\sigma_1)[1+o(1)] \text{ as } t\uparrow\omega.$$

The first of representations (9) follows from this relations by using (31). Taking into account the sign of the function $y'(t)$, we obtain conditions (8). The necessity is proved.

The sufficiency. Suppose that the function φ_1 satisfies the condition S and there take place conditions (3), (7), (8). Consider the twice continuously differentiable function $\widetilde{R}:]0;+\infty[\to]0;+\infty[$ that satisfies (31), just as in the proof of Theorem 1. We use the same function F with the same properties as in the proof of Theorem 1.

Taking

$$F(y'(t), y(t)) = \begin{pmatrix} \dfrac{|1-\sigma_0-\sigma_1|N(t)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\,[1+z_1(x)] \\ \dfrac{1}{\pi_\omega(t)}\,[1+z_2(x)] \end{pmatrix},$$

where

$$x = \beta\ln|\pi_\omega(t)|, \quad \beta = \begin{cases} 1 & \text{as } \omega = +\infty, \\ -1 & \text{as } \omega = +\infty, \end{cases}$$

we can reduce equation (1) to the system

$$\begin{cases} z_1' = \beta G_0(x)\Bigg[\dfrac{K_1(x,z_1,z_2)|1+z_2|^{\sigma_0}}{(1-\sigma_0-\sigma_1)}\Big(1-\sigma_0-\sigma_1-\dfrac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\Big) \\ \qquad -G_1(x)[1+z_1] - K_2(x,z_1,z_2)\Big(1+z_1+\dfrac{K_1(x,z_1,z_2)G_0(x)|1+z_2|^{\sigma_0}}{|1-\sigma_0-\sigma_1|}\Big) \\ \qquad\qquad +\dfrac{G_2(x)[1+z_1]}{R(|\ln|\pi_\omega(t)||)}\Bigg], \\ z_2' = \beta[1+z_2]\left[\dfrac{K_1(x,z_1,z_2)G_0(x)|1+z_2|^{\sigma_0}}{|1-\sigma_0-\sigma_1|[1+z_1]} - z_2\right], \end{cases} \tag{32}$$

where

$$G_0(x) = \widetilde{R}'\big(|\ln|\pi_\omega(t(x))||\big), \quad G_1(x) = \frac{\pi_\omega(t(x))N'(t(x))}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)N(t(x))},$$
$$G_2(x) = \frac{\widetilde{R}''\big(|\ln|\pi_\omega(t(x))||\big)\widetilde{R}(|\ln|\pi_\omega(t(x))||)}{(\widetilde{R}'(|\ln|\pi_\omega(t(x))||))^2},$$
$$\Psi_0(x,z_1,z_2) = F_0^{-1}\left(\frac{(1-\sigma_0-\sigma_1)N(t(x))}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)}[1+z_1], \frac{1}{\pi_\omega(t(x))}[1+z_2]\right),$$
$$\Psi_1(x,z_1,z_2) = F_1^{-1}\left(\frac{(1-\sigma_0-\sigma_1)N(t(x))}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)}[1+z_1], \frac{1}{\pi_\omega(t(x))}[1+z_2]\right),$$

$$K_1(x,z_1,z_2)=\frac{\Theta_0(\Psi_0(t(x),z_1,z_2))}{\Theta_0(|\pi_\omega(t(x))|)}\,,$$
$$K_2(x,z_1,z_2)=\frac{\widetilde{R}'(|\ln|\Psi_0(t(x),z_1,z_2)\Psi_1(t(x),z_1,z_2)||)}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)}\,.$$

We get

$$\lim_{k\to\infty}K_i(x,z_1,z_2)=1 \text{ uniformly by } (z_1,z_2)\in\Big[-\frac12;\frac12\Big]\times\Big[-\frac12;\frac12\Big], \tag{33}$$

as in the proof of Theorem 1.

By (3), it is clear that

$$\lim_{t\uparrow\omega}\frac{1}{\pi_\omega(t)}=Y_0^0.$$

Moreover, it follows from (7) and (8) that

$$\lim_{t\uparrow\omega}\frac{|1-\sigma_0-\sigma_1|N(t)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}=\Upsilon.$$

Therefore, we can choose $t_0\in[a,\omega[$ such that

$$\begin{pmatrix}\dfrac{(1-\sigma_0-\sigma_1)N(t)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}[1+z_1(x)]\\ \dfrac{1}{\pi_\omega(t)}[1+z_2(x)]\end{pmatrix}\in F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1}) \text{ as } t\in[t_0,\omega[,\quad |z_i|\le\frac12 \ (i=1,2).$$

Further, we consider system (32) on the set

$$\Omega=[x_0,+\infty[\times D, \text{ where } x_0=\beta\ln|t_0|,$$
$$D=\Big\{(z_1,z_2):\ |z_i|\le\frac12\ (i=1,2)\Big\}$$

and rewrite system (32) in the form

$$\begin{cases}z_1'=G_0(x)\big[A_{11}z_1+A_{12}z_1+R_1(x,z_1,z_2)+R_2(z_2)\big],\\ z_2'=A_{21}z_1+A_{22}z_2+R_3(x,z_1,z_2)+R_4(z_2),\end{cases}$$

where

$$A_{11}=A_{22}=-\beta,\quad A_{12}=\beta\sigma_0,\quad A_{21}=0,$$
$$\begin{aligned}R_1(x,z_1,z_2)=\beta\bigg(&(K_1(x,z_1,z_2)-1)|1+z_2|^{\sigma_0}-(K_2(x,z_1,z_2)-1)-G_1(x)[1+z_1]\\&+\frac{G_2(x)}{R(|\ln|\pi_\omega(x)||)[1+z_2]}-\frac{K_1(x,z_1,z_2)}{|1-\sigma_0-\sigma_1|}\,\frac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\\&-\frac{G_0(x)K_2(x,z_1,z_2)K_1(x,z_1,z_2)}{|1-\sigma_0-\sigma_1|}|1+z_2|^{\sigma_0}\bigg),\end{aligned}$$
$$R_2(z_2)=\big(|1+z_2|^{\sigma_0}-\sigma_0z_2-1\big),$$
$$R_3(x,z_1,z_2)=\beta\,\frac{G_0(x)K_1(x,z_1,z_2)}{|1-\sigma_0-\sigma_1|}\,\frac{|1+z_2|^{\sigma_0+1}}{1+z_1}\,,$$
$$R_4(z_2)=-\beta z_2^2.$$

It follows from (3) and (7) that

$$\lim_{x\to\infty}G_i(x)=0\ \ (i=0,1),\qquad \lim_{x\to\infty}G_2(x)=\frac{\mu-1}{\mu}\,.$$

By the character of the function G_0, it is clear that

$$\int\limits_{x_0}^{\infty} G_0(x)\,dx = \infty.$$

So, using (33), we have

$$\lim_{|z_1|+|z_2|\to 0} \frac{R_i(z_2)}{|z_1|+|z_2|} = 0 \ \text{ uniformly by } x:\ x \in]x_0, +\infty[$$

as $i = 2, 4$ and

$$\lim_{x\to+\infty} R_i(x, z_1, z_2) = 0 \ \text{ uniformly by } z_1, z_2:\ (z_1, z_2) \in D$$

as $i = 1, 3$.

Thus, for the system of differential equations (32) all conditions of Theorem 2.8 from [2] are fulfilled. According to this theorem, system (32) has at least one solution $\{z_i\}_{i=1}^2 : [x_1, +\infty[\to \mathbb{R}^2$ $(x_1 \geq x_0)$ that tends to zero as $x \to +\infty$. This solution corresponds to such solution y of equation (1) that admits asymptotic representations (9) as $t \uparrow \omega$. By our representations and (1), it is clear that the obtained solution is indeed the $P_\omega(Y_0, Y_1, \pm\infty)$-solution. $\square$

References

[1] M. A. Belozerova, Asymptotic representations of solutions with slowly varying derivatives of essential nonlinear differential equations of the second order. (Russian) *Bulletin of the Odessa National University. Mathematics and Mechanics* **20** (2015), no. 1(25), 7–19.

[2] V. M. Evtukhov and A. M. Samoilenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. (Russian) *Ukr. Mat. Zh.* **62** (2010), no. 1, 52–80; translation in *Ukr. Math. J.* **62** (2010), no. 1, 56–86.

[3] G. A. Gerzhanovskaya, Asymptotic representations of rapidly varying solutions of essentially nonlinear differential equations of the second order. *Nonlinear Oscil.* **20** (2017), no. 3, 328–345.

[4] E. Seneta, *Regularly Varying Functions*. Lecture Notes in Mathematics, Vol. 508. Springer-Verlag, Berlin–New York, 1976.

(Received 01.05.2018)

Authors' address:

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.

E-mail: `marbel@ukr.net`; `greta.odessa@gmail.com`