Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 39–78

Tengiz Buchukuri, Otar Chkadua, David Natroshvili

MIXED AND CRACK TYPE PROBLEMS OF THE THERMOPIEZOELECTRICITY THEORY WITHOUT ENERGY DISSIPATION

Abstract. In this paper, we study mixed and crack type boundary value problems of the linear theory of thermopiezoelectricity for homogeneous isotropic bodies possessing the inner structure and containing interior cracks. The model under consideration is based on the Green–Naghdi theory of thermopiezoelectricity without energy dissipation. This theory permits propagation of thermal waves at finite speed. Using the potential method and the theory of pseudodifferential equations on manifolds with boundary we prove existence and uniqueness of solutions and analyze their smoothness and asymptotic properties. We describe an efficient algorithm for finding the singularity exponents of the thermo-mechanical and electric fields near the crack edges and near the curves where different types of boundary conditions collide. By explicit calculations it is shown that the stress singularity exponents essentially depend on the material parameters, in general.

2010 Mathematics Subject Classification. 35B65, 35S15, 45M05, 47G30, 74A15, 74F05, 74F15, 74G40, 74G70.

Key words and phrases. Thermopiezoelectricity without energy dissipation, bodies with microstructure, mixed boundary value problem, crack problem, potential method, pseudodifferential equations, stress singularities.

რეზიუმე. ამ სტატიაში ჩვენ შევისწავლით თერმოპიეზოელექტროობის წრფივი თეორიის შერეულ და ბზარის ტიპის სასაზღვრო ამოცანებს შინაგანი სტრუქტურის მქონე ერთგვაროვანი იზოტროპული სხეულებისთვის, რომელთაც გააჩნია შინაგანი სტრუქტურა და შეიცავს შიდა ბზარებს. განხილული მოდელი ეფუძნება გრინ-ნახდის თერმოპიეზოელექტროობის თეორიას ენერგიის დისიპაციის გარეშე. ამ თეორიაში დასაშვებია თერმული ტალღების გავრცელება სასრული სიჩქარით. პოტენციალთა მეთოდისა და საზღვრიან მრავალსახეობებზე გავრცელებული ფსევდოდიფერენციულ განტოლებათა თეორიის გამოყენებით ჩვენ ვამტკიცებთ ამოცანების ამონახსნთა არსებობასა და ერთადერთობას, შევისწავლით მათ სიგლუვესა და ასიმპტოტურ თვისებებს. ჩვენ აღვწერთ ეფექტურ ალგორითმს თერმომექანიკური და ელექტრული ველების სინგულარობის ექსპონენტების გამოსათვლელად ბზარის კიდეების მახლობლობაში და ისეთი წირების მიდამოში, სადაც სხვადასხვა ტიპის სასაზღვრო პირობები ერთმანეთს ხვდება. პირდაპირი გამოთვლებით დგინდება, რომ ძაბვის სინგულარობის ექსპონენტები საზოგადოდ არსებითად არის დამოკიდებული მატერიალურ პარამეტრებზე.

1 Introduction

Theories of thermo-mechanics of continua consistent with a finite speed propagation of heat recently are attracting increasing attention. In contrast to the conventional heat transfer theory, these non-classical refined theories involve a hyperbolic-type heat transport equation, and are motivated by experiments exhibiting the actual occurrence of wave-type heat transport (second sound). Several authors have formulated these theories on different grounds, and a wide variety of problems revealing characteristic features of the theories has been investigated.

Green and Naghdi [13, 14] in 1993 developed a thermo-mechanical theory of thermoelastic bodies based on an entropy balance law rather than an entropy inequality (hereinafter we refer this theory as Green–Naghdi theory). The linearized form of this theory does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed. Moreover, the heat flux vector is determined by the same potential function that determines the stress. The thermal waves propagate with finite speeds and the solution has no dissipative term.

Almost complete historical and bibliographical notes to this direction can be found in the reference [16] where the dynamical equations of the thermopiezoelectricity without energy dissipation are derived on the basis of the Green–Naghdi theory established in [13,14] and Eringen's results obtained in [9,10].

In the present paper we consider the pseudo-oscillation equations obtained by the Laplace transform from the dynamical equations derived by Ieşan in [16] for homogeneous isotropic solids possessing thermopiezoelectricity properties without energy dissipation. We formulate the basic, mixed and crack type boundary value problems (BVP) and prove existence and uniqueness of solutions. Our main tools are the potential method and the theory of pseudodifferential equations. Solutions to the mixed and crack type boundary value problems have singularities near the crack edges and near the lines where the different types of boundary conditions collide, regardless of the smoothness of the boundary surfaces and given boundary data. Throughout the paper we shall refer to such lines as *exceptional curves*. We carry out a detailed theoretical investigation of regularity and asymptotic properties of thermo-mechanical and electric fields near the exceptional curves. By explicit calculations we show that the stress singularity exponents essentially depend on the material parameters, in general. We describe an efficient algorithm for finding the singularity exponents of the thermo-mechanical and electric fields. The obtained asymptotic formulas allow us to establish optimal regularity results for solutions.

2 Basic equations

Let $\Omega = \Omega^+$ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a simply connected piecewise smooth Lipschitz boundary $S = \partial\Omega$, and $\overline{\Omega} = \Omega \cup S$. Throughout the paper $n(x)$ stands for the outward unit normal vector to S at the point $x \in S$. We assume also that the origin of the co-ordinate system belongs to Ω.

By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to S from Ω and by $C_0^\infty(\Omega)$ the space of infinitely differentiable test functions with compact supports in $\Omega \subset \mathbb{R}^3$.

The symbols $\{\,\cdot\,\}_S^+$ and $\{\,\cdot\,\}_S^-$ designate one sided limits on S from Ω and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$, respectively. We drop the subscript S if it does not lead to misunderstanding.

By L_p, $L_{p,loc}$, W_p^r, $W_{p,loc}^r$, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) are denoted the Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [23]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

We use the notation $v_{i_1 \dots i_m}$ for the components of tensor v of order m and employ the usual Einstein summation convention where the subscripts range over the integers $\{1, 2, 3\}$. Partial derivatives with respect to spatial variable x_j we denote by $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$, while a superposed dot denotes partial differentiation with respect to the time variable t.

We consider an elastic body that at some instant occupies the region Ω of the Euclidean three-dimensional space and is bounded by a piecewise smooth Lipschitz surface S.

We restrict our consideration to the linear theory of homogeneous isotropic thermoelastic bodies developed by Green and Naghdi [13,14]. According to this theory the system of the governing equations consists of the following field equations [16]:

- The local form of the conservation law of linear momentum

$$\partial_j t_{ji} + \rho_0 f_i = \rho_0 \ddot{u}_i, \tag{2.1}$$

where t_{ji} is the stress tensor, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, f_i is the external body force per unit mass, and ρ_0 is the density in the reference configuration.

- The local form of the conservation law of the moment of momentum

$$\partial_j m_{ji} + \varepsilon_{ijk} t_{jk} + \rho_0 X_i = I_{ij} \ddot{\phi}_j, \tag{2.2}$$

where m_{ij} is the couple stress tensor, ε_{ijk} is the alternating Levi-Civita symbol, X_i is the external body couple per unit mass, I_{ij} are the coefficients of inertia, and ϕ_i is the microrotation vector.

- Maxwell's equations for the quasi-static electric fields

$$\partial_j D_j = f \ \text{ and } \ E_k = -\partial_k \psi, \tag{2.3}$$

where D is the electric displacement field, f is the density of free charge, E is the electric intensity, and ψ is the electric potential.

- The local form of energy balance

$$\rho_0 \dot{e} = t_{ij} \dot{e}_{ij} + m_{ij} \dot{\varkappa}_{ij} + \pi_i \dot{\zeta}_i + \epsilon \dot{\varphi} + \rho_0 s \theta + \partial_i(\Phi_i \theta) + E_i \dot{D}_i,$$

where e is the internal energy per unit mass, φ is the microstretch function, π_i is the microstretch stress vector, s is the external rate of supply of entropy per unit mass, θ is the absolute temperature, Φ_i are components of the entropy flux vector,

$$e_{ij} = \partial_i u_j + \varepsilon_{jik} \phi_k, \quad \varkappa_{ij} = \partial_i \phi_j, \quad \zeta_i = \partial_i \varphi \tag{2.4}$$

and

$$\epsilon = \partial_j \pi_j + \rho_0 \mathcal{F} - j_0 \ddot{\varphi}, \tag{2.5}$$

where j_0 is the microstretch inertia, and $\mathcal{F}$ is the microstretch body force.

- The equation of entropy

$$\rho_0 T_0 \dot{\eta} = q_{j,j} + \rho_0 Q, \tag{2.6}$$

where η is the entropy per unit mass and unit time, T_0 is the initial reference temperature, that is, the temperature in the natural state in the absence of deformation and electromagnetic field, q_i is the heat flux vector

$$q_i = T_0 \Phi_i,$$

and Q is the external rate of supply of heat per unit mass.

The quantities t_{ij}, m_{ij}, π_i, ϵ, D_i, q_i and $\rho_0\eta$ for homogeneous isotropic media can be expressed via u_i, ϕ_i, φ, ψ, ϑ by the following *constitutive relations* [16]:

$$t_{ij} = \lambda e_{rr} \delta_{ij} + (\mu + \varkappa) e_{ij} + \mu e_{ji} + \lambda_0 \varphi \delta_{ij} - \beta_0 T \delta_{ij}, \tag{2.7}$$
$$m_{ij} = \alpha \varkappa_{rr} \delta_{ij} + \beta \varkappa_{ji} + \gamma \varkappa_{ij} + b_0 \varepsilon_{ijk} \zeta_k + \lambda_1 \varepsilon_{jik} E_k + \nu_2 \varepsilon_{ijk} \partial_k \vartheta, \tag{2.8}$$
$$\pi_i = a_0 \zeta_i + \lambda_2 E_i + b_0 \varepsilon_{rsi} \varkappa_{rs} + \nu_1 \partial_i \vartheta, \tag{2.9}$$
$$\epsilon = \lambda_0 e_{rr} + \xi_0 \varphi - c_0 T, \tag{2.10}$$
$$D_i = -\lambda_1 \varepsilon_{ijk} \varkappa_{kj} - \lambda_2 \zeta_i - \nu_3 \partial_i \vartheta + \chi E_i, \tag{2.11}$$
$$q_i = T_0(\nu_2 \varepsilon_{rsi} \varkappa_{rs} + \nu_1 \zeta_i + k \partial_i \vartheta + \nu_3 E_i), \tag{2.12}$$
$$\rho_0 \eta = \beta_0 e_{rr} + c_0 \varphi + a T, \tag{2.13}$$

where ϑ is the temperature change to a reference temperature T_0,

$$T = \theta - T_0, \quad \vartheta = \int_{t_0}^{t} T\,dt,$$

δ_{ij} is the Kronecker delta and λ, μ, $\varkappa$, λ_0, β_0, α, β, γ, λ_1, ν_1, a_0, λ_2, ν_2, ξ_0, c_0, a, k, ν_3, and χ, are constitutive constants, then the field equations (2.1)–(2.3), (2.5), (2.6), read as [16]

$$(\mu+\varkappa)\partial_j\partial_j u_i + (\lambda+\mu)\partial_j\partial_i u_j + \varkappa\varepsilon_{ijk}\partial_j\phi_k + \lambda_0\partial_i\varphi - \beta_0\partial_i\dot{\vartheta} + \rho_0 f_i = \rho_0\ddot{u}_i, \tag{2.14}$$

$$\gamma\partial_j\partial_j\phi_i + (\alpha+\beta)\partial_j\partial_i\phi_j + \varkappa\varepsilon_{ijk}\partial_j u_k - 2\varkappa\phi_i + \rho_0 X_i = I_0\ddot{\phi}_i, \tag{2.15}$$

$$(a_0\partial_j\partial_j - \xi_0)\varphi - \lambda_2\partial_j\partial_j\psi + \nu_1\partial_j\partial_j\vartheta - \lambda_0\partial_j u_j + c_0\dot{\vartheta} + \rho_0\mathcal{F} = j_0\ddot{\varphi}, \tag{2.16}$$

$$\lambda_2\partial_j\partial_j\varphi + \chi\partial_j\partial_j\psi + \nu_3\partial_j\partial_j\vartheta = -f, \tag{2.17}$$

$$k\partial_j\partial_j\vartheta - \beta_0\partial_j\dot{u}_j - a\ddot{\vartheta} - c_0\dot{\varphi} + \nu_1\partial_j\partial_j\varphi - \nu_3\partial_j\partial_j\psi = -\frac{1}{T_0}\,\rho_0 Q, \tag{2.18}$$

Let $v = (e_{ij}, \varkappa_{ij}, \zeta_i, \varphi, T, \vartheta_i, E_i)$ and $v' = (e'_{ij}, \varkappa'_{ij}, \zeta'_i, \varphi', T', \vartheta'_i, E'_i)$. Introduce a symmetric bilinear form

$$\begin{aligned} B(v,v') := {} & \lambda e_{ii}e'_{jj} + (\mu+\varkappa)e_{ij}e'_{ij} + \mu e_{ji}e'_{ij} + \lambda_0(e_{jj}\varphi' + e'_{jj}\varphi) + \xi_0\varphi\varphi' \\ & + k\vartheta_j\vartheta'_j + \alpha\varkappa_{ii}\varkappa'_{jj} + \beta\varkappa_{ji}\varkappa'_{ij} + \gamma\varkappa_{ij}\varkappa'_{ij} + b_0\varepsilon_{ijk}(\varkappa_{ij}\zeta'_k + \varkappa'_{ij}\zeta_k) \\ & + \nu_2\varepsilon_{ijk}(\varkappa_{ij}\vartheta'_k + \varkappa'_{ij}\vartheta_k) + a_0\zeta_i\zeta'_i + \nu_1(\vartheta_i\zeta'_i + \vartheta'_i\zeta_i) + \chi E_iE'_i + aTT'. \end{aligned} \tag{2.19}$$

The corresponding quadratic form $B(v,v)$ can be represented as follows:

$$\begin{aligned} B(v,v) = {} & F_1(e_{11}, e_{22}, e_{33}, \varphi) + F_2(e_{12}, e_{21}, e_{13}, e_{31}, e_{23}, e_{32}) + F_3(\varkappa_{11}, \varkappa_{22}, \varkappa_{33}) \\ & + F_4(\varkappa_{12}, \varkappa_{13}, \varkappa_{21}, \varkappa_{23}, \varkappa_{31}, \varkappa_{32}, \zeta_1, \zeta_2, \zeta_3, \vartheta_1, \vartheta_2, \vartheta_3) + F_5(E_1, E_2, E_3, T), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} F_1(e_{11}, e_{22}, e_{33}, \varphi) = {} & (\lambda+2\mu+\varkappa)e_{11}e_{11} + \lambda e_{11}e_{22} + \lambda e_{11}e_{33} + \lambda_0 e_{11}\varphi + \lambda e_{22}e_{11} \\ & + (\lambda+2\mu+\varkappa)e_{22}e_{22} + \lambda e_{22}e_{33} + \lambda_0 e_{22}\varphi + \lambda e_{33}e_{11} + \lambda e_{33}e_{22} \\ & + (\lambda+2\mu+\varkappa)e_{33}e_{33} + \lambda_0 e_{33}\varphi + \lambda_0\varphi e_{11} + \lambda_0\varphi e_{22} + \lambda_0\varphi e_{33} + \xi_0\varphi^2, \\ F_2(e_{12}, e_{21}, e_{13}, e_{31}, e_{23}, e_{32}) = {} & (\mu+\varkappa)e_{12}e_{12} + \mu e_{12}e_{21} + (\mu+\varkappa)e_{13}e_{13} \\ & + \mu e_{13}e_{31} + \mu e_{21}e_{12} + (\mu+\varkappa)e_{21}e_{21} + \mu e_{23}e_{32} + (\mu+\varkappa)e_{23}e_{23} \\ & + \mu e_{31}e_{13} + (\mu+\varkappa)e_{31}e_{31} + \mu e_{32}e_{23} + (\mu+\varkappa)e_{32}e_{32}, \\ F_3(\varkappa_{11}, \varkappa_{22}, \varkappa_{33}) = {} & (\alpha+\beta+\gamma)\varkappa_{11}\varkappa_{11} + \alpha\varkappa_{11}\varkappa_{22} + \alpha\varkappa_{11}\varkappa_{33} + \alpha\varkappa_{22}\varkappa_{11} \\ & + (\alpha+\beta+\gamma)\varkappa_{22}\varkappa_{22} + \alpha\varkappa_{22}\varkappa_{33} + \alpha\varkappa_{33}\varkappa_{11} + \alpha\varkappa_{33}\varkappa_{22} + (\alpha+\beta+\gamma)\varkappa_{33}\varkappa_{33}, \\ F_4(\varkappa_{12}, \varkappa_{21}, \varkappa_{13}, \varkappa_{31}, \varkappa_{23}, \varkappa_{32}, \zeta_1, \zeta_2, \zeta_3, \vartheta_1, \vartheta_2, \vartheta_3) = {} & \varkappa_{12}(\gamma\varkappa_{12} + \beta\varkappa_{21} + b_0\zeta_3 + \nu_2\vartheta_3) \\ & + \varkappa_{21}(\beta\varkappa_{12} + \gamma\varkappa_{21} - b_0\zeta_3 - \nu_2\vartheta_3) + \varkappa_{13}(\gamma\varkappa_{13} + \beta\varkappa_{31} - b_0\zeta_2 - \nu_2\vartheta_2) \\ & + \varkappa_{31}(\beta\varkappa_{13} + \gamma\varkappa_{31} + b_0\zeta_2 + \nu_2\vartheta_2) + \varkappa_{23}(\gamma\varkappa_{23} + \beta\varkappa_{32} + b_0\zeta_1 + \nu_2\vartheta_1) \\ & + \varkappa_{32}(\beta\varkappa_{23} + \gamma\varkappa_{32} - b_0\zeta_1 - \nu_2\vartheta_1) + \zeta_1(b_0\varkappa_{23} - b_0\varkappa_{32} + a_0\zeta_1 + \nu_1\vartheta_1) \\ & + \zeta_2(-b_0\varkappa_{13} + b_0\varkappa_{31} + a_0\zeta_2 + \nu_1\vartheta_2) + \zeta_3(b_0\varkappa_{12} - b_0\varkappa_{21} + a_0\zeta_3 + \nu_1\vartheta_3) \\ & + \vartheta_1(\nu_2\varkappa_{23} - \nu_2\varkappa_{32} + \nu_1\zeta_1 + k\vartheta_1) + \vartheta_2(-\nu_2\varkappa_{13} + \nu_2\varkappa_{31} + \nu_1\zeta_2 + k\vartheta_2) \\ & + \vartheta_3(\nu_2\varkappa_{12} - \nu_2\varkappa_{21} + \nu_1\zeta_3 + k\vartheta_3), \\ F_5(E_1, E_2, E_3, T) = {} & \chi E_iE_i + aT^2. \end{aligned}$$

Throughout the paper we assume that $B(v,\overline{v})$ is a positive definite form with respect to the vector $v = (e_{ij}, \varkappa_{ij}, \zeta_j, \varphi, T, \vartheta_i, E_i)$,

$$B(v,\overline{v}) > 0 \ \text{ for all } \ v \neq 0. \tag{2.21}$$

From the positive-definiteness of the forms F_1, F_2, F_3, F_4, and F_5, by Sylvester's criterion we derive the following necessary and sufficient conditions for form (2.20) to be positive definite:

$$\begin{gathered}
\varkappa > 0, \ \ \varkappa + 2\mu > 0, \ \ \varkappa + 2\mu + 3\lambda > 0, \ \ \xi_0\,(\varkappa + 2\mu + 3\lambda) > 3\lambda_0^2, \\
\gamma > |\beta|, \ \ a_0 k - \nu_1^2 > 0, \ \ \beta + \gamma + 3\alpha > 0, \ \ \chi > 0, \ \ a > 0, \ \ k > 0, \ \ a_0 > 0, \\
a_0(\gamma - \beta) > 2b_0^2, \ \ (\gamma - \beta)(a_0 k - \nu_1^2) + 4b_0\nu_1\nu_2 - 2a_0\nu_2^2 - 2kb_0^2 > 0.
\end{gathered} \tag{2.22}$$

Further, we assume also that

$$\rho_0 > 0, \ \ I_0 > 0, \ \ j_0 > 0. \tag{2.23}$$

3 Equations of pseudo-oscillations

Let the sought functions u_i, ϕ_i, φ, ψ, ϑ, as well as the sources f_i, X_i, $\mathcal{F}$, f, Q involved in the system of equations (2.14)–(2.18), be harmonic time dependent, i.e.

$$\begin{gathered}
u_i(x,t)=e^{\tau t}u_i(x), \ \ \phi_i(x,t)=e^{\tau t}\phi_i(x), \ \ \varphi(x,t)=e^{\tau t}\varphi(x), \ \ \psi(x,t)=e^{\tau t}\psi(x), \ \ \vartheta(x,t)=e^{\tau t}\vartheta(x), \\
f_i(x,t)=e^{\tau t}f_i(x), \ \ X_i(x,t)=e^{\tau t}X_i(x), \ \ \mathcal{F}(x,t)=e^{\tau t}\mathcal{F}(x), \ \ f(x,t)=e^{\tau t}f(x), \ \ Q(x,t)=e^{\tau t}Q(x),
\end{gathered}$$

where $\tau = \sigma + \mathrm{i}\,\omega$ is a complex parameter, $\sigma, \omega \in \mathbb{R}$. Then equations (2.14)–(2.18) lead to the system

$$(\mu + \varkappa)\partial_j\partial_j u_i + (\lambda + \mu)\partial_j\partial_i u_j - \tau^2\rho_0 u_i + \varkappa\varepsilon_{ijk}\partial_j\phi_k + \lambda_0\partial_i\varphi - \tau\beta_0\partial_i\vartheta = -\rho_0 f_i, \tag{3.1}$$

$$\gamma\partial_j\partial_j\phi_i + (\alpha + \beta)\partial_j\partial_i\phi_j - \tau^2 I_0\phi_i + \varkappa\varepsilon_{ijk}\partial_j u_k - 2\varkappa\phi_i = -\rho_0 X_i, \tag{3.2}$$

$$(a_0\partial_j\partial_j - \xi_0)\varphi - \tau^2 j_0\varphi - \lambda_2\partial_j\partial_j\psi + \nu_1\partial_j\partial_j\vartheta + \tau c_0\vartheta - \lambda_0\partial_j u_j = -\rho_0\mathcal{F}, \tag{3.3}$$

$$\chi\partial_j\partial_j\psi + \lambda_2\partial_j\partial_j\varphi + \nu_3\partial_j\partial_j\vartheta = -f, \tag{3.4}$$

$$k\partial_j\partial_j\vartheta - \tau^2 a\vartheta - \tau\beta_0\partial_j u_j - \tau c_0\varphi + \nu_1\partial_j\partial_j\varphi - \nu_3\partial_j\partial_j\psi = -\frac{1}{T_0}\,\rho_0 Q. \tag{3.5}$$

If τ is a pure imaginary number, we obtain *the steady state oscillation equations*, and if $\tau = 0$, then we get *the equations of statics*.

Constitutive relations (2.7)–(2.13) for pseudo-oscillation state read as

$$t_{ij} = \lambda\partial_k u_k\delta_{ij} + (\mu + \varkappa)\partial_i u_j + \varkappa\varepsilon_{jik}\phi_k + \mu\partial_j u_i + \lambda_0\varphi\delta_{ij} - \tau\beta_0\vartheta\delta_{ij}, \tag{3.6}$$

$$m_{ij} = \alpha\partial_k\phi_k\delta_{ij} + \beta\partial_j\phi_i + \gamma\partial_i\phi_j + b_0\varepsilon_{ijk}\partial_k\varphi + \lambda_1\varepsilon_{ijk}\partial_k\psi + \nu_2\varepsilon_{ijk}\partial_k\vartheta, \tag{3.7}$$

$$\pi_i = a_0\partial_i\varphi - \lambda_2\partial_i\psi + b_0\varepsilon_{kli}\partial_k\phi_l + \nu_1\partial_i\vartheta, \tag{3.8}$$

$$\epsilon = \lambda_0\partial_k u_k + \xi_0\varphi - \tau c_0\vartheta, \tag{3.9}$$

$$D_i = -\lambda_1\varepsilon_{kli}\partial_k\phi_l - \lambda_2\partial_i\varphi - \nu_3\partial_i\vartheta - \chi\partial_i\psi, \tag{3.10}$$

$$q_i = T_0(\nu_2\varepsilon_{lki}\partial_l\phi_k + \nu_1\partial_i\varphi + k\partial_i\vartheta - \nu_3\partial_i\psi), \tag{3.11}$$

$$\rho_0\eta = \beta_0\partial_k u_k + c_0\varphi + \tau a\vartheta, \ \ i,j = 1,2,3. \tag{3.12}$$

Denote by

$$A(\partial,\tau) = \big[A_{ij}(\partial,\tau)\big]_{9\times 9}$$

the matrix differential operator generated by the left hand side expressions in (3.1)–(3.5),

$$\begin{gathered}
A_{ij}(\partial,\tau) = \delta_{ij}(\mu + \varkappa)\partial_l\partial_l + (\lambda + \mu)\partial_i\partial_j - \tau^2\rho_0\delta_{ij}, \quad A_{i,j+3}(\partial,\tau) = -\varkappa\varepsilon_{ijl}\partial_l, \\
A_{i7}(\partial,\tau) = \lambda_0\partial_i, \quad A_{i8}(\partial,\tau) = 0, \quad A_{i9}(\partial,\tau) = -\tau\beta_0\partial_i, \quad A_{i+3,j}(\partial,\tau) = -\varkappa\varepsilon_{ijl}\partial_l, \\
A_{i+3,j+3}(\partial,\tau)=\delta_{ij}\gamma\partial_l\partial_l+(\alpha+\beta)\partial_i\partial_j-(2\varkappa+\tau^2 I_0)\delta_{ij}, \quad A_{i+3,j+6}(\partial,\tau)=0, \quad A_{7,j}(\partial,\tau)=-\lambda_0\partial_j, \\
A_{7,j+3}(\partial,\tau)=0, \ \ A_{77}(\partial,\tau)=a_0\partial_l\partial_l-(\xi_0+\tau^2 j_0), \ \ A_{78}(\partial,\tau)=-\lambda_2\partial_l\partial_l, \ \ A_{79}(\partial,\tau)=\nu_1\partial_l\partial_l+\tau c_0, \\
A_{8j}(\partial,\tau) = 0, \quad A_{8,j+3}(\partial,\tau) = 0, \quad A_{87}(\partial,\tau) = \lambda_2\partial_l\partial_l, \quad A_{88}(\partial,\tau) = \chi\partial_l\partial_l, \\
A_{89}(\partial,\tau) = \nu_3\partial_l\partial_l, \quad A_{9j}(\partial,\tau) = -\tau\beta_0\partial_j, \quad A_{9,j+3}(\partial,\tau) = 0, \\
A_{97}(\partial,\tau) = \nu_1\partial_l\partial_l - \tau c_0, \quad A_{98}(\partial,\tau) = -\nu_3\partial_l\partial_l, \quad A_{99}(\partial,\tau) = k\partial_l\partial_l - \tau^2 a, \ \ i,j = 1,2,3.
\end{gathered}$$

Then we can rewrite system (3.1)–(3.5) in the matrix form

$$A(\partial,\tau)U=\Phi, \tag{3.13}$$

where

$$U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top,$$
$$\Phi=-\Big(\rho_0 f_1,\rho_0 f_2,\rho_0 f_3,\rho_0 X_1,\rho_0 X_2,\rho_0 X_3,\rho_0\mathcal{F},f,\frac{1}{T_0}\,\rho_0 Q\Big)^\top.$$

4 Generalized stress operator and Green's formulae

Let n be a unit vector field on $\overline{\Omega}$ coinciding with the outward unit normal vector to $\partial\Omega$. Introduce the generalized stress operator $\mathcal{T}(\partial,n,\tau)=[\mathcal{T}_{jk}(\partial,n,\tau)]_{9\times 9}$ defined by the relation

$$\mathcal{T}(\partial,n,\tau)(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top$$
$$=(t_{l1}n_l,t_{l2}n_l,t_{l3}n_l,m_{l1}n_l,m_{l2}n_l,m_{l3}n_l,\pi_l n_l,-D_l n_l,T_0^{-1}q_l n_l)^\top,$$

where $t_{ij},m_{ij},\pi_j,D_j,q_i$ are defined in (2.7)–(2.13). Entries of the matrix $\mathcal{T}(\partial,n,\tau)$ read as

$$\mathcal{T}_{ij}(\partial,n,\tau)=\lambda n_i\partial_j+\mu n_j\partial_i+\delta_{ij}(\mu+\varkappa)n_k\partial_k,\quad \mathcal{T}_{i,j+3}(\partial,n,\tau)=-\varkappa\varepsilon_{ijk}n_k,$$
$$\mathcal{T}_{i7}(\partial,n,\tau)=\lambda_0 n_i,\quad \mathcal{T}_{i8}(\partial,n,\tau)=0,\quad \mathcal{T}_{i,9}(\partial,n,\tau)=-\tau\beta_0 n_i,\quad \mathcal{T}_{i+3,j}(\partial,n)=0,$$
$$\mathcal{T}_{i+3,j+3}(\partial,n,\tau)=\alpha n_i\partial_j+\beta n_j\partial_i+\delta_{ij}\gamma n_k\partial_k,\quad \mathcal{T}_{i+3,7}(\partial,n,\tau)=b_0\varepsilon_{lik}n_l\partial_k,$$
$$\mathcal{T}_{i+3,8}(\partial,n,\tau)=\lambda_1\varepsilon_{lik}n_l\partial_k,\quad \mathcal{T}_{i+3,9}(\partial,n,\tau)=\nu_2\varepsilon_{lik}n_l\partial_k,\quad \mathcal{T}_{7j}(\partial,n,\tau)=0,$$
$$\mathcal{T}_{7,j+3}(\partial,n,\tau)=-b_0\varepsilon_{ljk}n_l\partial_k,\quad \mathcal{T}_{77}(\partial,n,\tau)=a_0 n_k\partial_k,\quad \mathcal{T}_{78}(\partial,n,\tau)=-\lambda_2 n_k\partial_k,$$
$$\mathcal{T}_{79}(\partial,n,\tau)=\nu_1 n_k\partial_k,\quad \mathcal{T}_{8j}(\partial,n,\tau)=0,\quad \mathcal{T}_{8,j+3}(\partial,n,\tau)=-\lambda_1\varepsilon_{ljk}n_l\partial_k,\quad \mathcal{T}_{87}(\partial,n,\tau)=\lambda_2 n_k\partial_k,$$
$$\mathcal{T}_{88}(\partial,n,\tau)=\chi n_k\partial_k,\quad \mathcal{T}_{89}(\partial,n,\tau)=\nu_3 n_k\partial_k,\quad \mathcal{T}_{9j}(\partial,n,\tau)=0,\quad \mathcal{T}_{9,j+3}(\partial,n,\tau)=-\nu_2\varepsilon_{ljk}n_l\partial_k,$$
$$\mathcal{T}_{97}(\partial,n,\tau)=\nu_1 n_k\partial_k,\quad \mathcal{T}_{98}(\partial,n,\tau)=-\nu_3 n_k\partial_k,\quad \mathcal{T}_{99}(\partial,n,\tau)=kn_l\partial_l,\ \ i,j=1,2,3.$$

For a domain with smooth boundary and smooth complex valued vector functions

$$U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top\in[C^2(\overline{\Omega})]^9,$$
$$U'=(u_1',u_2',u_3',\phi_1',\phi_2',\phi_3',\varphi',\psi',\vartheta')^\top\in[C^2(\overline{\Omega})]^9$$

the following Green formula holds

$$\int\limits_{\Omega}A(\partial,\tau)U\cdot U'\,dx=\int\limits_{\partial\Omega}\{\mathcal{T}(\partial,n,\tau)U\}^+\cdot\{U'\}^+\,dS-\int\limits_{\Omega}E(U,\overline{U'})\,dx, \tag{4.1}$$

where the overbar denotes complex conjugation operation, the central dot designates the scalar product in the complex space $\mathbb{C}^9$,

$$\begin{aligned}E(U,U')=&(\mu+\varkappa)\partial_j u_i\partial_j u_i'+\tau^2\rho_0 u_i u_i'+\lambda\partial_j u_j\partial_i u_i'+\mu\partial_i u_j\partial_j u_i'+\varkappa\varepsilon_{ijk}\phi_k\partial_j u_i'+\lambda_0\varphi\partial_i u_i'\\&-\tau\beta_0\vartheta\partial_i u_i'+\gamma\partial_j\phi_i\partial_j\phi_i'+(2\varkappa+\tau^2 I_0)\phi_i\phi_i'+\alpha\partial_j\phi_j\partial_i\phi_i'+\beta\partial_i\phi_j\partial_j\phi_i'\\&+\varkappa\varepsilon_{ijk}\partial_j u_i\phi_k'+b_0\varepsilon_{ijk}\partial_k\varphi\partial_i\phi_j'+\lambda_1\varepsilon_{ijk}\partial_k\psi\partial_i\phi_j'+\nu_2\varepsilon_{ijk}\partial_k\vartheta\partial_i\phi_j'+a_0\partial_j\varphi\partial_j\varphi'\\&+(\xi_0+\tau^2 j_0)\varphi\varphi'-\lambda_2\partial_j\psi\partial_j\varphi'+\nu_1\partial_j\vartheta\partial_j\varphi'-\tau c_0\vartheta\varphi'+\lambda_0\partial_j u_j\varphi'+b_0\varepsilon_{ijk}\partial_i\phi_j\partial_k\varphi'\\&+\chi\partial_j\psi\partial_j\psi'\lambda_2\partial_j\varphi\partial_j\psi'+\nu_3\partial_j\vartheta\partial_j\psi'-\lambda_1\varepsilon_{ijk}\partial_j\phi_k\partial_i\psi'+k\partial_j\vartheta\partial_j\vartheta'+\tau^2 a\vartheta\vartheta'\\&+\tau\beta_0\partial_j u_j\vartheta'+\nu_1\partial_j\varphi\partial_j\vartheta'+\tau c_0\varphi\vartheta'-\nu_3\partial_j\psi\partial_j\vartheta'+\nu_2\varepsilon_{ijk}\partial_j\phi_k\partial_i\vartheta'.\end{aligned} \tag{4.2}$$

By standard limiting procedure Green's formula (4.1) can be extended to Lipschitz domains and to vector-functions $U\in[W_p^1(\Omega)]^9$ and $U'\in[W_{p'}^1(\Omega)]^9$ with $A(\partial,\tau)U\in[L_p(\Omega)]^9$ $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$.

With the help of Green's formula (4.1) we can correctly determine *a generalized trace vector* $\{\mathcal{T}(\partial,n,\tau)U\}^+ \in [B_{p,p}^{-1/p}(\partial\Omega)]^9$ for a vector function $U\in[W_p^1(\Omega)]^9$ with $A(\partial,\tau)U\in[L_p(\Omega)]^9$ by the relation (cf. [20])

$$\big\langle\{\mathcal{T}(\partial,n,\tau)U\}^+,\{U'\}^+\big\rangle_{\partial\Omega}:=\int\limits_{\Omega}\big[A(\partial,\tau)U\cdot U'+E(U,\overline{U'})\big]\,dx, \tag{4.3}$$

where $U'\in[W_{p'}^1(\Omega)]^9$ is an arbitrary vector function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality between the function spaces $[B_{p,p}^{-1/p}(\partial\Omega)]^9$ and $[B_{p',p'}^{1/p}(\partial\Omega)]^9$ which extends the conventional L_2 inner product for complex valued vector functions,

$$\langle f,g\rangle_{\partial\Omega}=\int\limits_{\partial\Omega}\sum_{j=1}^{9}f_j(x)\overline{g_j(x)}\,dS\ \ \text{for}\ \ f,g\in[L_2(\partial\Omega)]^9.$$

Introduce the boundary operator $\widetilde{\mathcal{T}}(\partial,n,\tau)=[\widetilde{\mathcal{T}}(\partial,n,\tau)_{ij}]_{9\times9}$ associated with the formally adjoint differential operator $A^*(\partial,\tau)=A^\top(-\partial,\tau)$,

$$\begin{gathered}
2\widetilde{\mathcal{T}}_{ij}(\partial,n,\tau)=\lambda n_i\partial_j+\mu n_j\partial_i+\delta_{ij}(\mu+\varkappa)n_k\partial_k,\quad \widetilde{\mathcal{T}}_{i,j+3}(\partial,n,\tau)=-\varkappa\varepsilon_{ijk}n_k,\\
\widetilde{\mathcal{T}}_{i7}(\partial,n,\tau)=\lambda_0 n_i,\quad \widetilde{\mathcal{T}}_{i8}(\partial,n,\tau)=0,\quad \widetilde{\mathcal{T}}_{i,9}(\partial,n,\tau)=\tau\beta_0 n_i,\quad \widetilde{\mathcal{T}}_{i+3,j}(\partial,n,\tau)=0,\\
\widetilde{\mathcal{T}}_{i+3,j+3}(\partial,n,\tau)=\alpha n_i\partial_j+\beta n_j\partial_i+\delta_{ij}\gamma n_k\partial_k,\quad \widetilde{\mathcal{T}}_{i+3,7}(\partial,n,\tau)=b_0\varepsilon_{ilk}n_l\partial_k,\\
\widetilde{\mathcal{T}}_{i+3,8}(\partial,n,\tau)=\lambda_1\varepsilon_{lik}n_l\partial_k,\quad \widetilde{\mathcal{T}}_{i+3,9}(\partial,n,\tau)=\nu_2\varepsilon_{ilk}n_l\partial_k,\quad \widetilde{\mathcal{T}}_{7j}(\partial,n,\tau)=0,\\
\widetilde{\mathcal{T}}_{7,j+3}(\partial,n,\tau)=0,\quad \widetilde{\mathcal{T}}_{77}(\partial,n,\tau)=a_0 n_k\partial_k,\quad \widetilde{\mathcal{T}}_{78}(\partial,n,\tau)=\lambda_2 n_k\partial_k,\quad \widetilde{\mathcal{T}}_{79}(\partial,n,\tau)=\nu_1 n_k\partial_k,\\
\widetilde{\mathcal{T}}_{8j}(\partial,n,\tau)=0,\quad \widetilde{\mathcal{T}}_{8,j+3}(\partial,n,\tau)=0,\quad \widetilde{\mathcal{T}}_{87}(\partial,n,\tau)=-\lambda_2 n_k\partial_k,\quad \widetilde{\mathcal{T}}_{88}(\partial,n,\tau)=\chi n_k\partial_k,\\
\widetilde{\mathcal{T}}_{89}(\partial,n,\tau)=-\nu_3 n_k\partial_k,\quad \widetilde{\mathcal{T}}_{9j}(\partial,n,\tau)=0,\quad \widetilde{\mathcal{T}}_{9,j+3}(\partial,n,\tau)=0,\\
\widetilde{\mathcal{T}}_{97}(\partial,n,\tau)=\nu_1 n_k\partial_k,\quad \widetilde{\mathcal{T}}_{98}(\partial,n,\tau)=\nu_3 n_k\partial_k,\quad \widetilde{\mathcal{T}}_{99}(\partial,n,\tau)=k n_l\partial_l,\ \ i,j=1,2,3.
\end{gathered}$$

From (4.1) we deduce Green's second formula,

$$\begin{aligned}
&\int\limits_{\Omega}\big[A(\partial,\tau)U\cdot U'-U\cdot A^*(\partial,\tau)U'\big]\,dx\\
&\qquad=\int\limits_{\partial\Omega}\Big[\big\{\mathcal{T}(\partial,n,\tau)U\big\}^+\cdot\{U'\}^+-\big\{\widetilde{\mathcal{T}}(\partial,n,\tau)U'\big\}^+\cdot\{U'\}^+\Big]\,dS.
\end{aligned}\tag{4.4}$$

From Green's formulae (4.3) and (4.4) by standard limiting procedure we derive similar formulae for the exterior domain Ω^- provided vector functions $U,U'\in[W_{p,loc}^1(\Omega^-)]^9\cap\mathbf{Z}(\Omega^-)$ and $A(\partial,\tau)U$ is compactly supported. The class $\mathbf{Z}(\Omega^-)$ is defined as a set of functions U possesing the following asymptotic properties as $|x|\to\infty$:

$$\begin{gathered}
u_k(x)=\mathcal{O}(|x|^{-2}),\ \ \partial_j u_k(x)=\mathcal{O}(|x|^{-2}),\ \ \phi_k(x)=\mathcal{O}(|x|^{-2}),\ \ \partial_j\phi_k(x)=\mathcal{O}(|x|^{-2}),\\
\varphi(x)=\mathcal{O}(|x|^{-1}),\ \ \partial_j\varphi(x)=\mathcal{O}(|x|^{-2}),\ \ \psi(x)=\mathcal{O}(|x|^{-1}),\ \ \partial_j\psi(x)=\mathcal{O}(|x|^{-2}),\\
\vartheta(x)=\mathcal{O}(|x|^{-2}),\ \ \partial_j\vartheta(x)=\mathcal{O}(|x|^{-2}),\ \ k,j=1,2,3.
\end{gathered}\tag{4.5}$$

Note that the fundamental matrix of the operator $A(\partial_x,\tau)$ with $\tau=\sigma+i\,\omega$, $\sigma>\sigma_0\geq0$, possesses the decay properties (4.5) (see Appendix B).

If $A^*(\partial_x,\tau)\,U'$ is compactly supported as well and U' satisfies the decay conditions (4.5), then the following Green formulae hold for the exterior domain Ω^-:

$$\big\langle\{\mathcal{T}(\partial,n,\tau)U\}^-,\{U'\}^-\big\rangle_{\partial\Omega}=-\int\limits_{\Omega^-}\big[A(\partial,\tau)U\cdot U'+E(U,\overline{U'})\big]\,dx, \tag{4.6}$$

$$\int\limits_{\Omega^-} \big[A(\partial,\tau)U\cdot U' - U\cdot A^*(\partial,\tau)U'\big]\,dx$$
$$= -\int\limits_{\partial\Omega} \Big[\big\{\mathcal{T}(\partial,n,\tau)U\big\}^-\cdot\{U'\}^- - \{U\}^-\cdot\big\{\widetilde{\mathcal{T}}(\partial,n,\tau)U'\big\}^-\Big]\,dS.$$

We recall that the direction of the unit normal vector to $S=\partial\Omega$ is outward with respect to the domain $\Omega=\Omega^+$.

Denote by $\mathcal{E}(U,V)$ the sesquilinear form on $[H_2^1(\Omega)]^9\times[H_2^1(\Omega)]^9$

$$\mathcal{E}(U,V) := \int\limits_{\Omega} E(U,\overline{V})\,dx, \tag{4.7}$$

where $E(U,\overline{V})$ is defined by (4.2).

For $U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top$, $v=(e_{ij},\varkappa_{ij},\zeta_j,\varphi,T,\vartheta_i,E_i)$, where $e_{ij}=\partial_i u_j+\varepsilon_{jik}\phi_k$, $\varkappa_{ij}=\partial_i\phi_j$, $\zeta_i=\partial_i\varphi$, $T=\tau\vartheta$, $\vartheta_i=\partial_i\vartheta$, $E_i=-\partial_i\psi$, we have

$$\begin{aligned} E(U,\overline{U}) &= B(v,\overline{v}) + 2i\lambda_1\varepsilon_{ijk}\,\mathrm{Im}(\partial_i\phi_j\partial_k\overline{\psi}) + 2i\lambda_2\,\mathrm{Im}(\partial_j\varphi\partial_j\overline{\psi}) + 2i\nu_3\,\mathrm{Im}(\partial_j\psi\partial_j\overline{\vartheta}) \\ &\quad + 2i\tau\beta_0\,\mathrm{Im}(\partial_j u_j\overline{\vartheta}) + 2i\tau c_0\,\mathrm{Im}(\varphi\overline{\vartheta}) + \tau^2(\rho_0 u_i\overline{u}_i + I_0\phi\overline{\phi} + j_0\varphi\overline{\varphi} + a\vartheta\overline{\vartheta}). \end{aligned} \tag{4.8}$$

Therefore from (4.7), (4.8), (2.21), and (2.22) it follows that

$$\mathrm{Re}\,\mathcal{E}(U,U) \geq c_1\|U\|^2_{[H_2^1(\Omega)]^9} - c_2\|U\|^2_{[H_2^0(\Omega)]^9} \ \text{ for all }\ U\in[H_2^1(\Omega)]^9 \tag{4.9}$$

with some positive constants c_1 and c_2 depending on the material parameters and on the complex parameter τ, which shows that the sesquilinear form $\mathcal{E}(U,V)$ defined in (4.7) is coercive.

5 Boundary value problems and uniqueness theorems

Here we preserve the notation introduced in the previous subsections and formulate the boundary value problems for the pseudo-oscillation equation (3.13) assuming that

$$\tau=\sigma+i\omega,\quad \sigma>\sigma_0\geqslant 0,\ \ \omega\in\mathbb{R}.$$

Further, let S_m $(m=1,2,\dots,10)$ be proper sub-manifolds of $\partial\Omega$ such that $\overline{S}_1\cup S_2=\overline{S}_3\cup S_4=\overline{S}_5\cup S_6=\overline{S}_7\cup S_8=\overline{S}_9\cup S_{10}=\partial\Omega$, $S_1\cap S_2=S_3\cap S_4=S_5\cap S_6=S_7\cap S_8=S_9\cap S_{10}=\varnothing$.

We consider the following boundary value problems.

The general mixed boundary value problem $(\mathbf{G})^+_\tau$**:** *Find a solution*

$$U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top\in[W_p^1(\Omega)]^9$$

to the pseudo-oscillation equation (3.13) *with* $\Phi\in[L_p(\Omega)]^9$*,* $1<p<\infty$*, satisfying the boundary conditions*

$$\begin{gathered} u_i=\widetilde{u}_i\ \text{ on }\ S_1,\quad t_{ji}n_j=\widetilde{\epsilon}_i\ \text{ on }\ S_2,\quad \phi_i=\widetilde{\phi}_i\ \text{ on }\ S_3,\quad m_{ji}n_j=\widetilde{m}_i\ \text{ on }\ S_4, \\ \varphi=\widetilde{\varphi}\ \text{ on }\ S_5,\quad \pi_k n_k=\widetilde{\pi}\ \text{ on }\ S_6,\quad \psi=\widetilde{\psi}\ \text{ on }\ S_7,\quad D_j n_j=\widetilde{D}_i\ \text{ on }\ S_8, \\ \vartheta=\widetilde{\vartheta}\ \text{ on }\ S_9,\quad q_j n_j=\widetilde{q}\ \text{ on }\ S_{10},\quad i=1,2,3, \end{gathered} \tag{5.1}$$

where $\widetilde{u}_i$, $\widetilde{\phi}_i$, $\widetilde{\varphi}$, $\widetilde{\psi}$, $\widetilde{\vartheta}$, $\widetilde{\epsilon}_i$, $\widetilde{m}_i$, $\widetilde{\pi}$, $\widetilde{D}$ *and* $\widetilde{q}$ *are given functions. Here equation* (3.13) *is understood in the distributional sense, the Dirichlet type conditions are understood in the usual trace sense and the corresponding data belong to the space* $B^{1-1/p}_{p,p}$*, while the Neumann type conditions are understood in the generalized functional trace sense and the corresponding data belong to the space* $B^{-1/p}_{p,p}$*.*

The Dirichlet problem $(\mathbf{D})^+_\tau$**:** *Find a solution*

$$U=(u,\phi,\varphi,\psi,\vartheta)^\top\in[W_p^1(\Omega)]^9$$

to the pseudo-oscillation equation (3.13) *with* $\Phi \in [L_p(\Omega)]^9$, $1 < p < \infty$, *satisfying the Dirichlet type boundary condition*

$$\{U\}^+ = f \ \ on \ \ S, \tag{5.2}$$

where $f \in [B_{p,p}^{1-1/p}(S)]^9$ *is a given vector function.*

In the case when U satisfies the homogeneous equation

$$A(\partial_x, \tau)U = 0 \ \text{ in } \ \Omega, \tag{5.3}$$

we denote the corresponding problem by $\mathbf{(D)}^+_{\tau,0}$.

The Neumann problem $\mathbf{(N)}^+_\tau$: *Find a solution*

$$U = (u, \phi, \varphi, \psi, \vartheta)^\top \in [W_p^1(\Omega)]^9$$

to the pseudo-oscillation equation (3.13) *with* $\Phi \in [L_p(\Omega)]^9$, $1 < p < \infty$, *satisfying the Neumann type boundary condition*

$$\{\mathcal{T}(\partial_x, n, \tau)U\}^+ = F \ \ on \ \ S, \tag{5.4}$$

where $F \in [B_{p,p}^{-1/p}(S)]^9$ *is a given vector function.*

In the case when U satisfies the homogeneous equation (5.3) we denote the corresponding problem by $\mathbf{(N)}^+_{\tau,0}$.

Mixed boundary value problem for solids with interior cracks. Let us assume that a solid under consideration contains an interior crack. We identify the crack surface as a two-dimensional, two-sided manifold Σ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a proper part of a closed surface $S_0 \subset \Omega$ surrounding a domain $\overline{\Omega}_0 \subset \Omega$ and that Σ, S_0, and ℓ_c are C^∞-smooth. Denote $\Omega_\Sigma := \Omega \setminus \overline{\Sigma}$.

We write $v \in W_p^1(\Omega_\Sigma)$ if $v \in W_p^1(\Omega_0)$, $v \in W_p^1(\Omega \setminus \overline{\Omega_0})$, and $r_{_{S_0\setminus\overline{\Sigma}}}\{v\}^+ = r_{_{S_0\setminus\overline{\Sigma}}}\{v\}^-$.

Recall that throughout the paper $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega$ and $S_0 = \partial\Omega_0$. This agreement defines the positive direction of the normal vector on the crack surface Σ.

Further, we assume that S is dissected into two smooth subsurfaces, the Dirichlet part S_D and the Neumann part S_N, $S = \bar{S_D} \cap \bar{S_N}$, and consider the following mixed BVP $(\mathrm{MC})^+_\tau$:

(i) on the subsurface S_D there are given the displacement and the microrotation vectors, the microstretch function, the temperature and the electric potential functions (i.e., on S_D there are given the components of the vector $\{U\}^+$ - the Dirichlet data);

(ii) on the subsurface S_N there are prescribed the mechanical stress vector, the normal components of the microstretch stress vector, the heat flux, and the electric displacement vector (i.e., on S_N there are given the components of the vector $\{\mathcal{T}U\}^+$ – the Neumann data);

(iii) the crack surface Σ is mechanically traction free and we assume that the microstretch function, temperature, electric potential, and the normal components of the microstretch stress vector, heat flux, and the electric displacement vector are continuous across the crack surface.

Reducing the nonhomogeneous differential equation (3.13) to the corresponding homogeneous one, we can formulate the above mixed problem mathematically as follows: Find a vector function

$$U = (u, \phi, \varphi, \psi, \theta)^\top = (u_1, \dots, u_9)^\top \in [W_p^1(\Omega_\Sigma)]^9 \ \text{ with } \ 1 < p < \infty,$$

satisfying the homogeneous differential equation

$$A(\partial_x, \tau)\,U = 0 \ \text{ in } \ \Omega_\Sigma, \tag{5.5}$$

the crack conditions on Σ,

$$\big\{[\mathcal{T}U]_j\big\}^+ = F_j^+ \ \text{ on } \ \Sigma, \ \ j = \overline{1,6}, \tag{5.6}$$

$$\big\{[\mathcal{T}U]_j\big\}^- = F_j^- \ \text{ on } \ \Sigma, \quad j=\overline{1,6}, \tag{5.7}$$

$$\{u_7\}^+ - \{u_7\}^- = f_7 \ \text{ on } \ \Sigma, \tag{5.8}$$

$$\big\{[\mathcal{T}U]_7\big\}^+ - \big\{[\mathcal{T}U]_7\big\}^- = F_7 \ \text{ on } \ \Sigma, \tag{5.9}$$

$$\{u_8\}^+ - \{u_8\}^- = f_8 \ \text{ on } \ \Sigma, \tag{5.10}$$

$$\big\{[\mathcal{T}U]_8\big\}^+ - \big\{[\mathcal{T}U]_8\big\}^- = F_8 \ \text{ on } \ \Sigma, \tag{5.11}$$

$$\{u_9\}^+ - \{u_9\}^- = f_9 \ \text{ on } \ \Sigma, \tag{5.12}$$

$$\big\{[\mathcal{T}U]_9\big\}^+ - \big\{[\mathcal{T}U]_9\big\}^- = F_9 \ \text{ on } \ \Sigma, \tag{5.13}$$

and the mixed boundary conditions on $S=\overline{S}_D\cup\overline{S}_N$,

$$\{U\}^+ = g^{(D)} \ \text{ on } \ S_D, \tag{5.14}$$

$$\big\{\mathcal{T}U\big\}^+ = g^{(N)} \ \text{ on } \ S_N. \tag{5.15}$$

We require that the boundary data belong to the natural spaces,

$$f_7,f_8,f_9\in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma), \quad F_7,F_8,F_9\in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad g^{(D)}\in [B_{p,p}^{1-\frac{1}{p}}(S_D)]^9, \quad g^{(N)}\in [B_{p,p}^{-\frac{1}{p}}(S_N)]^9, \tag{5.16}$$

and the compatibility conditions

$$F_j^+ - F_j^- \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad j=\overline{1,6},$$

are satisfied.

Remark that if $U\in [W_p^1(\Omega_\Sigma)]^9$ solves the homogeneous differential equation (5.5) then actually we have the inclusion $U\in[C^\infty(\Omega_\Sigma)]^9$ due to the ellipticity of the corresponding differential operator. In fact, U is a complex valued analytic vector function of spatial real variables (x_1,x_2,x_3) in Ω_Σ.

Now we prove the uniqueness theorem (cf. [16, Theorem 3.1]).

Theorem 5.1. *Let conditions* (2.22) *and* (2.23) *be satisfied and let* $U=(u,\phi,\varphi,\psi,\vartheta)$ *be a solution of the problem* $(G)_\tau^+$ *for the homogeneous equation* (5.3) *satisfying the homogeneous boundary conditions* (5.1) *for* $p=2$*. Then,* $u=\phi=\varphi=\vartheta=0$*, and* $\psi=const$*. Moreover, if* $S_7\neq\varnothing$*, then* $\psi=0$ *as well.*

Proof. Due to (2.1), (2.2), we have the system of equations

$$\partial_j t_{ji} - \tau^2\rho_0 u_i = 0, \quad i=1,2,3, \tag{5.17}$$

$$\partial_j m_{ji} + \varepsilon_{ijk}t_{jk} - \tau^2 I_0\phi_i = 0, \quad i=1,2,3, \tag{5.18}$$

$$\partial_j\pi_j - \epsilon - \tau^2 j_0\varphi = 0, \tag{5.19}$$

$$\partial_j q_j - \tau\rho_0 T_0\eta = 0, \tag{5.20}$$

$$\partial_j D_j = 0, \tag{5.21}$$

where t_{ji}, m_{ji}, π_j, ϵ, q_j, η, D_j are defined from (3.6)–(3.11).

Multiply (5.17), (5.18), (5.19), (5.20), and (5.21) by $\overline{u}_i$, $\overline{\phi}_i$, $\overline{\varphi}$, $\overline{\vartheta}$, and $\overline{\psi}$, respectively, and integrate over Ω. In view of (2.4) and homogeneous boundary conditions we find

$$\int\limits_\Omega \Big(t_{ij}\overline{e}_{ij} + \varepsilon_{ijk}t_{ij}\overline{\phi}_k + \tau^2\rho_0 u_i\overline{u}_i\Big)\,dx = \int\limits_{\partial\Omega} n_j t_{ji}\overline{u}_i\,dS = 0, \tag{5.22}$$

$$\int\limits_\Omega \Big(m_{ij}\overline{\varkappa}_{ij} - \varepsilon_{ijk}t_{ij}\overline{\phi}_k + \tau^2 I_0\phi_i\overline{\phi}_i\Big)\,dx = \int\limits_{\partial\Omega} n_j m_{ji}\overline{\phi}_i\,dS = 0, \tag{5.23}$$

$$\int\limits_\Omega \Big(\pi_i\overline{\zeta}_i + \epsilon\overline{\varphi} + \tau^2 j_0\varphi\overline{\varphi}\Big)\,dx = \int\limits_{\partial\Omega} n_i\pi_i\overline{\varphi}\,dS = 0, \tag{5.24}$$

$$\frac{1}{T_0}\int\limits_{\Omega}\Big(q_i\partial_i\overline{\vartheta}+\tau\rho_0T_0\eta\overline{\vartheta}\Big)\,dx=\int\limits_{\partial\Omega}n_iq_i\overline{\vartheta}\,dS=0, \tag{5.25}$$

$$\int\limits_{\Omega}D_i\overline{E}_i\,dx=\int\limits_{\partial\Omega}n_iD_i\overline{\psi}\,dS=0. \tag{5.26}$$

By summing equalities (5.22)–(5.25) and complex conjugate of (5.26) we obtain

$$\int\limits_{\Omega}\Big(t_{ij}\overline{e}_{ij}+m_{ij}\overline{\varkappa}_{ij}+\pi_i\overline{\zeta}_i+\epsilon\overline{\varphi}+\frac{1}{T_0}\,q_i\partial_i\overline{\vartheta}+\tau\rho_0\eta\overline{\vartheta}+\overline{D}_iE_i+\tau^2\big(\rho_0u_i\overline{u}_i+I_0\phi\overline{\phi}+j_0\varphi\overline{\varphi}\big)\Big)\,dx=0. \tag{5.27}$$

By virtue of (2.19) the integrand in (5.27) can be rewritten as

$$\begin{aligned}
&\lambda e_{ii}\overline{e}_{jj}+(\mu+\varkappa)e_{ij}\overline{e}_{ij}+\mu e_{ji}\overline{e}_{ij}+\lambda_0\varphi\overline{e}_{jj}-\beta_0T\overline{e}_{jj}+\alpha\varkappa_{ii}\overline{\varkappa}_{jj}+\beta\varkappa_{ji}\overline{\varkappa}_{ij}+\gamma\varkappa_{ij}\overline{\varkappa}_{ij}\\
&\qquad+b_0\varepsilon_{ijk}\zeta_k\overline{\varkappa}_{ij}+\lambda_1\varepsilon_{jik}\overline{\varkappa}_{ij}E_k+\nu_2\varepsilon_{ijk}\overline{\varkappa}_{ij}\partial_k\vartheta+a_0\zeta_i\overline{\zeta}_i+\lambda_2E_i\overline{\zeta}_i\\
&\qquad+b_0\varepsilon_{ijk}\varkappa_{ij}\overline{\zeta}_k+\nu_1\partial_i\vartheta\overline{\zeta}_i+\lambda_0e_{jj}\overline{\varphi}+\xi_0\varphi\overline{\varphi}-c_0T\overline{\varphi}-\lambda_1\varepsilon_{jik}\overline{\varkappa}_{ij}E_k\\
&\qquad-\lambda_2\overline{\zeta}_iE_i-\nu_3\partial_i\overline{\vartheta}E_i+\chi\overline{E}_iE_i+\nu_2\varepsilon_{ijk}\varkappa_{ij}\partial_k\overline{\vartheta}+\nu_1\zeta_i\partial_i\overline{\vartheta}+k\partial_i\vartheta\partial_i\overline{\vartheta}\\
&\qquad+\nu_3E_i\partial_i\overline{\vartheta}+\tau\beta_0e_{jj}\overline{\vartheta}+\tau c_0\varphi\overline{\vartheta}+\tau aT\overline{\vartheta}+\tau^2(\rho_0u_i\overline{u}_i+I_0\phi\overline{\phi}+j_0\varphi\overline{\varphi})\\
&=B(v,\overline{v})+\tau\beta_0(e_{jj}\overline{\vartheta}-\overline{e}_{jj}\vartheta)+\tau c_0(\varphi\overline{\vartheta}-\overline{\varphi}\vartheta)+\tau^2(\rho_0u_i\overline{u}_i+I_0\phi\overline{\phi}+j_0\varphi\overline{\varphi}+a\vartheta\overline{\vartheta}),
\end{aligned}$$

where $B(v,v')$ is the bilinear form with respect to the variables $v=(e_{ij},\varkappa_{ij},\zeta_i,\varphi,T,\partial_i\vartheta,E_i)$ and $v'=(e'_{ij},\varkappa'_{ij},\zeta'_i,\varphi',T',\partial_i\vartheta',E'_i)$ defined in (2.19),

$$\begin{aligned}
B(v,v')=\lambda e_{ii}e'_{jj}&+(\mu+\varkappa)e_{ij}e'_{ij}+\mu e_{ji}e'_{ij}+\lambda_0(e_{jj}\varphi'+e'_{jj}\varphi)+\xi_0\varphi\varphi'+\alpha\varkappa_{ii}\varkappa'_{jj}\\
&+\beta\varkappa_{ji}\varkappa'_{ij}+\gamma\varkappa_{ij}\varkappa'_{ij}+b_0\varepsilon_{ijk}(\varkappa_{ij}\zeta'_k+\varkappa'_{ij}\zeta_k)+\nu_2\varepsilon_{ijk}(\varkappa_{ij}\partial_k\vartheta'+\varkappa'_{ij}\partial_k\vartheta)\\
&+a_0\zeta_i\zeta'_i+\nu_1(\partial_i\vartheta\zeta'_i+\partial_i\vartheta'\zeta_i)+\chi E_i\overline{E}_i+k\partial\vartheta\,\partial\vartheta'.
\end{aligned}$$

Due to (2.22) we have $B(v,\overline{v})>0$ for any complex valued vector $v\neq0$.

Let $\tau=\sigma+\mathrm{i}\,\omega$, $\sigma>0$. Separating the real and imaginary parts of (5.27) we get

$$\begin{aligned}
\int\limits_{\Omega}\Big(&B(v,\overline{v})-2\omega\beta_0\,\mathrm{Im}(e_{jj}\overline{\vartheta})-2\omega c_0\,\mathrm{Im}(\varphi\overline{\vartheta})\\
&+(\sigma^2-\omega^2)\big(\rho_0|u|^2+I_0|\phi|^2+j_0|\varphi|^2+a|\vartheta|^2\big)\Big)\,dx=0,
\end{aligned} \tag{5.28}$$

$$\int\limits_{\Omega}\Big(2\sigma\beta_0\,\mathrm{Im}(e_{jj}\overline{\vartheta})+2\sigma c_0\,\mathrm{Im}(\varphi\overline{\vartheta})+2\sigma\omega\big(\rho_0|u|^2+I_0|\phi|^2+j_0|\varphi|^2+a|\vartheta|^2\big)\Big)\,dx=0. \tag{5.29}$$

Multiply (5.29) by ω/σ and add to (5.28) to obtain

$$\int\limits_{\Omega}\Big(B(v,\overline{v})+(\sigma^2+\omega^2)\big(\rho_0|u|^2+I_0|\phi|^2+j_0|\varphi|^2+a|\vartheta|^2\big)\Big)\,dx=0,$$

implying $|u|=|\phi|=|\varphi|=|\vartheta|=0$ and $\int\limits_{\Omega}\chi|E|^2\,dx=0$. Whence $E=-\operatorname{grad}\psi=0$ and thus $\psi=const$. Evidently, if $S_7\neq\varnothing$, then $\psi=0$ follows, which completes the proof. □

From Theorem 5.1 the following uniqueness theorem follows directly.

Theorem 5.2. *Let S be Lipschitz surface and $\tau=\sigma+i\omega$ with $\sigma>\sigma_0\geqslant0$ and $\omega\in\mathbb{R}$.*

(i) *The basic Dirichlet BVP $(D)^+_\tau$ has at most one solution in the space $[W^1_2(\Omega)]^9$.*

(ii) *Solutions to the Neumann type BVP $(N)^+_\tau$ in the space $[W^1_2(\Omega)]^9$ are defined modulo a vector of type $U^{(N)}=(0,0,0,0,0,0,0,b,0)^\top$, where b is an arbitrary constant.*

(iii) *Mixed type boundary value problem $(MC)^+_\tau$ has at most one solution in the space $[W^1_2(\Omega_\Sigma)]^9$.*

Similar uniqueness result for $p\neq2$ will be proved later.

6 Properties of potentials and boundary operators

The full symbol of the pseudo-oscillation differential operator $A(\partial_x,\tau)$ with $\operatorname{Re}\tau \neq 0$ is non-singular, i.e.,

$$\det A(-i\,\xi,\tau) \neq 0 \ \ \forall\,\xi \in \mathbb{R}^3 \setminus \{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\,\xi,\tau)$ are locally integrable functions decaying at infinity as $\mathcal{O}(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x,\tau)=[\Gamma_{kj}(x,\tau)]_{9\times 9}$ of the operator $A(\partial_x,\tau)$ with the help of the Fourier transform technique,

$$\Gamma(x,\tau) = \mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\xi,\tau)].$$

The structure of the matrix $A^{-1}(-i\xi,\tau)$ allows to represent the fundamental matrix $\Gamma(x,\tau)$ in terms of elementary functions (see Appendix B). These explicit formulas imply that in a neighbourhood of the origin the fundamental matrix possesses the property $\Gamma(x,\tau)=\mathcal{O}(|x|^{-1})$, while the columns of $\Gamma(x,\tau)$ satisfy the decay conditions (4.5) at infinity.

Here we collect some necessary results for our analysis. Proofs of the theorems below are similar to the proofs of their counterparts in [2,3,8,17,18].

Let us introduce the single and double layer potentials:

$$V(h)(x) = V_S(h) = \int\limits_S \Gamma(x-y,\tau)\,h(y)\,d_yS,$$

$$W(h)(x) = W_S(h) = \int\limits_S \Big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(x-y,\tau)]^\top\Big]^\top h(y)\,d_yS,$$

where $h=(h_1,h_2,\dots,h_9)^\top$ is a density vector function.

Theorem 6.1. *Let $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. Then the single and double layer potentials can be extended to the continuous operators*

$$\begin{aligned}
V &: \big[B^s_{p,q}(S)\big]^9 \to \big[B^{s+1+\frac1p}_{p,q}(\Omega)\big]^9, &\qquad W &: [B^s_{p,q}(S)]^9 \to [B^{s+\frac1p}_{p,q}(\Omega)]^9,\\
&: [B^s_{p,q}(S)]^9 \to [B^{s+1+\frac1p}_{p,q,loc}(\Omega^-)]^9, & &: [B^s_{p,q}(S)]^9 \to [B^{s+\frac1p}_{p,q,loc}(\Omega^-)]^9,\\
&: [B^s_{p,p}(S)]^9 \to [H^{s+1+\frac1p}_{p}(\Omega)]^9, & &: [B^s_{p,p}(S)]^9 \to [H^{s+\frac1p}_{p}(\Omega)]^9,\\
&: [B^s_{p,p}(S)]^9 \to [H^{s+1+\frac1p}_{p,loc}(\Omega^-)]^9, & &: [B^s_{p,p}(S)]^9 \to [H^{s+\frac1p}_{p,loc}(\Omega^-)]^9.
\end{aligned}$$

Theorem 6.2. *Let $h^{(1)}\in[B^{-\frac1p}_{p,q}(S)]^9$, $h^{(2)}\in[B^{1-\frac1p}_{p,q}(S)]^9$, $1<p<\infty$, $1\le q\le\infty$. Then*

$$\big\{V(h^{(1)})(z)\big\}^\pm = \int\limits_S \Gamma(z-y,\tau)h^{(1)}(y)\,d_yS \ \ \text{on}\ \ S,$$

$$\big\{W(h^{(2)})(z)\big\}^\pm = \pm\frac12\,h^{(2)}(z) + \int\limits_S \Big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(z-y,\tau)]^\top\Big]^\top h^{(2)}(y)\,d_yS \ \ \text{on}\ \ S.$$

The equalities are understood in the sense of the space $[B^{1-1/p}_{p,q}(S)]^9$ (cf. [21])

Theorem 6.3. *Let $h^{(1)}\in[B^{-\frac1p}_{p,q}(S)]^9$, $h^{(2)}\in[B^{1-\frac1p}_{p,q}(S)]^9$, $1<p<\infty$, $1\le q\le\infty$. Then*

$$\big\{\mathcal{T}V(h^{(1)})(z)\big\}^\pm = \mp\frac12\,h^{(1)}(z) + \int\limits_S \mathcal{T}(\partial_z,n(z),\tau)\Gamma(z-y,\tau)\,h^{(1)}(y)\,d_yS \ \ \text{on}\ \ S,$$

$$\big\{\mathcal{T}W(h^{(2)})(z)\big\}^+ = \big\{\mathcal{T}W(h^{(2)})(z)\big\}^- \ \ \text{on}\ \ S,$$

where the equalities are understood in the sense of the space $[B^{-\frac1p}_{p,q}(S)]^9$.

We introduce the following notation for the boundary operators generated by the single and double layer potentials:

$$\mathcal{H}(h)(z)=\int_S \Gamma(z-y,\tau)h(y)\,d_yS,\quad z\in S, \tag{6.1}$$

$$\mathcal{K}(h)(z)=\int_S \mathcal{T}(\partial_z,n(z),\tau)\Gamma(z-y,\tau)h(y)\,d_yS,\quad z\in S, \tag{6.2}$$

$$\mathcal{N}(h)(z)=\int_S \Big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(z-y,\tau)]^\top\Big]^\top h(y)\,d_yS,\quad z\in S, \tag{6.3}$$

$$\mathcal{L}(h)(z)=\big\{\mathcal{T}W(h)(z)\big\}^+=\big\{\mathcal{T}W(h)(z)\big\}^-,\quad z\in S. \tag{6.4}$$

Note that $\mathcal{H}$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators (pseudodifferential operator of order 0), and $\mathcal{L}$ is a singular integro-differential operator (pseudodifferential operator of order 1). These operators possess the following mapping and Fredholm properties.

Theorem 6.4. *Let $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. Then the operators*

$$\begin{aligned} \mathcal{H}:[B^s_{p,q}(S)]^9\to[B^{s+1}_{p,q}(S)]^9, &\qquad \mathcal{H}:[H^s_p(S)]^9\to[H^{s+1}_p(S)]^9,\\ \mathcal{K},\mathcal{N}:[B^s_{p,q}(S)]^9\to[B^{s}_{p,q}(S)]^9, &\qquad \mathcal{K},\mathcal{N}:[H^s_p(S)]^9\to[H^{s}_p(S)]^9,\\ \mathcal{L}:[B^s_{p,q}(S)]^9\to[B^{s-1}_{p,q}(S)]^9, &\qquad \mathcal{L}:[H^s_p(S)]^9\to[H^{s-1}_p(S)]^9, \end{aligned}$$

are continuous.

The operators $\mathcal{H}$ and $\mathcal{L}$ are strongly elliptic pseudodifferential operators, while the operators $\pm\frac12\,I_9+\mathcal{K}$ and $\pm\frac12\,I_9+\mathcal{N}$ are elliptic, where I_9 stands for the 9×9 unit matrix.

Moreover, the operators $\mathcal{H}$, $\frac12\,I_9+\mathcal{N}$, and $\frac12\,I_9+\mathcal{K}$ are invertible, whereas the operators $-\frac12\,I_9+\mathcal{K}$, $-\frac12\,I_9+\mathcal{N}$, and $\mathcal{L}$ are Fredholm operators with zero index.

The following operator equalities hold in appropriate function spaces

$$\mathcal{L}\mathcal{H}=-\frac14\,I_9+\mathcal{K}^2,\quad \mathcal{H}\mathcal{L}=-\frac14\,I_9+\mathcal{N}^2. \tag{6.5}$$

7 Existence and regularity of solutions to mixed BVP $(\mathrm{MC})_\tau$

Before we start analysis of the mixed problem we present here existence results for the basic Dirichlet and Neumann boundary value problems. Using Theorem 6.4 and the fact that the null spaces of strongly elliptic pseudodifferential operators acting in Bessel potential $H^s_p(S)$ and Besov $B^s_{p,q}(S)$ spaces actually do not depend on the parameters s, p, and q, by quite the same arguments as in [3], we arrive at the following existence results.

Theorem 7.1. *Let $1<p<\infty$ and $f\in[B^{1-\frac1p}_{p,p}(S)]^9$. Then the pseudodifferential operator*

$$2^{-1}I_9+\mathcal{N}:[B^{1-\frac1p}_{p,p}(S)]^9\to[B^{1-\frac1p}_{p,p}(S)]^9$$

is continuously invertible, the interior Dirichlet BVP (5.3), (5.2) is uniquely solvable in the space $[W^1_p(\Omega)]^9$ and the solution is representable in the form of double layer potential $U=W(h)$ with the density vector function $h\in[B^{1-\frac1p}_{p,p}(S)]^9$ being a unique solution of the singular integral equation

$$[2^{-1}I_9+\mathcal{N}]h=f\ \ \text{on}\ \ S.$$

Theorem 7.2. *Let $1<p<\infty$ and a vector function $U\in[W^1_p(\Omega)]^9$ solves the homogeneous differential equation $A(\partial,\tau)U=0$ in Ω. Then it is uniquely representable in the form*

$$U(x)=V(\mathcal{H}^{-1}\{U\}^+)(x),\quad x\in\Omega,$$

where $\{U\}^+$ *is the trace of* U *on* S *from* Ω *and belongs to the space* $[B_{p,p}^{1-\frac{1}{p}}(S)]^9$. *Here* $\mathcal{H}^{-1}$ *is the inverse to the operator* $\mathcal{H} : B^{-\frac{1}{p}} \to B^{1-\frac{1}{p}}$.

Theorem 7.3. *Let* $1 < p < \infty$ *and* $F = (F_1, \dots, F_9)^\top \in [B_{p,p}^{-\frac{1}{p}}(S)]^9$.

(i) *The operator*

$$-2^{-1} I_9 + \mathcal{K} : [B_{p,p}^{-\frac{1}{p}}(S)]^9 \to [B_{p,p}^{-\frac{1}{p}}(S)]^9 \tag{7.1}$$

is an elliptic pseudodifferential operator with zero index and has a one-dimensional null space spanned by the vector function $h_0 = \mathcal{H}^{-1}\Psi$, *where*

$$\Psi := (0,0,0,0,0,0,0,1,0)^\top \ \text{ on } \ S. \tag{7.2}$$

(ii) *The null space of the operator adjoint to* (7.1),

$$-2^{-1} I_9 + \mathcal{K}^* : [B_{p',p'}^{\frac{1}{p}}(S)]^9 \to [B_{p',p'}^{\frac{1}{p}}(S)]^9, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

is the linear span of the vector $(0,0,0,0,0,0,0,1,0)^\top$.

(iii) *The equation*

$$[-2^{-1} I_9 + \mathcal{K}]h = F \ \text{ on } \ S, \tag{7.3}$$

is solvable if and only if

$$\int\limits_S F_8(x)\, dS = 0. \tag{7.4}$$

(iv) *If condition* (7.4) *holds, then solutions to equation* (7.3) *are defined modulo constant times* $h_0 = \mathcal{H}^{-1}\Psi$ *with* Ψ *defined in* (7.2).

(v) *If condition* (7.4) *holds, then the interior Neumann type boundary value problem* (5.3), (5.4) *is solvable in the space* $[W_p^1(\Omega)]^9$ *and its solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* $h \in [B_{p,p}^{-\frac{1}{p}}(S)]^9$ *is defined by equation* (7.3). *A solutions to the interior Neumann BVP in* Ω *is defined modulo summand* $C\,\Psi$ *with arbitrary constant* C *and* Ψ *given by* (7.2).

Now we start investigation of the mixed boundary value problem $(MC)_\tau$.

First let us note that the boundary conditions on the crack faces Σ, (5.6) and (5.7), can be transformed equivalently as

$$\big\{[\mathcal{T}U]_j\big\}^+ - \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ - F_j^- \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \ \ j = \overline{1,6},$$
$$\big\{[\mathcal{T}U]_j\big\}^+ + \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ + F_j^- \in B_{p,p}^{-\frac{1}{p}}(\Sigma), \ \ j = \overline{1,6}.$$

Therefore the boundary conditions (5.6)–(5.15) of the problem under consideration can be rewritten as

$$\big\{\mathcal{T}U\big\}^+ = g^{(N)} \ \text{ on } \ S_N, \tag{7.5}$$
$$\{U\}^+ = g^{(D)} \ \text{ on } \ S_D, \tag{7.6}$$
$$\big\{[\mathcal{T}U]_j\big\}^+ + \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ + F_j^- \ \text{ on } \ \Sigma, \ \ j = \overline{1,6}, \tag{7.7}$$
$$\{u_7\}^+ - \{u_7\}^- = f_7 \ \text{ on } \ \Sigma, \tag{7.8}$$
$$\{u_8\}^+ - \{u_8\}^- = f_8 \ \text{ on } \ \Sigma, \tag{7.9}$$
$$\{u_9\}^+ - \{u_9\}^- = f_9 \ \text{ on } \ \Sigma, \tag{7.10}$$
$$\big\{[\mathcal{T}U]_j\big\}^+ - \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ - F_j^- \ \text{ on } \ \Sigma, \ \ j = \overline{1,6}, \tag{7.11}$$
$$\big\{[\mathcal{T}U]_7\big\}^+ - \big\{[\mathcal{T}U]_7\big\}^- = F_7 \ \text{ on } \ \Sigma, \tag{7.12}$$

$$\big\{[\mathcal{T}U]_8\big\}^+ - \big\{[\mathcal{T}U]_8\big\}^- = F_8 \ \text{ on } \ \Sigma, \tag{7.13}$$

$$\big\{[\mathcal{T}U]_9\big\}^+ - \big\{[\mathcal{T}U]_9\big\}^- = F_9 \ \text{ on } \ \Sigma. \tag{7.14}$$

We look for a solution of the boundary value problem (5.5), (7.5)–(7.14) in the form

$$U = V(\mathcal{H}^{-1}\,h) + W_c(h^{(2)}) + V_c(h^{(1)}) \ \text{ in } \ \Omega_\Sigma, \tag{7.15}$$

where

$$V_c(h^{(1)})(x) := \int\limits_\Sigma \Gamma(x-y,\tau)h^{(1)}(y)\,d_yS,$$

$$W_c(h^{(2)})(x) := \int\limits_\Sigma \big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(x-y,\tau)]^\top\big]^\top h^{(2)}(y)\,d_yS,$$

$$V(\mathcal{H}^{-1}h)(x) := \int\limits_S \Gamma(x-y,\tau)(\mathcal{H}^{-1}h)(y)\,d_yS,$$

$h^{(i)} = (h_1^{(i)},\dots,h_9^{(i)})^\top$, $i=1,2$, and $h=(h_1,\dots,h_9)^\top$ are unknown densities,

$$h^{(1)} \in \big[\widetilde{B}_{p,p}^{-\frac1p}(\Sigma)\big]^9, \quad h^{(2)} \in \big[\widetilde{B}_{p,p}^{1-\frac1p}(\Sigma)\big]^9, \quad h \in \big[B_{p,p}^{1-\frac1p}(S)\big]^9. \tag{7.16}$$

Due to the above inclusions, clearly, in the potentials V_c and W_c we can take the closed surface S_0 as an integration manifold instead of the crack surface Σ. Recall that Σ is assumed to be a proper part of $S_0=\partial\Omega_0\subset\Omega$ (see Section 5).

The boundary and transmission conditions (7.5)–(7.14) lead to the equations:

$$r_{S_N}\mathcal{A}h + r_{S_N}\big[\mathcal{T}W_c(h^{(2)})\big] + r_{S_N}\big[\mathcal{T}V_c(h^{(1)})\big] = g^{(N)} \ \text{ on } \ S_N, \tag{7.17}$$

$$r_{S_D}h + r_{S_D}\big[W_c(h^{(2)})\big] + r_{S_D}V_c(h^{(1)}) = g^{(D)} \ \text{ on } \ S_D, \tag{7.18}$$

$$r_\Sigma\big[\mathcal{T}V(\mathcal{H}^{-1}h)\big]_j + r_\Sigma\big[\mathcal{L}_ch^{(2)}\big]_j + r_\Sigma\big[\mathcal{K}_c(h^{(1)})\big]_j = 2^{-1}(F_j^+ + F_j^-) \ \text{ on } \ \Sigma, \ \ j=\overline{1,6}, \tag{7.19}$$

where

$$h_7^{(2)} = f_7, \ \ h_8^{(2)} = f_8, \ \ h_9^{(2)} = f_9, \ \ h_j^{(1)} = F_j^- - F_j^+, \ \ j=\overline{1,6},$$
$$h_7^{(1)} = -F_7, \ \ h_8^{(1)} = -F_8, \ \ h_9^{(1)} = -F_9 \ \text{ on } \ \Sigma,$$

and $\mathcal{A} := (-2^{-1}I_9 + \mathcal{K})\mathcal{H}^{-1}$ is the Steklov–Poincaré type operator on S, and

$$\mathcal{L}_c(h^{(2)})(z) := \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^+ = \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^- \ \text{ on } \ \Sigma,$$

$$\mathcal{K}_c(h^{(1)})(z) := \int\limits_\Sigma \mathcal{T}(\partial_z,n(z),\tau)\Gamma(z-y,\tau)h^{(1)}(y)\,d_yS \ \text{ on } \ \Sigma.$$

As we see the sought for density $h^{(1)}$ and the last three components of the vector $h^{(2)}$ are determined explicitly by the data of the problem. Hence, it remains to find the density h and the first six components $\widetilde{h}^{(2)} = (h_1^{(2)},\dots,h_6^{(2)})^\top$ of the vector $h^{(2)}$.

The operator generated by the left hand side expressions of the above simultaneous equations (7.17)–(7.19), acting upon the unknown vector $(h,\widetilde{h}^{(2)})$, reads as

$$\mathcal{Q} := \begin{bmatrix} r_{S_N}\mathcal{A} & r_{S_N}[\mathcal{T}W_c]_{9\times6} \\ r_{S_D}I_9 & [r_{S_D}W_c]_{9\times6} \\ r_\Sigma[\mathcal{T}V(\mathcal{H}^{-1})]_{6\times9} & r_\Sigma[\mathcal{L}_c]_{6\times6} \end{bmatrix}_{24\times15},$$

where $[M]_{m\times n}$ denotes the upper left $m\times n$ dimensional block of a matrix M of dimension $m_0\times n_0$ with $m_0\geqslant m$ and $n_0\geqslant n$. This operator possesses the following mapping properties:

$$\begin{aligned}\mathcal{Q}:[H^s_p(S)]^9\times[\widetilde{H}^s_p(\Sigma)]^6\to[H^{s-1}_p(S_N)]^9\times[H^s_p(S_D)]^9\times[H^{s-1}_p(\Sigma)]^6,\\ \mathcal{Q}:[B^s_{p,q}(S)]^9\times[\widetilde{B}^s_{p,q}(\Sigma)]^6\to[B^{s-1}_{p,q}(S_N)]^9\times[B^s_{p,q}(S_D)]^9\times[B^{s-1}_{p,q}(\Sigma)]^6,\\ 1<p<\infty,\ \ 1\le q\le\infty,\ \ s\in\mathbb{R}.\end{aligned}\tag{7.20}$$

Our main goal is to establish invertibility of the operators (7.20). To this end, by introducing a new additional unknown vector we extend equation (7.18) from S_D onto the whole of S. We will do this in the following way. Denote by $g^{(D)}_0$ some fixed extension of $g^{(D)}$ from S_D onto the whole of S preserving the space. In particular, for the zero vector $g^{(D)}=0$ on S_D we always choose the fixed extension vector $g^{(D)}_0=0$ on S.

Introduce a new unknown vector w on S

$$w=h+r_{_S}[W_c(h^{(2)})]+r_{_S}V_c(h^{(1)})-g^{(D)}_0.\tag{7.21}$$

It is evident that $w\in[\widetilde{B}^{1-\frac1p}_{p,p}(S_N)]^9$ in accordance with (7.18), (7.16), (5.16), and the imbedding $g^{(D)}_0\in[B^{1-\frac1p}_{p,p}(S)]^9$. Moreover, the restriction of equation (7.21) on S_D coincides with equation (7.18). Therefore, we can replace equation (7.18) in system (7.17)–(7.19) by equation (7.21). Finally, we arrive at the following simultaneous equations with respect to unknowns h, w, and $\widetilde{h}^{(2)}$:

$$r_{_{S_N}}\mathcal{A}h+r_{_{S_N}}[\mathcal{T}W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(1)}\ \text{ on }\ S_N,\tag{7.22}$$

$$h-w+r_{_S}[W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(2)}\ \text{ on }\ S,\tag{7.23}$$

$$r_\Sigma\big[\mathcal{T}V(\mathcal{H}^{-1})\big]_{6\times9}(h)+r_\Sigma[\mathcal{L}_c]_{6\times6}(\widetilde{h}^{(2)})=g^{(3)}\ \text{ on }\ \Sigma,\tag{7.24}$$

where

$$\begin{aligned}g^{(1)}&=g^{(N)}-r_{_{S_N}}\big[\mathcal{T}V_c(h^{(1)})\big]-r_{_{S_N}}\Big[\mathcal{T}W_c\big(\big([0]_{1\times6},h^{(2)}_7,h^{(2)}_8,h^{(2)}_9\big)^\top\big)\Big],\\ g^{(2)}&=g^{(D)}_0-r_{_S}\big[V_c(h^{(1)})\big]-r_{_S}\Big[W_c\big(\big([0]_{1\times6},h^{(2)}_7,h^{(2)}_8,h^{(2)}_9\big)^\top\big)\Big],\\ g^{(3)}&=2^{-1}(F^++F^-)-r_{_\Sigma}[\mathcal{K}_c]_{6\times9}(h^{(1)})-r_{_\Sigma}\Big[\mathcal{L}_c\big(\big([0]_{1\times6},h^{(2)}_7,h^{(2)}_8,h^{(2)}_9\big)^\top\big)\Big],\end{aligned}$$

with $F^\pm=(F^\pm_1,\dots,F^\pm_6)^\top$.

Rewrite system (7.22)–(7.24) in the equivalent form

$$r_{_{S_N}}\mathcal{A}w+r_{_{S_N}}[\mathcal{T}W_c]_{9\times6}(\widetilde{h}^{(2)})-r_{_{S_N}}\mathcal{A}[r_{_S}W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(1)}-r_{_{S_N}}\mathcal{A}g^{(2)}\ \text{ on }\ S_N,\tag{7.25}$$

$$-w+h+r_{_S}[W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(2)}\ \text{ on }\ S,\tag{7.26}$$

$$r_\Sigma\big[\mathcal{T}V(\mathcal{H}^{-1})\big]_{6\times9}(h)+r_\Sigma[\mathcal{L}_c]_{6\times6}(\widetilde{h}^{(2)})=g^{(3)}\ \text{ on }\ \Sigma.\tag{7.27}$$

Remark 7.4. Systems (7.17)–(7.19) and (7.25)–(7.27) are equivalent in the following sense:

(i) if $(h,\widetilde{h}^{(2)})^\top$ solves system (7.17)–(7.19), then $(w,h,\widetilde{h}^{(2)})^\top$ with w given by (7.21) where $g^{(D)}_0$ is some fixed extension of the vector $g^{(D)}$ from S_D onto the whole of S involved in the right hand side of equation (7.26), solves system (7.25)–(7.27);

(ii) if $(w,h,\widetilde{h}^{(2)})^\top$ solves system (7.25)–(7.27), then $(h,\widetilde{h}^{(2)})^\top$ solves system (7.17)–(7.19).

The operator generated by the left hand sides of system (7.25)–(7.27) reads as

$$\mathcal{M}:=\begin{bmatrix}r_{_{S_N}}\mathcal{A} & [0]_{9\times9} & r_{_{S_N}}\mathcal{R}\\ -r_{_S}I_9 & r_{_S}I_9 & [r_{_S}W_c]_{9\times6}\\ [0]_{6\times9} & r_{_\Sigma}\big[\mathcal{T}V(\mathcal{H}^{-1})\big]_{6\times9} & r_{_\Sigma}[\mathcal{L}_c]_{6\times6}\end{bmatrix}_{24\times24},$$

where

$$\mathcal{R} = [\mathcal{T} W_c]_{9\times 6} - \mathcal{A}[r_{_S} W_c]_{9\times 6}.$$

This operator has the following mapping properties:

$$\begin{gathered}\mathcal{M} : [\widetilde{H}^s_p(S_N)]^9 \times [H^s_p(S)]^9 \times [\widetilde{H}^s_p(\Sigma)]^6 \to [H^{s-1}_p(S_N)]^9 \times [H^s_p(S)]^9 \times [H^{s-1}_p(\Sigma)]^6,\\ \mathcal{M} : [\widetilde{B}^s_{p,q}(S_N)]^9 \times [B^s_{p,q}(S)]^9 \times [\widetilde{B}^s_{p,q}(\Sigma)]^6 \to [B^{s-1}_{p,q}(S_N)]^9 \times [B^s_{p,q}(S)]^9 \times [B^{s-1}_{p,q}(\Sigma)]^6,\\ 1 < p < \infty,\ 1 \le q \le \infty,\ \ s \in \mathbb{R}.\end{gathered} \tag{7.28}$$

Due to the above agreement about the extension of the zero vector we see that if the right hand side functions of the system (7.17)–(7.19) vanish then the same holds for the system (7.25)–(7.27) and vice versa.

The uniqueness Theorem 5.2 and properties of the single and double layer potentials imply the following assertion.

Lemma 7.5. *The null spaces of the operators $\mathcal{Q}$ and $\mathcal{M}$ are trivial for $s = 1/2$ and $p = 2$.*

Now we start to analyse Fredholm properties of the operator $\mathcal{M}$.

For the principal part $\mathcal{M}_0$ of the operator $\mathcal{M}$ we have

$$\mathcal{M}_0 := \begin{bmatrix} r_{_{S_N}} \mathcal{A} & [0]_{9\times 9} & [0]_{9\times 6}\\ -r_{_S} I_9 & r_{_S} I_9 & [0]_{9\times 6}\\ [0]_{6\times 9} & [0]_{6\times 9} & r_{_\Sigma}\mathcal{L}^{(1)} \end{bmatrix}_{24\times 24}, \tag{7.29}$$

where $\mathcal{L}^{(1)} := [\mathcal{L}_c]_{6\times 6}$.

Clearly, the operator $\mathcal{M}_0$ has the same mapping properties as $\mathcal{M}$ and the difference $\mathcal{M} - \mathcal{M}_0$ is compact.

By the same arguments as in [3], we can establish that the operators $\mathcal{L}_c$ and $\mathcal{A}$ are strongly elliptic pseudodifferential operators of order 1, therefore $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator as well. Moreover, we have the following invertibility results.

Theorem 7.6. *Let $1 < p < \infty$, $1 \le q \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$. Then the operators*

$$r_{_\Sigma}\mathcal{L}^{(1)} : \big[\widetilde{H}^s_p(\Sigma)\big]^6 \to \big[H^{s-1}_p(\Sigma)\big]^6, \quad r_{_\Sigma}\mathcal{L}^{(1)} : \big[\widetilde{B}^s_{p,q}(\Sigma)\big]^6 \to \big[B^{s-1}_{p,q}(\Sigma)\big]^6 \tag{7.30}$$

are invertible.

Proof. With the help of the first equality in (6.5) we find that the principal homogeneous symbol matrix of the strongly elliptic pseudodifferential operator $\mathcal{L}_c$ reads as

$$\begin{aligned}\mathfrak{S}(\mathcal{L}_c; x,\xi) = \mathfrak{S}(\mathcal{L}_{S_0}; x,\xi) &:= \big[-4^{-1} I_9 + \mathfrak{S}^2(\mathcal{K}_{S_0}; x,\xi)\big]\big[\mathfrak{S}(\mathcal{H}_{S_0}; x,\xi)\big]^{-1}\\ &= \big[-4^{-1} I_9 + \mathfrak{S}^2(\mathcal{K}_c; x,\xi)\big]\big[\mathfrak{S}(\mathcal{H}_c; x,\xi)\big]^{-1},\ \ x \in \overline{\Sigma},\ \ \xi \in \mathbb{R}^2 \setminus \{0\},\end{aligned}$$

where $\mathcal{H}_{S_0}$ and $\mathcal{K}_{S_0}$ are integral operators given by (6.1) and (6.2) with S_0 for S.

One can show that the principal homogeneous symbol matrix of the operator $\mathcal{K}_c$ is an odd matrix function in ξ, whereas the principal homogeneous symbol matrix of the operator $\mathcal{H}_c$ is an even matrix function in ξ. Consequently, the matrix $\mathfrak{S}(\mathcal{L}_c; x,\xi)$ is even in ξ (for details see [3, Lemma C.2]).

From these results it follows that $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator with even principal homogeneous symbol. Therefore the matrix $[\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)]^{-1}\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1)$ is the unit matrix and the corresponding eigenvalues equal to 1. Now, from Theorem A.1 in Appendix A it follows that the operators (7.30) are Fredholm with zero index for $1 < p < \infty$, $1 \le q \le \infty$ and $1/p - 1/2 < s < 1/p + 1/2$. It remains to show that the corresponding null spaces are trivial. In turn, due to the same Theorem A.1, it suffices to prove that the operator $r_{_\Sigma}\,\mathcal{L}^{(1)} : [\widetilde{H}^{\frac12}_2(\Sigma)]^6 \to [H^{-\frac12}_2(\Sigma)]^6$ is injective, i.e, we have to prove that the homogeneous equation

$$r_{_\Sigma}\mathcal{L}^{(1)} g = 0 \ \text{ on } \ \Sigma \tag{7.31}$$

possesses only the trivial solution in the space $[\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^6$.

Let $g \in [\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^6$ solve equation (7.31) and construct the double layer potential

$$U = (u_1, \dots, u_9)^\top = W_c(\widetilde{g}), \quad \widetilde{g} = (g, 0, 0, 0)^\top.$$

In view of properties of the double layer potential and equation (7.31), it can easily be verified that the vector $U \in [W_2^1(\mathbb{R}^3 \setminus \overline{\Sigma})]^9$ is a solution to the following crack type boundary transmission problem:

$$\begin{aligned} A(\partial_x, \tau)U = 0 \ \text{ in } \ \mathbb{R}^3 \setminus \overline{\Sigma}, \\ \big\{[\mathcal{T}U]_j\big\}^+ = \big\{[\mathcal{T}U]_j\big\}^- = 0, \ \ j = \overline{1,6} \ \text{ on } \ \Sigma, \\ \{u_k\}^+ - \{u_k\}^- = 0, \ \ k = 7,8,9 \ \text{ on } \ \Sigma, \\ \big\{[\mathcal{T}U]_k\big\}^+ - \big\{[\mathcal{T}U]_k\big\}^- = 0, \ \ k = 7,8,9 \ \text{ on } \ \Sigma \end{aligned}$$

and satisfies the decay conditions (4.5) at infinity, i.e., $U \in \mathbf{Z}(\mathbb{R}^3 \setminus \overline{\Sigma})$.

Applying Green's identities (4.1), (4.6) by standard arguments we can show that $U = 0$ in $\mathbb{R}^3 \setminus \overline{\Sigma}$. Whence $g = (g_1, \dots, g_6)^\top = 0$ on Σ follows due to the equalities $\{u_j\}^+ - \{u_j\}^- = g_j$ on Σ, $j = \overline{1,6}$. This completes the proof. □

Due to (4.9) the operator $\mathcal{A}$ is coercive and consequently is elliptic. Moreover, it is strongly elliptic. Indeed, let $\mathcal{A}_x$ be the operator $\mathcal{A}$ written in some local coordinate system with origin at the frozen point $x \in S$. Denote by $\mathcal{A}_x^{(0)}$ the principal part of the operator $\mathcal{A}_x$ and let $\mathbb{R}^3(n)$ be the half-space $y_1 n_1(x) + y_2 n_2(x) + y_3 n_3(x) < 0$ with plane boundary $\mathbb{R}^2(n) = \partial\mathbb{R}^3(n)$. Evidently, $n(x)$ is the unit outward normal vector to $\mathbb{R}^3(n)$. From Green's formula (4.1) with $\Omega = \mathbb{R}^3(n)$, equality (4.8), and positive definiteness of form (4.1) it follows that for all $\varphi \in [C_0^\infty(\mathbb{R}^2)]^9$, $\varphi \neq 0$,

$$\operatorname{Re} \int\limits_{\mathbb{R}^2(n)} \mathcal{A}_x^{(0)}\varphi(y) \cdot \varphi(y)\, dy = \int\limits_{\mathbb{R}^2(n)} \operatorname{Re} \mathfrak{S}(\mathcal{A}; x, \xi)\psi(\xi) \cdot \psi(\xi)\, d\xi \geq 0, \quad \psi(\xi) = \mathcal{F}_{y \to \xi}(\varphi)(y),$$

(cf. [19, Theorem 17]) which ensures strong ellipticity property of the symbol $\mathfrak{S}(\mathcal{A}; x, \xi)$, that is, there exists a positive constant c such that $\operatorname{Re} \mathfrak{S}(\mathcal{A}; x, \xi)\zeta \cdot \zeta \geq c|\xi|\,|\zeta|^2$ for $x \in S$, $\xi \in \mathbb{R}^2$, $\zeta \in \mathbb{C}^9$.

Let $\widetilde{\lambda}_k$, $k = \overline{1,9}$, be the eigenvalues of the matrix $a_0(x) := [\mathfrak{S}(\mathcal{A}; x, 0, +1)]^{-1}\mathfrak{S}(\mathcal{A}; x, 0, -1)$, $x \in \ell_m = \partial S_D = \partial S_N$, where $\mathfrak{S}(\mathcal{A}; x, \xi)$ with $x \in \overline{S}_N$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$. As we will see below one of the eigenvalues ($\widetilde{\lambda}_9$ say) of the matrix $a_0(x)$ equals to 1.

Let us introduce the notation

$$\delta' = \inf_{\substack{1 \leq j \leq 9 \\ x \in \ell_m}} \frac{1}{2\pi} \arg \widetilde{\lambda}_j(x), \quad \delta'' = \sup_{\substack{1 \leq j \leq 9 \\ x \in \ell_m}} \frac{1}{2\pi} \arg \widetilde{\lambda}_j(x). \tag{7.32}$$

Due to strong ellipticity of the operator $\mathcal{A}$ and since one eigenvalue equals to 1, we deduce that $-1/2 < \delta' \leq 0 \leq \delta'' < 1/2$. Theorem A.1 in Appendix A implies the following assertion (cf. [3, Theorem 5.19]).

Theorem 7.7. *Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1/p - 1/2 + \delta'' < s < 1/p + 1/2 + \delta'$ with δ' and δ'' given by* (7.32). *Then the Steklov–Poincaré operators*

$$r_{S_N}\mathcal{A} : \big[\widetilde{H}_p^s(S_N)\big]^9 \to [H_p^{s-1}(S_N)]^9, \quad r_{S_N}\mathcal{A} : \big[\widetilde{B}_{p,q}^s(S_N)\big]^9 \to [B_{p,q}^{s-1}(S_N)]^9$$

are invertible.

In turn, Theorem 7.7 leads to the following invertibility result.

Theorem 7.8. *Let*

$$1 < p < \infty, \ \ 1 \leq q \leq \infty, \ \ \frac{1}{p} - \frac{1}{2} + \delta'' < s < \frac{1}{p} + \frac{1}{2} + \delta'. \tag{7.33}$$

Then operators (7.28) *are invertible.*

Proof. From Theorems 7.6 and 7.7 we conclude that for arbitrary p, q, and s satisfying conditions (7.33), the operators

$$\mathcal{M}_0 : [\widetilde{H}_p^s(S_N)]^9 \times [H_p^s(S)]^9 \times [\widetilde{H}_p^s(\Sigma)]^6 \to [H_p^{s-1}(S_N)]^9 \times [H_p^s(S)]^9 \times [H_p^{s-1}(\Sigma)]^6,$$
$$\mathcal{M}_0 : [\widetilde{B}_{p,q}^s(S_N)]^9 \times [B_{p,q}^s(S)]^9 \times [\widetilde{B}_{p,q}^s(\Sigma)]^6 \to [B_{p,q}^{s-1}(S_N)]^9 \times [B_{p,q}^s(S)]^9 \times [B_{p,q}^{s-1}(\Sigma)]^6,$$

with $\mathcal{M}_0$ defined in (7.29) are invertible. Therefore the operators (7.28) are Fredholm operators with index 0.

By Lemma 7.5 we conclude then that for $s = 1/2$ and $p = 2$ operator (7.28) is invertible. The null-spaces and indices of the operators (7.28) are the same for all values of the parameter $q \in [1, +\infty]$, provided p and s satisfy the inequalities (7.33) (see [1, Chapter 3, Proposition 10.6]). Therefore, for such values of the parameters p and s they are invertible. In particular, the nonhomogeneous system (7.25)–(7.27) is uniquely solvable in the corresponding spaces. Moreover, it can be easily shown that the solution vectors h, $\widetilde{h}^{(2)}$ do not depend on the extension of the vector $g^{(D)}$, while w does. However, the sum $w + g_0^{(D)}$ is defined uniquely. □

Due to Remark 7.4 we conclude that the operators (7.20) are invertible if p, q and s satisfy conditions (7.33).

With the help of this theorem we arrive at the following existence result for the original mixed BVP.

Theorem 7.9. *Let*

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'} \tag{7.34}$$

with δ' and δ'' given by (7.32). *Then the BVP* (5.5)–(5.15) *has a unique solution U in the space $[W_p^1(\Omega_\Sigma)]^9$, which can be represented as $U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)})$ in Ω_Σ, where h, $h^{(2)}$, and $h^{(1)}$ are defined by the system* (7.17)–(7.19).

Proof. The condition (7.34) follows from the inequality (7.33) with $s = 1 - 1/p$. Now existence of a solution $U \in [W_p^1(\Omega_\Sigma)]^9$ with p satisfying (7.34) follows from Theorem 7.8 and Remark 7.4. Due to the inequalities $-1/2 < \delta' \leq \delta'' < 1/2$ we have $p = 2 \in (\frac{4}{3-2\delta''}, \frac{4}{1-2\delta'})$. Therefore the unique solvability for $p = 2$ is a consequence of Theorem 5.2.

To show the uniqueness result for all other values of p from the interval (7.34) we proceed as follows. Let a vector $U \in [W_p^1(\Omega_\Sigma)]^9$ with p satisfying (7.34) be a solution to the homogeneous boundary value problem (5.5)–(5.15).

Then it is evident that

$$\{U\}_S^+ \in [B_{p,p}^{1-\frac{1}{p}}(S)]^9, \ \ \{\mathcal{T}U\}_S^+ \in [B_{p,p}^{-\frac{1}{p}}(S)]^9, \ \ \{U\}_\Sigma^\pm \in [B_{p,p}^{1-\frac{1}{p}}(\Sigma)]^9, \ \ \{\mathcal{T}U\}_\Sigma^\pm \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^9,$$
$$\{U\}_\Sigma^+ - \{U\}_\Sigma^- \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^9, \quad \{\mathcal{T}U\}_\Sigma^+ - \{\mathcal{T}U\}_\Sigma^- = 0 \ \text{ on } \ \Sigma.$$

By the general integral representation formula the vector U can be represented in Ω_Σ as

$$U = W_c(\{U\}_\Sigma^+ - \{U\}_\Sigma^-)V_c(\{\mathcal{T}U\}_\Sigma^+ - \{\mathcal{T}U\}_\Sigma^-) + W(\{U\}_S^+) - V(\{\mathcal{T}U\}_S^+),$$

i.e.,

$$U = U^* + W_c(h^{(2)}) + V_c(h^{(1)}) \ \text{ in } \ \Omega_\Sigma, \tag{7.35}$$

where

$$h^{(1)} = \{\mathcal{T}U\}_\Sigma^+ - \{\mathcal{T}U\}_\Sigma^-, \quad h^{(2)} := \{U\}_\Sigma^+ - \{U\}_\Sigma^- \ \text{ on } \Sigma,$$
$$U^* := W(\{U\}_S^+) - V\big(\{\mathcal{T}U\}_S^+\big) \in [W_p^1(\Omega)]^9.$$

Note that U^* solves the homogeneous equation

$$A(\partial, \tau)U^* = 0 \ \text{ in } \ \Omega.$$

Denote $h := \{U^*\}_S^+$. Clearly, $h \in [B_{p,p}^{1-1/p}(S)]^9$. Since the Dirichlet problem possesses a unique solution in the space $[W_p^1(\Omega)]^9$ for arbitrary $p \in [1, +\infty)$, due to Theorem 7.2 we can represent U^*

uniquely in the form of a single layer potential, $U^* = V(\mathcal{H}^{-1}h)$ in Ω (for details see [3, Chapter 5, Section 5.6]). Therefore from (7.35) we get

$$U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)}) \ \text{ in } \ \Omega_\Sigma.$$

Now, the homogeneous boundary and transmission conditions for U lead to the homogeneous system (cf. (7.17)–(7.19)) $\mathcal{Q}\Psi = 0$, where $\Psi = (h, h^{(2)}, h^{(1)})^\top$. Whence, $\Psi = 0$ follows immediately due to invertibility of $\mathcal{Q}$ (see Theorem 7.8 and Remark 7.4). Consequently, $U = 0$ in Ω_Σ. □

Let us now present some regularity results for solutions of the mixed boundary value problem (5.5)–(5.15).

Theorem 7.10. *Let $1 < t < \infty$, $1 \le q \le \infty$,*

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'}\,, \quad \frac{1}{t} - \frac{1}{2} + \delta'' < s < \frac{1}{t} + \frac{1}{2} + \delta'$$

with δ' and δ'' given by (7.32), and let $U \in [W^1_p(\Omega_\Sigma)]^9$ be the solution of the boundary value problem (5.5)–(5.15). Then the following regularity results hold:

(i) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,t}(\Sigma), \ \ F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,t}(\Sigma), \ \ j = \overline{1,6},$$
$$F_k \in \widetilde{B}^{s-1}_{t,t}(\Sigma), \ \ f_k \in \widetilde{B}^{s}_{t,t}(\Sigma), \ \ k = 7,8,9,$$
$$g^{(D)} \in [B^{s}_{t,t}(S_D)]^9, \ \ g^{(N)} \in [B^{s-1}_{t,t}(S_N)]^9,$$

then

$$U \in [H_t^{s+\frac{1}{t}}(\Omega_\Sigma)]^9;$$

(ii) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,q}(\Sigma), \ \ F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,q}(\Sigma), \ \ j = \overline{1,6},$$
$$F_k \in \widetilde{B}^{s-1}_{t,q}(\Sigma), \ \ f_k \in \widetilde{B}^{s}_{t,q}(\Sigma), \ \ k = 7,8,9,$$
$$g^{(D)} \in [B^{s}_{t,q}(S_D)]^9, \ \ g^{(N)} \in [B^{s-1}_{t,q}(S_N)]^9,$$

then

$$U \in [B_{t,q}^{s+\frac{1}{t}}(\Omega_\Sigma)]^9;$$

(iii) *If $\alpha > 0$ and*

$$F_j^+, F_j^- \in B^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ F_j^+ - F_j^- \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ j = \overline{1,6},$$
$$F_k \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ f_k \in C^{\alpha}(\overline{\Sigma}), \ \ r_{\ell_c} f_k = 0, \ \ k = 7,8,9,$$
$$g^{(D)} \in [C^{\alpha}(\overline{S}_D)]^9, \ \ g^{(N)} \in [B^{\alpha-1}_{\infty,\infty}(S_N)]^9,$$

then

$$U \in \bigcap_{\alpha'<\gamma} C^{\alpha'}(\overline{\Omega}_j), \ \ j = 0,1,$$

where $\gamma = \min\{\alpha, 1/2 + \delta'\}$, $-1/2 < \delta' \le 0$ and Ω_0 is an arbitrary proper subdomain of Ω such that $\Sigma \subset \partial\Omega_0 = S_0 \in C^\infty$ and $\Omega_1 = \Omega \setminus \overline{\Omega}_0$.

Moreover, in one-sided interior and exterior neighbourhoods of the surface S_0 the vector U has $C^{\gamma'-\varepsilon}$-smoothness with $\gamma' = \min\{\alpha, 1/2\}$, while in a one-sided interior neighbourhood of the surface S the vector U possesses $C^{\gamma''-\varepsilon}$-smoothness with $\gamma'' = \min\{\alpha, 1/2+\delta'\}$; here ε is an arbitrarily small positive number.

Proof. The proof is exactly the same as that of Theorem 5.22 in [3]. □

8 Asymptotic expansion of solutions

Here we investigate the asymptotic behaviour of solutions to the problem (5.5)–(5.15) near the exceptional curves ℓ_c and ℓ_m. For simplicity of description of the method applied below, we assume that the boundary data of the problem are infinitely smooth, $F_j^+, F_j^- \in C^\infty(\overline{\Sigma})$, $F_j^+ - F_j^- \in C_0^\infty(\overline{\Sigma})$, $j = \overline{1,6}$, $f_k, F_k \in C_0^\infty(\overline{\Sigma})$, $k = 7,8,9$, $g^{(D)} \in [C^\infty(\overline{S}_D)]^9$, $g^{(N)} \in [C^\infty(\overline{S}_N)]^9$, where $C_0^\infty(\overline{\Sigma})$ denotes a space of functions vanishing along with all tangential (to Σ) derivatives at $\ell_c = \partial\Sigma$.

In Section 7, we have shown that the boundary value problem (5.5)–(5.15) is uniquely solvable and the solution U can be represented by (7.15), where the densities are defined by equations (7.17)–(7.19) or by the equivalent system (7.25)–(7.27).

Let $\Phi := (w, h, \widetilde{h}^{(2)})^\top$ be a solution of the system (7.25)–(7.27): $\mathcal{M}\Phi = G$, where G is the vector constructed by the right hand sides of the system, $G \in [C^\infty(\overline{S}_N)]^9 \times [C^\infty(S)]^9 \times [C^\infty(\overline{\Sigma})]^6$. To establish the asymptotic behaviour of the vector U near the curves ℓ_c and ℓ_m, we rewrite (7.15) as follows:

$$U = V(\mathcal{H}^{-1}w) + W_c(\widetilde{g}) + \mathcal{R}, \tag{8.1}$$

where

$$\mathcal{R} := -V\big(\mathcal{H}^{-1}\big[r_{_S} W_c(h^{(2)}) + r_{_S} V_c(h^{(1)}) - g_0^{(D)}\big]\big) + W_c(f_0) + V_c(h^{(1)}),$$

with $f_0 = (0,0,0,0,0,0,f_7,f_8,f_9)^\top$.

Due to the relations

$$\begin{gathered} r_{_S} W_c(h^{(2)}) + r_{_S} V_c(h^{(1)}) - g_0^{(D)} \in [C^\infty(S)]^9, \\ h^{(1)} = \big(F_1^- - F_1^+, \dots, F_6^- - F_6^+, -F_7, -F_8, -F_9\big) \in [C_0^\infty(\overline{\Sigma})]^6, \\ h_7^{(2)} = f_7 \in C_0^\infty(\overline{\Sigma}), \ \ h_8^{(2)} = f_8 \in C_0^\infty(\overline{\Sigma}), \ \ h_9^{(2)} = f_9 \in C_0^\infty(\overline{\Sigma}). \end{gathered}$$

we deduce $r_{_{\overline{\Omega}_j}} \mathcal{R} \in [C^\infty(\overline{\Omega}_j)]^6$, where Ω_j, $j = 0,1$, are as in Theorem 7.10(iii).

The vector $\widetilde{g}$ involved in (8.1) is defined as follows: $\widetilde{g} = (\widetilde{h}^{(2)}, 0, 0, 0)^\top$, where $\widetilde{h}^2$ solves the pseudodifferential equation

$$r_{_\Sigma} \mathcal{L}^{(1)}\, \widetilde{h}^{(2)} = \Psi^{(1)} \ \text{ on } \ \Sigma \tag{8.2}$$

with $\Psi^{(1)} = (\Psi_1^{(1)}, \dots, \Psi_6^{(1)})^\top$. Evidently,

$$\Psi^{(1)} = g^{(3)} - r_{_\Sigma}[\mathcal{T}V(\mathcal{H}^{-1})]_{6\times 9}(h).$$

Finally, the vector w involved in (8.1) solves the pseudodifferential equation

$$r_{_{S_N}} \mathcal{A}w = \Psi^{(2)} \ \text{ on } \ S_N, \tag{8.3}$$

where

$$\Psi^{(2)} = g^{(1)} - r_{_{S_N}} \mathcal{A}g^{(2)} - r_{_{S_N}}\big([\mathcal{T}W_c]_{9\times 6}(\widetilde{h}^{(2)}) - \mathcal{A}[r_{_S} W_c]_{9\times 6}(\widetilde{h}^{(2)})\big) \in [C^\infty(\overline{S}_N)]^9.$$

As we have already mentioned, the principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)}; x, \xi)$, $x \in \overline{\Sigma}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even with respect to the variable ξ and therefore the matrix

$$\big[\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)\big]^{-1} \mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1), \ \ x \in \ell_c,$$

is the unit matrix I_6. Consequently, all eigenvalues of this matrix equal to one, $\widetilde{\lambda}_j(x) = 1$, $j = \overline{1,6}$, $x \in \ell_c$. Applying a partition of unity, natural local coordinate systems and local diffeomorphisms, we can rectify ℓ_c and Σ locally in a standard way. For simplicity, let us denote the local rectified images of ℓ_c and Σ under this diffeomorphisms by the same symbols. Then we identify a one-sided neighbourhood (on Σ) of an arbitrary point $\widetilde{x} \in \ell_c$ as a part of the half-plane $x_2 > 0$. Thus, we assume that $(x_1, 0) \in \ell_c$ and $(x_1, x_{2,+}) \in \Sigma$ for $0 < x_{2,+} < \varepsilon$. Clearly, $x_{2,+} = \operatorname{dist}(x, \ell_c)$.

Applying the results obtained in the references [6] and [7] we can derive the following asymptotic expansion for the solution $\widetilde{h}^{(2)}$ of the strongly elliptic pseudodifferential equation (8.2),

$$\widetilde{h}^{(2)}(x_1,x_{2,+}) = c_0(x_1)x_{2,+}^{\frac{1}{2}} + \sum_{k=1}^{M} c_k(x_1)x_{2,+}^{\frac{1}{2}+k} + \widetilde{h}^{(2)}_{M+1}(x_1,x_{2,+}), \tag{8.4}$$

where M is an arbitrary natural number, $c_k \in [C^\infty(\ell_c)]^6$, $k=0,1,\dots,M$, and the remainder term satisfies the inclusion

$$\widetilde{h}^{(2)}_{M+1} \in [C^{M+1}(\ell^+_{c,\varepsilon})]^6, \ \ \ell^+_{c,\varepsilon} = \ell_c \times [0,\varepsilon].$$

Note that, according to [7], the terms in expansion (8.4) do not contain logarithms, since the principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)};x,\xi)$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even in ξ.

To derive analogous asymptotic expansion for the solution vector w of equation (8.3), we apply the same local technique as above to a one-sided neighbourhood (in S_N) of the curve ℓ_m and preserve the same notation for the local coordinates.

Consider a 9×9 matrix $a_0(x_1)$ constructed by means of the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$,

$$a_0(x_1) := \big[\mathfrak{S}(\mathcal{A};x_1,0,+1)\big]^{-1}\,\mathfrak{S}(\mathcal{A};x_1,0,-1), \ \ (x_1,0)\in\ell_m. \tag{8.5}$$

Note that unlike to the above considered case, now (8.5) is not the unit matrix and therefore we proceed as follows.

Denote by $\widetilde{\lambda}_1(x_1),\dots,\widetilde{\lambda}_9(x_1)$ the eigenvalues of the matrix a_0. Let μ_j, $j=1,\dots,l$, $1\le l\le 9$, be the distinct eigenvalues and m_j be their algebraic multiplicities: $m_1+\cdots+m_l=9$. It is well known that the matrix $a_0(x_1)$ admits the decomposition (see, e.g., [12, Chapter 7, Section 7]) $a_0(x_1)=\mathcal{D}(x_1)\mathcal{J}_{a_0}(x_1)\mathcal{D}^{-1}(x_1)$, $(x_1,0)\in\ell_m$, where $\mathcal{D}$ is 9×9 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{a_0}$ has a block diagonal structure $\mathcal{J}_{a_0}(x_1) := \operatorname{diag}\{\mu_1(x_1)B^{(m_1)}(1),\dots,\mu_l(x_1)B^{(m_l)}(1)\}$. Here $B^{(\nu)}(t)$, $\nu\in\{m_1,\dots,m_l\}$, are upper triangular matrices:

$$B^{(\nu)}(t) = \|b^{(\nu)}_{jk}(t)\|_{\nu\times\nu}, \quad b^{(\nu)}_{jk}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}, & j<k,\\ 1, & j=k,\\ 0, & j>k,\end{cases}$$

i.e.,

$$B^{(\nu)}(t) = \begin{bmatrix} 1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{\nu-2}}{(\nu-2)!} & \dfrac{t^{\nu-1}}{(\nu-1)!}\\ 0 & 1 & t & \cdots & \dfrac{t^{\nu-3}}{(\nu-3)!} & \dfrac{t^{\nu-2}}{(\nu-2)!}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1 & t\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{\nu\times\nu}.$$

Denote

$$B_0(t) := \operatorname{diag}\big\{B^{(m_1)}(t),\dots,B^{(m_l)}(t)\big\}.$$

Again, applying the results from the reference [6] we derive the following asymptotic expansion for the solution ω of the strongly elliptic pseudodifferential equation (8.3):

$$\begin{aligned}\omega(x_1,x_{2,+}) &= \mathcal{D}(x_1)x_{2,+}^{\frac{1}{2}+\Delta(x_1)}B_0\Big(-\frac{1}{2\pi i}\,\log x_{2,+}\Big)\mathcal{D}^{-1}(x_1)b_0(x_1)\\ &\quad + \sum_{k=1}^{M}\mathcal{D}(x_1)x_{2,+}^{\frac{1}{2}+\Delta(x_1)+k}B_k(x_1,\log x_{2,+}) + \omega_{M+1}(x_1,x_{2,+}),\end{aligned} \tag{8.6}$$

where $b_0 \in [C^\infty(\ell_m)]^9$, $\omega_{M+1} \in [C^\infty(\ell^+_{m,\varepsilon})]^9$, $\ell^+_{m,\varepsilon} = \ell_m \times [0,\varepsilon]$, and

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)} t^j d_{kj}(x_1).$$

Here $m_0 = \max\{m_1,\dots,m_9\}$, the coefficients $d_{kj} \in [C^\infty(\ell_m)]^9$, $\Delta := (\Delta_1,\dots,\Delta_9)$, and

$$\Delta_j(x_1) = \frac{1}{2\pi i}\,\log\widetilde{\lambda}_j(x_1) = \frac{1}{2\pi}\,\arg\widetilde{\lambda}_j(x_1) + \frac{1}{2\pi i}\,\log|\widetilde{\lambda}_j(x_1)|,$$
$$-\pi < \arg\widetilde{\lambda}_j(x_1) < \pi,\ \ (x_1,0)\in\ell_m,\ \ j=\overline{1,9}.$$

Furthermore,

$$x_{2,+}^{\frac12+\Delta(x_1)} := \operatorname{diag}\Big\{x_{2,+}^{\frac12+\Delta_1(x_1)},\dots,x_{2,+}^{\frac12+\Delta_9(x_1)}\Big\}.$$

Now, having at hand formulae (8.4) and (8.6) with the help of the asymptotic expansion of potential-type functions obtained in [5] we can write the following spatial asymptotic expansions for the solution vector U of the boundary value problem (5.5)–(5.15) near the crack edge ℓ_c and near the collision curve ℓ_m.

(a) ***Asymptotic expansion near the crack edge ℓ_c:***

$$U(x) = \sum_{\mu=\pm1}\Bigg[\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j z_{s,\mu}^{\frac12-j} d_{sj}^{(c)}(x_1,\mu) + \sum_{\substack{k,l=0\\ k+l+j+p\geq1}}^{M+2}\ \sum_{j+p=0}^{M+2-l} x_2^l x_3^j z_{s,\mu}^{\frac12+p+k} d_{slkjp}^{(c)}(x_1,\mu)\Bigg] + U_{M+1}^{(c)}(x) \tag{8.7}$$

with the coefficients $d_{sj}^{(c)}(\,\cdot\,,\mu), d_{slkjp}^{(c)}(\,\cdot\,,\mu) \in [C^\infty(\ell_c)]^9$ and $U_{M+1}^{(c)} \in [C^{M+1}(\overline{\Omega}_j)]^9$, $j=0,1$. Here Ω_j, $j=0,1$, are as in Theorem 7.10(iii), and

$$z_{s,+1} = -(x_2 + x_3\zeta_{s,+1}),\ \ z_{s,-1} = x_2 - x_3\zeta_{s,-1},\ \ -\pi < \arg z_{s,\pm1} < \pi,\ \ \zeta_{s,\pm1}\in C^\infty(\ell_c), \tag{8.8}$$

where $\{\zeta_{s,\pm1}\}_{s=1}^{l_0}$ are the different roots in ζ of multiplicity n_s, $s=1,\dots,l_0$, of the polynomial $\det A^{(0)}([J_\varkappa^\top(x_1,0,0)]^{-1}\eta_\pm)$ with $\eta_\pm = (0,\pm1,\zeta)^\top$, satisfying the condition $\operatorname{Re}\zeta_{s,\pm1} < 0$. The matrix $J_\varkappa$ stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa$ related to the local co-ordinate system. Under this diffeomorphism ℓ_c and Σ are locally rectified and we assume that $(x_1,0,0)\in\ell_c$, $x_2 = \operatorname{dist}(x^{(\Sigma)},\ell_c)$, $x_3 = \operatorname{dist}(x,\Sigma)$, where $x^{(\Sigma)}$ is the projection of the reference point $x\in\Omega_\Sigma$ onto the plane corresponding to the image of Σ under the diffeomorphism $\varkappa$.

Note that the coefficients $d_{sj}^{(c)}(\,\cdot\,,\mu)$ can be expressed by the first coefficient c_0 in the asymptotic expansion (8.4) (for details see [5, Theorem 2.3]).

(b) ***Asymptotic expansion near the collision curve ℓ_m:***

$$\begin{aligned} U(x) = \sum_{\mu=\pm1}\Bigg\{&\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j\Big[d_{sj}^{(m)}(x_1,\mu) z_{s,\mu}^{\frac12+\Delta(x_1)-j} B_0\Big(-\frac{1}{2\pi i}\,\log z_{s,\mu}\Big)\Big]\widetilde{c}_j(x_1)\\ &+\sum_{\substack{k,l=0\\ k+l+j+p\geq1}}^{M+2}\ \sum_{j+p=0}^{M+2-l} x_2^l x_3^j d_{sljp}^{(m)}(x_1,\mu) z_{s,\mu}^{\frac12+\Delta(x_1)+p+k} B_{skjp}(x_1,\log z_{s,\mu})\Bigg\} + U_{M+1}^{(m)}(x), \end{aligned} \tag{8.9}$$

where $d_{sj}^{(m)}(\,\cdot\,,\mu)$ and $d_{sljp}^{(m)}(\,\cdot\,,\mu)$ are matrices with entries belonging to the space $C^\infty(\ell_m)$, $\widetilde{c}_j \in [C^\infty(\ell_m)]^9$, $U_{M+1}^{(m)} \in [C^{M+1}(\overline{\Omega}_1)]^9$, and

$$z_{s,\mu}^{\kappa+\Delta(x_1)} := \operatorname{diag}\big\{z_{s,\mu}^{\kappa+\Delta_1(x_1)},\dots,z_{s,\mu}^{\kappa+\Delta_9(x_1)}\big\},\ \ \kappa\in\mathbb{R},\ \ \mu=\pm1,\ \ x_1\in\ell_m;$$

$B_{skjp}(x_1,t)$ are polynomials with respect to the variable t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp} = k(2m_0-1)+m_0-1+j+p$, in general, where $m_0 = \max\{m_1,\dots,m_l\}$ and $m_1+\dots+m_l = 9$.

Note that the coefficients $d_{sj}^{(m)}(\,\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients $\widetilde{c}_j$ can be expressed by means of the first coefficient b_0 in the asymptotic expansion (8.6) (for details see [5, Theorem 2.3]).

9 Analysis of singularities of solutions

Let $x' \in \ell_c$ and $\Pi^{(c)}_{x'}$ be the plane passing through the point x' and orthogonal to the curve ℓ_c. We introduce the polar coordinates (r, α), $r \geq 0$, $-\pi \leq \alpha \leq \pi$, in the plane $\Pi^{(c)}_{x'}$ with pole at the point x'. Denote by $\Sigma^{\pm}$ the two different faces of the crack surface Σ. It is clear that $(r, \pm\pi) \in \Sigma^{\pm}$.

Denote the similar orthogonal plane to the curve ℓ_m by $\Pi^{(m)}_{x'}$ at the point $x' \in \ell_m$ and introduce there the polar coordinates (r, α), with pole at the point x'. The intersection of the plane $\Pi^{(m)}_{x'}$ and Ω_Σ can be identified with the half-plane $r \geq 0$ and $0 \leq \alpha \leq \pi$.

In these coordinate systems, the functions $z_{s,\pm 1}$ given by (8.8) read as follows:

$$z_{s,+1} = -r(\cos\alpha + \zeta_{s,+1}(x') \sin\alpha), \quad z_{s,-1} = r(\cos\alpha - \zeta_{s,-1}(x') \sin\alpha),$$

where $x' \in \ell_c \cup \ell_m$, $s = 1, \dots, l_0$. We can rewrite asymptotic expansions (8.7) and (8.9) in more convenient forms, in terms of the variables r and α. Moreover, we establish more refined asymptotic properties of the solution vector $U = (u, \phi, \varphi, \psi, \vartheta)^\top \in [C^\infty(\Omega_\Sigma)]^9$ near the exceptional curves.

(i) ***Asymptotic analysis of solutions near the crack edge ℓ_c.***

The asymptotic expansion (8.7) yields

$$U = (u, \phi, \varphi, \psi, \vartheta)^\top = a_0(x', \alpha)\, r^{1/2} + a_1(x', \alpha)\, r^{3/2} + \cdots, \tag{9.1}$$

where r is the distance from the reference point $x \in \Pi^{(c)}_{x'}$ to the curve ℓ_c, and $a_j = (a_{j1}, \dots, a_{j9})^\top$, $j = 0, 1, \dots$, are smooth vector functions of $x' \in \ell_c$.

From this representation it follows that in one-sided interior and exterior neighbourhoods of the surface $S_0 = \partial\Omega_0$ the vector $U = (u, \phi, \varphi, \psi, \vartheta)^\top$ has $C^{\frac{1}{2}}$-smoothness.

(ii) ***Asymptotic analysis of solutions near the curve ℓ_m.***

The asymptotic expansion (8.9) yields

$$U(x) = \sum_{\mu=\pm 1} \sum_{s=1}^{l_0} \sum_{j=0}^{n_s - 1} c_{sj\mu}(x', \alpha) r^{\gamma + i\delta} B_0\Big(-\frac{1}{2\pi i} \log r\Big) \widetilde{c}_{sj\mu}(x', \alpha) + \cdots, \tag{9.2}$$

where $x' \in \ell_m$,

$$\begin{gathered} r^{\gamma+i\delta} := \operatorname{diag}\big\{r^{\gamma_1 + i\delta_1}, \dots, r^{\gamma_9 + i\delta_9}\big\}, \\ \gamma_j = \frac{1}{2} + \frac{1}{2\pi} \arg \widetilde{\lambda}_j(x'), \quad \delta_j = \frac{1}{2\pi} \log|\widetilde{\lambda}_j(x')|, \quad j = \overline{1, 9}, \end{gathered} \tag{9.3}$$

and $\widetilde{\lambda}_j$, $j = \overline{1, 9}$, are eigenvalues of the matrix

$$a_0(x') = \big[\mathfrak{S}(\mathcal{A}; x', 0, +1)\big]^{-1} \mathfrak{S}(\mathcal{A}; x', 0, -1), \quad x' \in \ell_m.$$

Recall that here $\mathfrak{S}(\mathcal{A}; x', \xi)$ is the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A} = (-2^{-1} I_9 + \mathcal{K})\mathcal{H}^{-1}$. Moreover, the eigenvalues $\widetilde{\lambda}_j$, $j = \overline{1, 9}$, can be expressed in terms of the eigenvalues β_j, $j = \overline{1, 9}$, of the matrix $\mathfrak{S}(\mathcal{K}; x', 0, +1)$, where $\mathfrak{S}(\mathcal{K}; x', \xi)$ is the principal homogeneous symbol matrix of the singular integral operator $\mathcal{K}$ (see [4, Theorem 6.3]),

$$\widetilde{\lambda}_j = \frac{1 + 2\beta_j}{1 - 2\beta_j}, \quad j = \overline{1, 9}. \tag{9.4}$$

The symbol matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$ is calculated explicitly

$$\mathfrak{S}(\mathcal{K};x',0,+1)=\begin{bmatrix}0&0&0&0&0&0&0&0&0\\0&0&-ia&0&0&0&0&0&0\\0&ia&0&0&0&0&0&0&0\\0&0&0&0&0&0&ic&ip&iq\\0&0&0&0&0&-ib&0&0&0\\0&0&0&0&ib&0&0&0&0\\0&0&0&-\dfrac{ib_0}{2\gamma}&0&0&0&0&0\\0&0&0&-\dfrac{i\lambda_1}{2\gamma}&0&0&0&0&0\\0&0&0&-\dfrac{i\nu_2}{2\gamma}&0&0&0&0&0\end{bmatrix}_{9\times 9},$$

where

$$a=\frac{1}{4}\Big(\frac{\lambda}{\lambda+2\mu+\varkappa}-\frac{\mu}{\mu+\varkappa}\Big),\quad b=\frac{1}{4}\Big(\frac{\alpha}{\alpha+\beta+\gamma}-\frac{\beta}{\gamma}\Big),$$

$$c=b_0b_{11}+\lambda_1b_{21}+\nu_2b_{31},\quad p=b_0b_{12}+\lambda_1b_{22}+\nu_2b_{32},\quad q=b_0b_{13}+\lambda_1b_{23}+\nu_2b_{33},$$

$$\big[b_{jk}\big]_{3\times3}=\begin{bmatrix}a_0&-\lambda_2&\nu_1\\\lambda_2&\chi&\nu_3\\\nu_1&-\nu_3&k\end{bmatrix}^{-1}=(k\chi a_0+k\lambda_2^2-\chi\nu_1^2-2\lambda_2\nu_1\nu_3+a_0\nu_3^2)^{-1}$$
$$\times\begin{bmatrix}k\chi+\nu_3^2&k\lambda_2-\nu_1\nu_3&\chi\nu_1+\lambda_2\nu_3\\-k\lambda_2+\nu_1\nu_3&ka_0-\nu_1^2&-\nu a_0+\lambda_2\nu_1\\\chi\nu_1+\lambda_2\nu_3&-\lambda_2\nu_1+a_0\nu_3&\chi a_0+\lambda_2^2\end{bmatrix}.$$

The characteristic polynomial of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$ can be represented as

$$\det\big(\mathfrak{S}(\mathcal{K};x',0,+1)-\beta I\big)=-\frac{\beta^3(\beta^2-a^2)(\beta^2-b^2)(2\gamma\beta^2-cb_0-p\lambda_1-q\nu_2)}{2\gamma}\,.$$

Therefore we have the following expressions for eigenvalues of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$:

$$\beta_{1,2}=\mp\sqrt{d}\,,\ \ \beta_{3,4}=\mp a,\ \ \beta_{5,6}=\mp b,\ \ \beta_7=\beta_8=\beta_9=0,$$

where

$$|a|<\frac{1}{2}\,,\ \ |b|<\frac{1}{2}\,,\ \ d=\frac{cb_0+p\lambda_1+q\nu_2}{2\gamma}\,,\ \ \gamma>0. \tag{9.5}$$

Then due to (9.4) we have

$$\widetilde{\lambda}_1=\frac{1}{\widetilde{\lambda}_2}=\begin{cases}\dfrac{1-2i\sqrt{-d}}{1+2i\sqrt{-d}} & \text{if } d<0,\\[2mm] \dfrac{1-2\sqrt{d}}{1+2\sqrt{d}} & \text{if } d\geqslant 0,\end{cases}$$

$$\widetilde{\lambda}_3=\frac{1-2a}{1+2a}\,,\ \ \widetilde{\lambda}_4=\frac{1}{\widetilde{\lambda}_3}\,,\ \ \widetilde{\lambda}_5=\frac{1-2b}{1+2b}\,,\ \ \widetilde{\lambda}_6=\frac{1}{\widetilde{\lambda}_5}\,,\ \ \widetilde{\lambda}_7=\widetilde{\lambda}_8=\widetilde{\lambda}_9=1.$$

Note, that $\widetilde{\lambda}_3,\dots,\widetilde{\lambda}_9$ are positive eigenvalues, whereas $\widetilde{\lambda}_1$, and $\widetilde{\lambda}_2$ are positive if $d>0$ (see Appendix A) and $|\widetilde{\lambda}_1|=|\widetilde{\lambda}_2|=1$ if $d<0$.

Applying the above results we can explicitly write the exponents of the dominant terms in the asymptotic expansion (9.2)–(9.3):

$$\gamma_1=\frac{1}{2}-\frac{1}{\pi}\,\arctan 2\sqrt{-d}\,,\ \ \gamma_2=\frac{1}{2}+\frac{1}{\pi}\,\arctan 2\sqrt{-d}\,,\ \ \delta_1=\delta_2=0\ \text{ if } d<0, \tag{9.6}$$

$$\gamma_1 = \gamma_2 = \frac{1}{2}\,, \quad \delta_1 = \frac{1}{2\pi}\,\ln\frac{1-2\sqrt{d}}{1+2\sqrt{d}}\,, \quad \delta_2 = -\delta_1 \ \text{ if } \ d \geqslant 0, \tag{9.7}$$

and

$$\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = \frac{1}{2}\,,$$
$$\delta_3 = \frac{1}{2\pi}\,\ln\frac{1-2a}{1+2a}\,, \quad \delta_4 = -\delta_3, \quad \delta_5 = \frac{1}{2\pi}\,\ln\frac{1-2b}{1+2b}\,, \quad \delta_6 = -\delta_5, \quad \delta_7 = \delta_8 = \delta_9 = 0.$$

Note, that $B_0(t)$ has the form

$$B_0(t) = \begin{bmatrix} I_6 & [0]_{6\times 3} \\ [0]_{3\times 6} & B^{(3)}(t) \end{bmatrix}, \quad B^{(3)}(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \ \text{ if } \ d < 0,$$

and

$$B_0(t) = I_9 \ \text{ if } \ d \geqslant 0.$$

Now we can draw the conclusions concerning the asymptotic behaviour of solution U to the mixed problem near the exceptional curve ℓ_m:

- If $d < 0$, then the asymptotic expansion has the form

$$U = c_1 r^{\gamma_1} + c_2 r^{1/2+i\delta_3} + c_3 r^{1/2-i\delta_3} + c_4 r^{1/2+i\delta_5} + c_5 r^{1/2-i\delta_5} + c_6 r^{1/2}\ln r + c_7 r^{1/2}\ln^2 r + c_8 r^{1/2} + c_9 r^{\gamma_2} + \cdots .$$

  As we see from (9.5) and (9.6), the exponent γ_1 characterizing the behaviour of the solution near the line ℓ_m depends on the material constants and may take an arbitrary value from the interval $(0, \frac{1}{2})$. In this case the solution possesses C^{γ_1} smoothness in a neighbourhood of the line ℓ_m and since $\gamma_1 < \frac{1}{2}$ the first order derivatives of solutions have non-oscillating singularities near the exceptional curve ℓ_m.

- If $d \geqslant 0$, then

$$U = d_1 r^{1/2} + d_2 r^{1/2+i\delta_1} + d_3 r^{1/2-i\delta_1} + d_4 r^{1/2+i\delta_3} + d_5 r^{1/2-i\delta_3} + d_6 r^{1/2+i\delta_5} + d_7 r^{1/2-i\delta_5} + \mathcal{O}(r^{3/2-\varepsilon}),$$

  where ε is a sufficiently small positive number. In this case the solution possesses $C^{\frac{1}{2}}$-smoothness in a neighbourhood of the line ℓ_m.

10 Appendix A: Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary

Here we collect some results describing the Fredholm properties of strongly elliptic pseudodifferential operators on a compact manifold with boundary. They can be found in [1, 11, 15, 22]. We essentially use these results in Section 7 to prove the existence and regularity of solutions to the mixed boundary value problem for a solid with an interior crack.

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N\times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system $(x \in \overline{\mathcal{M}}, \xi \in \mathbb{R}^n \setminus \{0\})$.

Let $\widetilde{\lambda}_1(x), \dots, \widetilde{\lambda}_N(x)$ be the eigenvalues of the matrix

$$\big[\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, +1)\big]^{-1}\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, -1), \quad x \in \partial\overline{\mathcal{M}},$$

and let

$$\delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1}\ln\widetilde{\lambda}_j(x)\big], \quad j=1,\dots,N.$$

Here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty,0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strict inequality $-1/2<\delta_j(x)<1/2$ for $x\in\overline{\mathcal{M}}$. The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system at the point x. In particular, if the eigenvalue $\widetilde{\lambda}_j$ is real, then it is positive and consequently the corresponding $\delta_j=0$.

Note that when $\mathfrak{S}(\mathcal{A},x,\xi)$ is a positive definite matrix for every $x\in\overline{\mathcal{M}}$ and $\xi\in\mathbb{R}^n\setminus\{0\}$ or when it is an even matrix in ξ we have $\delta_j(x)=0$ for $j=1,\dots,N$, since all the eigenvalues $\widetilde{\lambda}_j(x)$ $(j=\overline{1,N})$ are positive numbers for any $x\in\overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators are characterized by the following theorem.

Theorem A.1. *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu\in\mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\big(\mathfrak{S}(\mathcal{A};x,\xi)\zeta\cdot\zeta\big)\ge c_0|\zeta|^2 \ \ \textit{for}\ \ x\in\overline{\mathcal{M}},\ \ \xi\in\mathbb{R}^n$$

with $|\xi|=1$, and $\zeta\in\mathbb{C}^N$. Then

$$\mathcal{A}:\widetilde{H}^s_p(\mathcal{M})\to H^{s-\nu}_p(\mathcal{M}),\quad \mathcal{A}:\widetilde{B}^s_{p,q}(\mathcal{M})\to B^{s-\nu}_{p,q}(\mathcal{M}), \tag{A.1}$$

are Fredholm operators with index zero if

$$\frac{1}{p}-1+\sup_{x\in\partial\mathcal{M},\,1\le j\le N}\delta_j(x)<s-\frac{\nu}{2}<\frac{1}{p}+\inf_{x\in\partial\mathcal{M},\,1\le j\le N}\delta_j(x). \tag{A.2}$$

Moreover, the null-spaces and indices of the operators (A.1) *are the same* (*for all values of the parameter $q\in[1,+\infty]$*) *provided p and s satisfy the inequality* (A.2).

11 Appendix B: Fundamental solution

Let Γ be the fundamental solution of the operator $A(\partial,\tau)$,

$$A(\partial,\tau)\Gamma(x)=\delta(x)I_9, \tag{B.1}$$

where $\delta(x)$ is Dirac's delta function and I_9 is the 9×9 unite matrix.

Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the direct and inverse Fourier transform operators in $\mathbb{R}^3$,

$$\mathcal{F}_{x\to\xi}[f]\equiv\widehat{f}(\xi)=\int\limits_{\mathbb{R}^3}e^{ix\cdot\xi}f(x)\,dx,\ \ \xi\in\mathbb{R}^3,$$

$$\mathcal{F}^{-1}_{\xi\to x}[g]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}e^{-i\xi\cdot x}g(\xi)\,d\xi,\ \ x\in\mathbb{R}^3.$$

Applying the Fourier operator $\mathcal{F}$ to both sides of equation (B.1) we get

$$A(-i\xi,\tau)\widehat{\Gamma}(\xi)=I_9,$$

whence

$$\widehat{\Gamma}(\xi)=[A(-i\xi,\tau)]^{-1}. \tag{B.2}$$

From (B.2) it follows that $\widehat{\Gamma}=(X^{(1)},\dots,X^{(9)})$, where $X^{(k)}=(X^{(k)}_1,\dots,X^{(k)}_9)^\top$, $k=1,\dots,9$, is a solution of the equation

$$A(-i\xi,\tau)X^{(k)}=B^{(k)} \tag{B.3}$$

with the right side $B^{(k)}=((C^{(k)})^\top,(F^{(k)})^\top,G^{(k)},H^{(k)},L^{(k)})^\top$, where

$$C^{(k)}=(\delta_{1k},\delta_{2k},\delta_{3k})^\top,\ \ F^{(k)}=(\delta_{4k},\delta_{5k},\delta_{6k})^\top,\ \ G^{(k)}=\delta_{7k},\ \ H^{(k)}=\delta_{8k},\ \ L^{(k)}=\delta_{9k}.$$

Introduce the notations

$$\widehat{u}^{(k)}=(x_1^{(k)},x_2^{(k)},x_3^{(k)})^\top,\ \ \widehat{\Phi}^{(k)}=(x_4^{(k)},x_5^{(k)},x_6^{(k)})^\top,\ \ \widehat{\varphi}^{(k)}=x_7^{(k)},\ \ \widehat{\psi}^{(k)}=x_8^{(k)},\ ;\ \widehat{\vartheta}^{(k)}=x_9^{(k)}. \tag{B.4}$$

Then equation (B.3) can be rewritten as

$$\begin{aligned}
\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\widehat{u}^{(k)}+(\lambda+\mu)\xi(\xi\cdot\widehat{u}^{(k)})+i\varkappa[\xi\times\widehat{\Phi}^{(k)}]+i\lambda_0\xi\widehat{\varphi}^{(k)}-i\tau\beta_0\xi\widehat{\vartheta}^{(k)}&=-C^{(k)},\\
\big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\widehat{\Phi}^{(k)}+(\alpha+\beta)\xi(\xi\cdot\widehat{\Phi}^{(k)})+i\varkappa\big[\xi\times\widehat{u}^{(k)}\big]&=-F^{(k)},\\
\big(a_0|\xi|^2+\xi_0+\tau^2j_0\big)\widehat{\varphi}^{(k)}+\lambda_2|\xi|^2\widehat{\psi}^{(k)}-\big(\nu_1|\xi|^2-\tau c_0\big)\widehat{\vartheta}^{(k)}-i\lambda_0(\xi\cdot\widehat{u}^{(k)})&=-G^{(k)},\\
\lambda_2|\xi|^2\widehat{\varphi}^{(k)}+\chi|\xi|^2\widehat{\psi}^{(k)}+\nu_3|\xi|^2\widehat{\vartheta}^{(k)}&=-H^{(k)},\\
\big(k|\xi|^2+\tau^2a\big)\widehat{\vartheta}^{(k)}-i\tau\beta_0(\xi\cdot\widehat{u}^{(k)})+\big(\nu_1|\xi|^2+\tau c_0\big)\widehat{\varphi}^{(k)}-\nu_3|\xi|^2\widehat{\psi}^{(k)}&=-L^{(k)},
\end{aligned} \tag{B.5}$$

Multiplying the first and second equations of (B.5) by $i\xi$ and denoting $\eta^{(k)}:=i\xi\cdot\widehat{u}^{(k)}$, $\zeta^{(k)}:=i\xi\cdot\widehat{\Phi}^{(k)}$, we get

$$\zeta^{(k)}=-\frac{i\xi_{k-3}}{(\alpha+\beta+\gamma)(|\xi|^2-k_1^2)}\,,\quad k_1^2=-\frac{\tau^2I_0+2\varkappa}{\alpha+\beta+\gamma}\,,$$

for $k=4,5,6$ and $\zeta^{(k)}=0$ otherwise, whereas the remaining equations constitute a system of four equations for unknowns $\eta^{(k)}$, $\widehat{\varphi}^{(k)}$, $\widehat{\psi}^{(k)}$, $\widehat{\vartheta}^{(k)}$,

$$\begin{aligned}
\big[(\lambda+2\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\eta^{(k)}-\lambda_0|\xi|^2\widehat{\varphi}^{(k)}+\tau\beta_0|\xi|^2\widehat{\vartheta}^{(k)}&=-i\xi\cdot C^{(k)},\\
\big(a_0|\xi|^2+\xi_0+\tau^2j_0\big)\widehat{\varphi}^{(k)}+\lambda_2|\xi|^2\widehat{\psi}^{(k)}-\big(\nu_1|\xi|^2-\tau c_0\big)\widehat{\vartheta}^{(k)}-\lambda_0\eta^{(k)}&=-G^{(k)},\\
\lambda_2|\xi|^2\widehat{\varphi}^{(k)}+\chi|\xi|^2\widehat{\psi}^{(k)}+\nu_3|\xi|^2\widehat{\vartheta}^{(k)}&=-H^{(k)},\\
\big(k|\xi|^2+\tau^2a\big)\widehat{\vartheta}^{(k)}-\tau\beta_0\eta+\big(\nu_1|\xi|^2+\tau c_0\big)\widehat{\varphi}^{(k)}-\nu_3|\xi|^2\widehat{\psi}^{(k)}&=-L^{(k)}.
\end{aligned} \tag{B.6}$$

Denote by $\widetilde{A}(|\xi|^2)$ the matrix of coefficients of system (B.6)

$$\widetilde{A}(|\xi|^2):=\begin{bmatrix}
\big[(\lambda+2\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big] & -\lambda_0|\xi|^2 & 0 & \tau\beta_0|\xi|^2\\
-\lambda_0 & \big(a_0|\xi|^2+\xi_0+\tau^2j_0\big) & \lambda_2|\xi|^2 & -\big(\nu_1|\xi|^2-\tau c_0\big)\\
0 & \lambda_2|\xi|^2 & \chi|\xi|^2 & \nu_3|\xi|^2\\
-\tau\beta_0 & \big(\nu_1|\xi|^2+\tau c_0\big) & -\nu_3|\xi|^2 & \big(k|\xi|^2+\tau^2a\big)
\end{bmatrix}.$$

Note, that

$$D(|\xi|^2):=\det\big(\widetilde{A}(|\xi|^2)\big)$$

can be factorized as

$$D(|\xi|^2)=d_0|\xi|^2\big(|\xi|^2-k_4^2\big)\big(|\xi|^2-k_5^2\big)\big(|\xi|^2-k_6^2\big),$$

where

$$d_0=(\lambda+2\mu+\varkappa)(a_0k\chi+a_0\nu_3^2+\chi\nu_1^2+2\lambda_2\nu_1\nu_3-k\lambda_2^2)$$

and k_4^2, k_5^2, k_6^2 are the roots of the polynomial

$$P(z)=z^3+p_1z^2+p_2z+p_3 \tag{B.7}$$

with

$$\begin{aligned}
p_1=\frac{\alpha+\beta+\gamma}{d_0}\Big\{&-k\chi\lambda_0^2-\tau^2\big[a(\varkappa+\lambda+2\mu)+\beta_0^2\big]\lambda_2^2-2\tau\chi\beta_0\lambda_0\nu_1-2\tau\beta_0\lambda_0\lambda_2\nu_3\\
&-\lambda_0^2\nu_3^2+(\varkappa+\lambda+2\mu)\tau^2j_0(k\chi+\nu_3^2)+k\varkappa\chi\xi_0+k\lambda\chi\xi_0+2k\mu\chi\xi_0+\varkappa\nu_3^2\xi_0+\lambda\nu_3^2\xi_0\\
&+2\mu\nu_3^2\xi_0+\tau^2(-k\lambda_2^2+\chi\nu_1^2+2\lambda_2\nu_1\nu_3)\rho_0+\tau^2a_0\big[a(\varkappa+\lambda+2\mu)\chi+\chi\beta_0^2+(k\chi+\nu_3^2)\rho_0\big]\Big\},\\
p_2=\frac{2(\alpha+\beta+\gamma)}{d_0}\,&(a\varkappa\tau^2\chi a_0+a\lambda\tau^2\chi a_0+2a\mu\tau^2\chi a_0+k\varkappa\tau^2\chi j_0+k\lambda\tau^2\chi j_0+2k\mu\tau^2\chi j_0
\end{aligned}$$

$$
\begin{aligned}
&+\tau^2\chi a_0\beta_0^2-k\chi\lambda_0^2-a\varkappa\tau^2\lambda_2^2-a\lambda\tau^2\lambda_2^2-2a\mu\tau^2\lambda_2^2-\tau^2\beta_0^2\lambda_2^2-2\tau\chi\beta_0\lambda_0\nu_1-2\tau\beta_0\lambda_0\lambda_2\nu_3\\
&+\varkappa\tau^2 j_0\nu_3^2+\lambda\tau^2 j_0\nu_3^2+2\mu\tau^2 j_0\nu_3^2-\lambda_0^2\nu_3^2+k\varkappa\chi\xi_0+k\lambda\chi\xi_0+2k\mu\chi\xi_0+\varkappa\nu_3^2\xi_0+\lambda\nu_3^2\xi_0\\
&+2\mu\nu_3^2\xi_0+k\tau^2\chi a_0\rho_0-k\tau^2\lambda_2^2\rho_0+\tau^2\chi\nu_1^2\rho_0+2\tau^2\lambda_2\nu_1\nu_3\rho_0+\tau^2a_0\nu_3^2\rho_0),
\end{aligned}
$$

$$
p_3=\frac{\alpha+\beta+\gamma}{d_0}\,\tau^4\chi\big[-c_0^2+a(\tau^2j_0+\xi_0)\big]\rho_0, \tag{B.8}
$$

From (B.6) for $\eta^{(k)}$, $\widehat{\varphi}^{(k)}$, $\widehat{\psi}^{(k)}$, $\widehat{\vartheta}^{(k)}$ we have

$$
\big(\eta^{(k)},\widehat{\varphi}^{(k)},\widehat{\psi}^{(k)},\widehat{\vartheta}^{(k)}\big)^\top=-\widetilde{A}^{-1}\big(|\xi|^2\big)\big(iC^{(k)}\cdot\xi,G^{(k)},H^{(k)},L^{(k)}\big)^\top,
$$

implying

$$
\begin{aligned}
\eta^{(1)}=&-i|\xi|^2\Big(\chi\Big(-\tau^2c_0^2+\big(|\xi|^2k+a\tau^2\big)\big(\xi_0+\tau^2j_0\big)\Big)-|\xi|^2\Big(|\xi|^2k\chi a_0+a\tau^2\chi a_0+|\xi|^2k\lambda_0\lambda_2\\
&-a\tau^2\lambda_0\lambda_2+|\xi|^2\chi\nu_1^2+\big(\tau c_0\big(\lambda_0-\lambda_2\big)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\nu_3^2\Big)\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,\\
\eta^{(2)}=&-i|\xi|^2\Big(\chi\Big(-\tau^2c_0^2+\big(|\xi|^2k+a\tau^2\big)(\xi_0+\tau^2j_0)\Big)-|\xi|^2\Big(|\xi|^2k\chi a_0+a\tau^2\chi a_0+|\xi|^2k\lambda_0\lambda_2\\
&-a\tau^2\lambda_0\lambda_2+|\xi|^2\chi\nu_1^2+\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\nu_3^2\Big)\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,\\
\eta^{(3)}=&-i|\xi|^2\Big(\chi\Big(-\tau^2c_0^2+\big(|\xi|^2k+a\tau^2\big)(\xi_0+\tau^2j_0)\Big)-|\xi|^2\Big(|\xi|^2k\chi a_0+a\tau^2\chi a_0+|\xi|^2k\lambda_0\lambda_2\\
&-a\tau^2\lambda_0\lambda_2+|\xi|^2\chi\nu_1^2+\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+(\xi_0+|\xi|^2a_0+\tau^2j_0)\nu_3^2\Big)\Big)\,\frac{\xi_3}{D(|\xi|^2)}\,,\\
\eta^{(4)}=&\;\eta^{(5)}=\eta^{(6)}=0,\\
\eta^{(7)}=&-|\xi|^4\Big(\tau^2\chi c_0\beta_0+|\xi|^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)+\lambda_0\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\eta^{(8)}=&\;|\xi|^4\Big(\big(|\xi|^2k+a\tau^2\big)\lambda_0^2+|\xi|^2\lambda_0\nu_1(\tau\beta_0-\nu_3)\\
&+\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\eta^{(9)}=&\;|\xi|^4\Big(\tau\beta_0\Big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)-|\xi|^2\lambda_0\lambda_2-\lambda_0(-\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_0\nu_3)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\varphi}^{(1)}=&-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1+\lambda_0\big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\big)\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,\\
\widehat{\varphi}^{(2)}=&-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1+\lambda_0\Big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\Big)\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,\\
\widehat{\varphi}^{(3)}=&-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1+\lambda_0\Big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\Big)\Big)\,\frac{\xi_3}{D(|\xi|^2)},\\
\widehat{\varphi}^{(4)}=&\;\widehat{\varphi}^{(5)}=\widehat{\varphi}^{(6)}=0,\\
\widehat{\varphi}^{(7)}=&-|\xi|^2\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\varphi}^{(8)}=&\;|\xi|^2\Big(\big(\tau c_0-|\xi|^2\nu_1\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&+\lambda_0\Big(|\xi|^2\tau^2\beta_0^2+|\xi|^2\tau\beta_0\nu_3+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\varphi}^{(9)}=&\;|\xi|^2\Big(\tau\chi c_0\Big(|\xi|^2(\varkappa+\lambda+2\mu)-\tau^2\rho_0-\tau\chi\beta_0\lambda_0\\
&+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\psi}^{(1)}=&\;i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)
\end{aligned}
$$

$$+\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,$$

$$\widehat{\psi}^{(2)}=i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)$$
$$+\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,$$

$$\widehat{\psi}^{(3)}=i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)$$
$$+\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_3}{D(|\xi|^2)}\,,$$

$$\widehat{\psi}^{(4)}=\widehat{\psi}^{(5)}=\widehat{\psi}^{(6)}=0,$$

$$\widehat{\psi}^{(7)}=|\xi|^2\Big(\lambda_2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$-\nu_3\Big(-|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0+|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\psi}^{(8)}=-|\xi|^2\tau\beta_0\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0+\lambda_0\big(\tau c_0-|\xi|^2\nu_1\big)\Big)$$
$$-\big(|\xi|^2k+a\tau^2\big)\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$+\big(\tau c_0+|\xi|^2\nu_1\big)\Big(|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0-|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\psi}^{(9)}=\widehat{\vartheta}^{(8)}=|\xi|^2\Big(\nu_3\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$-\lambda_2\Big(|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0-|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\vartheta}^{(1)}=-i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)-|\xi|^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,$$

$$\widehat{\vartheta}^{(2)}=-i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)-|\xi|^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,$$

$$\widehat{\vartheta}^{(3)}=-i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)-|\xi|^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_3}{D(|\xi|^2)}\,,$$

$$\widehat{\vartheta}^{(4)}=\widehat{\vartheta}^{(5)}=\widehat{\vartheta}^{(6)}=0,$$

$$\widehat{\vartheta}^{(7)}=-|\xi|^2\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$-|\xi|^2\Big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_2\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\vartheta}^{(8)}=-\nu_3\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$-\lambda_0\Big(-|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0+|\xi|^2\nu_1\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{|\xi|^2}{D(|\xi|^2)}\,,$$

$$\widehat{\vartheta}^{(9)}=-|\xi|^2\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\chi\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)(|\xi|^2)}\,.$$

Rewrite the first two equations of (B.5) as follows:

$$\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\widehat{u}^{(k)}+i\varkappa\big[\xi\times\widehat{\Phi}^{(k)}\big]=-C^{(k)}+i(\lambda+\mu)\eta^{(k)}\xi-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)},\qquad\text{(B.9)}$$
$$\big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\widehat{\Phi}^{(k)}+i\varkappa\big[\xi\times\widehat{u}^{(k)}\big]=-F^{(k)}+i(\alpha+\beta)\zeta^{(k)}\xi.\qquad\text{(B.10)}$$

Taking cross product of ξ with both sides of (B.9) and employ the identity

$$\big[\xi\times[\xi\times a]\big]=(\xi\cdot a)\xi-|\xi|^2a$$

we get

$$\begin{gathered}\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\,\big[\xi\times\widehat{u}^{(k)}\big]-i\varkappa|\xi|^2\widehat{\Phi}^{(k)}=-\big[C^{(k)}\times\xi\big]-\varkappa\zeta^{(k)}\xi,\\ \big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\widehat{\Phi}^{(k)}+i\varkappa\big[\xi\times\widehat{u}^{(k)}\big]=-F^{(k)}+i(\alpha+\beta)\zeta^{(k)}\xi.\end{gathered}$$

Hence

$$\widehat{\Phi}^{(k)}(\xi)=\frac{i\varkappa(\zeta^{(k)}\varkappa\xi+[C^{(k)}\times\xi])-(F^{(k)}-i(\alpha+\beta)\zeta^{(k)}\xi)((\varkappa+\mu)|\xi|^2+\tau^2\rho_0)}{\Theta(\xi)}$$

with

$$\Theta(\xi)=\big(2\varkappa+\gamma|\xi|^2+\tau^2I_0\big)\big((\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big)-\varkappa^2|\xi|^2.$$

Similarly, if we take cross product of ξ with both sides of (B.10),

$$\begin{gathered}\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\widehat{u}^{(k)}+i\varkappa\big[\xi\times\widehat{\Phi}^{(k)}\big]=-C^{(k)}+i(\lambda+\mu)\eta^{(k)}\xi-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)},\\ \big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\,\big[\xi\times\widehat{\Phi}^{(k)}\big]-i\varkappa|\xi|^2\widehat{u}^{(k)}=-[F^{(k)}\times\xi]-\varkappa\eta^{(k)}\xi,\end{gathered}$$

we find

$$\begin{aligned}\widehat{u}^{\,(k)}(\xi)=\frac{1}{\Theta(\xi)}\Big[\Big(&i(\lambda+\mu)\eta^{(k)}\xi-C^{(k)}-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)}\Big)\\ &\times\big(\gamma|\xi|^2+2\varkappa+\tau^2I_0\big)+i\varkappa\big([F^{(k)}\times\xi]+\varkappa\eta^{(k)}\xi\big)\Big].\end{aligned}$$

Let k_2^2 and k_3^2 be the roots of the quadratic polynomial

$$Q(z)=(2\varkappa+\gamma z+\tau^2I_0)\big((\varkappa+\mu)z+\tau^2\rho_0\big)-\varkappa^2z=\gamma(\varkappa+\mu)z^2+q_1z+q_2, \tag{B.11}$$

where

$$q_1=\gamma\tau^2\rho_0+(\varkappa+\mu)(2\varkappa+\tau^2I_0)-\varkappa^2,\ \ q_2=\tau^4\rho_0I_0,$$

then

$$\begin{gathered}k_2^2=\frac{-q_1-\sqrt{q_1^2-4\gamma(\varkappa+\mu)q_2}}{2\gamma(\varkappa+\mu)}\,,\quad k_3^2=\frac{-q_1+\sqrt{q_1^2-4\gamma(\varkappa+\mu)q_2}}{2\gamma(\varkappa+\mu)}\,,\\ \frac{1}{Q(|\xi|^2)}=\frac{1}{\gamma(\varkappa+\mu)(k_2^2-k_3^2)}\Big(\frac{1}{|\xi|^2-k_2^2}-\frac{1}{|\xi|^2-k_3^2}\Big),\end{gathered}$$

and

$$\widehat{\Phi}^{(k)}(\xi)=\frac{1}{Q(|\xi|^2)}\Big[i\varkappa\big(\zeta^{(k)}\varkappa\xi+[C^{(k)}\times\xi]\big)-\big(F^{(k)}-i(\alpha+\beta)\zeta^{(k)}\xi\big)\big((\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big)\Big], \tag{B.12}$$

$$\begin{aligned}\widehat{u}^{(k)}(\xi)=\frac{1}{Q(|\xi|^2)}\Big[&\big(-C^{(k)}+i(\lambda+\mu)\eta^{(k)}\xi-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)}\big)\big(\gamma|\xi|^2+2\varkappa+\tau^2I_0\big)\\ &+i\varkappa\big([F^{(k)}\times\xi]+\varkappa\eta^{(k)}\xi\big)\Big].\end{aligned} \tag{B.13}$$

From (B.12)–(B.13) we obtain

$$\begin{aligned}\widehat{\Phi}_j^{(m)}&=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}+\frac{(\varkappa^2+(\alpha+\beta)((\varkappa+\mu)|\xi|^2+\tau^2\rho_0))}{\alpha+\beta+\gamma}\cdot\frac{\xi_j\xi_m}{(|\xi|^2-k_1^2)Q(|\xi|^2)}\,,\ \ j,m=1,2,3,\\ \widehat{\Phi}_j^{(m+3)}&=-\delta_{mj}\big[(\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big]\,\frac{1}{Q(|\xi|^2)}\,,\ \ m,j=1,2,3,\\ \widehat{\Phi}_j^{(m)}&=0,\ \ j=1,2,3;\ \ m=7,8,9,\\ \widehat{u}_j^{(m)}&=\Big[\big(\gamma|\xi|^2+2\varkappa+\tau^2I_0\big)\Big(-1+|\xi|^2\xi_j\xi_m\Big(-\lambda_0^2\big((k|\xi|^2+a\tau^2)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\big)\\ &\quad+\tau\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0\tau\beta_0-|\xi|^2\lambda_0\big(\tau\chi\beta_0\nu_1+\big(\chi\nu_1+\lambda_2(\tau\beta_0+\nu_3)\big)\tau\beta_0\big)\Big)\Big)\Big]\,\frac{1}{Q(|\xi|^2)}\end{aligned}$$

$$
\begin{aligned}
&+\Big[\Big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big)\Big(|\xi|^2 k\xi_0\chi+a\xi_0\tau^2\chi-\tau^2\chi c_0^2+|\xi|^2 k\tau^2\chi j_0\\
&\qquad+a\tau^4\chi j_0-|\xi|^4 k\lambda_0\lambda_2-a|\xi|^2\tau^2\lambda_0\lambda_2+|\xi|^4\chi\nu_1^2\\
&\qquad+|\xi|^2\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+|\xi|^2(\xi_0+\tau^2 j_0)\nu_3^2\\
&\qquad+|\xi|^2 a_0\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^2\xi_j\xi_m}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad j,m=1,2,3,
\end{aligned}
$$

$$
\widehat{u}_j^{(m+3)}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}\,,\quad j,m=1,2,3,
$$

$$
\begin{aligned}
\widehat{u}_j^{(7)}&=i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0\\
&\quad+\lambda_0\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\\
&\qquad\times\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)-|\xi|^2\tau\chi\beta_0\tau\beta_0\Big)\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}\\
&\quad-i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\tau^2\chi c_0\beta_0+|\xi|^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)\\
&\qquad+\lambda_0\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad j=1,2,3,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{u}_j^{(8)}&=-i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0\\
&\quad+\lambda_0^2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2 k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\qquad+|\xi|^2\tau\beta_0(\nu_3-\tau\beta_0)-|\xi|^2\nu_3\tau\beta_0\Big)\\
&\quad-\lambda_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big(|\xi|^2\nu_1(\nu_3-\tau\beta_0-\tau c_0(\nu_3+\tau\beta_0))\big)\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}\\
&\quad+i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\big(|\xi|^2 k+a\tau^2\big)\lambda_0^2+|\xi|^2\lambda_0\nu_1(\tau\beta_0-\nu_3)\\
&\qquad+\tau\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)},\quad j=1,2,3,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{u}_j^{(9)}&=i\Bigg[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\lambda_0\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\quad+|\xi|^2\big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\\
&\quad-\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\quad+\chi\Big(-|\xi|^2\lambda_0^2+(\xi_0+|\xi|^2 a_0+\tau^2 j_0)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\tau\beta_0\Big)\Bigg]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}\\
&\quad+i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\tau\beta_0\big(\chi(\xi_0+|\xi|^2 a_0+\tau^2 j_0)-|\xi|^2\lambda_0\lambda_2\big)\\
&\quad-\lambda_0\big(-\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_0\nu_3\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad j=1,2,3.
\end{aligned}
$$

From (B.4) it follows that the Fourier transform of the entries of the fundamental solution matrix have the form

$$
\begin{aligned}
\widehat{\Gamma}_{jm}&=\Bigg\{\Bigg[\big(\gamma|\xi|^2+2\varkappa+\tau^2 I_0\big)\Big(-1+|\xi|^2\xi_j\xi_m\Big(-\lambda_0^2\big((k|\xi|^2+a\tau^2)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\big)\\
&\quad+\tau\chi\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)\beta_0\tau\beta_0-|\xi|^2\lambda_0\big(\tau\chi\beta_0\nu_1+\big(\chi\nu_1+\lambda_2(\tau\beta_0+\nu_3)\big)\tau\beta_0\big)\Big)\Big)\Bigg]\\
&\quad+\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(|\xi|^2 k\xi_0\chi+a\xi_0\tau^2\chi-\tau^2\chi c_0^2+|\xi|^2 k\tau^2\chi j_0\\
&\quad+a\tau^4\chi j_0-|\xi|^4 k\lambda_0\lambda_2-a|\xi|^2\tau^2\lambda_0\lambda_2+|\xi|^4\chi\nu_1^2
\end{aligned}
$$

$$+|\xi|^2\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+|\xi|^2(\xi_0+\tau^2 j_0)\nu_3^2$$
$$+|\xi|^2 a_0\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\bigg]\,\frac{|\xi|^2\xi_j\xi_m}{D(|\xi|^2)}\bigg\}\,\frac{1}{Q(|\xi|^2)},\ \ j,m=1,2,3,$$

$$\widehat{\Gamma}_{j(m+3)}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}\,,\ \ j,m=1,2,3,$$

$$\widehat{\Gamma}_{j7}=i\bigg[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0$$
$$+\lambda_0\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)$$
$$\times\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)-|\xi|^2\tau\chi\beta_0\tau\beta_0\Big)\Big)\bigg]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$-i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\tau^2\chi c_0\beta_0+|\xi|^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)$$
$$+\lambda_0\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)},\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{j8}=-i\bigg[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0$$
$$+\lambda_0^2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2 k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+|\xi|^2\tau\beta_0(\nu_3-\tau\beta_0)-|\xi|^2\nu_3\tau\beta_0\Big)$$
$$-\lambda_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big(|\xi|^2\nu_1(\nu_3-\tau\beta_0)-\tau c_0(\nu_3+\tau\beta_0)\big)\Big)\bigg]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$+i\Big[\Big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big)\Big(\big(|\xi|^2 k+a\tau^2\big)\lambda_0^2+|\xi|^2\lambda_0\nu_1(\tau\beta_0-\nu_3)$$
$$+\tau\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{j9}=i\bigg[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\lambda_0\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+|\xi|^2\big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)$$
$$-\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\chi\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\tau\beta_0\Big)\bigg]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$+i\Big[\Big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big)\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2 a_0+\tau^2 j_0\big)-|\xi|^2\lambda_0\lambda_2\big)$$
$$-\lambda_0\big(-\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_0\nu_3\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)},\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{j+3,m}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}+\frac{(\varkappa^2+(\alpha+\beta)((\varkappa+\mu)|\xi|^2+\tau^2\rho_0))}{\alpha+\beta+\gamma}\cdot\frac{\xi_j\xi_m}{(|\xi|^2-k_1^2)Q(|\xi|^2)}\,,\ \ j,m=1,2,3,$$

$$\widehat{\Gamma}_{j+3,m+3}=-\delta_{mj}\big[(\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big]\,\frac{1}{Q(|\xi|^2)}\,,\ \ j=1,2,3,\ \ m=1,\dots,6,$$

$$\widehat{\Gamma}_{7j}=-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1$$
$$+\lambda_0\Big(\big(|\xi|^2 k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\Big)\Big)\,\frac{\xi_j}{D(|\xi|^2)}\,,\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{74}=\widehat{\Gamma}_{75}=\widehat{\Gamma}_{76}=0,$$

$$\widehat{\Gamma}_{77}=-|\xi|^2\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{78}=|\xi|^2\Big(\big(\tau c_0-|\xi|^2\nu_1\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\lambda_0\Big(|\xi|^2\tau^2\beta_0^2+|\xi|^2\tau\beta_0\nu_3+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{79}=|\xi|^2\Big(\tau\chi c_0\Big(|\xi|^2(\varkappa+\lambda+2\mu)-\tau^2\rho_0-\tau\chi\beta_0\lambda_0$$
$$+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)},$$

$$\widehat{\Gamma}_{8j}=i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)$$
$$+\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_j}{D(|\xi|^2)},\quad j=1,2,3,$$

$$\widehat{\Gamma}_{84}=\widehat{\Gamma}_{85}=\widehat{\Gamma}_{86}=0,$$

$$\widehat{\Gamma}_{87}=\widehat{\psi}^{(7)}=|\xi|^2\Big(\lambda_2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$-\nu_3\Big(-|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0+|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{88}=\widehat{\psi}^{(8)}=-|\xi|^2\tau\beta_0\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0+\lambda_0\big(\tau c_0-|\xi|^2\nu_1\big)\Big)$$
$$-\big(|\xi|^2k+a\tau^2\big)\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$+\big(\tau c_0+|\xi|^2\nu_1\big)\Big(|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0-|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{89}=\widehat{\psi}^{(9)}=|\xi|^2\Big(\nu_3\big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$-\lambda_2\Big(|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0-|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{9j}=\widehat{\vartheta}^{(1)}=-i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)-|\xi|^2\lambda_0\lambda_2\big)$$
$$-\lambda_0\Big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_j}{D(|\xi|^2)}\,,\quad j=1,2,3,$$

$$\widehat{\Gamma}_{94}=\widehat{\Gamma}_{95}=\widehat{\Gamma}_{96}=0,$$

$$\widehat{\Gamma}_{97}=-|\xi|^2\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$-|\xi|^2\Big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_2\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{98}=-\nu_3\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)$$
$$-\lambda_0\Big(-|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0+|\xi|^2\nu_1\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{|\xi|^2}{D(|\xi|^2)}\,,$$

$$\widehat{\Gamma}_{99}=-|\xi|^2\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\chi\big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,.$$

Remark B.1. To perform the inverse Fourier thransform, for simplicity, now we assume that the polynomials $P(z)=z^3+p_1z^2+p_2z+p_3$ and $Q(z)=\gamma(\varkappa+\mu)z^2+q_1z+q_2$ defined in (B.7) and (B.11) respectively have distinct non-negative roots in z. Note that this assumption does not follow from conditions (2.22) and (2.23). Indeed, let $\tau>0$ and choose λ_2 and c_0, which are not involved in conditions (2.22) and (2.23), sufficiently large. We will have $p_3>0$ in view of (B.8) and therefore the polynomial $P(z)$ will have at least one negative root without violating conditions (2.22) and (2.23).

In what follows we will find an explicit representation of the fundamental matrix in terms of elementary functions by inverting the Fourier transform

$$\Gamma(x)=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} e^{-i\xi\cdot x}\,\widehat{\Gamma}(\xi)\,d\xi. \tag{B.14}$$

To this end, let us note that the functions

$$\frac{1}{Q(|\xi|^2)}\,,\quad \frac{1}{D(|\xi|^2)}\,,\quad \frac{1}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad \frac{1}{(|\xi|^2-k_1^2)Q(|\xi|^2)}$$

can be expanded as follows:

$$\begin{gathered}
\frac{1}{Q(|\xi|^2)}=\sum_{\alpha=2}^{3}\frac{c_\alpha^{(1)}}{|\xi|^2-k_\alpha^2}\,,\quad \frac{1}{(|\xi|^2-k_1^2)Q(|\xi|^2)}=\sum_{\alpha=1}^{3}\frac{c_\alpha^{(2)}}{|\xi|^2-k_\alpha^2}\,,\\
\frac{1}{D(|\xi|^2)}=c_0^{(3)}\,\frac{1}{|\xi|^2}+\sum_{\alpha=4}^{6}\frac{c_\alpha^{(3)}}{|\xi|^2-k_\alpha^2}\,,\quad \frac{1}{D(|\xi|^2)Q(|\xi|^2)}=c_0^{(4)}\,\frac{1}{|\xi|^2}+\sum_{\alpha=2}^{6}\frac{c_\alpha^{(4)}}{|\xi|^2-k_\alpha^2}\,,
\end{gathered} \tag{B.15}$$

where

$$\begin{gathered}
c_2^{(1)}=-c_3^{(1)}=\big(\gamma(\varkappa+\mu)(k_2^2-k_3^2)\big)^{-1},\quad c_1^{(2)}=\big(\gamma(\varkappa+\mu)(k_1^2-k_2^2)(k_1^2-k_3^2)\big)^{-1},\\
c_3^{(2)}=\big(\gamma(\varkappa+\mu)(k_3^2-k_1^2)(k_3^2-k_2^2)\big)^{-1},\quad c_0^{(3)}=-\big(d_0k_4^2k_5^2k_6^2\big)^{-1},\\
c_4^{(3)}=\big(d_0k_4^2(k_4^2-k_5^2)(k_4^2-k_6^2)\big)^{-1},\quad c_5^{(3)}=\big(d_0k_5^2(k_5^2-k_4^2)(k_5^2-k_6^2)\big)^{-1},\\
c_6^{(3)}=\big(d_0k_6^2(k_6^2-k_4^2)(k_6^2-k_5^2)\big)^{-1},\quad c_0^{(4)}=\Big(\gamma(\varkappa+\mu)d_0\prod_{j=2}^{6}k_j^2\Big)^{-1},\\
c_2^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_2^2(k_2^2-k_3^2)(k_2^2-k_4^2)(k_2^2-k_5^2)(k_2^2-k_6^2)\Big)^{-1},\\
c_3^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_3^2(k_3^2-k_4^2)(k_3^2-k_5^2)(k_3^2-k_5^2)(k_3^2-k_6^2)\Big)^{-1},\\
c_4^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_4^2(k_4^2-k_2^2)(k_4^2-k_3^2)(k_4^2-k_5^2)(k_4^2-k_6^2)\Big)^{-1},\\
c_5^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_5^2(k_5^2-k_2^2)(k_5^2-k_3^2)(k_5^2-k_4^2)(k_5^2-k_6^2)\Big)^{-1},\\
c_6^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_6^2(k_6^2-k_2^2)(k_6^2-k_3^2)(k_6^2-k_4^2)(k_6^2-k_5^2)\Big)^{-1}.
\end{gathered}$$

Let $k_0=0$. Choose k_p, $p=1,\dots,6$ so, that $-\pi<\arg(k_p)\le 0$ and denote by K_p, $p=0,\dots,6$, the functions

$$K_p(x)=\frac{\exp(-ik_p|x|)}{4\pi|x|}\,,\ \ p=0,\dots,6. \tag{B.16}$$

Then K_p belongs to the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions in $\mathbb{R}^3$ and

$$(\Delta+k_p^2)K_p(x)=-\delta(x),\ \ \big(|\xi|^2-k_p^2\big)\widehat{K}_p(\xi)=1,\ \ \mathcal{F}^{-1}_{\xi\to x}\big(|\xi|^2\widehat{K}_p(\xi)\big)=\delta(x)+k_p^2K_p(x),\ \ p=0,\dots,6,$$

where $\widehat{K}_p(\xi)=\mathcal{F}_{x\to\xi}(K_p)(\xi)$.

From (B.15) we get

$$\begin{gathered}
\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{Q(|\xi|^2)}\Big)=\sum_{p=2}^{3}c_p^{(1)}K_p(x),\quad \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{(|\xi|^2-k_1^2)Q(|\xi|^2)}\Big)=\sum_{p=1}^{3}c_p^{(2)}K_p(x),\\
\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{D(|\xi|^2)}\Big)=\sum_{p=0}^{6}c_p^{(3)}K_p(x),\quad \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{D(|\xi|^2)Q(|\xi|^2)}\Big)=\sum_{p=0}^{6}c_p^{(4)}K_p(x),
\end{gathered} \tag{B.17}$$

where $c_p^{(3)}=0$, $p=1,2,3$, $c_1^{(4)}=0$.

To obtain the expression of the fundamental solution Γ, we have to evaluate the inverse Fourier transform (B.14) of the Fourier image $\widehat{\Gamma}$. Note that due to ellipticity of the operator $A(\partial,\tau)$ its fundamental solution Γ belongs to $C^\infty(\mathbb{R}^3\setminus\{0\})\cap L^1_{loc}(\mathbb{R}^3)$ and therefore terms containing $\delta(x)$ are canceled. Taking into consideration relations (B.16)–(B.17) and properties of the inverse Fourier transform operator we arrive at the following expressions for the components of the fundamental solution matrix:

$$\begin{aligned}
\Gamma_{jm}(x) &= \sum_{p=2}^{3} c_p^{(1)}\Big[(\gamma k_p^2+2\varkappa+\tau^2 I_0)\Big(-1+k_p^2\xi_j\xi_m\Big(-\lambda_0^2\big((kk_p^2+a\tau^2)\chi+k_p^2\nu_3(\tau\beta_0+\nu_3)\big)\\
&\quad +\tau\chi(\xi_0+k_p^2a_0+\tau^2 j_0)\beta_0\tau\beta_0-k_p^2\lambda_0\big(\tau\chi\beta_0\nu_1+\big(\chi\nu_1+\lambda_2(\tau\beta_0+\nu_3)\big)\tau\beta_0\big)\Big)\Big)\Big]K_p(x)\\
&\quad -\sum_{p=0}^{6} c_p^{(1)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2 I_0)\big)\Big(k_p^2 k\xi_0\chi+a\xi_0\tau^2\chi-\tau^2\chi c_0^2+k_p^2 k\tau^2\chi j_0\\
&\quad +a\tau^4\chi j_0-k_p^4 k\lambda_0\lambda_2-ak_p^2\tau^2\lambda_0\lambda_2+k_p^4\chi\nu_1^2+k_p^2\big(\tau c_0(\lambda_0-\lambda_2)+k_p^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3\\
&\quad +k_p^2(\xi_0+\tau^2 j_0)\nu_3^2+k_p^2a_0(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\Big)\Big]k_p^2\partial_j\partial_m K_p(x),\quad j,m=1,2,3,\\
\Gamma_{j\,(m+3)}(x) &= \varkappa\varepsilon_{jmk}\sum_{p=2}^{3} c_p^{(1)}\partial_k K_p(x),\quad j,m=1,2,3,\\
\Gamma_{j7}(x) &= \sum_{p=2}^{3} c_p^{(1)}\Big[(k_p^2\gamma+2\varkappa+\tau^2 I_0)\Big((\tau\chi c_0+k_p^2\chi\nu_1+k_p^2\lambda_2\nu_3)\\
&\qquad\times\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0+\lambda_0\Big(k_p^2\tau^2\chi\beta_0^2+(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\\
&\qquad\times\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)-k_p^2\tau\chi\beta_0\tau\beta_0\Big)\Big)\Big]k_p^2\partial_j K_p(x)\\
&\quad +\sum_{p=0}^{6} c_p^{(4)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2 I_0)\big)\Big(\tau^2\chi c_0\beta_0+k_p^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)\\
&\qquad +\lambda_0(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\Big)\Big]k_p^4\partial_j K_p(x),\quad j=1,2,3,\\
\Gamma_{j8}(x) &= -\sum_{p=2}^{3} c_p^{(1)}\Big[(k_p^2\gamma+2\varkappa+\tau^2 I_0)\Big((\xi_0+k_p^2a_0+\tau^2 j_0)\nu_3\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0\\
&\quad +\lambda_0^2\Big(k_p^2\tau^2\beta_0^2+(k_p^2k+a\tau^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)+k_p^2\tau\beta_0(\nu_3-\tau\beta_0)-k_p^2\nu_3\tau\beta_0\Big)\\
&\qquad -\lambda_0\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big(k_p^2\nu_1(\nu_3-\tau\beta_0)-\tau c_0(\nu_3+\tau\beta_0)\big)\Big)\Big]k_p^2\partial_j K_p(x)\\
&\quad +\sum_{p=0}^{6} c_p^{(4)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2 I_0)\big)\Big((k_p^2k+a\tau^2)\lambda_0^2+k_p^2\lambda_0\nu_1(\tau\beta_0-\nu_3)\\
&\qquad +\tau(\xi_0+k_p^2a_0+\tau^2 j_0)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\Big]k_p^4\partial_j K_p(x),\quad j=1,2,3,\\
\Gamma_{j9}(x) &= \sum_{p=2}^{3} c_p^{(1)}\Big[(k_p^2\gamma+2\varkappa+\tau^2 I_0)\Big(\lambda_0\Big(-\tau\chi c_0\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\qquad +k_p^2\big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_0\nu_3)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\\
&\quad -\Big(-k_p^2\lambda_0\lambda_2\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)
\end{aligned}$$

$$+\chi\Big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\tau\beta_0\Big)\Big]k_p^2\partial_jK_p(x)$$

$$+\sum_{p=0}^{6}c_p^{(4)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2I_0)\big)\Big(\tau\beta_0\big(\chi(\xi_0+k_p^2a_0+\tau^2j_0)-k_p^2\lambda_0\lambda_2\big)$$

$$-\lambda_0(-\tau\chi c_0+k_p^2\chi\nu_1+k_p^2\lambda_0\nu_3)\Big)\Big]k_p^4\partial_jK_p(x),\quad j=1,2,3,$$

$$\Gamma_{j+3,m}(x)=\sum_{p=2}^{3}c_p^{(1)}\varkappa\varepsilon_{jmk}\partial_kK_p(x)$$

$$+\sum_{p=1}^{3}c_p^{(2)}\,\frac{(\varkappa^2+(\alpha+\beta)((\varkappa+\mu)k_p^2+\tau^2\rho_0))}{\alpha+\beta+\gamma}\,\partial_j\partial_mK_p(x),\quad j,m=1,2,3,$$

$$\Gamma_{j+3,m+3}(x)=-\sum_{p=2}^{3}c_p^{(1)}\delta_{mj}\big[(\varkappa+\mu)k_p^2+\tau^2\rho_0\big]K_p(x),\quad j=1,2,3,\quad m=1,\dots,6,$$

$$\Gamma_{7j}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(-\tau^2\chi c_0\beta_0+k_p^2\tau\chi\beta_0\nu_1$$

$$+\lambda_0\big((k_p^2k+a\tau^2)\chi+k_p^2\nu_3(\tau\beta_0+\nu_3)\big)\Big)\Big]\partial_jK_p(x),\quad j=1,2,3,$$

$$\Gamma_{74}(x)=\Gamma_{75}(x)=\Gamma_{76}(x)=0,$$

$$\Gamma_{77}(x)=-\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\big(k_p^2\tau^2\chi\beta_0^2+(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big]K_p(x),$$

$$\Gamma_{78}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big((\tau c_0-k_p^2\nu_1)\nu_3\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$

$$+\lambda_0\big(k_p^2\tau^2\beta_0^2+k_p^2\tau\beta_0\nu_3+(k_p^2k+a\tau^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{79}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\tau\chi c_0\big(k_p^2(\varkappa+\lambda+2\mu)-\tau^2\rho_0-\tau\chi\beta_0\lambda_0$$

$$+(\chi\nu_1+\lambda_0\nu_3)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{8j}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\lambda_2\big(k_p^2k\lambda_0+\tau(-\tau c_0\beta_0+a\tau\lambda_0+k_p^2\beta_0\nu_1)\big)$$

$$+\big(\tau(\xi_0+k_p^2a_0+\tau^2j_0)\beta_0-\lambda_0(\tau c_0+k_p^2\nu_1)\big)\nu_3\Big)\Big]\partial_jK_p(x),\quad j=1,2,3,$$

$$\Gamma_{84}(x)=\Gamma_{85}(x)=\Gamma_{86}(x)=0,$$

$$\Gamma_{87}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\lambda_2\big(k_p^2\tau^2\beta_0^2+(k_p^2k+a\tau^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)$$

$$-\nu_3\big(-k_p^2\tau\beta_0\lambda_0+(\tau c_0+k_p^2\nu_1)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{88}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[-k_p^2\tau\beta_0\big(\tau(\xi_0+k_p^2a_0+\tau^2j_0)\beta_0+\lambda_0(\tau c_0-k_p^2\nu_1)\big)$$

$$-(k_p^2k+a\tau^2)\big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)$$

$$+(\tau c_0+k_p^2\nu_1)\big(k_p^2\tau\beta_0\lambda_0+(\tau c_0-k_p^2\nu_1)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big]K_p(x),$$

$$\Gamma_{89}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\nu_3\big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)$$

$$
\begin{aligned}
&\qquad - \lambda_2\big(k_p^2\tau\beta_0\lambda_0 + (\tau c_0 - k_p^2\nu_1)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\Big]K_p(x),\\
\Gamma_{9j}(x) &= \sum_{p=0}^{6} c_p^{(3)}\Big[k_p^2\big(\tau\beta_0\big(\chi(\xi_0+k_p^2a_0+\tau^2j_0)-k_p^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+k_p^2\chi\nu_1+k_p^2\lambda_2\nu_3\big)\big)\Big]\partial_j K_p(x),\\
&\qquad\qquad j=1,2,3,\\
\Gamma_{94}(x) &= \Gamma_{95}(x)=\Gamma_{96}(x)=0,\\
\Gamma_{97}(x) &= \sum_{p=0}^{6} c_p^{(3)}\Big[-k_p^2\big(-\tau\chi c_0(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\\
&\qquad - k_p^2(-\tau\chi\beta_0\lambda_0 + (\chi\nu_1+\lambda_2\nu_3)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big))\big)\Big]K_p(x),\\
\Gamma_{98}(x) &= \sum_{p=0}^{6} c_p^{(3)}\Big[-\nu_3\big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\\
&\qquad - \lambda_0\big(-k_p^2\tau\beta_0\lambda_0+(\tau c_0+k_p^2\nu_1)\big(k_p^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big]k_p^2K_p(x),\\
\Gamma_{99}(x) &= \sum_{p=0}^{6} c_p^{(3)}\Big[-k_p^2\Big(-k_p^2\lambda_0\lambda_2\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\qquad + \chi\big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\Big]K_p(x).
\end{aligned}
$$

Acknowledgement

This research was supported by the Shota Rustaveli National Science Foundation (SRNSF) Grant # FR/286/5-101/13.

References

[1] M. S. Agranovich, Elliptic singular integro-differential operators. (Russian) *Uspehi Mat. Nauk* **20** (1965), no. 5 (125), 3–120.

[2] T. Buchukuri, O. Chkadua, R. Duduchava and D. Natroshvili, Interface crack problems for metallic-piezoelectric composite structures. *Mem. Differ. Equ. Math. Phys.* **55** (2012), 1–150.

[3] T. Buchukuri, O. Chkadua and D. Natroshvili, Mathematical problems of generalized thermo-electro-magneto-elasticity theory. *Mem. Differ. Equ. Math. Phys.* **68** (2016), 1–166.

[4] T. Buchukuri, O. Chkadua and D. Natroshvili, Mixed boundary value problems of pseudo-oscillations of generalized thermo-electro-magneto-elasticity theory for solids with interior cracks. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 3, 308–351.

[5] O. Chkadua and R. Duduchava, Asymptotics of functions represented by potentials. *Russ. J. Math. Phys.* **7** (2000), no. 1, 15–47.

[6] O. Chkadua and R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic. *Math. Nachr.* **222** (2001), 79–139.

[7] M. Costabel, M. Dauge and R. Duduchava, Asymptotics without logarithmic terms for crack problems. *Comm. Partial Differential Equations* **28** (2003), no. 5-6, 869–926.

[8] R. Duduchava, The Green formula and layer potentials. *Integral Equations Operator Theory* **41** (2001), no. 2, 127–178.

[9] A. C. Eringen, *Microcontinuum Field Theories.* I. *Foundations and Solids.* Springer-Verlag, New York, 1999.

[10] A. C. Eringen, Electromagnetic theory of microstretch elasticity and bone modeling. *Internat. J. Engrg. Sci.* **42** (2004), no. 3-4, 231–242.

[11] G. I. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, 52. American Mathematical Society, Providence, R.I., 1981.

[12] F. R. Gantmacher, *Theory of Matrices.* (Russian) Nauka, Moscow, 1967.

[13] A. E. Green and P. M. Naghdi, A re-examination of the basic postulates of thermomechanics. *Proc. Roy. Soc. London Ser. A* **432** (1991), no. 1885, 171–194.

[14] A. E. Green and P. M. Naghdi, Thermoelasticity without energy dissipation. *J. Elasticity* **31** (1993), no. 3, 189–208.

[15] G. Grubb, Pseudo-differential boundary problems in L_p spaces. *Comm. Partial Differential Equations* **15** (1990), no. 3, 289–340.

[16] D. Ieşan, Thermopiezoelectricity without energy dissipation. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **464** (2008), no. 2091, 631–656.

[17] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge, 2000.

[18] D. Natroshvili, T. Buchukuri and O. Chkadua, Mathematical modelling and analysis of interaction problems for piezoelastic composites. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* **30** (2006), fasc 1, 159–190.

[19] D. G. Natroshvili, O. O. Chkadua and E. M. Shargorodsky, Mixed problems for homogeneous anisotropic elastic media. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **39** (1990), 133–181.

[20] J. Nečas, *Direct Methods in the Theory of Elliptic Equations.* Springer Monographs in Mathematics. Springer, Heidelberg, 2012.

[21] R. T. Seeley, Singular integrals and boundary value problems. *Amer. J. Math.* **88** (1966), 781–809.

[22] E. Shargorodsky, An L_p-analogue of the Vishik-Eskin theory. *Mem. Differential Equations Math. Phys.* **2** (1994), 41–146.

[23] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam–New York, 1978.

(Received 17.02.2017)

Authors' addresses:

Tengiz Buchukuri

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

E-mail: t-buchukuri@yahoo.com

Otar Chkadua

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. Sokhumi State University, 9 A. Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: chkadua@rmi.ge

David Natroshvili

1. Georgian Technical University, 77 M. Kostava St., Tbilisi 0175, Georgia;

2. I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.

E-mail: natrosh@hotmail.com