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ASYMPTOTIC REPRESENTATIONS OF ONE CLASS OF SOLUTIONS OF n-th ORDER NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Asymptotic representations of some classes of solutions of non-autonomous ordinary differential n-th order equations which somewhat are close to linear equations are established.

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1 Introduction

Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) y |\ln|y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$[1].

A solution y of the equation (1.1), which is nonzero on the interval $[t_y, \omega[\subset [a, \omega[$, is said to be a $P_\omega(\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty \end{cases} \qquad (k = \overline{0, n-1}), \qquad \lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda_0. \tag{1.2}$$

We notice that the differential equation (1.1) is a special case of the differential equation of a more general form

$$y^{(n)} = \alpha_0 p(t) \varphi(y),$$

where α_0 and p are the same as in the equation (1.1) and $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a continuous and regularly varying function as $y \to Y_0$ of the order γ, Y_0 is equal either to zero or to $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0.

The differential equation (1.1) belongs to the class of two-term non-autonomous equations with regularly varying nonlinear function $\varphi(y)$ as $y \to 0$ and $y \to \pm\infty$. In recent decades, the asymptotic theory of such equations has been studied by many authors (see, e.g., monograph by V. Maric [8] and the references therein concerning the second order equation; see also the papers by V. M. Evtukhov, A. M. Samoilenko [6] and by V. M. Evtukhov, A. M. Klopot [4] for differential equations of order n).

In [6] and [4], for the two-term differential equations of n-th order with regularly varying nonlinear function $\varphi(y)$ as $y \to 0$ and $y \to \pm\infty$, the authors obtained asymptotic representation for all possible types of $P_\omega(\lambda_0)$-solutions and their derivatives up to the order $n-1$, inclusive. However, the results of these works do not cover the case where $\varphi(y) = y|\ln|y||^{\sigma}$ is a regularly varying function of order one. By such nonlinearity of the equation (1.1), not being a substantially non-linear, and due to the asymptotic relation $\varphi(y) = y^{1+o(1)}$ as $y \to 0$ $(\pm\infty)$, the differential equation is asymptotically close to the linear differential equation

$$y^{(n)} = \alpha_0 p(t) y, \tag{1.3}$$

and therefore is of theoretical interest.

In [3], for the equation (1.1), the asymptotic behavior of $P_\omega(\lambda_0)$-solutions as $t \uparrow \omega$ was investigated when $\lambda_0 \in R \setminus \{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}\}$.

The aim of the present paper is to establish the existence conditions of $P_\omega(\lambda_0)$-solutions of the equation (1.1) in case $\lambda_0 = 0$, and to obtain asymptotic representations as $t \uparrow \omega$ for all such solutions and their derivatives up to order $n-1$, inclusive.

2 Auxiliary statements

To obtain our main results we need two lemmas, the first one is related to a priori asymptotic properties of $P_\omega(0)$-solutions and the other is about the existence of vanishing at a singular point solutions of a system of quasi-linear differential equations.

To state the first one, we introduce the function

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t - \omega & \text{if } \ \omega < +\infty. \end{cases}$$

From Lemma 10.6 introduced in [2, Ch. 3, § 10, pp. 143–144] we get the following statement.

[1] We assume that $a > 1$ for $\omega = +\infty$ and $\omega - a < 1$ for $\omega < +\infty$.

Lemma 2.1. *If $n \geq 2$, then each $P_\omega(0)$-solution of the differential equation* (1.1) *satisfies the following asymptotic relation as $t \uparrow \omega$:*

$$y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\, y^{(n-2)}(t) \ \ (k = 1, \dots, n-2), \quad y^{(n-1)}(t) = o\Big(\frac{y^{(n-2)}(t)}{\pi_\omega(t)}\Big), \tag{2.1}$$

and in case $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$ (finite or equal to $\pm\infty$) exists, the following relation holds:

$$y^{(n)}(t) \sim -\frac{y^{(n-1)}(t)}{\pi_\omega(t)} \ \ \text{as} \ \ t \uparrow \omega. \tag{2.2}$$

Next, we consider a system of quasi-linear differential equations

$$\begin{cases} v_k' = h(t)\Big[f_k(t, v_1, \dots, v_n) + \sum\limits_{i=1}^{n} c_{ki} v_i\Big] & (k = \overline{1, n-1}), \\ v_n' = H(t)\Big[f_n(t, v_1, \dots, v_n) + \sum\limits_{i=1}^{n} c_{ni} v_i\Big], \end{cases} \tag{2.3}$$

in which $c_{ki} \in \mathbb{R}$ $(k, i = \overline{1, n})$, $h, H : [t_0, \omega[\to \mathbb{R} \setminus \{0\}$ are continuously differentiable functions, and $f_k : [t_0, \omega[\times \mathbb{R}^n_{\frac{1}{2}}$ $(k = \overline{1, n})$ are continuous functions satisfying the condition

$$\lim_{t\uparrow\omega} f_k(t, v_1, \dots, v_n) = 0 \ \text{ uniformly in } \ (v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}, \tag{2.4}$$

where

$$\mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1, \dots, v_n) \in \mathbb{R}^n : \ \ |v_i| \leq \frac{1}{2} \ \ (i = \overline{1, n})\Big\}.$$

By Theorem 2.6 from [5] for the system of differential equations (2.3) the following lemma holds.

Lemma 2.2. *Let the functions h and H satisfy the conditions*

$$\lim_{t\uparrow\omega} \frac{H(t)}{h(t)} = 0, \quad \int_{t_0}^{\omega} H(\tau)\, d\tau = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{1}{H(t)} \Big(\frac{H(t)}{h(t)}\Big)' = 0.$$

Moreover, suppose the matrices $C_n = (c_{ki})_{k,i=1}^{n}$ and $C_{n-1} = (c_{ki})_{k,i=1}^{n-1}$ are such that $\det C_n \neq 0$ and C_{n-1} has no eigenvalues with zero real part. Then the system of differential equations (2.3) *has at least one solution $(v_k)_{k=1}^{n} : [t_1, \omega[\, [\mathbb{R}^n_{\frac{1}{2}}$ $(t_1 \in [t_0, \omega[)$ that tends to zero as $t \uparrow \omega$. Furthermore, if among the eigenvalues of matrix C_{n-1} there are m eigenvalues (taking into account the multiplicity) whose real parts have a sign opposite to that of the function $h(t)$ on the interval $[t_0, \omega[$, then if the inequality $H(t)(\det C_n)(\det C_{n-1}) > 0$ holds on $[t_0, \omega[$, there exist m-parameter solutions of the system* (2.3)*, and there exists an $m+1$-parameter family when the opposite inequality holds.*

3 Main results

In order to formulate the main results, let us introduce the following auxiliary functions:

$$P_1(t) = \int_{A_1}^{t} p(\tau)\, d\tau, \quad P_2(t) = \int_{A_2}^{t} P_1(\tau)\, d\tau,$$

$$J_A(t) = \int_{A}^{t} p(\tau) \pi_\omega^{n-2}(\tau) |\ln|\pi_\omega(\tau)||^\sigma\, d\tau, \quad I(t) = \int_{a}^{t} J_A(\tau)\, d\tau,$$

where

$$A_1 = \begin{cases} a, & \text{if } \int\limits_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega p(\tau)\,d\tau < \infty, \end{cases} \qquad A_2 = \begin{cases} a, & \text{if } \int\limits_a^\omega |P_1(\tau)|\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega |P_1(\tau)|\,d\tau < \infty. \end{cases}$$

$$A = \begin{cases} a, & \text{if } \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|^{n-2}|\ln|\pi_\omega(\tau)||^\sigma\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|^{n-2}|\ln|\pi_\omega(\tau)||^\sigma\,d\tau < +\infty. \end{cases}$$

When $n = 2$, i.e., in the case of a second order differential equation, the conditions of the existence and asymptotic behavior of $P_\omega(0)$-solutions were obtained in [1].

Theorem 3.1. *Let $n = 2$ and $\sigma \neq 1$, then the differential equation* (1.1) *has $P_\omega(0)$-solutions if and only if the following conditions hold:*

$$\lim_{t\uparrow\omega} |P_2(t)|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega} \frac{P_1^2(t)|P_2(t)|^{\frac{\sigma}{1-\sigma}}}{p(t)} = 0, \tag{3.1}$$

Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$:

$$\ln|y(t)| = \mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}[1+o(1)], \quad \frac{y'(t)}{y(t)} = \alpha_0 P_1(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \tag{3.2}$$

where $\mu = \alpha_0 \operatorname{sign}[(1-\sigma)P_2(t)]$. Furthermore, if the conditions (3.1) *are valid, then the differential equation* (1.1) *has a one-parametric (two-parametric) family of such solutions in the case where $A_1 = \omega$ ($A_1 = a$).*

For the case $n > 2$, the following theorem holds.

Theorem 3.2. *Let $n \geq 3$ and suppose that*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} \tag{3.3}$$

exists (*finite or equal to $\pm\infty$*)*. Then the differential equation* (1.1) *has $P_\omega(0)$-solutions if and only if the following conditions hold:*

$$\lim_{t\uparrow\omega} \pi_\omega(t)J_A(t) = 0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} = -1, \quad \lim_{t\uparrow\omega} I(t) = \pm\infty, \tag{3.4}$$

and each of these solutions admits the following asymptotic representations as $t \uparrow \omega$:

$$\frac{y^{(k-1)}(t)}{y^{(n-2)}(t)} = \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\,[1+o(1)] \quad (k = \overline{1, n-2}), \tag{3.5}$$

$$\ln|y^{(n-2)}(t)| = \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,I(t)[1+o(1)], \tag{3.6}$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,J_A(t)[1+o(1)]. \tag{3.7}$$

Moreover, when the conditions (3.4) *are satisfied, the differential equation* (1.1) *has an $n-1$-parametric family of solutions that admits asymptotic representations* (3.5)*–*(3.7) *as $t \uparrow \omega$ in case $\omega = +\infty$, and it has two-parametric family of solutions with such representations in case $\omega < +\infty$.*

Proof. Necessity. Let $y : [t_y, \omega[\to \mathbb{R}$ be an arbitrary $P_\omega(0)$-solution of the equation (1.1). Then by the definition of $P_\omega(\lambda_0)$-solution there exists $t_0 \in [t_y, \omega[$ such that $\ln|y(t)| \neq 0$ on the interval $[t_0, \omega[$ and, by Lemma 2.1, the asymptotic relations (2.1) hold. According to the first asymptotic relation of (2.1), we have the asymptotic representations (3.4) from which, in particular, we get

$$y(t) \sim \frac{\pi_\omega^{n-2}(t)}{(n-2)!}\, y^{(n-2)}(t), \quad y'(t) \sim \frac{\pi_\omega^{n-3}(t)}{(n-3)!}\, y^{(n-2)}(t) \ \text{ as } \ t \uparrow \omega.$$

This implies that

$$\frac{y'(t)}{y(t)} \sim \frac{n-2}{\pi_\omega(t)} \ \text{ as } \ t \uparrow \omega$$

and therefore

$$\ln|y(t)| \sim (n-2)\ln|\pi_\omega(t)| \ \text{ as } \ t \uparrow \omega.$$

By virtue of these asymptotic relations, from (1.1) we get

$$y^{(n)}(t) = \frac{\alpha_0}{(n-2)!} p(t)\pi_\omega^{n-2}(t)|(n-2)\ln|\pi_\omega(t)||^\sigma y^{(n-2)}(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

i.e.,

$$\frac{y^{(n)}(t)}{y^{(n-2)}(t)} = \frac{\alpha_0|n-2|^\sigma p(t)\pi_\omega^{n-2}(t)}{(n-2)!}\,|\ln|\pi_\omega(t)||^\sigma[1+o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.8}$$

Since

$$\Big(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\Big)' = \frac{y^{(n)}(t)}{y^{(n-2)}(t)}\Big[1 - \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)}\Big]$$

and, by the definition of $P_\omega(0)$-solution,

$$\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)} = 0,$$

we have

$$\Big(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\Big)' \sim \frac{y^{(n)}(t)}{y^{(n-2)}(t)} \ \text{ as } \ t \uparrow \omega.$$

Therefore, the asymptotic relation (3.8) can be written as

$$\Big(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\Big)' = \frac{\alpha_0|n-2|^\sigma p(t)\pi_\omega^{n-2}(t)}{(n-2)!}\,|\ln|\pi_\omega(t)||^\sigma[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Integrating this relation from t_0 to t, we obtain

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = c_0 + \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\int_{t_0}^{t} p(\tau)\pi_\omega^{n-2}(\tau)|\ln|\pi_\omega(\tau)||^\sigma[1+o(1)]\,d\tau, \tag{3.9}$$

where c_0 is a constant, or taking into account the choice of limit integration A in the function J_A, we get

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = c + \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\, J_A(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

where

$$c = c_0 + \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\int_{t_0}^{A} p(\tau)\pi_\omega^{n-2}(\tau)|\ln|\pi_\omega(\tau)||^\sigma[1+o(1)]\,d\tau.$$

In the case where $A = a$, the integral on the right-hand side of (3.9) tends to $\pm\infty$ as $t \uparrow \omega$, and then (3.9) can be written as

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\, J_A(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.10}$$

We will show that in case $A = \omega$, when the integral on the right-hand side of (3.9) tends to zero as $t \uparrow \omega$, the relation (3.10) also holds, i.e., $c = 0$. Indeed, if $c \neq 0$, then from (3.9) we have

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = c + o(1) \ \text{ as } \ t \uparrow \omega.$$

This representation for $\omega = +\infty$ (i.e., $\pi_\omega(t) = t$) contradicts the last relation of (2.1), and if $\omega < +\infty$, by integration we obtain

$$\ln |y^{(n-2)}(t)| = c_1 + o(1) \ \text{ as } \ t \uparrow \omega \quad (c_1 = const),$$

which is in contradiction with the first condition of (2.1) (when $k = n-2$).

Therefore, in each of two possible cases under consideration the asymptotic relation (3.10) holds, that is, (3.7) holds, and by the use of the last asymptotic relation of (2.1), the first condition of (3.4) is satisfied.

Moreover, from (3.10) and (3.8) it follows that

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{J_A'(t)}{J_A(t)}\,[1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

Then

$$\frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{\pi_\omega(t) J_A'(t)}{J_A(t)}\,[1 + o(1)] \ \text{ as } \ t \uparrow \omega \tag{3.11}$$

and, by virtue of the existence of the limit (3.3) (finite or equal to $\pm\infty$) and using Lemma 2.1, we conclude that (2.2) holds, whereby from (3.11) follows the validity of the second condition of (3.4).

Finally, integrating (3.10) from t_0 to t we get

$$\ln |y^{(n-2)}(t)| = c + \frac{\alpha_0 |n-2|^\sigma}{(n-2)!} \int\limits_{t_0}^{t} J_A(\tau)[1 + o(1)]\, d\tau.$$

Since, by the definition of $P_\omega(0)$-solutions, $\lim\limits_{t\uparrow\omega} \ln |y^{(n-2)}(t)| = \pm\infty$, the third condition of (3.4) is fulfilled and it can be written as (3.6).

Sufficiency. Let $n \geq 3$ and the conditions (3.4) hold. We will show that in this case the differential equation (1.1) has $P_\omega(0)$-solutions admitting asymptotic representations (3.5)–(3.7) as $t \uparrow \omega$, and we find out the quantities of solutions with such representations.

Since

$$\pi_\omega(t) J_A(t) = \frac{\pi_\omega(t) J_A(t)}{I(t)}\, I(t),$$

from the conditions (3.4) we get

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_A(t)}{I(t)} = 0. \tag{3.12}$$

Moreover, by the L'Hospital rule,

$$\lim_{t\uparrow\omega} \frac{I(t)}{\ln |\pi_\omega(t)|} = \lim_{t\uparrow\omega} \pi_\omega(t) J_A(t) = 0. \tag{3.13}$$

Applying now to the equation (1.1) transformations

$$\begin{aligned}
\frac{y^{(k-1)}(t)}{y^{(n-2)}(t)} &= \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\,[1 + v_k(t)] \ \ (k = \overline{1, n-2}),\\
\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} &= \frac{\alpha_0 |n-2|^\sigma}{(n-2)!}\, J_A(t)[1 + v_{n-1}(t)],\\
\ln |y^{(n-2)}(t)| &= \frac{\alpha_0 |n-2|^\sigma}{(n-2)!}\, I(t)[1 + v_n(t)],
\end{aligned} \tag{3.14}$$

we obtain the system of differential equations

$$\begin{aligned}
v_k' &= \frac{n-k-1}{\pi_\omega(t)}\,(v_{k+1}-v_k) - \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,J_A(t)(1+v_k)(1+v_{n-1}) \ \ (k=\overline{1,n-3}),\\
v_{n-2}' &= -\frac{v_{n-2}}{\pi_\omega(t)} - \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,J_A(t)(1+v_{n-2})(1+v_{n-1}),\\
v_{n-1}' &= -\frac{J_A'(t)}{J_A(t)}\,(1+v_{n-1}) - \frac{\alpha_0|n-2|^\sigma}{(n-2)!} J_A(t)(1+v_{n-1})^2\\
&\quad + \frac{J_A'(t)}{J_A(t)}\,(1+v_1)\,\frac{|\ln|\frac{\pi_\omega^{n-2}(t)}{(n-2)!}\,(1+v_1)||^\sigma}{|n-2|^\sigma|\ln|\pi_\omega(t)||^\sigma}\,\Big|1+\frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,\frac{I(t)(1+v_n)}{\ln|\frac{\pi_\omega^{n-2}(t)}{(n-2)!}\,(1+v_1)|}\Big|^\sigma,\\
v_n' &= \frac{J_A(t)}{I(t)}\,(1+v_{n-1}) - \frac{J_A(t)}{I(t)}\,(1+v_n).
\end{aligned}$$

We set

$$h(t)=\frac{1}{\pi_\omega(t)},\quad H(t)=\frac{J_A(t)}{I(t)}\,,$$

$$\delta_1(t)=\frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,\pi_\omega(t)J_A(t),\quad \delta_2(t)=\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}+1,$$

$$\delta_3(t)=\frac{\alpha_0|n-2|^\sigma}{(n-2)!(n-2)}\,\frac{I(t)}{\ln|\pi_\omega(t)|}\,,\quad \delta_4(t,v_1)=\frac{\ln|\frac{1+v_1}{(n-2)!}|}{(n-2)\ln|\pi_\omega(t)|}\,,$$

and rewrite this system in the form

$$\begin{cases}
v_k' = h(t)\big[f_k(t,v_1,\dots,v_n)-(n-k-1)v_k+(n-k-1)v_{k+1}\big] \quad (k=\overline{1,n-3}),\\
v_{n-2}' = h(t)\big[f_{n-2}(t,v_1,\dots,v_n)-v_{n-2}\big],\\
v_{n-1}' = h(t)[f_{n-1}(t,v_1,\dots,v_n)-v_1+v_{n-1}],\\
v_n' = H(t)[v_{n-1}-v_n],
\end{cases} \tag{3.15}$$

where

$$\begin{aligned}
f_k(t,v_1,\dots,v_n) &= \delta_2(t)(1+v_k)(1+v_{n-1}) \ \ (k=\overline{1,n-3}),\\
f_{n-2}(t,v_1,\dots,v_n) &= \delta_1(t)(1+v_{n-1})^2-\delta_2(t)(1+v_{n-1}),\\
f_{n-1}(t,v_1,\dots,v_n) &= \delta_1(t)(1+v_{n-1})(1+v_{n-1})-\delta_2(t)(1+v_{n-1})\\
&\quad + (1+v_1)\Big[1+\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}|1+\delta_4(t,v_1)|^\sigma\Big|1+\frac{\delta_3(t)(1+v_n)}{1+\delta_4(t,v_1)}\Big|^\sigma\Big].
\end{aligned}$$

Here, by the conditions (3.4) and (3.13),

$$\lim_{t\uparrow\omega}\delta_i(t)=0 \ \ (i=1,2,3) \tag{3.16}$$

and

$$\lim_{t\uparrow\omega}\delta_4(t,v_1)=0 \ \text{ uniformly in } \ v_1\in\Big[-\frac12,\frac12\Big]. \tag{3.17}$$

Taking into account these limit relations, we choose a number $t_0\in]a,\omega[$ such that for $t\in[t_0,\omega[$ and $|v_1|\le\frac12$, $|v_n|\le\frac12$ the inequalities

$$|\delta_4(t,v_1)|\le\frac12\,,\quad \Big|\frac{\delta_3(t)(1+v_n)}{1+\delta_4(t,v_1)}\Big|\le\frac12$$

hold. Next, we consider the system (3.15) on the set

$$\Omega=[t_0,\omega[\times\mathbb{R}^n_{\frac12},\ \text{ where } \ \mathbb{R}^n_{\frac12}=\Big\{(v_1,\dots,v_n)\in\mathbb{R}^n:\ |v_i|\le\frac12\,,\ i=\overline{1,n}\Big\}.$$

The right-hand sides of (3.15) are continuous on this set, the functions h, H are continuously differentiable on the interval $[t_0,\omega[$, and by the conditions (3.16), (3.17),

$$\lim_{t\uparrow\omega} f_k(t,v_1,\dots,v_n)=0 \ \text{ uniformly in } \ (v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}.$$

Hence, the system of differential equations (3.15) is a quasilinear system of differential equations of the type (2.3).

We show that for (3.15) all conditions of Lemma 2.2 are satisfied.

By virtue of the definition of functions I and J_A,

$$\int\limits_{t_0}^{t} H(\tau)\,d\tau \sim \ln|J_A(t)| \longrightarrow \pm\infty \ \text{ as } \ t\uparrow\omega.$$

Moreover,

$$\frac{H(t)}{h(t)}=\frac{\pi_\omega(t)J_A(t)}{I(t)}\,,\qquad \frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)'=1+\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}-\frac{\pi_\omega(t)J_A(t)}{I(t)}$$

and therefore, in view of the second conditions of (3.4) and (3.12), we obtain

$$\lim_{t\uparrow\omega}\frac{H(t)}{h(t)}=0,\quad \lim_{t\uparrow\omega}\frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)'=0.$$

Thus the conditions (2.4) of Lemma 2.2 are satisfied for the system (3.15).

The matrices C_{n-1} and C_n of dimension $(n-1)\times(n-1)$ and $n\times n$ (respectively) from Lemma 2.2, in the case of the system of differential equations (3.15), have the form

$$C_{n-1}=\begin{pmatrix} -(n-2) & n-2 & 0 & \dots & 0 & 0 & 0\\ 0 & -(n-3) & n-3 & \dots & 0 & 0 & 0\\ 0 & 0 & -(n-4) & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -2 & 2 & 0\\ 0 & 0 & 0 & \dots & 0 & -1 & 0\\ -1 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix},\qquad C_n=\begin{pmatrix} C_{n-1} & 0_{n-1}\\ e_{n-1} & -1\end{pmatrix},$$

where 0_{n-1} is a zero column vector of dimension $n-1$ and e_{n-1} is a unit row vector of dimension $n-1$ with the last component equal to one.

These matrices are such that

$$\det C_{n-1}=(-1)^{n-2}(n-2)!,\quad \det C_n=(-1)^{n-1}(n-2)!$$

and

$$\det[C_{n-1}-\rho E_{n-1}]=(-1)^{n-1}(\rho+n-2)(\rho+n-3)\cdots(\rho+1)(\rho-1),$$

where E_{n-1} is the identity matrix of dimension $(n-1)\times(n-1)$. Hence, in particular, we get that the matrix C_{n-1} has $n-1$ nonzero real eigenvalues from which $n-2$ are negative and one is positive.

Thus, for (3.15) the conditions of Lemma 2.2 are satisfied. According to this lemma, (3.15) has at least one solution $(v_k)_{k=1}^n:[t_1,\omega[\to R^n$ $(t_1\in[t_0,\omega[)$, which tends to zero as $t\uparrow\omega$. Moreover, among the eigenvalues of the matrix C_{n-1} we have $n-2$ positive and one negative, and $\det C_n\det C_{n-1}<0$. By Lemma 2.2, if the inequality $h(t)>0$ (resp., $h(t)<0$) holds on the interval $[t_0,\omega[$, then (3.15) has $(n-2)$-parametric (resp., one-parametric) family of solutions vanishing at ω in case $H(t)<0$ on $[t_0,\omega[$, and $n-1$-parametric (resp., two-parametric) family of solutions in case $H(t)>0$ on $[t_0,\omega[$.

For the final conclusion on a number of vanishing solutions, as $t\uparrow\omega$, of the system (3.15) it is necessary to determine the signs of functions h and H on $[t_0,\omega[$.

Since $h(t)=\pi_\omega^{-1}(t)$, by the definition of π_ω we have

$$\operatorname{sign} h(t)=\begin{cases}1 & \text{if } \ \omega=+\infty,\\ -1 & \text{if } \ \omega<+\infty.\end{cases}$$

For the function H, according to the definition of I we have

$$H(t) = \frac{J_A(t)}{I(t)} = \frac{|J_A(t)|}{\int_a^t |J_A(\tau)|\, d\tau} > 0 \quad \text{if} \ \ t \in [t_0, \omega[\,.$$

Using the obtained sign conditions for the functions h and H, we arrive at the following final conclusions about a number of vanishing solutions as $t \uparrow \omega$ for the system of differential equations (3.15):

(1) if $\omega = +\infty$, then the system of differential equations (3.15) has $n-1$-parametric family of vanishing solutions as $t \to +\infty$;

(2) if $\omega < +\infty$, then the system of differential equations (3.15) has two-parametric family of vanishing solutions as $t \uparrow \omega$.

Using the substitution (3.14), every solution $(v_k)_{k=1}^n : [t_1, \omega[\to \mathbb{R}^n$ of (3.15) which tends to zero corresponds to a solution $y : [t_1, \omega[\to \mathbb{R}$ of the differential equation (1.1) which admits as $t \uparrow \omega$ the asymptotic representations (3.5)–(3.7). Using these representations and the condition (3.4), it is not difficult to see that each such solution is $P_\omega(\frac{n-i-1}{n-i})$-solution of (1.1). □

Remark 3.3. When checking the fulfillment of the conditions (3.4), we may consider that owing to the first of these conditions, the second and third conditions are equivalent, respectively, to

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^n(t)|\ln|\pi_\omega(t)||^\sigma = 0 \quad \text{and} \quad \int\limits_a^\omega p(t)|\pi_\omega(t)|^{n-1}|\ln|\pi_\omega(t)||^\sigma\, dt = +\infty.$$

Finally, pay attention to the fact that Theorem 3.2 covers the case $\sigma = 0$, that is, when the equation (1.1) is a linear differential equation of the form (1.3).

For (1.3), by Theorem 3.2 and with regard for Remark 3.3, the following corollary holds.

Corollary 3.4. *Let $n \geq 3$ and suppose that the limit* (3.3) *exists* (*finite or equal to* $\pm\infty$). *Then the linear differential equation* (1.3) *has $P_\omega(0)$-solutions if and only if the following conditions hold:*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega^{n-1}(t)p(t)}{\int\limits_A^t \pi_\omega^{n-2}(\tau)p(\tau)\, d\tau} = -1, \quad \int\limits_a^\omega |\pi_\omega(\tau)|^{n-1}p(\tau)\, d\tau = +\infty, \quad \lim_{t\uparrow\omega} \pi_\omega^n(t)p(t) = 0, \tag{3.18}$$

and for each such solution the following asymptotic representations take place as $t \uparrow \omega$:

$$\frac{y^{(k-1)}(t)}{y^{(n-2)}(t)} = \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\,[1+o(1)] \ \ (k = \overline{1, n-2}), \tag{3.19}$$

$$\ln|y^{(n-2)}(t)| = -\frac{\alpha_0}{(n-2)!}\int\limits_a^t p(\tau)\pi_\omega^{n-1}(\tau)\, d\tau[1+o(1)], \tag{3.20}$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = -\frac{\alpha_0}{(n-2)!}\, p(t)\pi_\omega^{n-1}(t)[1+o(1)]. \tag{3.21}$$

Moreover, when the conditions (3.18) *are satisfied, the differential equation* (1.3) *has $n-1$-parametric family of $P_\omega(0)$-solutions with the representations* (3.19)–(3.21) *in case $\omega = +\infty$, and in case $\omega < \infty$* (1.3) *has two-parametric family.*

This corollary in case $\omega = +\infty$ complements the results for linear differential equations with asymptotically small coefficients given in [7, Ch. 1, Section 6, pp. 184–186].

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