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ON OSCILLATORY AND MONOTONE SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. The nonlinear functional differential system with deviating arguments

$$u_1^{(n_1)}(t) = f_1\big(t, u_2(\tau_1(t))\big), \quad u_2^{(n_2)}(t) = f_2\big(t, u_1(\tau_2(t))\big)$$

is considered, where $f_i : [a, +\infty[\times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ and $\tau_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ are continuous functions, and $\tau_i(t) \to +\infty$ as $t \to +\infty$ $(i = 1, 2)$. Conditions are found under which any proper solution of that system is, respectively: a) oscillatory, b) either oscillatory or Kneser solution, c) either oscillatory or rapidly increasing.

რეზიუმე. განხილულია გადახრილარგუმენტებიანი არაწრფივი ფუნქციონალურ-დიფერენციალური სისტემა

$$u_1^{(n_1)}(t) = f_1\big(t, u_2(\tau_1(t))\big), \quad u_2^{(n_2)}(t) = f_2\big(t, u_1(\tau_2(t))\big),$$

სადაც $f_i : [a, +\infty[\times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ და $\tau_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ უწყვეტი ფუნქციებია და $\tau_i(t) \to +\infty$, როცა $t \to +\infty$ $(i = 1, 2)$. ნაპოვნია პირობები, რომელთა შესრულებისას ამ სისტემის ნებისმიერი წესიერი ამონახსნი სათანადოდ არის: ა) რხევადი, ბ) ან რხევადი, ან კნეზერული, გ) ან რხევადი, ან სწრაფად ზრდადი.

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The present paper is devoted to the investigation of asymptotic properties of solutions of the nonlinear functional differential system

$$u_1^{(n_1)}(t) = f_1\big(t, u_2(\tau_1(t))\big), \quad u_2^{(n_2)}(t) = f_2\big(t, u_1(\tau_2(t))\big). \tag{1}$$

Here, $n_1 \geq 1$, $n_2 \geq 2$, $a > 0$, while $f_i : [a, +\infty[\times \mathbb{R} \to \mathbb{R}$ and $\tau_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ are continuous functions. Moreover,

$$\lim_{t \to +\infty} \tau_i(t) = +\infty \quad (i = 1, 2),$$

and one of the following two conditions

$$f_i(t, 0) = 0, \quad (-1)^{i-1} f_i(t, x) \leq (-1)^{i-1} f_i(t, y) \ \text{ for } \ t > a, \ \ x < y \ \ (i = 1, 2); \tag{2}$$

$$f_i(t, 0) = 0, \quad f_i(t, x) \leq f_i(t, y) \ \text{ for } \ t \geq a, \ \ x < y \ \ (i = 1, 2) \tag{3}$$

is satisfied.

Asymptotic (including oscillatory) properties of solutions of the system (1) previously have been investigated mainly in the cases where this system can be reduced to one $n_1 + n_2$-order functional differential equation, or in the cases where $n_1 = n_2 = 1$ (see [1–7, 11, 12, 15–19] and the references therein). The case, where $n_1 + n_2 > 2$, $\tau_i(t) \not\equiv t$ $(i = 1, 2)$, and the system (1) cannot be reduced to one equation, still remains practically unstudied. The results of the present paper concern namely this case.

Let $a_0 \geq a$. A vector function $(u_1, u_2) : [a_0, +\infty[\to \mathbb{R}^2$ is said to be a **solution of the system** (1) if u_1 and u_2 are, respectively, n_1-times and n_2-times continuously differentiable functions, and there exist continuous functions $v_i :]-\infty, a_0] \to \mathbb{R}$ $(i = 1, 2)$ such that on $[a_0, +\infty[$ the equalities (1) are fulfilled, where

$$u_i(t) = v_i(t) \ \text{ for } \ t \leq a_0 \ \ (i = 1, 2).$$

A solution (u_1, u_2) of the system (1), defined on some interval $[a_0, +\infty[\subset [a, +\infty[$, is said to be **proper** if it is not identically zero in any neighborhood of $+\infty$.

A proper solution of the system (1) is said to be **oscillatory** if at least one of its components changes the sign in any neighborhood of $+\infty$.

Note that if one of the conditions (2) and (3) is satisfied, then both components of every oscillatory solution of the system (1) change the sign in any neighborhood of $+\infty$.

A nontrivial solution $(u_1,u_2):[a_0,+\infty[\,\to\mathbb{R}$ of the system (1) is said to be a **Kneser solution** if on $[a_0,+\infty[$ it satisfies the inequalities

$$(-1)^i u_1^{(i)}(t)u_1(t)\geq 0 \ \ (i=1,\dots,n_1),$$
$$(-1)^k u_2^{(k)}(t)u_2(t)\geq 0 \ \ (k=1,\dots,n_2),$$

and it is said to be **rapidly increasing** if

$$\lim_{t\to+\infty}|u_i^{(n_i-1)}(t)|>0 \ \ (i=1,2).$$

Let

$$n=n_1+n_2,$$

and following I. Kiguradze [8, 9] introduce the definitions.

Definition 1. The system (1) has the **property** A_0 if every its proper solution for n even is oscillatory, and for n odd either is oscillatory or is a Kneser solution.

Definition 2. The system (1) has the **property** B_0 if every its proper solution for n even either is oscillatory, or is a Kneser solution, or is rapidly increasing, and for n odd either is oscillatory or is rapidly increasing.

I. T. Kiguradze [8, 9] has established unimprovable in a certain sense conditions under which the differential system

$$u_1^{(n_1)}(t)=f_1(t,u_2(t)), \quad u_2^{(n_2)}(t)=f_2(t,u_1(t))$$

has the property A_0 (the property B_0). The theorems below are the generalizations of those results for the system (1).

If m is a natural number, then by $\mathcal{N}_m^0$ we denote the set of those $k\in\{1,\dots,m\}$ for which $m+k$ is even.

For any natural k, we put

$$\varphi_k(t,x)=x\bigg[|\tau_2(t)|^{n_1-1}+\int_a^{\tau_2(t)}(\tau_2(t)-s)^{n_1-1}\big|f_1\big(t,x|\tau_1(s)|^{k-1}\big)\big|\,ds\bigg].$$

Theorem 1. *Let the condition* (2) *hold and let for any* $x\neq 0$ *and* $k\in\mathcal{N}_{n_2-1}^0$ *the equalities*

$$\int_a^{+\infty}|f_1(t,x)|\,dt=+\infty, \quad \int_a^{+\infty}t^{n_2-1}|f_2(t,x)|\,dt=+\infty, \tag{4}$$
$$\int_a^{+\infty}t^{n_2-k-1}\big|f_2(t,\varphi_k(t,x))\big|\,dt=+\infty \tag{5}$$

be satisfied. Then the system (1) *has the property* A_0.

Theorem 2. *Let* $n_2>2$ $(n_2=2)$ *and the condition* (3) *hold. If, moreover, for any* $x\neq 0$ *and* $k\in\mathcal{N}_{n_2-2}^0$ *the equalities* (4) *and* (5) *are satisfied* (*for any* $x\neq 0$ *the equalities* (4) *are satisfied*), *then the system* (1) *has the property* B_0.

Remark 1. For the equality (5) to be satisfied for any $x\neq 0$ and $k\in\mathcal{N}_{n_2-1}^0$ it is sufficient that the equality

$$\int_a^{+\infty}\big|f_2\big(t,x|\tau_2(t)|^{n_1-1}\big)\big|\,dt=+\infty$$

be satisfied for any $x\neq 0$.

The conditions of Theorems 1 and 2 do not guarantee the existence of proper solutions appearing in the definitions of the properties A_0 and B_0. The problem on the existence of such solutions needs additional investigation. In particular, for the system (1) we have to study the initial problem

$$u_i^{(k-1)}(a)=c_{ik} \ \ (k=1,\dots,n_i; \ \ i=1,2), \tag{6}$$

the Kneser problem

$$\sum_{i=1}^{2}\sum_{k=1}^{n_i}|u_i^{(k-1)}(a)|=c_0,\ \ (-1)^{k-1}u_i^{(k-1)}(t)u_i(t)>0 \ \text{ for }\ t\geq a\ \ (k=1,\dots,n_i;\ \ i=1,2), \tag{7}$$

and the Kiguradze problem [10]

$$\begin{gathered} u_1^{(k-1)}(a)=\alpha_{1k}u_2^{(n_2-1)}(a)+c_{1k}\ \ (k=1,\dots,n_1),\\ u_2^{(k-1)}(a)=\alpha_{2k}u_1^{(n_2-1)}(a)+c_{2k}\ \ (k=1,\dots,n_2-1),\quad \liminf_{t\to+\infty}|u_2^{(n_2-1)}(t)|<+\infty. \end{gathered} \tag{8}$$

The following lemma is valid.

Lemma 1. *If the conditions*

$$a\leq\tau_i(t)<t,\ \ f_i(t,x)\neq 0\ \ \textit{for}\ \ t>a,\ \ x\neq 0\ \ (i=1,2),$$

and

$$\sum_{i=1}^{2}\sum_{k=1}^{n_i}|c_{ik}|>0$$

are fulfilled, then the problem (1), (6) *is solvable and every its solution is proper.*

On the basis of the methods proposed in [13] and [14], the following lemmas can be proved.

Lemma 2. *If* $c_0>0$,

$$\tau_i(t)>t\ \ \textit{for}\ \ t>a\ \ (i=1,2),$$

and

$$f_1(t,x)x>0,\ \ (-1)^{n_1+n_2}f_2(t,x)x>0\ \ \textit{for}\ \ t>a,\ \ x\neq 0,$$

then the problem (1), (7) *is solvable.*

Lemma 3. *Let the conditions*

$$a\leq\tau_i(t)<t,\ \ f_i(t,x)x>0\ \ \textit{for}\ \ t\geq a,\ \ x\neq 0\ \ (i=1,2),$$

$$f_1(t,x)\leq f_1(t,y)\ \ \textit{for}\ \ t\geq a,\ \ x\leq y,$$

and

$$\int_a^{+\infty}\big|f_1\big(t,x|\tau_1(t)|^{n_2-1}\big)\big|\,dt=+\infty\ \ \textit{for}\ \ x\neq 0$$

hold. If, moreover,

$$\alpha_{1j}>0,\ \ \alpha_{2k}>0\ \ (j=1,\dots,n_1;\ \ k=1,\dots,n_2-1),\quad \sum_{j=1}^{n_1}|c_{1j}|+\sum_{k=1}^{n_2-1}|c_{2k}|>0,$$

then the problem (1), (8) *is solvable and every its solution is proper.*

Theorem 1 and Lemmas 1 and 2 yield the following propositions.

Theorem 3. *Let* n_1+n_2 *be even and along with* (2) *the condition*

$$\tau_i(t)<t,\ \ f_i(t,x)\neq 0\ \ \textit{for}\ \ t\geq a,\ \ x\neq 0\ \ (i=1,2) \tag{9}$$

be satisfied. If, moreover, for any $x\neq 0$ *and* $k\in\mathcal{N}_{n_2-1}^0$ *the equalities* (4) *and* (5) *are fulfilled, then the system* (1) *has an infinite set of proper solutions and every such solution is oscillatory.*

Theorem 3′. *Let* n_1+n_2 *be odd and along with* (2) *the condition*

$$\tau_i(t)>t,\ \ f_i(t,x)\neq 0\ \ \textit{for}\ \ t>a,\ \ x\neq 0\ \ (i=1,2) \tag{10}$$

hold. If, moreover, for any $x\neq 0$ *and* $k\in\mathcal{N}_{n_2-1}^0$ *the equalities* (4) *and* (5) *are satisfied, then:*

(i) *the system* (1) *has an infinite set of proper Kneser solutions and every such solution is vanishing at infinity;*

(ii) *an arbitrary nontrivial solution* (u_1, u_2) *of the system* (1), *defined on some interval* $[a_0, +\infty[\, \subset [a, +\infty[$ *and satisfying the inequality*

$$\min\big\{(-1)^k u_i^{(k)}(a_0)u_i(a_0):\ k = 1, \dots, n_i - 1;\ i = 1, 2\big\} \le 0,$$

is oscillatory.

On the basis of Theorem 2 and Lemma 3 the following theorem can be proved.

Theorem 4. *Let* $n_1 + n_2$ *be odd and the conditions* (3) *and* (9) *hold. If, moreover,* $n_2 > 2$ $(n_2 = 2)$ *and for any* $x \neq 0$ *and* $k \in \mathcal{N}^0_{n_2-2}$ *the equalities* (4) *and* (5) *are satisfied* (*for any* $x \neq 0$ *the equalities* (4) *are satisfied*), *then the system* (1) *has infinite sets of oscillatory and rapidly increasing solutions.*

Remark 2. If $n_1 + n_2$ is even and the conditions (3) and (10) hold, then by Lemma 3 the system (1) has an infinite set of proper Kneser solutions. However, in this case the problem on the existence of oscillatory and rapidly increasing solutions of that system remains open.

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References

1. R. P. Agarwal, M. Bohner, and W.-T. Li, Nonoscillation and oscillation: theory for functional differential equations. *Monographs and Textbooks in Pure and Applied Mathematics*, 267. *Marcel Dekker, Inc., New York*, 2004.
2. R. P. Agarwal, S. R. Grace, and D. O'Regan, Oscillation theory for difference and functional differential equations. *Springer Science & Business Media*, 2013.
3. R. P. Agarwal and D. O'Regan, Infinite interval problems for differential, difference and integral equations. *Kluwer Academic Publishers, Dordrecht*, 2001.
4. Z. Došlá and I. Kiguradze, On vanishing at infinity solutions of second order linear differential equations with advanced arguments. *Funkcial. Ekvac.* **41** (1998), no. 2, 189–205.
5. Z. Došlá and I. Kiguradze, On boundedness and stability of solutions of second order linear differential equations with advanced arguments. *Adv. Math. Sci. Appl.* **9** (1999), no. 1, 1–24.
6. L. Erbe, Q. Kong, and B.-G. Zhang, Oscillation theory for functional differential equations. Vol. 190. *CRC Press*, 1994.
7. J. R. Graef, R. Koplatadze, and G. Kvinikadze, Nonlinear functional differential equations with Properties A and B. *J. Math. Anal. Appl.* **306** (2005), no. 1, 136–160.
8. I. Kiguradze, On oscillatory solutions of higher order nonlinear nonautonomous differential equations and systems. *Czech-Georgian Workshop on Boundary Value Problems – WBVP-2016, Brno, Czech Republic*, 2016; `http://users.math.cas.cz/ sremr/wbvp2016/abstracts/kiguradze1.pdf`.
9. I. Kiguradze, Oscillatory solutions of higher order nonlinear nonautonomous differential systems. *Mem. Differential Equations Math. Phys.* **69** (2016), 123–127.
10. I. Kiguradze, On boundary value problems with the condition at infinity for systems of higher order nonlinear differential equations. *International Workshop QUALITDE–2015*, 79–80, *Tbilisi, Georgia*, 2015; `http://rmi.tsu.ge/eng/QUALITDE-2015/Kiguradze_workshop_2015.pdf.`
11. I. T. Kiguradze and D. V. Izyumova, Oscillatory properties of a class of differential equations with deviating argument. (Russian) *Differ. Uravn.* **21** (1985), no. 4, 588–596; translation in *Differ. Equations* **21** (1985), 384–391.
12. I. Kiguradze and N. Partsvania, On the Kneser problem for two-dimensional differential systems with advanced arguments. *J. Inequal. Appl.* **7** (2002), no. 4, 453–477.
13. I. T. Kiguradze and I. Rachůnková, On the solvability of a nonlinear Kneser type problem. (Russian) *Differ. Uravn.* **15** (1979), no. 10, 1754–1765; translation in *Differential Equations* **15** (1980), 1248–1256.
14. I. Kiguradze and Z. Sokhadze, On a boundary value problem on an infinite interval for nonlinear functional differential equations. *Georgian Math. J.* **24** (2017) (to appear).
15. I. T. Kiguradze and I. P. Stavroulakis, On the existence of proper oscillating solutions of advanced differential equations. (Russian) *Differ. Uravn.* **34** (1998), no. 6, 751–757; translation in *Differential Equations* **34** (1998), no. 6, 748–754.
16. I. Kiguradze and I. P. Stavroulakis, On the oscillation of solutions of higher order Emden-Fowler advanced differential equations. *Appl. Anal.* **70** (1998), no. 1-2, 97–112.
17. R. Koplatadze, On oscillatory properties of solutions of functional-differential equations. *Mem. Differential Equations Math. Phys.* **3** (1994), 179 pp.

18. R. G. Koplatadze and T. A. Chanturia, Oscillation properties of differential equations with deviating argument. (Russian) *Izdat. Tbilis. Univ., Tbilisi*, 1977.
19. G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, Oscillation theory of differential equations with deviating arguments. *Marcel Dekker, Inc., New York*, 1987.

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