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THE SCREEN TYPE DIRICHLET BOUNDARY VALUE PROBLEMS FOR ANISOTROPIC PSEUDO-MAXWELL'S EQUATIONS

Dedicated to Professor Boris Khvedelidze
on the occasion of his 100th birthday anniversary

Abstract. We investigate the Dirichlet type boundary value problems for anisotropic pseudo-Maxwell's equations in screen type problems. It is shown that the problems with tangent Dirichlet traces are well-posed in tangent Sobolev spaces and they can equivalently be reduced to the Dirichlet boundary value problems in usual Sobolev spaces. Using the potential method and theory if pseudeodifferential equations the uniqieness and existence theorems are proved. Asymptotic expansions of solutions near the screen edge are derived and used to establish the best Hölder smoothness for solutions.

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რეზიუმე. გამოკვლეულია ეკრანის ტიპის დირიხლეს სასაზღვრო ამოცანები ანიზოტროპული ფსევდო-მაქსველის განტოლებებისათვის. ნაჩვენებია, რომ ამოცანები დირიხლეს მხები კვალებით კორექტულად არიან დასმული მხებ სობოლევის სივრცეებში და ისინი ეკვივალენტურად დაიყვანება სობოლევის სივრცეებში დასმულ დირიხლეს ამოცანებზე. პოტენციალთა მეთოდისა და ფსევდო-დიფერენციალურ განტოლებათა მეთოდის გამოყენებით დამტკიცებულია არსებობისა და ერთადერთობის თეორემები. მიღებულია ამონახსნის ასიმპტოტური დაშლა ეკრანის საზღვრის მახლობლობაში, რომლის გამოყენებით დადგენილია ამონახსნის ჰელდერული უწყვეტობის საუკეთესო მაჩვენებელი.

1. Introduction

The study of boundary value problems in electromagnetism naturally leads us to the pseudo-Maxwell's equations with inherited tangent boundary conditions, which are in some sense non-standard for the system of elliptic equations, cf. the works of Buffa, Costabel, Christiansen, Dauge, Hazard, Lenoir, Mitrea, Nicaise and others. Due to the presence of tangent boundary conditions the usage of the potential methods for the investigation is complicated and the case of tangent Dirichlet type boundary condition is mostly studied by variational methods. Our goal is to investigate well-posedness of the screen type Dirichlet boundary value problems for pseudo-Maxwell's equations

$$A(D)\boldsymbol{U} := \operatorname{curl}\,\mu^{-1}\operatorname{curl}\,\boldsymbol{U} - s\varepsilon\operatorname{grad}\operatorname{div}(\varepsilon\,\boldsymbol{U}) - \omega^2\varepsilon\,\boldsymbol{U} = 0 \ \text{ in } \ \mathbb{R}^3\setminus\overline{\mathscr{C}} \quad (1.1)$$

with the help of the potential method and tools of pseudodifferential equations; here, $\mathscr{C} \subset \mathbb{R}^3$ denotes a screen which is a compact, orientable and non self-intersecting surface with the boundary.

The present investigation covers the anisotropic case when the coefficients in (1.1) are real-valued and constant matrices

$$\varepsilon = [\varepsilon_{jk}]_{3\times 3}, \quad \mu = [\mu_{jk}]_{3\times 3}, \tag{1.2}$$

which are symmetric and positive definite,

$$\langle \varepsilon\xi, \xi\rangle \geq c|\xi|^2, \quad \langle \mu\xi, \xi\rangle \geq d|\xi|^2, \ \ \forall\, \xi \in \mathbb{R}^3\,,$$

for some positive constants $c > 0$, $d > 0$, where

$$\langle \eta, \xi\rangle := \sum_{j=1}^{3} \eta_j\overline{\xi}_j, \ \ \eta, \xi \in \mathbb{C}^3,$$

s in (1.2) is a positive real number and the frequency parameter ω is assumed to be non-zero and complex valued, i.e., $\operatorname{Im}\omega \neq 0$.

2. Formulation of the Problems

From now on throughout the paper, unless stated otherwise, Ω denotes either a bounded $\Omega^+ \subset \mathbb{R}^3$ or an unbounded $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}^+$ domain with the smooth, non-self-intersecting boundary $\mathscr{S} := \partial\Omega^+$ and $\boldsymbol{\nu}$ is the outer unit normal vector field to $\mathscr{S}$. Whenever necessary, we will specify the case.

By $\mathscr{C}$ we denote a subsurface of $\mathscr{S}$ (a screen) with a boundary $\partial\mathscr{C}$, which has two faces $\mathscr{C}^-$ and $\mathscr{C}^+$ and inherits the orientation from $\mathscr{S}$: $\mathscr{C}^+$ borders the inner domain Ω^+ and $\mathscr{C}^-$ borders the outer domain Ω^-. The unbounded domain with a screen configuration is denoted by

$$\mathbb{R}^3_{\mathscr{C}} := \mathbb{R}^3 \setminus \overline{\mathscr{C}}.$$

The space $\widetilde{\mathbb{H}}^r(\mathscr{C})$ comprises those functions $\varphi \in \mathbb{H}^r(\mathscr{S})$ which are supported in $\overline{\mathscr{C}}$ (functions with the "vanishing traces on the boundary"). For the detailed definitions and properties of these spaces we refer, e.g., to [13, 14, 16, 17]).

It is well-known that $\mathbb{H}^{r-1/2}(\mathscr{S})$ is a trace space for $\mathbb{H}^r(\Omega)$, provided that $r > 1/2$ and the corresponding trace operator is denoted by $\gamma_{\mathscr{S}}$. For the detailed definitions and properties of these spaces we refer, e.g., to [17].

Let us note that since $\mathscr{S}$ is smooth, the Dirichlet trace $\gamma_{\mathscr{S}}\boldsymbol{U}$, the tangential (Dirichlet) traces $\gamma_\tau\boldsymbol{U} = \gamma_{\mathscr{S}}(\boldsymbol{\nu}\times\boldsymbol{U})$ and $\gamma_\pi\boldsymbol{U} = \gamma_{\mathscr{C}}[(\boldsymbol{\nu}\times\boldsymbol{U})\times\boldsymbol{\nu}]$, the normal (Dirichlet) traces $\gamma_n\boldsymbol{U} = \langle\boldsymbol{\nu}, \gamma_{\mathscr{S}}\boldsymbol{U}\rangle$ (i.e., $\gamma_n\boldsymbol{U} = \boldsymbol{\nu}\cdot\gamma_{\mathscr{S}}\boldsymbol{U}$) are well defined for the elements of $\mathbb{H}^1(\Omega)$ and $\gamma_\tau\boldsymbol{U}, \gamma_\pi\boldsymbol{U}$ belong to the Sobolev space

$$\mathbb{H}^{\frac{1}{2}}_t(\mathscr{S}) := \left\{\boldsymbol{U}\in(H^{\frac{1}{2}}(\Gamma))^3 :\ \boldsymbol{\nu}\cdot\boldsymbol{U} = 0 \text{ on } \mathscr{S}\right\}$$

of tangential vector fields of order 1/2 on the surface $\mathscr{S}$, while $\gamma_n\boldsymbol{U}\in H^{\frac{1}{2}}(\mathscr{S})$ and $\gamma_{\mathscr{S}}\boldsymbol{U}\in\mathbb{H}^{\frac{1}{2}}(\mathscr{S})$.

First, for the smooth functions, using the Gauß formula (integration by parts), we obtain the following Green's formulae:

$$\begin{aligned}\big(\boldsymbol{A}(D)\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+} = \big(\boldsymbol{\nu}\times\mu^{-1}\operatorname{curl}\boldsymbol{U}, \boldsymbol{V}_\pi\big)_{\mathscr{S}} - \big(s\operatorname{div}(\varepsilon\boldsymbol{U}), \varepsilon\boldsymbol{\nu}\cdot\boldsymbol{V}\big)_{\mathscr{S}}\\ + \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - \omega^2\big(\varepsilon\,\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+},\end{aligned} \tag{2.1}$$

where $\boldsymbol{a}_{\varepsilon,\mu}$ is the natural bilinear differential form associated with Green's formulae (2.1)

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_\Omega := \big(\mu^{-1}\operatorname{curl}\boldsymbol{U}, \operatorname{curl}\boldsymbol{V}\big)_\Omega + s\,\big(\operatorname{div}(\varepsilon\boldsymbol{U}), \operatorname{div}(\varepsilon\boldsymbol{V})\big)_\Omega. \tag{2.2}$$

and $\boldsymbol{V}_\pi := \boldsymbol{V} - \langle\boldsymbol{\nu}, \boldsymbol{V}\rangle\boldsymbol{\nu}$.

Note that Green's formula (2.1) allows us to define the Neumann's trace

$$\boldsymbol{T}(D, \boldsymbol{\nu})\boldsymbol{U} := s\operatorname{div}(\varepsilon\boldsymbol{U})\varepsilon\boldsymbol{\nu} - \boldsymbol{\nu}\times\mu^{-1}\operatorname{curl}\boldsymbol{U}, \tag{2.3}$$

for an arbitrary vector $\boldsymbol{U}\in\mathbb{H}^1(\Omega^+)$ provided that $\boldsymbol{A}(D)\boldsymbol{U}\in\mathbb{L}_2(\Omega^+)$ by the duality as follows

$$\big(\boldsymbol{T}(D, \boldsymbol{\nu})\boldsymbol{U}, \boldsymbol{V}\big)_{\mathscr{S}} = \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - \big(\boldsymbol{A}(D)\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+} - \omega^2\big(\varepsilon\,\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+}, \tag{2.4}$$

for all $\boldsymbol{V}\in\mathbb{H}^1(\Omega^+)$.

Theorem 2.1 (cf. [6]). *In* (1.1)*, the operator*

$$\boldsymbol{A}(D)\boldsymbol{U} := \operatorname{curl}\mu^{-1}\operatorname{curl}\boldsymbol{U} - s\,\varepsilon\operatorname{grad}\operatorname{div}(\varepsilon\boldsymbol{U}) - \omega^2\varepsilon\boldsymbol{U}$$

is elliptic, has a positive definite principal symbol and is self-adjoint.

Now we are ready to formulate the screen type Dirichlet boundary value problems (BVPs) for anisotropic pseudo-Maxwell's equations:

The Dirichlet boundary value problem D:

Find $\boldsymbol{U}\in\mathbb{H}^1(\mathbb{R}^3_{\mathscr{C}})$ such that

$$\begin{cases}\boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathscr{C}},\\ \gamma^\pm(\boldsymbol{U}) = \mathbf{g}^\pm & \text{on } \mathscr{C},\end{cases} \tag{2.5}$$

where the given data $\mathbf{g}^\pm$ satisfy the conditions

$$\mathbf{g}^\pm\in\mathbb{H}^{1/2}(\mathscr{C}), \quad \mathbf{g}^+ - \mathbf{g}^-\in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}). \tag{2.6}$$

The Dirichlet boundary value problem D_τ:

Find $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathscr{C}}) := \big\{\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathscr{C}}) : \ \langle \varepsilon\boldsymbol{\nu}, \gamma_{\mathscr{C}^\pm}\boldsymbol{U}\rangle = 0 \text{ on } \mathscr{C}\big\}$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathscr{C}}, \\ \gamma^\pm_\tau(\boldsymbol{U}) = \mathbf{f}^\pm & \text{on } \mathscr{C}, \end{cases} \tag{2.7}$$

where the given data $\mathbf{f}^\pm$ satisfy the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{1/2}_t(\mathscr{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{1/2}_t(\mathscr{C}). \tag{2.8}$$

The Dirichlet boundary value problem D_π:

Find $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathscr{C}})$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathscr{C}}, \\ \gamma^\pm_\pi(\boldsymbol{U}) = \mathbf{f}^\pm & \text{on } \mathscr{C}, \end{cases} \tag{2.9}$$

where the given data $\mathbf{f}^\pm$ satisfy the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{1/2}_t(\mathscr{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{1/2}_t(\mathscr{C}). \tag{2.10}$$

Before we proceed it is worth to note that tangent boundary conditions in *Problems* D_τ and D_π are motivated by tight connections between boundary value problems for pseudo-Maxwell's equation and Maxwell's equation, where the boundary operators γ_τ and γ_π are natural, cf. [1–3,7] and others. However, since we consider smooth screens there is a connection between the traces γ_τ and γ_π established by the geometric operation $\boldsymbol{\nu} \times \cdot$ which is in fact a rotation operator and therefore from the uniqueness, existence and regularity results for the *Problem* D_τ we get the same results for the *Problem* D_π, and vice versa. Moreover, the uniqueness, existence and regularity results for these problems are an easy consequence of the results obtained for the *Problem D* below due to the following formula:

$$\mathbf{g} = (\boldsymbol{\nu} \times \mathbf{g}) \times \boldsymbol{\nu} + \frac{\langle \varepsilon\boldsymbol{\nu}, \mathbf{g}\rangle - \langle \varepsilon\boldsymbol{\nu}, (\boldsymbol{\nu} \times \mathbf{g}) \times \boldsymbol{\nu}\rangle}{\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}\,\boldsymbol{\nu}, \tag{2.11}$$

which holds true for the smooth vector field $\boldsymbol{\nu}$ and any $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\mathscr{S})$. Indeed, first, from the decomposition

$$\mathbf{g} = \boldsymbol{\nu} \times (\mathbf{g} \times \boldsymbol{\nu}) + \langle \boldsymbol{\nu}, \mathbf{g}\rangle \boldsymbol{\nu} \tag{2.12}$$

we have

$$\langle \varepsilon\boldsymbol{\nu}, \mathbf{g}\rangle = \langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu} \times (\mathbf{g} \times \boldsymbol{\nu})\rangle + \langle \boldsymbol{\nu}, \mathbf{g}\rangle\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle. \tag{2.13}$$

Now, by expressing $\langle \boldsymbol{\nu}, \mathbf{g}\rangle$ from (2.13) and inserting it into (2.12), we get (2.11). Further, if $\boldsymbol{U}$ is a unique solution of the *Problem D* with the boundary data

$$\mathbf{g}^\pm = \mathbf{f}^\pm \times \boldsymbol{\nu} - \frac{\langle \varepsilon\boldsymbol{\nu}, \mathbf{f}^\pm \times \boldsymbol{\nu}\rangle}{\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}\,\boldsymbol{\nu},$$

where $\mathbf{f}^{\pm}$ satisfy the conditions (2.8) (therefore $\mathbf{g}^{\pm}$ satisfy the conditions (2.6)), we need to show that $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathscr{C}})$ and $\gamma^{\pm}_{\tau}(\boldsymbol{U}) = \mathbf{f}^{\pm}$. Clearly, we have

$$\langle \varepsilon\boldsymbol{\nu}, \gamma_{\mathscr{C}^{\pm}}\boldsymbol{U}\rangle = \langle \varepsilon\boldsymbol{\nu}, \mathbf{g}^{\pm}\rangle = \langle \varepsilon\boldsymbol{\nu}, \mathbf{f}^{\pm}\times\boldsymbol{\nu}\rangle - \frac{\langle \varepsilon\boldsymbol{\nu}, \mathbf{f}^{\pm}\times\boldsymbol{\nu}\rangle}{\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle = 0$$

and

$$\gamma^{\pm}_{\tau}(\boldsymbol{U}) = \boldsymbol{\nu}\times(\mathbf{f}^{\pm}\times\boldsymbol{\nu}) - \frac{\langle \varepsilon\boldsymbol{\nu}, \mathbf{f}^{\pm}\times\boldsymbol{\nu}\rangle}{\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}(\boldsymbol{\nu}\times\boldsymbol{\nu}) = \boldsymbol{\nu}\times(\mathbf{f}^{\pm}\times\boldsymbol{\nu}) = \mathbf{f}^{\pm},$$

since $\mathbf{f}^{\pm} \in \mathbb{H}^{1/2}_t(\mathscr{C})$. Thus it is sufficient to study the *Problem D*.

3. Vector Potentials

The elliptic operator $\boldsymbol{A}(D)$ in (1.1) has the fundamental solution (cf. [13])

$$\mathbf{F}_{\boldsymbol{A}}(x) := \mathscr{F}^{-1}_{\xi\to x}\big[\mathscr{A}^{-1}(\xi)\big] = \mathscr{F}^{-1}_{\xi'\to x'}\bigg[\pm\frac{1}{2\pi}\int\limits_{\mathscr{L}} e^{-i\tau x_3}\mathscr{A}^{-1}(\xi',\tau)\,d\tau\bigg],$$
$$\xi' = (\xi_1,\xi_2)^{\top} \in \mathbb{R}^2, \ \ x = (x', x_3) \in \mathbb{R}^3,$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform and $\mathscr{A}(\xi)$ is the full symbol of the operator $\boldsymbol{A}(D)$:

$$\mathscr{A}(\xi) := \sigma_{\mathrm{curl}}(\xi)\mu^{-1}\sigma_{\mathrm{curl}}(\xi) + s\,\varepsilon\big[\xi_j\xi_k\big]_{3\times3}\varepsilon - \omega^2\varepsilon, \ \ \xi = (\xi_1,\xi_2,\xi_3)^{\top} \in \mathbb{R}^3,$$

where

$$\sigma_{\mathrm{curl}}(\xi) := \begin{bmatrix} 0 & i\xi_3 & -i\xi_2 \\ -i\xi_3 & 0 & i\xi_1 \\ i\xi_2 & -i\xi_1 & 0 \end{bmatrix}.$$

If $x_3 < 0$ (if, respectively, $x_3 > 0$), we fix the sign "+" (the sign "−") and a contour $\mathscr{L}$ in the upper (in the lower) complex half-plane, which encloses all roots of the polynomial equation $\det\mathscr{A}(\xi) = 0$ in the corresponding half-planes.

Let us consider, respectively, the *single-layer* and *double-layer* potential operators

$$\mathbf{V}\boldsymbol{U}(x) := \oint\limits_{\mathscr{S}} \mathbf{F}_{\boldsymbol{A}}(x-\tau)\boldsymbol{U}(\tau)\,dS, \tag{3.1}$$

$$\mathbf{W}\boldsymbol{U}(x) := \oint\limits_{\mathscr{S}} \Big[\big(\boldsymbol{T}(D,\boldsymbol{\nu}(\tau))\mathbf{F}_{\boldsymbol{A}}\big)(x-\tau)\Big]^{\top}\boldsymbol{U}(\tau)\,dS, \ \ x \in \Omega, \tag{3.2}$$

related to pseudo-Maxwell's equations in (1.1). Obviously,

$$\boldsymbol{A}(D)\mathbf{V}\boldsymbol{U}(x) = \boldsymbol{A}(D)\mathbf{W}\boldsymbol{U}(x) = 0, \ \ \forall\,\boldsymbol{U} \in \mathbb{L}_1(\mathscr{S}), \ \ \forall\,x \in \Omega. \tag{3.3}$$

For the next Propositions 3.1–3.4 and for their proofs we refer, e.g., to [9, 11, 15].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a domain with the smooth boundary $\mathscr{S} = \partial\Omega$.*

The potential operators above map continuously the spaces

$$\begin{aligned} \mathbf{V} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r+3/2}(\Omega), \\ \mathbf{W} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r+1/2}(\Omega), \ \ \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.4}$$

The direct values $\mathbf{V_{-1}}$, $\mathbf{W_0}$ and $\mathbf{V_{+1}}$ of the potential operators $\mathbf{V}$, $\mathbf{W}$ and $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}$ are pseudodifferential operators of order -1, 0 and 1, respectively, and map continuously the spaces

$$\begin{aligned} \mathbf{V_{-1}} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r+1}(\mathscr{S}), \\ \mathbf{W_0} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r}(\mathscr{S}), \\ \mathbf{V_{+1}} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r-1}(\mathscr{S}), \ \ \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Proposition 3.2. *The potential operators on an open, compact, smooth surface $\mathscr{C} \subset \mathbb{R}^3$ have the following mapping properties:*

$$\begin{aligned} \mathbf{V} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r+3/2}(\mathbb{R}^3_{\mathscr{C}}), \\ \mathbf{W} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r+1/2}(\mathbb{R}^3_{\mathscr{C}}), \ \ \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.6}$$

The direct values $\mathbf{V_{-1}}$, $\mathbf{W_0}$ and $\mathbf{V_{+1}}$ of the potential operators $\mathbf{V}$, $\mathbf{W}$ and $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}$ are pseudodifferential operators of order -1, 0 and 1, respectively, and have the following mapping properties:

$$\begin{aligned} \mathbf{V_{-1}} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r+1}(\mathscr{C}), \\ \mathbf{W_0} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r}(\mathscr{C}), \\ \mathbf{V_{+1}} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r-1}(\mathscr{C}), \ \ \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.7}$$

Proposition 3.3. *For the traces of potential operators we have the following Plemelji formulae:*

$$(\gamma_{\mathscr{S}^-}\mathbf{V}\boldsymbol{U})(x) = (\gamma_{\mathscr{S}^+}\mathbf{V}\boldsymbol{U})(x) = \mathbf{V_{-1}}\boldsymbol{U}(x), \tag{3.8}$$

$$(\gamma_{\mathscr{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}\boldsymbol{U})(x) = \mp\frac{1}{2}\boldsymbol{U}(x) + (\mathbf{W_0})^*(x,D)\boldsymbol{U}(x), \tag{3.9}$$

$$(\gamma_{\mathscr{S}^\pm}\mathbf{W}\boldsymbol{U})(x) = \pm\frac{1}{2}\boldsymbol{U}(x) + \mathbf{W_0}(x,D)\boldsymbol{U}(x), \tag{3.10}$$

$$(\gamma_{\mathscr{S}^-}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}\boldsymbol{U})(x) = (\gamma_{\mathscr{S}^+}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}\boldsymbol{U})(x) = \mathbf{V_{+1}}\boldsymbol{U}(x), \tag{3.11}$$
$$x \in \mathscr{S}, \ \ \boldsymbol{U} \in \mathbb{H}^s_p(\mathscr{S}),$$

where $(\mathbf{W_0})^*(x,D)$ is the adjoint to the pseudodifferential operator $\mathbf{W_0}(x,D)$, the direct value of the potential operator $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}$ on the boundary $\mathscr{S}$.

Proposition 3.4. *Let the boundary $\mathscr{S} = \partial\Omega^\pm$ be a compact smooth surface. Solutions to pseudo-Maxwell's equations with anisotropic coefficients ε and μ are represented as*

$$\boldsymbol{U}(x) = \pm\mathbf{W}(\gamma_{\mathscr{S}^\pm}\boldsymbol{U})(x) \mp \mathbf{V}(\gamma_{\mathscr{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U})(x), \ \ x \in \Omega^\pm, \tag{3.12}$$

where $\gamma_{\mathscr{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\Psi$ *is Neumann's trace operator (see* (2.3)*) and* $\gamma_{\mathscr{S}^\pm}\Psi$ *is Dirichlet's trace operator.*

If $\mathscr{C}\subset\mathbb{R}^3$ *is an open compact smooth surface, then a solution to pseudo-Maxwell's equations with anisotropic coefficients* ε *and* μ *is represented as*

$$\boldsymbol{U}(x)=\mathbf{W}([\boldsymbol{U}])(x)-\mathbf{V}([\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}])(x),\quad x\in\mathbb{R}^3_{\mathscr{C}},$$
$$[\boldsymbol{U}]:=\gamma_{\mathscr{C}^+}\boldsymbol{U}-\gamma_{\mathscr{C}^-}\boldsymbol{U},\quad \big[\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}\big]:=\gamma_{\mathscr{C}^+}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}-\gamma_{\mathscr{C}^-}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}.$$

As a consequence of the representation formula (3.12) we derive the following

Corollary 3.5. *For a complex valued frequency, a solution to the screen type boundary value problems for pseudo-Maxwell's equations decays at infinity exponentially, i.e.,*

$$\boldsymbol{U}(x)=\mathscr{O}\big(e^{-\alpha|x|}\big)\ \ \textit{as}\ \ |x|\to\infty\ \ \textit{provided that}\ \operatorname{Im}\omega\neq 0 \tag{3.13}$$

for some $\alpha>0$.

Theorem 3.6. *The Problem D has at most one solution.*

Proof. The proof is standard and uses Green's formula (cf. (2.1)–(2.4)). Let R be a sufficiently large positive number and $B(R)$ be the ball centered at the origin with radius R. Set $\Omega_R:=\mathbb{R}^3_{\mathscr{C}}\cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R including both sides of $\mathscr{C}$.

Let $\boldsymbol{U}$ be a solution of the homogeneous problem. Then applying Green's formula for $\boldsymbol{V}=\boldsymbol{U}$ in Ω_R and passing to the limit $R\to\infty$, taking into account the estimate

$$\boldsymbol{U}(x)=\mathscr{O}\big(e^{-\alpha|x|}\big)\ \text{ as }\ |x|\to\infty\ \text{ for }\ \alpha>0,$$

we get

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3}-\omega^2(\varepsilon\,\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3}=0.$$

Since ε and μ^{-1} are positive definite constant matrices, $s>0$, and $\operatorname{Im}\omega\neq 0$, it follows that

$$(\varepsilon\,\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3}=0,$$

and therefore $\boldsymbol{U}\equiv 0$ in $\mathbb{R}^3$. $\square$

4. The Screen Type Dirichlet Problem

Let $\ell\mathbf{f}^+\in\mathbb{H}^{-1/2}(\mathscr{S})$ be a fixed extension of the function $\mathbf{f}^+\in\mathbb{H}^{-1/2}(\mathscr{C})$ up to the entire closed surface $\mathscr{S}$ and let $\ell_0(\mathbf{f}^+-\mathbf{f}^-)\in\mathbb{H}^{-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathscr{S})$ be an extension by zero of the function $\mathbf{f}^+-\mathbf{f}^-\in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{-1/2}(\mathscr{C})$, cf. (2.6). Then any extension of the function $\mathbf{f}^+\in\mathbb{H}^{-1/2}(\mathscr{C})$ onto $\mathscr{S}$ is given as

$$\ell^+\mathbf{f}^+=\ell\mathbf{f}^+ + \ ,$$

where   is an arbitrary element of the space $\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}^c)$, $\mathscr{C}^c:=\mathscr{S}\setminus\overline{\mathscr{C}}$. Therefore, any extension of the function $\mathbf{f}^-\in\mathbb{H}^{1/2}(\mathscr{C})$ onto $\mathscr{S}$ is defined as

$\ell^- \mathbf{f}^- := \ell^+ \mathbf{f}^+ - \ell_0(\mathbf{f}^+ - \mathbf{f}^-) \in \mathbb{H}^{1/2}(\mathscr{S})$ and we have

$$\begin{gathered} r_{\mathscr{C}} \ell^- \mathbf{f}^- = \mathbf{f}^+ - (\mathbf{f}^+ - \mathbf{f}^-) = \mathbf{f}^-, \\ r_{\mathscr{C}^c} \ell^+ \mathbf{f}^+ = r_{\mathscr{C}^c} \ell^- \mathbf{f}^-. \end{gathered} \tag{4.1}$$

We look for a solution of the screen type Dirichlet problem (2.5)-(2.6) in the form of single-layer potentials:

$$\boldsymbol{U}(x) = \begin{cases} \mathbf{V}(\mathbf{V_{-1}})^{-1} \ell^+ \mathbf{f}^+(x), & x \in \Omega^+, \\ \mathbf{V}(\mathbf{V_{-1}})^{-1} \ell^- \mathbf{f}^-(x), & x \in \Omega^-. \end{cases} \tag{4.2}$$

Then $\boldsymbol{U}$ satisfies the basic differential equation (1.1) in the domains $\Omega^\pm$, as well as the boundary conditions on $\mathscr{C}$. From the ellipticity of the differential operator $\boldsymbol{A}(D)$ it follows that a generalized solution of the equation $\boldsymbol{A}(D)\boldsymbol{U} = 0$ is analytic in $\mathbb{R}^3_{\mathscr{C}}$ and following continuity conditions

$$\begin{cases} r_{\mathscr{C}^c} \gamma_{\mathscr{S}^+} \boldsymbol{U} - r_{\mathscr{C}^c} \gamma_{\mathscr{S}^-} \boldsymbol{U} = 0, \\ r_{\mathscr{C}^c} \gamma_{\mathscr{S}^+} (\boldsymbol{T}(D, \boldsymbol{\nu}) \boldsymbol{U}) - r_{\mathscr{C}^c} \gamma_{\mathscr{S}^-} (\boldsymbol{T}(D, \boldsymbol{\nu}) \boldsymbol{U}) = 0 \end{cases} \tag{4.3}$$

hold across the complementary surface $\mathscr{C}^c$. It is clear that by our construction the first equation in (4.3) is satisfied, cf. (3.8) and (4.1). From the second equation, by applying (3.9) and (4.1) we derive the equation

$$r_{\mathscr{C}^c} \Big(-\frac{1}{2} \mathbf{I} + (\mathbf{W_0})^* \Big) (\mathbf{V_{-1}})^{-1} \ell^+ \mathbf{f}^+ - r_{\mathscr{C}^c} \Big(\frac{1}{2} \mathbf{I} + (\mathbf{W_0})^* \Big) (\mathbf{V_{-1}})^{-1} \ell^- \mathbf{f}^- = 0,$$

which is a strongly elliptic pseudo-differential equation on the surface $\mathscr{C}$

$$-r_{\mathscr{C}^c} (\mathbf{V_{-1}})^{-1} \ = \mathbf{F}, \tag{4.4}$$

with the known right-hand side

$$\mathbf{F} := r_{\mathscr{C}^c} (\mathbf{V_{-1}})^{-1} \ell \mathbf{f}^+ - r_{\mathscr{C}^c} \Big(\frac{1}{2} \mathbf{I} + (\mathbf{W_0})^* \Big) (\mathbf{V_{-1}})^{-1} \ell_0 (\mathbf{f}^+ - \mathbf{f}^-) \in \mathbb{H}^{\frac{1}{2}}(\mathscr{C}^c).$$

The principal homogeneous symbol $\sigma_{-(\mathbf{V_{-1}})^{-1}}(x, \xi)$ of the operator $-(\mathbf{V_{-1}})^{-1}$ is even with respect to ξ for all $x \in \overline{\mathscr{C}}$. This implies that the matrix

$$\big(\sigma_{-(\mathbf{V_{-1}})^{-1}}(x', 0, 0, -1) \big)^{-1} \sigma_{-(\mathbf{V_{-1}})^{-1}}(x', 0, 0, +1) = I, \ \ x' \in \partial\mathscr{C}, \tag{4.5}$$

has trivial eigenvalues. Using the equality (4.5) analogously to Lemma 3.12 from [6] we can prove the following theorem.

Theorem 4.1. *The operator*

$$-r_{\mathscr{C}^c} (\mathbf{V_{-1}})^{-1} : \widetilde{\mathbb{H}}^s(\mathscr{C}^c) \to \mathbb{H}^{s-1}(\mathscr{C}^c)$$

is invertible for all $0 < s < 1$.

From Theorem 4.1 the following existence result follows immediately.

Theorem 4.2. *The Problem D possesses a unique solution $\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathscr{C}})$ which can be represented by single- layer potentials*

$$\boldsymbol{U} = \begin{cases} \mathbf{V}(\mathbf{V_{-1}})^{-1}(\ell\mathbf{f}^+ + \quad) & \textit{in } \Omega^+, \\ \mathbf{V}(\mathbf{V_{-1}})^{-1}\big(\ell\mathbf{f}^+ + \quad - \ell_0(\mathbf{f}^+ - \mathbf{f}^-)\big) & \textit{in } \Omega^-, \end{cases}$$

where is a solution of the uniquely solvable pseudo-differential equation (4.4).

Moreover, if the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{\frac{1}{2}+s}(\mathscr{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{\frac{1}{2}+s}(\mathscr{C}).$$

for the data in (2.6) *hold, a solution $\boldsymbol{U}$ of the screen type Dirichlet problem belongs to the space $\mathbb{H}^{1+s}(\mathbb{R}^3_{\mathscr{C}})$ for all $s \in [0, 1/2)$.*

Finally, we characterize the asymptotic behaviour of solutions of the problem D-I near the screen edge $\partial\mathscr{C}$.

Let $x' \in \partial\mathscr{C}$ and $\Pi_{x'}$ be the plane passing through the point x' and orthogonal to the curve $\partial\mathscr{C}$. We introduce the polar coordinates (r, α), $z \geq 0$, $-\pi \leq \alpha \leq \pi$, on the plane $\Pi_{x'}$, with pole at the point x', such that the points $(r, \pm\pi)$ describe the faces of the screen $\mathscr{C}$ in the vicinity of the boundary $\partial\mathscr{C}$. We assume that the boundary data $\mathbf{f}^\pm$ are infinitely smooth. Applying the results obtained in [4, 5, 8, 12], near the screen edge we obtain the following asymptotic expansion:

$$\boldsymbol{U}(x', r, \alpha) = \mathbf{d}_0(x', \alpha) r^{\frac{1}{2}} + \sum_{k=1}^{M} \mathbf{d}_k(x', \alpha) r^{\frac{1}{2}+k} + \boldsymbol{U}_{M+1}(x', r, \alpha), \tag{4.6}$$

where $\mathbf{d}_k \in (C^\infty(\partial\mathscr{C} \times [-\pi, \pi]))^3$, $k = 0, \ldots, M$, $\boldsymbol{U}_{M+1} \in C^{M+1}(\overline{\Omega}^\pm)$.

Note that from asymptotic expansion (4.6) it follows that $\boldsymbol{U}$ has $C^{\frac{1}{2}}$-smoothness in the tubular neighbourhood of the screen edge $\partial\mathscr{C}$.

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