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EXISTENCE, UNIQUENESS AND CONTINUOUS DEPENDENCE ON PARAMETERS OF SOLUTIONS TO NEURAL FIELD EQUATIONS

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. We obtain conditions for the existence and uniqueness of solutions to generalized neural field equations involving parameterized measure. We study continuous dependence of these solutions on the spatiotemporal integration kernel, delay effects, firing rate, external input and measure. We also construct the connection between the delayed Amari and Hopfield network models.

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რეზიუმე. მიღებულია განზოგადებული ნეირონული ველის განტოლებების ამონახსნების არსებობის და ერთადერთობის პირობები, რომლებიც შეიცავს პარამეტრიზებულ ზომას. შესწავლილია ამ ამონახსნების უწყვეტად დამოკიდებულება სივრცე-დროის ინტეგრირების გულზე, დაგვიანების ეფექტებზე, სიგნალების გენერირების სიხშირეზე, გარედან შეტანილ მონაცემებზე და ზომაზე. ჩვენ ასევე ვამყარბთ კავშირს და დაგვიანებულ ამარისა და ჰოპფულის ქსელის მოდელებს შორის.

Introduction

The main object of our study is the following parameterized integral equation involving integration with respect to an arbitrary measure:

$$
\begin{aligned}
&u(t,x,\lambda) \\
&= \int_{-\infty}^{t} ds \int_{\Omega} W(t,s,x,y,\lambda) f\big(u(s-\tau(s,x,y,\lambda),y,\lambda),\lambda\big)\nu(dy,\lambda) \\
&\quad + I(t,x,\lambda), \quad t>a, \quad x\in\Omega, \quad \lambda\in\Lambda \qquad (1)
\end{aligned}
$$

with the initial (prehistory) condition

$$u(\xi,x,\lambda)=\varphi(\xi,x,\lambda), \quad \xi\le a, \quad x\in\Omega, \quad \lambda\in\Lambda. \qquad (2)$$

Here, the function u represents the activity of a neural element at time t and position x. The generalized spatio-temporal connectivity kernel W determines the time-dependent coupling between elements at positions x and y. The non-negative activation function f gives the firing rate of a neuron with activity u. The non-negative function τ represents the time-dependent axonal delay effects in the neural field, which require a prehistory condition given by the function φ. The function $I(t,x)$ represents a variable external input. All the above functions involve a parametrization by the parameter λ which, as well as introducing of an arbitrary parameterized measure $\nu(\,\cdot\,,\lambda)$, gives us some investigation advantages.

The equation (1) covers a wide variety of neural field models:

The most well-known Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int_R \omega(x-y) f(u(t,y))dy + I(t,x), \quad t\ge 0, \quad x\in R,$$

can be obtained from the equation (1) by taking

$$
\begin{aligned}
W(t,s,x,y,\lambda) &= \exp\big(-(t-s)\big)\omega(x-y), \\
\tau(t,x,y,\lambda) &= \varphi(\xi,x,\lambda)\equiv 0.
\end{aligned}
$$

The two-population Amari model (see [2], [16])

$$
\begin{aligned}
\begin{pmatrix}\partial_t u_e \\ \alpha\partial_t u_i\end{pmatrix}(t,x) &= -\begin{pmatrix}u_e \\ u_i\end{pmatrix}(t,x) \\
&\quad + \int_R \begin{pmatrix}\omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii}\end{pmatrix}(x-y)\begin{pmatrix}f_e(u_e(t,x)) \\ f_i(u_i(t,x))\end{pmatrix} dy \\
&\quad + \begin{pmatrix}I_e \\ I_i\end{pmatrix}(t,x), \quad t\ge 0, \quad x\in R,
\end{aligned}
$$

can be obtained from the equation (1) by taking

$$W(t,s,x,y,\lambda)$$
$$= \operatorname{diag}\Big(\exp(-(t-s)), \exp\big(-(t-s)/\alpha\big)/\alpha\Big)\begin{pmatrix}\omega_{ee} & -\omega_{ei}\\ \omega_{ie} & -\omega_{ii}\end{pmatrix}(x-y),$$
$$\tau(t,x,y,\lambda) = \varphi(\xi,x,\lambda) \equiv 0.$$

The delayed Amari model (see e.g. [5])

$$\partial_t u(t,x) = -Lu(t,x) + \int\limits_{\Omega} \omega(t,x,y) f\big(u(t-\tau(x,y),y)\big)\,dy + I(t,x),$$
$$t \in \Big[-\max_{x,y\in\overline{\Omega}} \tau(x,y), \infty\Big), \quad x \in \Omega \subset B_{R^m}(0,r), \quad L = \operatorname{diag}(l_1,\dots,l_n), \quad l_i > 0$$

with a time-dependent connectivity kernel is also a special case of the model (1) with

$$W(t,s,x,y,\lambda) = \operatorname{diag}\Big(l_1\exp\big(-l_1(t-s)\big),\dots,l_n\exp\big(-l_n(t-s)\big)\Big)\omega(t,x,y),$$
$$\tau(t,x,y,\lambda) = \tau(x,y), \quad \varphi(\xi,x,\lambda) \equiv 0.$$

Another special case of the equation (1) arises in models that take into account the microstructure of the neural field (see [4, 9, 13])

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int\limits_{R^m} \omega^{\varepsilon}(x-y) f(u(t,y))\,dy,\\ \omega^{\varepsilon}(x) = \omega(x, x/\varepsilon), \quad 0<\varepsilon\ll 1,\\ t\ge 0, \quad x\in R^m.\end{gathered} \tag{3}$$

If the microstructure is periodic, then, as the heterogeneity parameter $\varepsilon \to 0$, the above model converges (see e.g. [12]) to the homogenized Amari model

$$\begin{gathered}\partial_t u(t,x_c,x_f)\\ = -u(t,x_c,x_f) + \int\limits_{R^m}\int\limits_{\mathcal{Y}} \omega(x_c-y_c, x_f-y_f) f\big(u(t,y_c,y_f)\big)\,dy_c\,dy_f,\\ t>0, \quad x_c\in R^m, \quad x_f\in\mathcal{Y}\subset R^k,\end{gathered} \tag{4}$$

where x_c and x_f are the coarse-scale and fine-scale spatial variables, respectively. Taking

$$\Omega = R^m \times \mathcal{Y} \ \ (\mathcal{Y} \text{ is some } k\text{-dimensional torus [15]}),$$
$$x = (x_c, x_f), \quad y = (y_c, y_f),$$
$$W(t,s,x,y,\lambda) = \exp\big(-(t-s)\big)\omega(x_c-y_c, x_f-y_f)$$

in (1) with

$$\tau(t,x,y,\lambda) = \varphi(\xi,x,\lambda) \equiv 0,$$

we get the model (4). It should be pointed out here that the case of non-periodic microstructure in the model (3) that leads (see [12]) to non-Lebesgue measure in (4) is also covered by (1). It is more realistic to assume some small deviations from the periodicity in the neural networks structure reflected in the properties of the connectivity kernel with respect to the second argument. Hence, it is natural to ask whether the solution of the model (3) with a non-periodic perturbation of the periodic connectivity kernel in some sense is "close" to the solution in the non-perturbed case. One possible answer to this question is suggested in Appendix. The answer is based on the main result of the paper which is the existence, uniqueness and continuous dependence of solutions to (1) on the model parameters.

Another application of the main result is the possibility to connect the models in use in the neural field theory to the well-known Hopfield network model [8] utilizing the parameterized measure involved in (1). As the network models of the Hopfield type are used for numerical simulations of the neural fields, our results thus justify implementation of such numerical schemes.

The paper is organized in the following way. In Section 1 a special case (that is relevant in the neural field theory) of the general statement on the solvability and continuous dependence on a parameter of solutions to the Volterra operator equation from the paper [3] is given. Based on this theorem, analogous results are obtained in Section 2 for the generalized neural field model (1). Section 3 is devoted to the connection between the delayed Amari and Hopfield network models. In addition, a mathematical justification of the two known numerical schemes is offered, which illustrates a generality of the methods suggested in the paper. Finally, Appendix contains a short informal description of the homogenization procedure for the neural field equations with non-periodic microstructure based on the convergence of Banach algebras with mean value.

1. Preliminaries

In this section we provide an overview of the notation used in the paper, introduce the main definitions and formulate a fixed point theorem for locally contracting Volterra operators.

Let us introduce the following notations:

- R^m is the m-dimensional real vector space with the norm $|\cdot|$;
- Λ is some metric space;
- $B_\Lambda(\lambda_0, r)$ is the ball in the space Λ of the radius $r > 0$ centered at the point $\lambda_0 \in \Lambda$;
- Ω is a closed subset of R^m;
- $\partial\Omega$ is the boundary of the Ω;
- $\Omega_r = \Omega \cap B_{R^m}(0, r)$;

– $BC(\Omega, R^n)$ is the space of bounded continuous functions $\vartheta : \Omega \to R^n$ with the norm $\|\vartheta\|_{BC(\Omega,R^n)} = \sup_{x\in\Omega} |\vartheta(x)|$;
– $C_{comp}(\Omega, R^n)$ is the locally convex space of continuous functions $\vartheta : \Omega \to R^n$, with a compact support, equipped with the topology of uniform convergence on compact subsets;
– $Y(\mathbb{I}) = C(\mathbb{I}, BC(\Omega, R^n))$ consists of all continuous functions $\upsilon : \mathbb{I} \to BC(\Omega, R^n)$, with the norm $\|\upsilon\|_{Y(\mathbb{I})} = \max_{t\in\mathbb{I}} \|\upsilon(t)\|_{BC(\Omega,R^n)}$ if $\mathbb{I}$ is compact; if $\mathbb{I}$ is not compact, then $Y(\mathbb{I})$ is a locally convex linear space equipped with the topology of uniform convergence on compact subsets of $\mathbb{I}$;

Let $[a, b]$ be a compact subinterval of the real line. In the three forthcoming definitions we use the following notation: $Y = Y([a, b])$, $Y_\xi = Y([a, a + \xi])$ for any $\xi \in (0, b-a)$.

Definition 1. An operator $\Psi : Y \to Y$ is said to be a Volterra operator if for any $\xi \in (0, b-a)$ and any $y_1, y_2 \in Y$ the equality $y_1(t) = y_2(t)$ on $[a, a+\xi]$ implies that $(\Psi y_1)(t) = (\Psi y_2)(t)$ on $[a, a+\xi]$.

Choosing an arbitrary $\xi \in (0, b-a)$, we introduce the following three important operators. Let $E_\xi : Y \to Y_\xi$ be defined as $(E_\xi y)(t) = y_\xi(t)$, $t \in [a, a+\xi]$, where $y_\xi(t)$ is a restriction of the function $y(t)$ to the subinterval $[a, a + \xi]$; conversely, to each $y_\xi \in Y_\xi$ the operator $P_\xi : Y_\xi \to Y$ assigns one of the extensions $y \in Y$ of the element y_ξ (P_ξ may not be uniquely defined); the operator $\Psi_\xi : Y_\xi \to Y_\xi$ is given by $\Psi_\xi y_\xi = E_\xi \Psi P_\xi y_\xi$. Note that for any Volterra operator $\Psi : Y \to Y$ the operator $\Psi_\xi : Y_\xi \to Y_\xi$ is also a Volterra operator and is independent of the choice of P_ξ.

Definition 2. A Volterra operator $\Psi : Y \to Y$ is called locally contracting if there exist $q < 1$, $\theta > 0$, such that for all elements $y_1, y_2 \in Y$ the following two conditions are satisfied:

$\mathfrak{q}_1$) $\|E_\theta \Psi y_1 - E_\theta \Psi y_2\|_{Y_\theta} \le q\|E_\theta y_1 - E_\theta y_2\|_{Y_\theta}$,

$\mathfrak{q}_2$) for any $\gamma \in [0, b-a-\theta]$, the equality $E_\gamma y_1 = E_\gamma y_2$ implies that

$$\left\|E_{\gamma+\theta} \Psi y_1 - E_{\gamma+\theta} \Psi y_2\right\|_{Y_{\gamma+\theta}} \le q\left\|E_{\gamma+\theta} y_1 - E_{\gamma+\theta} y_1\right\|_{Y_{\gamma+\theta}}. \tag{5}$$

Definition 3. If there exists $\gamma \in (0, b-a]$ and a function $y_\gamma \in Y_\gamma$, which satisfies the equation $\Psi_\gamma y_\gamma = y_\gamma$, then we call y_γ a local solution of the Volterra equation

$$y(t) = (\Psi y)(t), \quad t \in [a, b]. \tag{6}$$

In the case if $\gamma = b - a$, we call this solution global (relative to the interval $[a, b]$).

To study continuous dependence on a parameter, we need some more definitions.

Definition 4. Let $F(\,\cdot\,,\,\cdot\,) : Y \times \Lambda \to Y$ be a family of Volterra operators depending on a parameter $\lambda \in \Lambda$. This family is called uniformly locally contracting if for each $\lambda \in \Lambda$ the operator $F(\,\cdot\,,\lambda)$ is locally contracting and the constants $q \geq 0$ and $\theta > 0$ from Definition 3, are independent of $\lambda \in \Lambda$.

The following theorem concerning the well-posedness of the operator equation

$$y(t) = (F(y, \lambda))(t), \quad t \in [a, b], \quad \lambda \in \Lambda, \tag{7}$$

is a special case of Theorem 1 in Burlakov, et al [3]. It represents the main theoretical tool for the problems to be studied in this paper.

Theorem 1. *Assume that for some $\lambda_0 \in \Lambda$ and $r_0 > 0$, the family of Volterra operators $F(\,\cdot\,,\lambda) : Y \to Y$ ($\lambda \in B_\Lambda(\lambda_0, r_0)$) is uniformly locally contracting and the mapping $F(\,\cdot\,,\,\cdot\,) : Y \times \Lambda \to Y$ is continuous at (y, λ_0) for all $y \in Y$.*

Then there exists $r > 0$, such that the equation (7) has a unique global solution $y(t, \lambda)$ for all $\lambda \in B_\Lambda(\lambda_0, r)$, and

$$\|y(\,\cdot\,,\lambda) - y(\,\cdot\,,\lambda_0)\|_Y \to 0 \ \text{ as } \ \lambda \to \lambda_0.$$

Moreover, for each $\lambda \in B_\Lambda(\lambda_0, r)$, any local solution of the equation (7) is also unique and is a restriction of the corresponding global solution.

2. The Main Result

In this section we justify the property of well-posedness for the generalized neural field equation (1).

The following assumptions will be imposed on the functions involved:

(**A1**) The function $f : R^n \times \Lambda \to R^n$ is continuous, bounded and Lipschitz one in the first variable uniformly with respect to $\lambda \in \Lambda$.

(**A2**) For any $b \in R$ and $r > 0$, the delay function $\tau : (-\infty, b] \times \Omega \times \Omega_r \times \Lambda_c \to [0, \infty)$ is uniformly continuous, where Λ_c is some compact subset of Λ.

(**A3**) The initial (prehistory) function $\varphi : (-\infty, a] \times \Omega \times \Lambda_c \to R^n$ is uniformly continuous.

(**A4**) The external input function $I : [a, \infty) \times \Omega \times \Lambda \to R^n$ generates a continuous mapping $\lambda \mapsto I(\,\cdot\,,\,\cdot\,,\lambda)$ from Λ to the space $Y[a, \infty)$.

(**A5**) For any $b > a$ and $r > 0$, the kernel function $W : [a, b] \times [-r, r] \times \Omega \times \Omega_r \times \Lambda_c \to R^n$ is uniformly continuous.

(**A6**) The complete σ-additive measures $\nu(\,\cdot\,,\lambda)$ ($\lambda \in \Lambda$) are finite on compact subsets of Ω and weakly continuous with respect to $\lambda \in \Lambda$ i.e. the measures can be interpreted as linear functionals on the separable locally convex space $C_{comp}(\Omega, R^n)$.

(**A7**) For any $b > a$,

$$\max_{t\in[a,b]} \Bigg(\int_{-\infty}^{t} ds \sup_{x\in\Omega,\lambda\in\Lambda} \int_{\Omega} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) \Bigg) < \infty.$$

(**A8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{t\in[a,b],\ x\in\Omega,\ \lambda\in\Lambda} \int_{-\infty}^{t} ds \int_{\Omega-\Omega_r} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) = 0.$$

Definition 5. Let $\lambda \in \Lambda$. We define a local solution to the problem (1), (2) on $[a, a+\gamma]\times R^n$, $\gamma \in (0,\infty)$, to be a function $u_\gamma \in Y([a, a+\gamma])$ that satisfies the equation (1) on $[a, a+\gamma]$ and the prehistory condition (2). We define a global solution to the problem (1), (2) to be a function $u \in Y([a,\infty))$, whose restriction u_γ to $[a, a+\gamma]$ is its local solution for any $\gamma \in (0,\infty)$.

Theorem 2. *Suppose that the assumptions* (**A1**)–(**A8**) *are fulfilled. Then the initial value problem* (1), (2) *has a unique continuous solution* $u(\,\cdot\,,\,\cdot\,,\lambda) \in Y([a,\infty))$ *for any* $\lambda \in \Lambda$*, and the correspondence* $\lambda \mapsto u(\,\cdot\,,\,\cdot\,,\lambda)$ *is a continuous mapping from* Λ *to* $Y([a,\infty))$*. Moreover, for each* $\lambda \in \Lambda$*, any local solution of the problem* (1), (2) *is also unique and it is a restriction of the corresponding global solution.*

Proof. Due to the definition of the topology in $Y([a,\infty))$, it suffices to prove this result for the case of an arbitrary compact interval $[a,b] \subset [a,\infty)$. In what follows we therefore keep fixed an arbitrary $b > a$ and keep the notation Y for the space $Y([a,b])$.

For each $\lambda \in \Lambda$ and $\varphi(\xi, x, \lambda)$ satisfying the assumption (**A3**) we define the following integral operator

$$\begin{aligned}(F(u,\lambda))(t,x) &= I_1(t,x,\lambda) + I_2(t,x,\lambda) \\ &+ \int_a^t ds \int_\Omega W(t,s,x,y,\lambda) f\big((S(u,\lambda))(t,s,x,y,\lambda),\lambda\big)\nu(dy,\lambda), \end{aligned} \tag{8}$$

where

$$\begin{aligned}&(S(u,\lambda))(t,x,y,\lambda) \\ &\qquad = \begin{cases} \varphi(t-\tau(t,x,y,\lambda),x,\lambda) & \text{if } t-\tau(t,x,y,\lambda) < a, \\ u(t-\tau(t,x,y,\lambda),y,\lambda) & \text{if } t-\tau(t,x,y,\lambda) \geq a, \end{cases}\end{aligned} \tag{9}$$

and

$$I_1(t,x,\lambda) = \varphi(a,x,\lambda) + I(t,x,\lambda),$$

$$I_2(t,x,\lambda) = \int_{-\infty}^{a} ds \int_\Omega W(t,s,x,y,\lambda) f\big(\varphi(s-\tau(s,x,y,\lambda),x,\lambda),\lambda\big)\nu(dy,\lambda).$$

Below we assume that $|f(u)| \le M$ for all $u \in R^n$.

We have to apply Theorem 1. Towards this end, we need to show that the operator family $F(\,\cdot\,,\lambda)$ $(\lambda \in \Lambda)$ satisfies the assumptions of this theorem.

At the first step of the proof we will show that $F(u,\lambda) \in Y$ for each $u \in Y$, $\lambda \in \Lambda$. Applying the assumption (**A8**) for the given $\varepsilon > 0$, we find $r > 0$ such that

$$\sup_{t\in[a,b],\ x\in\Omega,\ \lambda\in\Lambda} \int\limits_{-\infty}^{t} ds \int\limits_{\Omega-\Omega_r} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) < \frac{\varepsilon}{M}\,. \tag{10}$$

For this r and a fixed $\lambda \in \Lambda$, we find a positive $\delta = \delta(\lambda)$ (u is kept fixed) such that

$$\begin{aligned}\Big|&W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\\ &\qquad - W(t_0,s_0,x_0,y_0,\lambda)f\big((S(u,\lambda))(s_0,x_0,y_0,\lambda),\lambda\big)\Big|\\ &\qquad\qquad\qquad < \frac{\varepsilon}{((b-a)\nu(\Omega_r,\lambda))}\end{aligned} \tag{11}$$

for all $t, t_0, s, s_0 \in [a,b]$, $x, x_0 \in \Omega$, $y, y_0 \in \Omega_r$, satisfying

$$|t-t_0| < \delta,\quad |s-s_0| < \delta,\quad |x-x_0| < \delta,\quad |y-y_0| < \delta.$$

We show first that $F(\,\cdot\,,\lambda) : Y \to Y$ for each $\lambda \in \Lambda$. In other words, we have to prove that the mapping $t \mapsto (F(u,\lambda))(t,\,\cdot\,)$ is a continuous function from $[a,b]$ to $BC(\Omega,R^n)$.

As the assumptions (**A3**), (**A4**) imply $\varphi(a,\,\cdot\,,\lambda) \in BC(\Omega,R^n)$ and $I(\,\cdot\,,\,\cdot\,,\lambda) \in Y$ $(\lambda \in \Lambda)$, we only need to check that $I_2(\,\cdot\,,\,\cdot\,,\lambda) \in Y$ and $F_0(u,\lambda) \in Y$ for all $u \in Y$ and $\lambda \in \Lambda$, where

$$(F_0(u,\lambda))(t,x) = \int\limits_a^t ds \int\limits_\Omega W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\nu(dy,\lambda).$$

The proofs are similar, so we concentrate on the more involved case of F_0.

For any $t \in [a,b]$, we have

$$\begin{aligned}\big|&(F_0(u,\lambda))(t,x) - (F_0(u,\lambda))(t,x_0)\big|\\ &\le \int\limits_a^t ds \int\limits_{\Omega_r} \Big|W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\\ &\quad - W(t,s,x_0,y,\lambda)f\big((S(u,\lambda))(s,x_0,y,\lambda),\lambda\big)\Big|\nu(dy,\lambda)\\ &+ \int\limits_a^b ds \int\limits_{\Omega-\Omega_r} \Big(\big|W(t,s,x,y,\lambda)\big| + \big|W(t,s,x_0,y,\lambda)\big|\Big)\nu(dy,\lambda) < 3\varepsilon\end{aligned}$$

as long as $|x - x_0| < \delta = \delta(\lambda)$ due to the estimates (10) and (11). This proves the continuity of $(F_0(u,\lambda))(t,x)$ in x.

The boundedness of this function for each $t \in [a,b]$ follows from the assumption (**A7**) and boundedness of the function $f : R^n \to R^n$.

Finally, we check that $t \mapsto (F_0(u,\lambda))(t,\,\cdot\,)$ is a continuous mapping from $[a,b]$ to $BC(\Omega, R^n)$ if $u \in Y$:

$$\sup_{x\in\Omega}\big|(F_0(u,\lambda))(t,x) - (F_0(u,\lambda))(t_0,x)\big|$$

$$\leq \sup_{x\in\Omega}\Big|\int_a^t ds \int_\Omega W(t,s,x,y,\lambda) f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)$$

$$-\int_a^{t_0} ds \int_\Omega W(t_0,s,x,y,\lambda) f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\Big|\nu(dy,\lambda)$$

$$\leq \int_{t_0}^t ds \sup_{x\in\Omega}\int_\Omega \big|W(t,s,x,y,\lambda)\big| M\nu(dy,\lambda) < \varepsilon$$

as long as $t - t_0 < \delta$. (Here we have assumed that $t > t_0$ and again used the assumption (**A7**).) We have therefore proved that $F_0(\,\cdot\,,\lambda), F(\,\cdot\,,\lambda) : Y \to Y$ for each $\lambda \in \Lambda$.

At the second step of the proof we show that the Volterra operator (8) is a local contraction in the first variable, uniformly with respect to the parameter λ.

We choose arbitrary constants $q < 1$, $\gamma \in [0, b-a)$ and $\lambda \in \Lambda$. Let $\widetilde{f}$ be the Lipschitz constant for the function f. Since the space Y consists of continuous functions, we can unify the two properties from Definition 2 into a single one and prove that $u_1(t,\,\cdot\,) = u_2(t,\,\cdot\,)$, $t \in [a, a+\gamma)$, where $u_1, u_2 \in Y$, implies the inequality (5) for the chosen $q < 1$ and some $\theta > 0$. Indeed,

$$\big\|F(u_1,\lambda) - F(u_2,\lambda)\big\|_Y$$

$$= \sup_{t\in[a,a+\gamma+\theta],\ x\in\Omega}\Big|\int_a^t ds \int_\Omega W(t,s,x,y,\lambda) f\big((S(u_1,\lambda))(s,x,y,\lambda)\big)\nu(dy,\lambda)$$

$$-\int_a^t ds \int_\Omega W(t,s,x,y,\lambda) f\big((S(u_2,\lambda))(s,x,y,\lambda)\big)\nu(dy,\lambda)\Big|$$

$$\leq \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega}\Big|\int_{a+\gamma}^t ds \int_\Omega W(t,s,x,y,\lambda)\Big(f\big((S(u_1,\lambda))(s,x,y,\lambda)\big)$$

$$-f\big((S(u_2,\lambda))(s,x,y,\lambda)\big)\Big)\nu(dy,\lambda)\Big|$$

$$\le \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \int\limits_{a+\gamma}^{t} ds \int\limits_{\Omega} \big|W(t,s,x,y,\lambda)\big|\widetilde{f}\nu(dy,\lambda)\|u_1-u_2\|_Y$$
$$\le \widetilde{q}\|u_1-u_2\|_Y,$$

where

$$\widetilde{q} = \widetilde{f} \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \int\limits_{a+\gamma}^{t} ds \int\limits_{\Omega} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda).$$

Using the assumption (**A7**), we can always find a $\theta > 0$ such that $\widetilde{q} \le q < 1$. This proves the property of local contractivity of the operator $F(\,\cdot\,,\lambda) : Y \to Y$ for any $\lambda \in \Lambda$. Moreover, we easily obtain from $\gamma \in [0, b-a)$ the estimate on $\widetilde{q}$ that this property is uniform with respect to γ and λ, i.e. $\theta > 0$ and $q < 1$ can be chosen to be independent of $\gamma \in [0, b-a)$ and $\lambda \in \Lambda$.

At the third and final step of the proof we show the continuity of the mapping $F : Y \times \Lambda \to Y$. We pick arbitrary $\lambda_0 \in \Lambda$, $u_0 \in Y$, where continuity will be examined, and arbitrary sequences $\lambda_N \to \lambda_0$, $u_N \to u_0$ $(N \to \infty)$.

We start with estimation of the following difference:

$$\begin{aligned}
&\big|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\big| \\
&\qquad \le \big|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_N)\big)(s,x,y,\lambda_0)\big| \\
&\qquad\qquad + \big|(S(u_0,\lambda_N)\big)(s,x,y,\lambda_0) - (S(u_0,\lambda_0)\big)(s,x,y,\lambda_0)\big|.
\end{aligned}$$

The first term on the right-hand side of this inequality is less than $\varepsilon/2$ for all $s \in (-\infty, b]$, $x, y \in \Omega$, $N \ge N_1$ as $u_N \to u_0$ $(N \to \infty)$. By virtue of the assumptions (**A2**) and (**A3**), the second term on the right-hand side is less than $\varepsilon/2$ for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_r$, $N \ge N_2(r)$. Thus, for any $r > 0$, we have

$$\big|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\big| \le \varepsilon \tag{12}$$

for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_r$, $N \ge N_3(r)$.

Then, choosing $\varepsilon > 0$, we find a number $r_0 > 0$ such that the estimate (10) holds true. Increasing, if necessary, the value of r_0, we may, in addition, assume without loss of generality that $\nu(\Omega_{r_0}, \lambda_0) > 0$ and $\nu(\partial\Omega_{r_0}, \lambda_0) = 0$, so that

$$\lim_{N\to\infty} \nu(\Omega_{r_0}, \lambda_N) = \nu(\Omega_{r_0}, \lambda_0)$$

(see e.g. [7, Chapter VI, Theorem 2]).

Using this r_0, we estimate the following difference:

$$\begin{aligned}
&\Big|f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big| \\
&\le \Big|f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) - f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\Big| \\
&\quad + \Big|f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_0\big) - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big|.
\end{aligned}$$

By virtue of the assumption (**A1**), the first term on the right-hand side of the inequality is less than ε for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_4(r_0)$. Using the assumption (**A1**) and the estimate (12), we get that the second term on the right-hand side of the inequality is less than ε for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_3(r_0)$. Thus, taking into account (**A1**) and (**A7**), we obtain the inequality

$$\Big| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_N)\Big(f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) \\ - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\Big| < \varepsilon \quad (13)$$

for all $t \in [a,b]$, $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_5(r_0)$.

The assumption (**A5**) yields

$$\big|W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big| < \frac{\varepsilon}{M((b-a)\nu(\Omega_r,\lambda))} \quad (14)$$

for all $t \in [a,b]$, $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_6(r_0)$.

Using the assumptions (**A3**), (**A4**), and (**A6**), we find a natural number $N_7(r_0)$ such that

$$\begin{gathered} \sup_{t\in[a,b],\ x\in\Omega} \Big| \int\limits_{\Omega_{r_0}} \Phi(t,x,y)\big(\nu(dy,\lambda_N) - \nu(dy,\lambda_0)\big)\Big| < \varepsilon, \\ \nu(\Omega_{r_0},\lambda_N) \leq 2\nu(\Omega_{r_0},\lambda_0), \\ \sup_{x\in\Omega} \big|\varphi(a,x,\lambda_N) - \varphi(a,x,\lambda_0)\big| < \varepsilon, \\ \sup_{t\in[a,b],\ x\in\Omega} \big|I(t,x,\lambda_N) - I(t,x,\lambda_0)\big| < \varepsilon, \ \ |\lambda_N - \lambda_0| < \delta \end{gathered} \quad (15)$$

for all $N \geq N_7(r_0)$. Here, the function

$$\Phi(t,x,y) = \int\limits_{-\infty}^{t} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\, ds$$

is uniformly continuous on the set $[a,b] \times \Omega \times \Omega_{r_0}$, so that

$$\int\limits_{\Omega_{r_0}} \Phi(t,x,y)\nu(dy,\lambda_N) \longrightarrow \int\limits_{\Omega_{r_0}} \Phi(t,x,y)\nu(dy,\lambda_0)$$

as $n \to \infty$ uniformly with respect to the variables $t \in [a,b]$, $x \in \Omega$.

Next, we estimate

$$\sup_{t\in[a,b],\ x\in\Omega}\big|I_2(t,x,\lambda_N)-I_2(t,x,\lambda_0)\big|$$

$$\leq \sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int\limits_{-\infty}^{t} ds\int\limits_{\Omega} W(t,s,x,y,\lambda_N)$$

$$\times f\Big(\varphi\big(s-\tau(s,x,y,\lambda_N),x,\lambda_N\big),\lambda_N\Big)\nu(dy,\lambda_N)$$

$$-\int\limits_{-\infty}^{t} ds\int\limits_{\Omega} W(t,s,x,y,\lambda_0)f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg|$$

$$\leq \sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int\limits_{-\infty}^{t} ds\int\limits_{\Omega-\Omega_{r_0}} W(t,s,x,y,\lambda_N)$$

$$\times f\big(\varphi(s-\tau(s,x,y,\lambda_N),x,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N)$$

$$-\int\limits_{-\infty}^{t} ds\int\limits_{\Omega-\Omega_{r_0}} W(t,s,x,y,\lambda_0)f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg|$$

$$+\sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int\limits_{-\infty}^{t} ds\int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_N)\Big(f\big(\varphi(s-\tau(s,x,y,\lambda_N),x,\lambda_N\big),\lambda_N\Big)$$

$$-f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\Bigg|$$

$$+\sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int\limits_{-\infty}^{t} ds\int\limits_{\Omega_{r_0}} \big(W(t,s,x,y,\lambda_N)-W(t,s,x,y,\lambda_0)\big)$$

$$\times f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)\Bigg|$$

$$+\sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int\limits_{-\infty}^{t} ds\int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_0)$$

$$\times f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)$$

$$-\int\limits_{-\infty}^{t} ds\int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_0)f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg|.$$

The first term on the right-hand side of the inequality is less than 2ε as the estimate (10) and the assumption (**A1**) hold true. Each of the second and the third terms on the right-hand side of the inequality is less than

ε due to (13) and (**A1**), (**A7**), (14), respectively, for all $N > N_8(r_0) = \max\{N_5(r_0), N_6(r_0)\}$. The estimate (15) yields the last term on the right-hand side of the inequality is less than ε for all $N > N_7(r_0)$.

Thus, we get that

$$\sup_{t\in[a,b],\ x\in\Omega} \big|I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\big| < 5\varepsilon \tag{16}$$

for all $N \geq N_9(r_0) = \max\{N_7(r_0), N_8(r_0)\}$.

Finally, taking into account the estimates (10), (11), (13)–(16) and the assumption (**A7**), we obtain

$$\begin{aligned}
&\big\|F(u_N,\lambda_N) - F(u_0,\lambda_0)\big\|_Y \leq \sup_{x\in\Omega}\big|\varphi(a,x,\lambda_N) - \varphi(a,x,\lambda_0)\big| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega}\big|I(t,x,\lambda_N) - I(t,x,\lambda_0)\big| \\
&\qquad\qquad + \sup_{t\in[a,b],\ x\in\Omega}\big|I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\big| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_\Omega W(t,s,x,y,\lambda_N) \\
&\qquad\qquad \times f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int_a^t ds\int_\Omega W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\bigg| \\
&\quad \leq 7\varepsilon + \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\qquad\qquad \times f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\bigg| + 2\varepsilon \\
&\quad \leq 9\varepsilon + \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\qquad \times \Big(f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) \\
&\qquad\qquad - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\bigg| \\
&+ \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} \big(W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big)
\end{aligned}$$

$$
\begin{aligned}
&\times f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)\Big| \\
&+\sup_{t\in[a,b],x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) \\
&\qquad\times f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N) \\
&-\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0)f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\bigg| \\
&\le 10\varepsilon+(b-a)\nu(\Omega_{r_0},\lambda_N)\,\frac{\varepsilon}{((b-a)\nu(\Omega_{r_0},\lambda_0))} \\
&\quad+\sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_{\Omega_{r_0}}\Phi(t,x,y)(\nu(dy,\lambda_N)-\nu(dy,\lambda_0))\bigg|<13\varepsilon
\end{aligned}
$$

for all $N\ge N_9(r_0)$.

The proof is complete. $\square$

Remark 1. If Ω is compact, then the assumption (**A8**) is fulfilled automatically and can therefore be omitted, while the assumptions (**A2**)–(**A5**) only require continuity of the corresponding functions instead of their uniform continuity in the variable x.

3. The Hopfield Model with Delay

In this section we prove convergence of the generalized Hopfield network to the Amari neural field equation.

Consider the following delayed Hopfield network model (see e.g. [14])

$$
\dot z_i(t,N)=-\alpha z_i(t,N)+\sum_{j=1}^{N}\omega_{ij}(N)f\big(z_j(t-\tau_{ij}(t,N),N)\big)+\mathrm{J}_i(t,N),\quad t>a,\ \ i=1,\dots,N, \tag{17}
$$

parameterized by a natural parameter N. Here at each natural N, $z_i(\,\cdot\,,N)$ are n-dimensional vector functions, $\omega_{ij}(N)$ are real $n\times n$-matrices (connectivities), $\tau_{ij}(\,\cdot\,,N)$ are nonnegative real-valued continuous functions (axonal delays), $f:R^n\to R^n$ are firing rate functions which are Lipschitz and bounded and $\mathrm{J}_i(\,\cdot\,,N)$ are continuous external input n-dimensional vector functions.

The initial conditions for (17) are given as

$$
z_i(\xi,N)=\varphi_i(\xi,N),\ \ \xi\le a,\ \ i=1,\dots,N. \tag{18}
$$

We use the general well-posedness result from the previous section to justify the convergence of a sequence of the delayed Hopfield equations (17) (with the initial conditions (18)) to the Amari equation involving a spatio-temporal delay

$$\partial_t u(t,x) = -\alpha u(t,x) + \int_\Omega \omega(x,y) f(u(s-\tau(t,x,y),y))\nu(dy) + J(t,x), \quad t > a, \ \ x \in \Omega, \tag{19}$$

with the initial (prehistory) condition

$$u(\xi,x) = \varphi(\xi,x), \ \ \xi \le a, \ \ x \in \Omega. \tag{20}$$

On the above functions we impose the following assumptions:

(**B1**) The function $f : R^n \to R^n$ is continuous, bounded and Lipschitz one.

(**B2**) The spatio-temporal delay $\tau : R \times \Omega \times \Omega \to [0,\infty)$ is continuous.

(**B3**) The initial (prehistory) function $\varphi : (-\infty, a] \times \Omega \to R^n$ is continuous.

(**B4**) For any $b > a$, the external input function $J : [a,b] \times \Omega \to R^n$ is uniformly continuous and bounded with respect to the second variable.

(**B5**) The kernel function $\omega : \Omega \times \Omega \to R^n$ is continuous.

(**B6**) $\nu(\,\cdot\,)$ is the Lebesgue measure on Ω.

(**B7**) For any $b > a$,

$$\sup_{x\in\Omega} \int_\Omega |\omega(x,y)|\nu(dy) < \infty.$$

(**B8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{x\in\Omega} \int_{\Omega-\Omega_r} |\omega(x,y)|\nu(dy) = 0.$$

The following theorem represents the main result of this section.

Theorem 3. *For each natural number N let $\{\Delta_i(N),\ i = 1,\dots,N\}$ be a finite family of open subsets of Ω satisfying the conditions*

$$\bigcup_{i=1}^N \overline{\Delta}_i(N) = \Omega_N \quad \textit{and} \quad \lim_{N\to\infty} \textit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i = 1,\dots,N)$ be arbitrary points in $\Delta_i(N)$. Finally, let the assumptions (**B1**)–(**B8**) *be fulfilled. Then the sequence of the solutions $z_i(t,N)$ $(t \in R)$ of the initial value problem* (17), (18), *where the coefficients are defined by*

$$\begin{gathered} \omega_{ij}(N) = \beta_i(N)\omega(y_i(N), y_j(N)), \quad \textit{where} \ \ \beta_i(N) = \nu(\Delta_i(N)), \\ \tau_{ij}(t,N) = \tau(t, y_i(N), y_j(N)), \quad \mathrm{J}_i(t,N) = J(t, y_i(N)), \end{gathered} \tag{21}$$

converges for any $b > a$ to the solution $u(t,x)$ ($t \in R$, $x \in \Omega$) of the initial value problem (19), (20) *as $N \to \infty$, in the following sense:*

$$\lim_{N\to\infty} \sup_{t\in[a,b]} \Big(\sup_{1\le i\le N} \Big(\sup_{x\in\Delta_i(N)} |u(t,x) - z_i(t,N)| \Big)\Big) = 0. \tag{22}$$

In order to prove this theorem, we will need to use the following statement.

Lemma 1. *Assume that for each natural number N we have a finite family of open subsets $\{\Delta_i(N),\ i = 1,\dots,N\}$ of Ω satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \quad \textit{and} \quad \lim_{N\to\infty} \textit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ ($i = 1,\dots,N$) be arbitrary points in $\Delta_i(N)$, $\mathfrak{D}_i(N)$ be the Dirac measures at $y_i(N)$ and $\beta_i(N) = \nu(\Delta_i(N))$. Then the sequence of the discrete weighted measures

$$\nu_N = \sum_{i=1}^{N} \beta_i(N)\mathfrak{D}_i(N) \tag{23}$$

weakly converges (in the sense of the weak topology on the dual space to $C_{comp}(\Omega)$) to the Lebesgue measure on Ω.

Proof. We simply observe that for any continuous and compactly supported function $\Phi(x)$, $x \in \Omega$, we get

$$\int_\Omega \Phi(x)\nu_N(dx) = \sum_{i=1}^{N} \Phi\big(y_i(N)\big)\beta_i(N)$$
$$= \sum_{i=1}^{N} \Phi\big(y_i(N)\big)\nu\big(\Delta_i(N)\big) \longrightarrow \int_\Omega \Phi(x)\nu(dx), \tag{24}$$

as $N \to \infty$, due to the properties of the Riemann–Stiltjes integrals (see e.g. Chapter 2 in [11]). ☐

Proof of the Theorem 3. In order to apply Theorem 2, we first of all define the metric space $\Lambda = \{\lambda_N,\ N = 0,1,2,\dots\}$, where $\lambda_0 = \infty$, $\lambda_N = N$ for natural numbers N, and the distance is given by $d(\lambda_N,\lambda_M) = |1/N - 1/M|$ ($N, M \neq 0$) and $d(\lambda_N,\lambda_0) = 1/N$ ($N \neq 0$), so that $\lambda_N \to \lambda_0$ simply means that $N \to \infty$. Multiplication by the function $\eta(t-s)$, where $\eta(\sigma) = \exp(-\alpha\sigma)$, followed by integration, converts the equation (19) into the equation (1), where f, τ,

$$W(t,s,x,y) = \exp(-\alpha(t-s))\omega(x,y),$$
$$I(t,x) = \int_a^t \exp(-\alpha(t-s))J(s,x)\,ds$$

are all independent of λ, and the measures are defined as $\nu(\,\cdot\,,\lambda_N) = \nu_N$ (see (23)) and $\nu(\,\cdot\,,\lambda_0) = \nu$, respectively.

The assumptions (**A1**)–(**A5**) of Theorem 2 are trivial, the assumption (**A6**) is fulfilled due to Lemma 1 and the above definition of convergence in Λ.

Taking into account that

$$\max_{t\in[a,b]} \int\limits_{-\infty}^{t} \exp(-\alpha(t-s))\,ds = \frac{1}{\alpha}\,,$$

it is straightforward to check the assumptions (**A7**) and (**A8**).

From Theorem 2 it now follows that the solutions $u(t,x,N)$ of the initial boundary value problems

$$\partial_t u(t,x,N) = -\alpha u(t,x,N) \\ + \int\limits_{\Omega} \omega(x,y) f\big(u(s-\tau(t,x,y),y,N)\big)\nu_N(dy) + J(t,x),\;\; t>a,\;\; x\in\Omega, \quad (25)$$

with the initial (prehistory) condition

$$u(\xi,x,N) = \varphi(\xi,x),\;\; \xi\le a,\;\; x\in\Omega, \quad (26)$$

converge to the solution $u(t,x)$ ($t\in R$, $x\in\Omega$) of the initial value problem (19), (20), as $N\to\infty$, uniformly on $[a,b]\times\Omega$ for any $b>a$. Evidently, replacing x by $y_i(N)$ in the equation (25) and in the initial condition (26) yields the initial value problem (17), (18). It remains therefore to notice that the set $z_i(t,N) = u(t,y_i(N),N)$ ($i=1,\dots,N$) is a (unique) solution of the latter problem. $\square$

The theoretical results of this section can be applied to justify numerical integration schemes. For example, Faye et al [5] considered discretization of the following delayed Amari model

$$\partial_t u(t,x) = -\alpha u(t,x) + \int\limits_{\Omega} \omega\big(|x-y|\big) f\Big(u\Big(t-\frac{|x-y|}{v},y\Big)\Big)\,dy \quad (27)$$

in the cases

I. $u(t,x)\in R$, $\Omega=[-L,L]$,

II. $u(t,x)\in R^2$, $\Omega=[-L,L]$,

III. $u(t,x)\in R$, $\Omega=[-L,L]^2$.

Faye et al have justified their numerical schemes using convergence of the trapezoidal integration rule and the rectangular method to the corresponding integrals. We will show how our results can be applied for the more involved case III:

$$\partial_t u_{ij}(t) = -\alpha u_{ij}(t) + \sum_{k=1}^{M}\sum_{l=1}^{M} \omega\big(|(x_i^1, x_j^2) - (x_k^1, x_l^2)|\big) \times f\Big(u_{kl}\Big(t - \frac{|(x_i^1, x_j^2) - (x_k^1, x_l^2)|}{v}\Big)\Big)\, dy. \quad (28)$$

Here,

$$x = (x^1, x^2), \;\; u_{ij}(t) = u\big(t, (x_i^1, x_j^2)\big), \;\; i, j = 1, \dots, M.$$

Denoting

$$z_i(t) = u_{ij}(t), \quad \omega_{ij} = \omega\big(|(x_i^1, x_j^2) - (x_k^1, x_l^2)|\big),$$
$$\tau_{ij}(t) = \frac{|(x_i^1, x_j^2) - (x_k^1, x_l^2)|}{v},$$
$$i = iM + j, \;\; j = kM + l, \;\; N = M^2,$$

in (28), we get the Hopfield network model (17). Applying Theorem 3, we prove convergence of the numerical scheme (28) to the equation (27).

Rankin et al [10] discretize the Amari model (27) for

$$u(t, x) \in R, \quad \Omega = [-L, L]^2, \quad v = \infty,$$

also by substituting Ω with the grid $\{(x_i^1, x_j^2),\, i, j = 1, \dots, M\}$ and then use a combination of the Fourier transform and the inverse Fourier transform to obtain the solution numerically. Discretization of the Amari model on a hyperbolic disc $\Omega = \{x = (r, \theta),\, r \in [0, r_0],\, r_0 \in R,\, \theta \in [0, 2\pi)\}$ using the rectangular rule for the quadrature $\{(r_i, \theta_j),\, i = 1, \dots, M,\, j = i = 1, \dots, N\}$ was implemented in [6] to study of the localized solutions. As it easy to conclude from Theorem 3, the solutions obtained in both these cases converge to the corresponding analytical solutions as $M \to \infty$ and $N \to \infty$.

We emphasize here that Theorem 3 also allows one to justify discretization schemes on unbounded domains for equations involving spatio-temporal-dependent delay as well.

Appendix

In this section we consider the following neural field model with a general (i.e. non-periodic) microstructure:

$$\begin{gathered} \partial_t u(t, x) = -u(t, x) + \int\limits_{R^m} \omega_i^\varepsilon(x - y) f(u(t, y))\, dy, \\ \omega_i^\varepsilon(x) = \omega_i(x, x/\varepsilon), \;\; 0 < \varepsilon \ll 1, \\ t \geq 0, \;\; x \in R^m. \end{gathered} \quad (29)$$

which is a parametrized version of (3).

Question: What can we say about behavior of the solutions u_n to the equation (29) as $\omega_i^\varepsilon \to \omega_0^\varepsilon$ uniformly ($i \to \infty$), where ω_0^ε is periodic with respect to the second argument?

Following the idea of homogenization of the equation (3) (see [12])), we first look at the family of homogenized problems

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f)$$
$$+\int\limits_{R^m}\int\limits_{K_n} \omega_i(x_c-y_c,x_f-y_f)f(u(t,y_c,y_f))\,dy_c\,\nu_n(dy_f), \tag{30}$$
$$t>0,\ \ x_c\in R^m,\ \ x_f\in K_i\subset R^k$$

and the corresponding limit problem as $i\to\infty$

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f)$$
$$+\int\limits_{R^m}\int\limits_{K_0} \omega_0(x_c-y_c,x_f-y_f)f(u(t,y_c,y_f))\,dy_c\,\nu_0(dy_f), \tag{31}$$
$$t>0,\ \ x_c\in R^m,\ \ x_f\in K_0\subset R^k.$$

As in [12], we assume that for each $i=0,1,2,\dots$, the connectivity kernel $\omega_i(x,\cdot)$ $(x\in R^m)$ belongs to A_i, where $A_i=C(K_i)$ are some Banach algebras of continuous functions defined on the compact sets $K_i\subset R^k$ and equipped with the mean values $\mathfrak{M}_i$ (which give rise to the finite measure ν_i defined on K_i). Further, we assume that there is a compact $\overline{K}$ such that $\bigcup\limits_{i=0}^{\infty} K_i\subseteq\overline{K}$, so we can extend the measures ν_i corresponding to the mean values $\mathfrak{M}_i$ $(i=0,1,2,\dots)$, to the compact $\overline{K}$ by putting $\nu_i(\overline{K}\setminus K_i)=0$. Finally, we assume that convergence of the connectivity kernels is a consequence of a convergence of the associated Banach algebras with mean. More precisely, we suppose that:

1) the compacts K_i converge to the compact K_0 in the Hausdorff metric;
2) $\mathfrak{M}_n(\chi\big|_{K_n})\to\mathfrak{M}_0(\chi\big|_{K_0})$ for any function $\chi\in C(\overline{K})$ (here $\chi\big|_{K_i}$ denotes the restriction of the function $\chi\in C(\overline{K})$ to the set K_i).

Thus, we get

$$\int\limits_{K_n}\chi(x)\nu_n(dx)\longrightarrow\int\limits_{K_0}\chi(x)\nu_0(dx)$$

for any $\chi\in C(\overline{K})$, which means that the sequence of measures ν_n weakly converges to the measure ν_0. Hence, we can apply Theorem 2 to the problems (30) and (31) and get uniform convergence of the corresponding solutions. This approach can serve as a possible answer to the above-formulated question.

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