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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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In the present note we give sufficient conditions of boundedness and vanishing at infinity solutions of the differential equation

$$u'' + (l(t) + p(t))u = q(t), \tag{1}$$

where $l : [a, +\infty[\to]0, +\infty[$ is the function with bounded variation on every finite interval, and p and $q : [a, +\infty[\to R$ are measurable functions such that

$$\int_a^{+\infty} \frac{|p(t)|}{\sqrt{l(t)}}\, dt < +\infty \tag{2}$$

and

$$\int_a^{+\infty} \frac{|q(t)|}{\sqrt{l(t)}}\, dt < +\infty. \tag{3}$$

The use will be made of the following notation and definitions.

M is the set of functions $l : [0, +\infty[\to]0, +\infty[$ admitting the representation

$$l(t) = l_0(t) + \lambda(t), \tag{4}$$

where $l_0 : [0, +\infty[\to]0, +\infty[$ is the nondecreasing function and $\lambda : [0, +\infty[\to R$ is a locally absolutely continuous function such that

$$\lim_{t\to\infty} \frac{\lambda(t)}{l_0(t)} = 0, \quad \int_0^{+\infty} \frac{|\lambda'(t)|}{l_0(t)}\, dt < +\infty. \tag{5}$$

$$M^\infty = \{l \in M : \lim_{t\to+\infty} l(t) = +\infty\}.$$

$\dim X$ is the dimension of a linear space X.

We say that a function $l : [0, +\infty[\to]0, +\infty[$ belongs to the set H if it tends monotonically to $+\infty$ as $t \to +\infty$, and there exists $\varepsilon > 0$ such that for any increasing unbounded sequence of positive numbers $(t_k)_{k=1}^\infty$ satisfying the conditions

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$$\pi - \varepsilon < \lim_{k\to+\infty} \int_{t_{2k}}^{t_{2k+1}} \sqrt{l(t)}\, dt < \pi$$

and

$$0 < \liminf_{k\to\infty} \sqrt{l(t_{2k})}\,(t_{2k} - t_{2k-1}) \le \limsup_{k\to\infty} \sqrt{l(t_{2k})}\,(t_{2k} - t_{2k-1}) < \varepsilon,$$

the equality

$$\sum_{k=1}^{\infty} \Big[\lg l(t_{2k+1}) - \lg l(t_{2k}) \Big] = +\infty.$$

holds.

M_H is the set of functions $l : [0,+\infty[\to]0,+\infty[$ admitting the representation (4), where $l_0 \in H$, and $\lambda : [0,+\infty[\to R$ is a locally absolutely continuous function satisfying (5).

The solution u of equation (1) is said to be bounded if

$$\sup \Big\{ |u(t)| : 0 \le t < +\infty \Big\} < +\infty,$$

and vanishing at infinity if

$$\lim_{t\to+\infty} u(t) = 0.$$

Along with (1), we consider linear homogeneous equations

$$u'' = l(t)u \tag{6}$$

and

$$u'' = (l(t) + p(t))u, \tag{7}$$

whose spaces of vanishing at infinity solutions will be denoted by $Z(l)$ and $Z(l+p)$, respectively.

Theorem -3. *If $l \in M$ and the conditions* (2) *and* (3) *are satisfied, then every solution of equation* (1) *is bounded.*

Corollary 1 (Z. Opial [9]). *If $l \in M$ and the condition* (2) *is satisfied, then every solution of equation* (7) *is bounded.*

H. Milloux [7] and Z. Opial [8] have proved that if $l \in M^{\infty}$, then

$$\dim Z(l) \ge 1.$$

The question, whether the dimension of the space $Z(l)$ is invariant with respect to the perturbation of p satisfying (2), remained open.

The following theorem answers this question.

Theorem -2. *If $l \in M^{\infty}$ and the condition* (2) *is satisfied, then*

$$\dim Z(l+p) = \dim Z(l).$$

Generalizing earlier known results on vanishing at infinity solutions of equation (6) (see [1, 6, 10, 11, 12]), P. Hartman [3] and T. Chanturia [2] have respectively proved that

$$l \in H \Longrightarrow \dim Z(l) = 2$$

and

$$l \in M_H \Longrightarrow \dim Z(l) = 2.$$

Therefore from Theorem 2 it follows

Corollary 2. *If $l \in M_H$ and the condition* (2) *is satisfied, then*

$$\dim Z(p+l) = 2.$$

Corollary 3 (Kiguradze–Chanturia [4]). * *Let $(m_j)_{j=1}^{\infty}$ be a sequence of natural numbers and $(r_j)_{j=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that*

$$\lim_{j\to\infty} r_j = \infty \quad and \quad t_j = \pi \sum_{i=1}^{j-1} \frac{m_i}{r_i} \to \infty \quad for \quad j \to \infty.$$

In addition, let

$$l(t) = r_j^2 \quad for \quad t_j \le t < t_{j+1} \quad (j = 1, 2, \dots),$$

where $t_1 = 0$, and let (2) *be satisfied. Then*

$$\dim Z(l+p) = 1.$$

Theorem -1. *If*

$$l \in M^{\infty},\ \dim Z(l) = 2$$

and the conditions (2) *and* (3) *are satisfied, then every solution of equation* (1) *is vanishing at infinity.*

Corollary 4. *If*

$$l \in M_H$$

and the conditions (2) *and* (3) *are satisfied, then every solution of equation* (1) *is vanishing at infinity.*

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*See also [6], Theorem 4.10.

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