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ON PERTURBED MULTI-POINT PROBLEMS FOR NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. For nonlinear functional differential systems unimprovable conditions of solvability of perturbed multi-point boundary value problems are established.

რეზიუმე. არაწრფივი ფუნქციონალურ დიფერენციალური სისტემებისათვის დადგენილია შეშფოთებული მრავალწერტილოვანი სასაზღვრო ამოცანების ამოხსნადობის არაგაუმჯობესებადი პირობები.

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Consider the boundary value problem

$$\frac{dx_i(t)}{dt} = f_i(x_1, \dots, x_n)(t) \quad (i = 1, \dots, n), \tag{1}$$

$$x_i(t_i) = \varphi_i(x_1, \dots, x_n)(t) \quad (i = 1, \dots, n), \tag{2}$$

where $t_1, \dots, t_n$ are points from the segment $I = [a, b]$, while $f_i : C(I; \mathbb{R}^n) \to L(I; \mathbb{R})$ $(i = 1, \dots, n)$ and $\varphi_i : C(I; \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ are, respectively, continuous operators and functionals.

A vector function $(x_i)_{i=1}^n : I \to \mathbb{R}^n$ with absolutely continuous components $x_i : I \to \mathbb{R}$ $(i = 1, \dots, n)$ is said to be a solution of the system (1) if it satisfies this system almost everywhere on I.

A solution of the system (1), satisfying the boundary conditions (2), is said to be a solution of the problem (1),(2).

Particular cases of (1) are systems of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_{0i}\big(t, x_1(t), \dots, x_n(t)\big) \quad (i = 1, \dots, n) \tag{3}$$

and systems of differential equations with deviated arguments

$$\frac{dx_i(t)}{dt} = g_i\big(t, x_1(\tau_i(t)), \dots, x_n(\tau_n(t)), x_i(t)\big) \quad (i = 1, \dots, n), \tag{4}$$

where $f_{0i} : I \times \mathbb{R}^n \to \mathbb{R}$ and $g_i : I \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions from the Carathéodory class, and $\tau_i : I \to I$ $(i = 1, \dots, n)$ are measurable functions.

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Particular cases of (2) are the boundary conditions of periodic type

$$x_i(a) = \alpha_i x_i(b) \quad (i = 1, \dots, n) \tag{2_1}$$

and the multi-point boundary conditions

$$x_i(t_i) = \sum_{k=1}^{n} \sum_{j=1}^{m} \ell_{ijk} x_k(t_{ijk}) + c_i \quad (i = 1, \dots, n). \tag{2_2}$$

Boundary value problems for systems of the type (1) have been investigating intensively and are the subject of numerous works (see, e.g., [1]–[5], [12] and the references therein).

In the case where $\varphi_i = c_i = const$ $(i = 1, \dots, n)$, the problem $(3), (2)$, i.e. the system (3) with the boundary conditions

$$x_i(t_i) = c_i \quad (i = 1, \dots, n)$$

is called the Cauchy–Nicoletti problem. Optimal, in a certain sense, sufficient conditions for the solvability and unique solvability of that problem are contained in the papers [6], [7], [14].

In the paper [8] I. Kiguradze proposed a new method of investigation of boundary value problems of the type $(3), (2)$ which is based on a priori estimates of solutions of systems of one-sided differential inequalities. This method allows us to study from the unified viewpoint a sufficiently large class of perturbed multi-point boundary value problems and the periodic problem (see [8] and [10]).

In our paper, new sufficient conditions for the solvability of boundary value problems of the type $(1), (2)$ are given, which, in contrast to previous results, cover the cases where the system (1) is superlinear or sublinear or some equations of this system are superlinear, while others are sublinear.

Throughout the paper, the use will be made of the following notation:

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$;

$\mathbb{R}^n$ is the n-dimensional real Euclidian space;

$y = (y_i)_{i=1}^n$ and $Y = (y_{ik})_{i,k=1}^n$ are an n-dimensional column vector and an $n \times n$-matrix with elements y_i and $y_{ik} \in \mathbb{R}$ $(i = 1, \dots, n)$;

Y^{-1} is the inverse matrix to Y; $r(Y)$ is the spectral radius of Y;

E is the unit matrix;

$C(I; \mathbb{R}^n)$ is the space of n-dimensional continuous vector functions $x = (x_i)_{i=1}^n : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{ \sum_{i=1}^{n} |x_i(t)| : \ t \in I \Big\};$$

$L(I; \mathbb{R})$ is the space of Lebesgue integrable functions $x : I \to \mathbb{R}$ with the norm $\|x\|_L = \int\limits_a^b |x(s)|\, ds$;

$L(I; \mathbb{R}_+)$ is the set all nonnegative functions from $L(I; \mathbb{R})$.

We will say that the operator $p : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$ belongs to the Carathéodory class if it is continuous and

$$\sup\Big\{|p(x)(\cdot)| : \ x \in C(I;\mathbb{R}^n), \ \|x\|_C \le \rho\Big\} \in L(I;\mathbb{R}_+) \ \text{ for } \ \rho \in \mathbb{R}_+.$$

Everywhere below, when we discuss the boundary value problem (1), (2), it is supposed that the operators $f_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$ $(i = 1,\dots,n)$ belong to the Carathéodory class, and the functionals $\varphi_i : C(I;\mathbb{R}^n) \to \mathbb{R}$ $(i = 1,\dots,n)$ are continuous.

We will consider the case where for arbitrary $(x_i)_{i=1}^n \in C(I;\mathbb{R}^n)$ and for almost all $t \in I$ the inequalities

$$f_i(x_1,\dots,x_n)(t)\,\mathrm{sgn}\,\big((t-t_i)x_i(t)\big) \le$$
$$\le p_i(x_1,\dots,x_n)(t)\Big(-|x_i(t)| + \sum_{k=1}^n h_{ik}\|x_k\|_C + h_i\Big) +$$
$$+\delta_i(x_1,\dots,x_n)\Big(\sum_{k=1}^n q_{ik}(t)\|x_k\|_C + q_i(t)\Big) \ \ (i = 1,\dots,n), \tag{5}$$
$$\big|\varphi_i(x_1,\dots,x_n)\big| \le$$
$$\le \varphi_{0i}(|x_i|) + \delta_i(x_1,\dots,x_n)\Big(\sum_{k=1}^n \ell_{ik}\|x_k\|_C + \ell_i\Big) \ \ (i = 1,\dots,n) \tag{6}$$

are satisfied, where $p_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}_+)$ $(i = 1,\dots,n)$, $\delta_i : C(I;\mathbb{R}^n) \to \mathbb{R}_+$ are any nonlinear operators and functionals; $\varphi_{0i} : C(I;\mathbb{R}) \to \mathbb{R}$ $(i = 1,\dots,n)$ are linear non-negative functionals; h_{ik}, h_i, ℓ_{ik} and ℓ_i are non-negative constants;

$$q_{ik} \in L(I;\mathbb{R}_+), \ \ q_i \in L(I;\mathbb{R}_+) \ \ (i,k = 1,\dots,n).$$

Suppose

$$\widetilde{p}_i(x_1,\dots,x_n)(t) = \exp\bigg(-\bigg|\int_{t_i}^{t} p_i(x_1,\dots,x_n)(s)\,ds\bigg|\bigg) \ \ (i = 1,\dots,n), \tag{7}$$
$$H = \Big(h_{ik} + (1+\varphi_{0i}(1))\|q_{ik}\|_L + \ell_{ik}\Big)_{i,k=1}^n. \tag{8}$$

Theorem 1. *Let along with* (5) *and* (6) *the conditions*

$$\varphi_{0i}(1) \le 1, \quad 1 - \varphi_{0i}\big(\widetilde{p}_i(x_1,\dots,x_n)\big) \ge \delta_i(x_1,\dots,x_n) \ \ (i = 1,\dots,n), \tag{9}$$
$$r(H) < 1 \tag{10}$$

be fulfilled, where $\widetilde{p}_i$ $(i=1,\dots,n)$ *and* H *are operators and a matrix, given by the equalities* (7) *and* (8)*. Then the problem* (1), (2) *has at least one solution.*

Consider now the boundary value problem of periodic type (1), (2_1), where $\alpha_1,\dots,\alpha_n$ are arbitrary real constants. In particular, if $\alpha_1 = \dots = \alpha_n = 1$, then (1), (2_1) is a periodic problem.

The following theorem is valid.

Theorem 2. *Let there exist operators $p_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}_+)$ $(i = 1,\dots,n)$ and numbers $\sigma_i \in \{-1,1\}$, $h_{ik} \geq 0$, $h_i \geq 0$ such that for any $(x_i)_{i=1}^n \in C(I;\mathbb{R}^n)$ and for almost all $t \in I$ the inequalities*

$$f_i(x_1,\dots,x_n)\operatorname{sgn}(\sigma_i x_i(t)) \leq$$

$$\leq p_i(x_1,\dots,x_n)(t)\Big(-|x_i(t)| + \sum_{k=1}^{n} h_{ik}\|x_k\|_C + h_i\Big) \ \ (i = 1,\dots,n),$$

$$\int_a^b p_i(x_1,\dots,x_n)(s)\,ds > 0 \ \ (i = 1,\dots,n)$$

hold. If, moreover, the numbers α_i, σ_i satisfy the inequalities

$$(1 - |\alpha_i|)\sigma_i \geq 0 \ \ (i = 1,\dots,n),$$

and the matrix $H = (h_{ik})_{i,k=1}^n$ satisfies the condition (10)*, then the problem* (1), (2$_1$) *has at least one solution.*

Note that in both Theorems 1 and 2 the condition (10) is unimprovable in the sense that it cannot be replaced by the non-strict inequality

$$r(H) \leq 1. \tag{10$'$}$$

Indeed, it is clear that the periodic problem

$$\frac{dx_i(t)}{dt} = -\sigma_i x_i(t) + \|x_i\|_C + 1, \quad x_i(a) = x_i(b) \ \ (i = 1,\dots,n)$$

has no solution, though for this problem all the conditions of Theorem 2 are satisfied except the condition (10), instead of which the inequality (10$'$) holds, since in that case $H = E$, $r(H) = 1$.

Let us now consider the boundary value problem (4), (2$_1$).

For this problem from Theorem 2 we get

Corollary 1. *Let on the set $I \times \mathbb{R}^{n+1}$ the inequalities*

$$g_i(t,y_1,\dots,y_n,y_{n+1})\operatorname{sgn}(\sigma_i y_{n+1}) \leq$$

$$\leq p_i(t,y_1,\dots,y_n,y_{n+1})\Big(-|y_{n+1}| + \sum_{k=1}^{n} h_{ik}|y_k| + h_i\Big) \ \ (i = 1,\dots,n)$$

hold, where $p_i : I \times \mathbb{R}^{n+1} \to]-\infty,[$ $(i = 1,\dots,n)$ are functions from the Carathéodory class, h_{ik} and h_i are non-negative constants, and $\sigma_i \in \{-1,1\}$. If, moreover, the inequalities

$$(1 - |\alpha_i|)\sigma_i \geq 0 \ \ (i = 1,\dots,n), \quad r(H) < 1$$

are satisfied, where $H = (h_{ik})_{i,k=1}^n$, then the problem (4), (2$_1$) *has at least one solution.*

As it is noted above, the theorems proven by us cover the cases where the system (1) is superlinear or sublinear or some of equations of these systems are superlinear, and others are sublinear.

Indeed, suppose the equalities

$$g_i(t, y_1, \dots, y_n, y_{n+1}) =$$
$$= p_i(t) \exp\Big(\beta_i \sum_{k=1}^{n+1} |y_k|\Big)\big(- \sigma_i y_{n+1} + g_{0i}(t, y_1, \dots, y_n, y_{n+1})\big) \ \ (i = 1, \dots, n)$$

hold, where $\beta_i \in \mathbb{R}$, $\sigma_i \in \{-1, 1\}$, $p_i \in L(I; R_+)$ $(i = 1, \dots, n)$, and $g_{0i} : I \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous bounded functions. If, moreover, $\sigma_i(1 - |\alpha_i| \geq 0$ $(i = 1, \dots, n)$, then according to Corollary 1 the problem $(4), (2_1)$ has at least one solution. On the other hand, the i-th equation of the system (1) is superlinear if $\beta_i > 0$, and sublinear if $\beta_i < 0$. Note that in these cases the problem $(4), (2_1)$, generally speaking, is a problem at resonance since if $\alpha_i = 1$ for some $i \in \{1, \dots, n\}$, then the linear homogeneous problem $\frac{dx_i(t)}{dt} = 0, \quad x_i(a) = \alpha_i x_i(b) \ \ (i = 1, \dots, n)$ has an infinite set of solutions.

Finally, consider the problem (1),(2_2), where $t_{ijk} \in I$, $\ell_{ij} \in R$, $c_i \in R$. Put

$$\ell_{ik} = \sum_{j=1}^{n} |\ell_{ijk}|.$$

For this problem Theorem 1 takes the form

Theorem 3. *Let there exist operators $p_i : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}_+)$ $(i = 1, \dots, n)$, non-negative numbers h_{ik}, h_i $(i = 1, \dots, n)$, and functions q_{ik} and $q_i \in L(I; \mathbb{R}_+)$ $(i, k = 1, \dots, n)$ such that for any $(x_k)_{k=1}^n \in C(I; \mathbb{R}^n)$ almost everywhere on I, the inequalities*

$$f_i(x_1, \dots, x_n) \operatorname{sgn}(\sigma_i x_i(t)) \leq p_i(x_1, \dots, x_n)(t)\Big(- |x_i(t)| +$$
$$+ \sum_{k=1}^{n} h_{ik} \|x_k\|_C + h_i\Big) + \sum_{k=1}^{n} q_{ik}(t) \|x_k\|_C + q_i(t) \ \ (i = 1, \dots, n)$$

hold. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L + \ell_{ik})_{i,k=1}^n$ satisfies the condition (10)*, then the problem* $(1), (2_2)$ *has at least one solution.*

For the boundary value problem $(4), (2_2)$ this theorem yields

Corollary 2. *Let on the set $I \times \mathbb{R}^n$ the inequalities*

$$g_i(t, y_1, \dots, y_n, y_{n+1}) \operatorname{sgn}\big((t - t_i) y_{n+1}\big) \leq p_i(t, y_1, \dots, y_n, y_{n+1})\Big(- |y_{n+1}| +$$
$$+ \sum_{k=1}^{n} h_{ik} \|y_k\| + h_i\Big) + \sum_{k=1}^{n} q_{ik}(t) \|y_k\| + q_i(t) \ \ (i = 1, \dots, n)$$

be fulfilled, where $p_i : I \times \mathbb{R}^n \to]-\infty, 0[$ $(i = 1, \dots, n)$ are functions from the Carathéodory class, h_{ik} and h_i are non-negative constants, q_{ik} and $q_i \in L(I; \mathbb{R}_+)$. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L + \ell_{ik})_{i,k=1}^n$ satisfies the condition (10)*, then the problem* $(1), (2_2)$ *has at least one solution.*

The above-formulated theorems are a generalization of I. Kiguradze's results [10] for the system (1). They are proved using the results of the papers [9], [11], [13].

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