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MATHEMATICAL PROBLEMS OF THERMOELASTICITY FOR HEMITROPIC SOLIDS

Dedicated to Mikheil Basheleishvili on the occasion of his 80-th birthday

Abstract. The purpose of this paper is to construct explicitly, in terms of elementary functions, fundamental matrices of solutions to the differential equations of the elasticity theory of hemitropic materials with regard to thermal stresses. We consider the differential equations corresponding to the static equilibrium case, and also to the pseudo-oscillations and steady state oscillations cases. We derive the corresponding Green's formulas and construct the integral representation formulas of solutions by means of generalized layer and Newtonian potentials. We formulate the basic boundary value problems in appropriate function spaces and prove the uniqueness theorems. We develop the potential method and prove the existence and regularity theorems for basic and mixed type boundary value problems.

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[illegible]

1. Introduction

The main goal of our investigation is to study the basic boundary value and initial boundary value problems of the theory of elasticity for bodies with complex inner structure. Technological and industrial developments, and also great success in biological and medical sciences require to use more generalized and refined models for elastic bodies. In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or *Cosserat solids*. They model composites with a complex inner structure whose material particles have 6 degree of freedom (3 displacement components and 3 microrotation components). Recall that the classical elasticity theory allows only 3 degrees of freedom (3 displacement components).

Experiments have shown that micropolar materials possess quite different properties in comparison with the classical elastic materials (see, e.g., [2], [3], [6], [13], [29], and the references therein). For example, in noncentrosymmetric micropolar materials (which are called also as hemitropic or chiral materials) there propagate the left-handed and right-handed elastic waves. Moreover, the twisting behaviour under an axial stress is a purely hemitropic (chiral) phenomenon and has no counterpart in classical elasticity. Note that hemitropic materials are not isotropic with respect to inversion, i.e., they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules - DNA, as well as on a large scale, as in composites with helical or screw–shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. For a wider overview of the subject concerning different areas of applications we refer to the references [2], [3], [13], [22], [55], [52], [28], [29], [31], [32], [40], [48], [11], [60].

Mathematical models describing the chiral properties of elastic hemitropic materials have been proposed by Aero and Kuvshinski [2], [3] (for the historical notes see also [13], [48], [11], and the references therein).

Our main goal is to investigate mathematical problems of the *elasticity theory for hemitropic solids with regard to thermal effects*. In this case, beside the above mentioned displacement and microrotation vectors the field equations contain the *temperature distribution scalar function* (see, e.g., [11]). Note that in the theory of hemitropic elasticity there are introduced the asymmetric *force stress tensor* and *moment stress tensor*, which are related with the asymmetric *strain tensor, torsion* (*curvature*) *tensor* and temperature function via the constitutive equations. All these mechanical quantities are expressed in terms of the components of the displacement vector, microrotation vector and temperature distribution. In turn the displacement vector, microrotation vector and temperature function satisfy a coupled complex system of second order partial differential equations of dynamics. When the mechanical characteristics do not depend on time variable t we have the differential equations of statics. If the characteristics are time harmonic dependent (i.e., they are represented as the product of the time dependent exponential function $\exp\{-i\sigma t\}$ and a function of the spatial variable $x \in \mathbb{R}^3$) then we have the steady state oscillation equations. Here σ is a real frequency parameter. Note that if $\sigma = 0$, then we obtain the equations of statics. If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter, then we have the so called pseudo-oscillation equations (which are related to the dynamical equations via Laplace transform). The corresponding simultaneous equations generate 7×7 strongly elliptic, formally non-self-adjoint differential operators with constant coefficients involving 14 material parameters.

In this paper, first we collect the field equations and introduce the corresponding matrix operators. Afterwards, we derive the corresponding Green's formulas and formulate the basic boundary value problems. Further, we construct the matrices of fundamental solutions explicitly, in terms of elementary functions, for the differential operators of statics, steady state oscillations and pseudo-oscillations. We formulate the generalized Sommerfeld–Kupradze type radiation conditions which play a crucial role to establish the uniqueness results in the case of exterior boundary value problems (BVP). Applying the theory of pseudodifferential equations and the potential method we investigate the basic and mixed type BVPs and prove the corresponding uniqueness and existence theorems in Hölder, Bessel potential and Besov spaces. We study the smoothness properties of solutions and derive almost the best regularity results for mixed type BVPs.

The basic boundary value problems (BVPs) corresponding to the model when thermal effects are not taken into consideration are well investigated for general domains of arbitrary shape and the uniqueness and existence theorems are proved, and regularity results for solutions are established by potential methods as well as by variational methods (see [11], [51], [44], [45], [46], and the references therein).

2. Field Equations

2.1. **Constitutive relations and basic differential equations.** Denote by $\mathbb{R}^3$ the three-dimensional Euclidean space and let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a boundary $S := \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$. Further, let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We assume that $\overline{\Omega} \in \{\overline{\Omega^+}, \overline{\Omega^-}\}$ is filled with an elastic material possessing the hemitropic properties.

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the displacement vector and the microrotation vector, respectively. By ϑ we denote the temperature increment – temperature distribution function. Here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the microrotation vector in the hemitropic elasticity theory is kinematically distinct from the macrorotation vector $\frac{1}{2}$ curl u.

Throughout the paper the central dot denotes the real scalar product, i.e., $a \cdot b := \sum\limits_{k=1}^{N} a_k\, b_k$ for $a, b \in \mathbb{C}^N$.

The force stress $\{\tau_{pq}\}$ and the couple stress $\{\mu_{pq}\}$ tensors in the linear theory of hemitropic thermoelasticity read as follows (the constitutive equations)

$$\tau_{pq} = \tau_{pq}(U) := (\mu+\alpha)\partial_p u_q + (\mu-\alpha)\partial_q u_p + \lambda\delta_{pq}\operatorname{div} u + \delta\delta_{pq}\operatorname{div}\omega + \\ + (\varkappa+\nu)\partial_p\omega_q + (\varkappa-\nu)\partial_q\omega_p - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k - \delta_{pq}\eta\vartheta, \tag{2.1}$$

$$\mu_{pq} = \mu_{pq}(U) := \delta\delta_{pq}\operatorname{div} u + (\varkappa+\nu)\Big[\partial_p u_q - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\Big] + \beta\delta_{pq}\operatorname{div}\omega + \\ + (\varkappa-\nu)\Big[\partial_q u_p - \sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\Big] + (\gamma+\varepsilon)\partial_p\omega_q + (\gamma-\varepsilon)\partial_q\omega_p - \delta_{pq}\zeta\vartheta, \tag{2.2}$$

where $U = (u, \omega, \vartheta)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi–Civitá) symbol, and α, β, γ, δ, λ, μ, ν, $\varkappa$, and ε are the material constants, while $\eta > 0$ and $\zeta > 0$ are constants describing the coupling of mechanical and thermal fields (see [2], [11]), $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$.

The linear equations of dynamics of the thermoelasticity theory of hemitropic materials have the form (see, e.g., [11])

$$\sum_{p=1}^{3}\partial_p\tau_{pq}(x,t) + \varrho F_q(x,t) = \varrho\,\frac{\partial^2 u_q(x,t)}{\partial t^2}, \quad q = 1,2,3,$$

$$\sum_{p=1}^{3}\partial_p\mu_{pq}(x,t) + \sum_{l,r=1}^{3}\varepsilon_{qlr}\tau_{lr}(x,t) + \varrho G_q(x,t) = \mathcal{I}\,\frac{\partial^2\omega_q(x,t)}{\partial t^2}, \quad q = 1,2,3,$$

$$\kappa'\Delta\vartheta(x,t) - \eta\,\frac{\partial}{\partial t}\operatorname{div} u(x,t) - \zeta\,\frac{\partial}{\partial t}\operatorname{div}\omega(x,t) - \kappa''\,\frac{\partial}{\partial t}\vartheta(x,t) + Q(x,t) = 0,$$

where t is the time variable, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the body force and body couple vectors per unit mass, Q is the heat source density, ϱ is the mass density of the elastic material, and $\mathcal{I}$ is a constant characterizing the so called spin torque corresponding to the interior microrotations (i.e., the moment of inertia per unit volume); here $\kappa' = \frac{\lambda_0}{T_0}$ and $\kappa'' = \frac{c_0}{T_0}$, where $\lambda_0 > 0$ is the heat conduction coefficient, $T_0 > 0$ is a natural state temperature and $c_0 > 0$ is the specific heat coefficient.

Using the relations (2.1)–(2.2) we can rewrite the above dynamic equations as

$$\begin{gathered}(\mu+\alpha)\Delta u(x,t)+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,u(x,t)+(\varkappa+\nu)\Delta\,\omega(x,t)+\\ +(\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,\omega(x,t)+2\alpha\,\mathrm{curl}\,\omega(x,t)-\\ -\eta\,\mathrm{grad}\,\vartheta(x,t)+\varrho F(x,t)=\varrho\,\frac{\partial^2 u(x,t)}{\partial t^2}\,,\\ (\varkappa+\nu)\Delta u(x,t)+(\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,u(x,t)+2\alpha\,\mathrm{curl}\,u(x,t)+\\ +(\gamma+\varepsilon)\Delta\omega(x,t)+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\omega(x,t)+4\nu\,\mathrm{curl}\,\omega(x,t)-\\ -4\alpha\,\omega(x,t)-\zeta\,\mathrm{grad}\,\vartheta(x,t)+\varrho G(x,t)=\mathcal{I}\,\frac{\partial^2\omega(x,t)}{\partial t^2}\,,\\ \kappa'\Delta\vartheta(x,t)-\eta\,\frac{\partial}{\partial t}\,\mathrm{div}\,u(x,t)-\zeta\,\frac{\partial}{\partial t}\,\mathrm{div}\,\omega(x,t)-\\ -\kappa''\,\frac{\partial}{\partial t}\,\vartheta(x,t)+Q(x,t)=0\,,\end{gathered}\tag{2.3}$$

where Δ is the Laplace operator.

If all the quantities involved in these equations are harmonic time dependent, i.e., $u(x,t)=u(x)\exp\{-it\sigma\}$, $\omega(x,t)=\omega(x)\exp\{-it\sigma\}$, $\vartheta(x,t)=\vartheta(x)\exp\{-it\sigma\}$, $F(x,t)=F(x)\exp\{-it\sigma\}$, $G(x,t)=G(x)\exp\{-it\sigma\}$ and $Q(x,t)=Q(x)\exp\{-it\sigma\}$ with $\sigma\in\mathbb{R}$ and $i=\sqrt{-1}$, we obtain the *steady state oscillation equations* of the hemitropic theory of thermoelasticity:

$$\begin{gathered}(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,u(x)+(\varkappa+\nu)\Delta\omega(x)+\\ +(\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,\omega(x)+2\alpha\,\mathrm{curl}\,\omega(x)-\\ -\eta\,\mathrm{grad}\,\vartheta(x)+\varrho\sigma^2 u(x)=-\varrho F(x),\\ (\varkappa+\nu)\Delta u(x)+(\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,u(x)+2\alpha\,\mathrm{curl}\,u(x)+\\ +(\gamma+\varepsilon)\Delta\omega(x)+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\omega(x)+4\nu\,\mathrm{curl}\,\omega(x)-\\ -\zeta\,\mathrm{grad}\,\vartheta(x)+(\mathcal{I}\sigma^2-4\alpha)\omega(x)=-\varrho G(x),\\ (\kappa'\Delta+i\sigma\kappa'')\vartheta(x)+i\eta\sigma\,\mathrm{div}\,u(x)+i\zeta\sigma\,\mathrm{div}\,\omega(x)=-Q(x);\end{gathered}\tag{2.4}$$

here u, ω, F, and G are complex-valued vector functions, while ϑ and Q are complex-valued scalar functions, and σ is a frequency parameter.

If $\sigma=\sigma_1+i\sigma_2$ is a complex parameter with $\sigma_2\neq 0$, then the above equations are called the *pseudo–oscillation equations*, while for $\sigma=0$ they represent the *equilibrium equations of statics*.

Let us introduce the block wise 7×7 matrix differential operator corresponding to the system (2.4):

$$L(\partial,\sigma):=\begin{bmatrix}L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) & L^{(5)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma) & L^{(6)}(\partial,\sigma)\\ L^{(7)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(9)}(\partial,\sigma)\end{bmatrix}_{7\times 7}\,,\tag{2.5}$$

where

$$\begin{gathered}
L^{(1)}(\partial,\sigma) := \big[(\mu+\alpha)\Delta+\varrho\sigma^2\big]I_3+(\lambda+\mu-\alpha)Q(\partial),\\
L^{(2)}(\partial,\sigma)=L^{(3)}(\partial,\sigma) := (\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial)+2\alpha R(\partial),\\
L^{(4)}(\partial,\sigma) := \big[(\gamma+\varepsilon)\Delta+(\mathcal{I}\sigma^2-4\alpha)\big]I_3+(\beta+\gamma-\varepsilon)Q(\partial)+4\nu R(\partial),\\
L^{(5)}(\partial,\sigma) := -\eta\nabla^\top,\ \ L^{(6)}(\partial,\sigma) := -\zeta\nabla^\top,\ \ L^{(7)}(\partial,\sigma) := i\eta\sigma\nabla,\\
L^{(8)}(\partial,\sigma) := i\zeta\sigma\nabla,\ \ L^{(9)}(\partial,\sigma) := \kappa'\Delta+i\sigma\kappa''.
\end{gathered} \tag{2.6}$$

Here and in the sequel I_k stands for the $k\times k$ unit matrix and

$$\begin{gathered}
R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2\\ \partial_3 & 0 & -\partial_1\\ -\partial_2 & \partial_1 & 0\end{bmatrix}_{3\times3},\\
Q(\partial) := [\partial_k\partial_j]_{3\times3},\quad \nabla\equiv\nabla(\partial) := [\partial_1,\partial_2,\partial_3].
\end{gathered} \tag{2.7}$$

It is easy to see that for $v=(v_1,v_2,v_3)^\top$

$$R(\partial)v = \begin{bmatrix}\partial_2v_3-\partial_3v_2\\ \partial_3v_1-\partial_1v_3\\ \partial_1v_2-\partial_2v_1\end{bmatrix} = \operatorname{curl} v,\quad Q(\partial)v=\operatorname{grad}\operatorname{div} v, \tag{2.8}$$

$$\begin{gathered}
R(-\partial)=-R(\partial)=[R(\partial)]^\top,\quad Q(\partial)R(\partial)=R(\partial)Q(\partial)=0,\\
Q(\partial)=[Q(\partial)]^\top,\quad [R(\partial)]^2=Q(\partial)-\Delta I_3,\quad [Q(\partial)]^2=Q(\partial)\Delta.
\end{gathered} \tag{2.9}$$

Due to the above notation, the system (2.4) can be rewritten in matrix form as

$$L(\partial,\sigma)U(x)=\Phi(x),\ \ U=(u,\omega,\vartheta)^\top,\ \ \Phi=(-\varrho F,-\varrho G,-Q)^\top. \tag{2.10}$$

Note that $L(\partial,\sigma)$ is not formally self-adjoint. Further, let us remark that the differential operator

$$L(\partial) := L(\partial,0) \tag{2.11}$$

corresponds to the static equilibrium case, while the formally self-adjoint differential operator

$$L_0(\partial) := \begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial) & [0]_{3\times1}\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial) & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & \kappa'\Delta\end{bmatrix}_{7\times7} \tag{2.12}$$

with

$$\begin{gathered}
L_0^{(1)}(\partial) := (\mu+\alpha)\Delta I_3+(\lambda+\mu-\alpha)Q(\partial),\\
L_0^{(2)}(\partial)=L_0^{(3)}(\partial) := (\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial),\\
L_0^{(4)}(\partial) := (\gamma+\varepsilon)\Delta I_3+(\beta+\gamma-\varepsilon)Q(\partial),
\end{gathered} \tag{2.13}$$

represents the principal homogeneous part of the operators (2.5) and (2.11).

Denote

$$\begin{aligned}\widetilde{L}(\partial,\sigma) &:= \begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma)\end{bmatrix}_{6\times 6},\\ \widetilde{L}_0(\partial) &:= \begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial) \\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial)\end{bmatrix}_{6\times 6}.\end{aligned} \tag{2.14}$$

These operators correspond to the equations of hemitropic elasticity when thermal effects are not taken into consideration ([44]). It is clear that the operator $L_0(\partial)$, $\widetilde{L}(\partial,\sigma)$ and $\widetilde{L}_0(\partial)$ are formally self-adjoint.

2.2. **Generalized stress operators.** The components of the force stress vector $\tau^{(n)}$ and the coupled stress vector $\mu^{(n)}$, acting on a surface element with a normal vector $n=(n_1,n_2,n_3)$, read as

$$\tau_q^{(n)} = \sum_{p=1}^3 \tau_{pq} n_p, \quad \mu_q^{(n)} = \sum_{p=1}^3 \mu_{pq} n_p, \quad q=1,2,3. \tag{2.15}$$

It is also well known that the normal component of the heat flux vector across a surface element with a normal vector $n=(n_1,n_2,n_3)$ is expressed by the normal derivative of the temperature function

$$\kappa' n\cdot\nabla\vartheta = \kappa'\sum_{p=1}^3 n_p\partial_p\vartheta = \kappa'\partial_n\vartheta, \tag{2.16}$$

where $\partial_n=\partial/\partial n$ denotes the usual normal derivative.

Throughout the paper we will refer the six vector $(\tau^{(n)},\mu^{(n)})^\top$ as the *mechanical thermo-stress vector*, while the seven vector $(\tau^{(n)},\mu^{(n)},\kappa'\partial_n\vartheta)^\top$ as *generalized thermo-stress vector*.

Let us introduce the generalized thermo-stress operators

$$\mathcal{T}(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \end{bmatrix}_{6\times 7}, \tag{2.17}$$

$$\mathcal{P}(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}, \tag{2.18}$$

where

$$\begin{aligned} T^{(j)} &= [T_{pq}^{(j)}]_{3\times 3}, \quad j=\overline{1,4}, \quad n^\top=(n_1,n_2,n_3)^\top,\\ T_{pq}^{(1)}(\partial,n) &= (\mu+\alpha)\delta_{pq}\partial_n + (\mu-\alpha)n_q\partial_p + \lambda n_p\partial_q,\\ T_{pq}^{(2)}(\partial,n) &= (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha\sum_{k=1}^3 \varepsilon_{pqk}n_k,\\ T_{pq}^{(3)}(\partial,n) &= (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q,\\ T_{pq}^{(4)}(\partial,n) &= (\gamma+\varepsilon)\delta_{pq}\partial_n + (\gamma-\varepsilon)n_q\partial_p + \beta n_p\partial_q - 2\nu\sum_{k=1}^3 \varepsilon_{pqk}n_k. \end{aligned} \tag{2.19}$$

It can be easily checked that for an arbitrary vector $U=(u,\omega,\vartheta)^\top$ we have

$$\mathcal{T}(\partial,n)U=(\tau^{(n)},\mu^{(n)})^\top,\quad \mathcal{P}(\partial,n)U=(\tau^{(n)},\mu^{(n)},\kappa'\partial_n\vartheta)^\top,$$

i.e., the six vector $\mathcal{T}(\partial,n)U$ corresponds to the mechanical thermo-stress vector and the seven vector $\mathcal{P}(\partial,n)U$ corresponds to the generalized thermo-stress vector.

Further, let us introduce the associated boundary operators which occur in Green's formulas and are related to the adjoint differential operator $L^*(\partial,\sigma):=L^\top(-\partial,\sigma)$:

$$\mathcal{T}^*(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta_0 n^\top\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta_0 n^\top\end{bmatrix}_{6\times 7}, \tag{2.20}$$

$$\mathcal{P}^*(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta_0 n^\top\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta_0 n^\top\\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n\end{bmatrix}_{7\times 7}. \tag{2.21}$$

Note that the principal homogeneous parts of the operators $\mathcal{T}(\partial,n)$ and $\mathcal{T}^*(\partial,n)$ are the same, as well as the principal homogeneous parts of the operators $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$.

Note that when the thermal effects are not taken into consideration the hemitropic stress operator reads as [44]

$$T(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6}. \tag{2.22}$$

Evidently, for $U=(u,\omega,0)$ and $\widetilde{U}=(u,\omega)$ we have $\mathcal{T}(\partial,n)U=T(\partial,n)\widetilde{U}$ in view of (2.17) and (2.22).

2.3. **Green's formulae.** For vector functions $\widetilde{U}=(u,\omega)^\top$, $\widetilde{U}'=(u',\omega')^\top\in[C^2(\overline{\Omega^+})]^6$, we have the following Green formula [44]

$$\int\limits_{\Omega^+}\big[\widetilde{U}'\cdot\widetilde{L}(\partial,0)\widetilde{U}+E(\widetilde{U}',\widetilde{U})\big]\,dx=\int\limits_{\partial\Omega^+}\{\widetilde{U}'\}^+\cdot\{T(\partial,n)\widetilde{U}\}^+\,dS, \tag{2.23}$$

where the operators $\widetilde{L}(\partial,0)$ and $T(\partial,n)$ are given by (2.14) and (2.22) respectively, $\partial\Omega^+$ is a piecewise smooth manifold, n is the outward unit normal vector to $\partial\Omega^+$, the symbols $\{\,\cdot\,\}^\pm$ denote the limiting values on $\partial\Omega^\pm$ from $\Omega^\pm$ respectively, $E(\cdot\,,\,\cdot)$ is the so called *energy bilinear form*,

$$\begin{aligned}E(\widetilde{U}',\widetilde{U})=E(\widetilde{U},\widetilde{U}')=\sum_{p,q=1}^{3}\Big\{&(\mu+\alpha)u'_{pq}u_{pq}+(\mu-\alpha)u'_{pq}u_{qp}+\\ &+(\varkappa+\nu)(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq})+(\varkappa-\nu)(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp})+(\gamma+\varepsilon)\omega'_{pq}\omega_{pq}+\\ &+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp}+\delta(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp})+\lambda u'_{pp}u_{qq}+\beta\omega'_{pp}\omega_{qq}\Big\}\end{aligned} \tag{2.24}$$

with

$$u_{pq}=\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k,\ \ \omega_{pq}=\partial_p\omega_q,\ \ p,q=1,2,3. \tag{2.25}$$

In what follows the over bar denotes complex conjugation. The necessary and sufficient conditions for the quadratic form $E(\widetilde{U},\overline{\widetilde{U}})$ to be positive definite are the following inequalities (see [3], [11], [17])

$$\begin{gathered}\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\\ \lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\\ (\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\ (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\ \ (\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0,\\ \mu\big[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2\big]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\\ \mu\big[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2\big]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.\end{gathered} \tag{2.26}$$

Let us note that, if the condition $3\lambda+2\mu>0$ is fulfilled, which is very natural in the classical elasticity, then the above conditions are equivalent to the following simultaneous inequalities

$$\begin{gathered}\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ 3\lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\\ (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0.\end{gathered} \tag{2.27}$$

For simplicity in what follows we assume that $3\lambda+2\mu>0$ and therefore the conditions (2.27) imply positive definiteness of the energy quadratic form $E(\widetilde{U},\overline{\widetilde{U}})$ defined by (2.24).

From (2.27) it follows that

$$\begin{gathered}\gamma>0,\ \ \varepsilon>0,\ \ \lambda+\mu>0,\ \ \beta+\gamma>0,\\ d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\ d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0.\end{gathered} \tag{2.28}$$

The formula (2.24) can be rewritten as

$$\begin{aligned}E(\widetilde{U},\widetilde{U}')=&\frac{3\lambda+2\mu}{3}\Big(\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div}u'+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega'\Big)+\\ &+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega')+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega\cdot\operatorname{curl}\omega'+\\ &+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big]\Big[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\Big)\Big]+\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big]\Big[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\Big)\Big]+\end{aligned}$$

$$+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\Big)+$$
$$+\frac{1}{3}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\Big)\Big]+$$
$$+\alpha\Big(\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big)\cdot\Big(\operatorname{curl}u'+\frac{\nu}{\alpha}\operatorname{curl}\omega'-2\omega'\Big). \tag{2.29}$$

In particular,

$$E(\widetilde{U},\overline{\widetilde{U}})=\frac{3\lambda+2\mu}{3}\Big|\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big|^2+$$
$$+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)|\operatorname{div}\omega|^2+$$
$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\Big|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big|^2+$$
$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big|^2+$$
$$+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{1}{2}\Big|\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big|^2+\frac{1}{3}\Big|\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big|^2\Big]+$$
$$+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)|\operatorname{curl}\omega|^2+\alpha\Big|\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big|^2. \tag{2.30}$$

We formulate here the following technical lemma.

Lemma 2.1. *Let $\widetilde{U}=(u,\omega)^\top\in[C^1(\Omega^+)]^6$ be a complex–valued vector and $E(\widetilde{U},\overline{\widetilde{U}})=0$ in Ω^+. Then*

$$u(x)=[a\times x]+b,\qquad \omega(x)=a,\quad x\in\Omega^+, \tag{2.31}$$

where a and b are arbitrary three-dimensional constant complex vectors.

Moreover,

- (i) *for an arbitrary vector $\widetilde{U}=(u,\omega)^\top$ defined by formulas* (2.31) *and an arbitrary unit vector $n=(n_1,n_2,n_3)$ the generalized hemitropic stress vector $T(\partial,n)\widetilde{U}$ vanishes identically, i.e., $T(\partial,n)\widetilde{U}(x)=0$ for all $x\in\Omega^+$.*
- (ii) *for an arbitrary vector $U:=(\widetilde{U},0)^\top=(u,\omega,0)^\top$, where u and ω are given by formulas* (2.31)*, and an arbitrary unit vector $n=(n_1,n_2,n_3)$ the generalized hemitropic thermo-stress vector $\mathcal{P}(\partial,n)U$ vanishes identically, i.e., $\mathcal{P}(\partial,n)U(x)=0$ for all $x\in\Omega^+$.*

Proof. The first part of the lemma is shown in [44]. The second part easily follows from the first part and from the formulas (2.17), (2.18), (2.22). □

Throughout the paper L_p, W^s_p, H^s_p, and $B^s_{p,q}$ (with $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$) denote the well-known Lebesgue, Sobolev–Slobodetski, Bessel potential, and Besov spaces, respectively (see, e.g., [56], [57], [36]). We recall that $H_2^s = W_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Further, let $\mathcal{M}_0$ be a Lipschitz surface without boundary. For a Lipschitz sub-manifold $\mathcal{M} \subset \mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M}) = \big\{ g: \; g \in H_p^s(\mathcal{M}_0), \; \operatorname{supp} g \subset \overline{\mathcal{M}} \big\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M}) = \big\{ g: \; g \in B_{p,q}^s(\mathcal{M}_0), \; \operatorname{supp} g \subset \overline{\mathcal{M}} \big\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M}) = \big\{ r_{\mathcal{M}} f: \; f \in H_p^s(\mathcal{M}_0) \big\},$$
$$B_{p,q}^s(\mathcal{M}) = \big\{ r_{\mathcal{M}} f: \; f \in B_{p,q}^s(\mathcal{M}_0) \big\}.$$

Here $r_{\mathcal{M}}$ is the restriction operator.

If $\widetilde{U} = \widetilde{U}^{(1)} + i\widetilde{U}^{(2)}$ is a complex–valued vector, where $\widetilde{U}^{(j)} = (u^{(j)}, \omega^{(j)})^\top$ $(j = 1, 2)$ are real-valued vectors, then

$$E(\widetilde{U}, \overline{\widetilde{U}}) = E(\widetilde{U}^{(1)}, \widetilde{U}^{(1)}) + E(\widetilde{U}^{(2)}, \widetilde{U}^{(2)}),$$

and, due to the positive definiteness of the energy form for real–valued vector functions, we have

$$E(\widetilde{U}, \overline{\widetilde{U}}) \geq c^* \sum_{p,q=1}^{3} \Big[(u_{pq}^{(1)})^2 + (u_{pq}^{(2)})^2 + (\omega_{pq}^{(1)})^2 + (\omega_{pq}^{(2)})^2 \Big], \tag{2.32}$$

where c^* is a positive constant depending only on the material constants, and $u_{pq}^{(j)}$ and $\omega_{pq}^{(j)}$ are defined by formulae (2.25) with $u^{(j)}$ and $\omega^{(j)}$ for u and ω.

From the positive definiteness of the energy form $E(\cdot,\cdot)$ with respect to the variables (2.25) it easily follows that there exist positive constants c_1 and c_2 such that for an arbitrary real–valued vector $U \in [C^1(\overline{\Omega^+})]^6$

$$\begin{aligned} \widetilde{\mathcal{B}}(\widetilde{U}, \widetilde{U}) := \int_{\Omega^+} E(\widetilde{U}, \overline{\widetilde{U}})\, dx \geq \\ \geq c_1 \int_{\Omega^+} \bigg\{ \sum_{p,q=1}^{3} \big[(\partial_p u_q)^2 + (\partial_p \omega_q)^2 \big] + \sum_{p=1}^{3} [u_p^2 + \omega_p^2] \bigg\}\, dx - \\ - c_2 \int_{\Omega^+} \sum_{p=1}^{3} [u_p^2 + \omega_p^2]\, dx, \end{aligned} \tag{2.33}$$

i.e., the following Korn's type inequality holds (cf. [15, Part I, § 12], [37, Ch. 10])

$$\widetilde{\mathcal{B}}(\widetilde{U}, \widetilde{U}) \geq c_1 \|\widetilde{U}\|^2_{[H^1(\Omega^+)]^6} - c_2 \|\widetilde{U}\|^2_{[H^0(\Omega^+)]^6}, \tag{2.34}$$

where $\|\cdot\|_{[H^s(\Omega^+)]^6}$ denotes the norm in the Sobolev space $[H^s(\Omega^+)]^6$.

These results imply that the differential operators $\widetilde{L}(\partial,\sigma)$ and $\widetilde{L}_0(\partial)$ are *strongly elliptic* and the following inequality (*the accretivity condition*) holds (cf., e.g., [15, Part I, § 5], [37, Ch. 4, Lemma 4.5])

$$c_2'|\xi|^2|\eta|^2 \geq \widetilde{L}_0(\xi)\eta\cdot\eta = \sum_{k,j=1}^{6} \{\widetilde{L}_0(\xi)\}_{kj}\eta_j\overline{\eta_k} \geq c_1'|\xi|^2|\eta|^2 \tag{2.35}$$

with some constants $c_k' > 0$ $(k=1,2)$ for arbitrary $\xi\in\mathbb{R}^3$ and arbitrary complex vector $\eta\in\mathbb{C}^6$.

Consequently, in view of (2.12) and (2.35) the differential operator $L(\partial,\sigma)$ is strongly elliptic as well, since

$$C_2'|\xi|^2|\eta|^2 \geq L_0(\xi)\eta\cdot\eta = \sum_{k,j=1}^{6} \{L_0(\xi)\}_{kj}\eta_j\overline{\eta_k} \geq C_1'|\xi|^2|\eta|^2 \tag{2.36}$$

with some constants $C_k' > 0$ $(k=1,2)$ for arbitrary $\xi\in\mathbb{R}^3$ and arbitrary complex vector $\eta\in\mathbb{C}^7$.

Now let $U = (\widetilde{U},\vartheta)^\top = (u,\omega,\vartheta)^\top \in [C^2(\overline{\Omega^+})]^7$, $U' = (\widetilde{U}',\vartheta')^\top = (u',\omega',\vartheta')^\top \in [C^2(\overline{\Omega^+})]^7$. With the help of relation (2.23) and standard manipulations we can show that the following Green's formulas hold

$$\begin{gathered}\int\limits_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx = \int\limits_{\partial\Omega^+} \{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+\,dS- \\ -\int\limits_{\Omega^+}\Big[E(\widetilde{U}',\widetilde{U}) - \varrho\sigma^2u'\cdot u - \mathcal{I}\sigma^2\omega'\cdot\omega - \eta\vartheta\,\mathrm{div}\,u' - \zeta\vartheta\,\mathrm{div}\,\omega'- \\ - i\eta\sigma\vartheta'\,\mathrm{div}\,u - i\zeta\sigma\vartheta'\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\vartheta' + \kappa'\,\mathrm{grad}\,\vartheta'\cdot\mathrm{grad}\,\vartheta\Big]\,dx,\end{gathered} \tag{2.37}$$

$$\begin{gathered}\int\limits_{\Omega^+}\big[U'\cdot L(\partial,\sigma)U - L^*(\partial,\sigma)U'\cdot U\big]\,dx = \\ = \int\limits_{\partial\Omega^+}\Big[\{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+ - \{\mathcal{P}^*(\partial,n)U'\}^+\cdot\{U\}^+\Big]\,dS,\end{gathered} \tag{2.38}$$

where the differential operator $L(\partial,\sigma)$ is given by (2.5), $L^*(\partial,\sigma)=L^\top(-\partial,\sigma)$ is the formally adjoint operator to $L(\partial,\sigma)$, the boundary operators $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$ are defined by (2.18) and (2.21) respectively. The proof of (2.37) and (2.38) easily follows from (2.23) and the identity

$$\begin{gathered}U'\cdot L(\partial,\sigma)U = \widetilde{U}'\cdot\widetilde{L}(\partial,0)\widetilde{U} + \varrho\sigma^2u'\cdot u - \eta\,\mathrm{grad}\,\vartheta\cdot u' + \mathcal{I}\sigma^2\omega'\cdot\omega- \\ -\zeta\,\mathrm{grad}\,\vartheta\cdot\omega' + \kappa'\vartheta'\Delta\vartheta + i\eta\sigma\vartheta'\,\mathrm{div}\,u + i\sigma\zeta\vartheta'\,\mathrm{div}\,\omega + i\sigma\kappa''\vartheta\vartheta'.\end{gathered}$$

By the standard limiting approach, Green's formula (2.37) can be extended to Lipschitz domains (see, e.g., [47], [37]) and to the case of complex-valued vector functions $U \in [W^1_p(\Omega^+)]^7$ and $U' \in [W^1_{p'}(\Omega^+)]^7$ with $1/p + 1/p' = 1$, $1 < p < \infty$, and $L(\partial,\sigma)U \in [L_p(\Omega^+)]^7$ (cf. [36], [7], [37])

$$\big\langle \{U'\}^+, \{\mathcal{P}(\partial,n)U\}^+\big\rangle_{\partial\Omega^+} = \int\limits_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx +$$
$$+\int\limits_{\Omega^+}\Big[E(\widetilde{U}',\widetilde{U}) - \varrho\sigma^2 u'\cdot u - \mathcal{I}\sigma^2\omega'\cdot\omega - \eta\vartheta\,\mathrm{div}\,u' - \zeta\vartheta\,\mathrm{div}\,\omega' -$$
$$- i\eta\sigma\vartheta'\,\mathrm{div}\,u - i\zeta\sigma\vartheta'\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\vartheta' + \kappa'\,\mathrm{grad}\,\vartheta'\cdot\mathrm{grad}\,\vartheta\Big]\,dx, \quad (2.39)$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the spaces $[B^{1/p}_{p,p}(\partial\Omega^+)]^7$ and $[B^{-1/p}_{p',p'}(\partial\Omega^+)]^7$, which extends the usual real L_2-scalar product for regular vector–functions, i.e., for $f,g \in [L_2(S)]^7$ we have

$$\langle f,g\rangle_S = \sum_{k=1}^{7}\int\limits_S f_k\,g_k\,dS = (f,g)_{[L_2(S)]^7}. \quad (2.40)$$

Clearly, the generalized trace functional $\{\mathcal{P}(\partial,n)U\}^+ \in [B^{-1/p}_{p,p}(\partial\Omega^+)]^7$ is correctly determined by the relation (2.39).

Let us introduce the sesquilinear form related to the operator $L(\partial,\sigma)$

$$\mathcal{B}(U,U') := \int\limits_{\Omega^+}\Big[E(\widetilde{U},\overline{\widetilde{U}'}) - \varrho\sigma^2 u\cdot\overline{u'} - \mathcal{I}\sigma^2\omega\cdot\overline{\omega'} - \eta\vartheta\,\mathrm{div}\,\overline{u'} - \zeta\vartheta\,\mathrm{div}\,\overline{\omega'} -$$
$$- i\eta\sigma\overline{\vartheta'}\,\mathrm{div}\,u - i\zeta\sigma\overline{\vartheta'}\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\overline{\vartheta'} + \kappa'\,\mathrm{grad}\,\vartheta\cdot\mathrm{grad}\,\overline{\vartheta'}\Big]\,dx. \quad (2.41)$$

With the help of (2.34) we derive the inequality

$$\mathcal{B}(U,U) \geq C_1\|U\|^2_{[H^1(\Omega^+)]^7} - C_2\|U\|^2_{[H^0(\Omega^+)]^7}, \quad (2.42)$$

with some positive constants C_1 and C_2. □

2.4. **Basic BVPs and uniqueness theorems for bounded domains.** We start with the formulation of the basic interior and exterior boundary value problems for the domains Ω^+ and $\Omega^- = \mathbb{R}\setminus\overline{\Omega^+}$. Let the boundary $S = \partial\Omega^\pm$ be divided into two disjoint submanifolds S_D and S_N such that $S_D \cap S_N = \varnothing$ and $\overline{S_D}\cup\overline{S_N} = S$. Put $\ell := \partial S_D = \partial S_N$.

Problem $(I^{(\sigma)})^\pm$ (Dirichlet problem). Find a solution vector $U = (u,\omega,\vartheta)^\top$ to the differential equation

$$L(\partial,\sigma)U(x) = \Phi^{(\pm)}(x), \quad x \in \Omega^\pm, \quad (2.43)$$

satisfying the boundary conditions

$$\{U(x)\}^\pm = f(x), \quad x \in S. \quad (2.44)$$

Problem $(II^{(\sigma)})^{\pm}$ (Neumann problem). Find a solution vector $U = (u, \omega, \vartheta)^{\top}$ to the equation (2.43) satisfying the boundary condition

$$\{\mathcal{P}(\partial, n)U(x)\}^{\pm} = F(x), \quad x \in S. \tag{2.45}$$

Problem $(III^{(\sigma)})^{\pm}$ (Mixed problem). Find a solution vector $U = (u, \omega, \vartheta)^{\top}$ to the equation (2.43) satisfying the boundary conditions

$$\{U(x)\}^{\pm} = f^{(D)}(x), \quad x \in S_D, \tag{2.46}$$

$$\{\mathcal{P}(\partial, n)U(x)\}^{\pm} = F^{(N)}(x), \quad x \in S_N. \tag{2.47}$$

Note that in contrast to the Dirichlet and Neumann BVPs, solutions to mixed BVPs, even for given C^{∞}-regular data, in general, are not in the Hölder space $[C^{\alpha}(\overline{\Omega^{+}})]^7$ with $\alpha > 1/2$ at the collision curve ℓ, while they are infinitely differentiable elsewhere. Therefore we investigate the mixed BVP in the Sobolev space $[W_p^1(\Omega^{+})]^7$. In the case of such generalized formulation we assume that the data of the BVPs belong to the natural function spaces,

$$\begin{gathered} \Phi^{(+)} \in [L_p(\Omega^{+})]^7, \;\; \Phi^{(-)} \in [L_{p,comp}(\Omega^{-})]^7, \\ f \in [B_{p,p}^{1-\frac{1}{p}}(S)]^7, \;\; f^{(D)} \in [B_{p,p}^{1-\frac{1}{p}}(S_D)]^7, \\ F \in [B_{p,p}^{-\frac{1}{p}}(S)]^7, \;\; F^{(N)} \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^7. \end{gathered} \tag{2.48}$$

The differential equation (2.43) is understood in the distributional or in the weak sense, the Dirichlet type conditions (2.44) and (2.46) are understood in the trace sense, and finally the Neumann type conditions (2.45) and (2.47) are understood in the generalized trace functional sense defined with the help of Green's identity (2.39).

In the case of the exterior problems for the domain Ω^{-} the solution vectors should satisfy some decay conditions at infinity. Namely, for pseudo-oscillation BVPs with $\Im\sigma = \sigma_2 > 0$ we assume that for sufficiently large $|x|$, i.e., as $|x| \to \infty$, the solution vectors and their derivatives are polynomially bounded. As we shall see below in Subsection 3.5, any solution of the differential equation (2.43) with compactly supported $\Phi^{(-)}$ actually decrease exponentially as $|x| \to \infty$. For the exterior BVPs of statics (i.e., when $\sigma = 0$) the conditions at infinity will be specified later in Section 4.

Now we prove the following uniqueness results.

Theorem 2.2. *Let $\sigma = \sigma_1 + i\sigma_2$ with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then the homogeneous boundary value problems $(I^{(\sigma)})^{+}$, $(II^{(\sigma)})^{+}$ and $(III^{(\sigma)})^{+}$ have only the trivial solution in the space $[W_2^1(\Omega^{+})]^7$.*

Proof. Let $U = (u, \omega, \vartheta)^{\top} \in [W_2^1(\Omega^{+})]^7$ be a solution of the homogeneous boundary value problem $(K^{(\sigma)})^{+}$, $K = I, II, III$. Since $L(\partial, \sigma)U = 0$ we can apply Green's formula of type (2.39). In particular, let us multiply the first vector equations in (2.4) by $\overline{u}$, the second equation by $\overline{\omega}$, the complex conjugate of the third scalar equation by ϑ and integrate their sum over Ω^{+}.

Taking into account the relations (2.5) and (2.6), with the help of Gauss formula and evident manipulations we obtain

$$\int\limits_{\Omega^+}\Big[L^{(1)}(\partial,\sigma)u\cdot\overline{u}+L^{(2)}(\partial,\sigma)\omega\cdot\overline{u}+L^{(5)}(\partial,\sigma)\vartheta\cdot\overline{u}+$$
$$+L^{(3)}(\partial,\sigma)u\cdot\overline{\omega}+L^{(4)}(\partial,\sigma)\omega\cdot\overline{\omega}+L^{(6)}(\partial,\sigma)\vartheta\cdot\overline{\omega}+$$
$$+C_0\overline{L^{(7)}(\partial,\sigma)u}\vartheta+C_0\overline{L^{(8)}(\partial,\sigma)\omega}\vartheta+C_0\overline{L^{(9)}(\partial,\sigma)\vartheta}\vartheta\Big]\,dx=$$
$$=\int\limits_{\Omega^+}\Big[\widetilde{L}(\partial,0)\widetilde{U}\cdot\overline{\widetilde{U}}+\varrho\sigma^2|u|^2+\mathcal{I}\sigma^2|\omega|^2-\eta\nabla\vartheta\cdot\overline{u}-\zeta\nabla\vartheta\cdot\overline{\omega}-$$
$$-C_0\big(i\eta\overline{\sigma}\vartheta\,\mathrm{div}\,\overline{u}+i\zeta\overline{\sigma}\vartheta\,\mathrm{div}\,\overline{\omega}-\kappa'\vartheta\Delta\overline{\vartheta}+i\overline{\sigma}\kappa''|\vartheta|^2\big)\Big]\,dx=$$
$$=\int\limits_{\Omega^+}\big[-E(\widetilde{U},\overline{\widetilde{U}})+\varrho\sigma^2|u|^2+\mathcal{I}\sigma^2|\omega|^2\big]\,dx+\int\limits_{\partial\Omega^+}T(\partial,n)\widetilde{U}\cdot\overline{\widetilde{U}}\,dS-$$
$$-\int\limits_{\partial\Omega^+}(\eta\vartheta n\cdot\overline{u}+\zeta\vartheta n\cdot\overline{\omega})\,dS+$$
$$+\int\limits_{\Omega^+}(1-iC_0\overline{\sigma})\big(\eta\vartheta\,\mathrm{div}\,\overline{u}+\zeta\vartheta\,\mathrm{div}\,\overline{\omega}\big)\,dx-$$
$$-C_0\int\limits_{\Omega^+}\big(\kappa'|\nabla\vartheta|^2+i\overline{\sigma}\kappa''|\vartheta|^2\big)\,dx+\kappa'C_0\int\limits_{\partial\Omega^+}\vartheta\partial_n\overline{\vartheta}\,dS$$

with $\widetilde{U}=(u,\omega)^\top\in[W_2^1(\Omega^+)]^6$ and an arbitrary constant C_0. In view of (2.17) and choosing

$$C_0=-\frac{i}{\overline{\sigma}}=\frac{\sigma_2-i\sigma_1}{|\sigma|^2}\,, \tag{2.49}$$

we arrive at the relation

$$\int\limits_{\Omega^+}\Big[L^{(1)}(\partial,\sigma)u\cdot\overline{u}+L^{(2)}(\partial,\sigma)\omega\cdot\overline{u}+L^{(5)}(\partial,\sigma)\vartheta\cdot\overline{u}+$$
$$+L^{(3)}(\partial,\sigma)u\cdot\overline{\omega}+L^{(4)}(\partial,\sigma)\omega\cdot\overline{\omega}+L^{(6)}(\partial,\sigma)\vartheta\cdot\overline{\omega}+$$
$$+C_0\overline{L^{(7)}(\partial,\sigma)u}\vartheta+C_0\overline{L^{(8)}(\partial,\sigma)\omega}\vartheta+C_0\overline{L^{(9)}(\partial,\sigma)\vartheta}\vartheta\Big]\,dx=$$
$$=\int\limits_{\Omega^+}\Big[-E(\widetilde{U},\overline{\widetilde{U}})+\varrho\sigma^2|u|^2+\mathcal{I}\sigma^2|\omega|^2-\kappa'C_0|\nabla\vartheta|^2-\kappa''|\vartheta|^2\big]\,dx+$$
$$+\int\limits_{\partial\Omega^+}\big[\mathcal{T}(\partial,n)U\cdot\overline{\widetilde{U}}+\kappa'C_0\vartheta\partial_n\overline{\vartheta}\Big]\,dS. \tag{2.50}$$

Since U solves the homogeneous BVP problem $(K^{(\sigma)})^+$ we see that the left hand side expression and the surface integral in the right hand side in (2.50) vanish, and we get

$$\int\limits_{\Omega^+} \Big[E(\widetilde{U},\overline{\widetilde{U}}) - \varrho\sigma^2|u|^2 - \mathcal{I}\sigma^2|\omega|^2 + \kappa' C_0|\nabla\vartheta|^2 + \kappa''|\vartheta|^2\Big]\,dx = 0. \tag{2.51}$$

The imaginary part of this equation reads as

$$\sigma_1 \int\limits_{\Omega^+} \Big[2\sigma_2\varrho|u|^2 + 2\sigma_2\mathcal{I}|\omega|^2 + \frac{\kappa'}{|\sigma|^2}|\nabla\vartheta|^2\Big]\,dx = 0. \tag{2.52}$$

Whence, for $\sigma_1 \neq 0$ we have $u = 0$, $\omega = 0$ and $\vartheta = const$ in Ω^+ since $\sigma_2 > 0$ and $\kappa' > 0$. From (2.51) we then conclude $\vartheta = 0$ in Ω^+.

If $\sigma_1 = 0$, then from (2.51) and (2.49) it follows

$$\int\limits_{\Omega^+} \Big[E(\widetilde{U},\overline{\widetilde{U}}) + \sigma_2^2\varrho|u|^2 + \sigma_2^2\mathcal{I}|\omega|^2 + \frac{\kappa'}{\sigma_2}|\nabla\vartheta|^2 + \kappa''|\vartheta|^2\Big]\,dx = 0.$$

Therefore, $u = 0$, $\omega = 0$ and $\vartheta = 0$ in Ω^+. □

Note that in the case of static problems, i.e., when $\sigma = 0$, without loss of generality we can assume that solution vectors to the basic BVPs are real valued. Moreover, the differential equation and the corresponding boundary conditions for the temperature function become uncoupled and we have the following uniqueness theorem.

Theorem 2.3. *The Dirichlet and mixed boundary value problems of statics $(I^{(0)})^+$ and $(III^{(0)})^+$ have at most one solution in the space $[W_2^1(\Omega^+)]^7$. A solution $U = (u,\omega,\vartheta)^\top$ to the Neumann BVP $(II^{(0)})^+$ is defined modulo the vector $U_0 = \vartheta_0(u_0,\omega_0,1)^\top + (\widetilde{\Psi},0)^\top$, where $\widetilde{\Psi}$ is an arbitrary generalized rigid displacement vector, i.e.,*

$$\widetilde{\Psi}(x) = ([a\times x] + b, a)^\top \tag{2.53}$$

with arbitrary three-dimensional real constant vectors a and b, ϑ_0 is an arbitrary real constant, $u_0 = (u_{01},u_{02},u_{03})^\top$ and $\omega_0 = (\omega_{01},\omega_{02},\omega_{03})^\top$ are such that $\widetilde{V}_0 = (u_0,\omega_0)^\top$ is a particular solution of the problem

$$\begin{gathered} \widetilde{L}(\partial,0)\widetilde{V}_0 = 0, \quad x\in\Omega^+, \\ \{T(\partial,n)\widetilde{V}_0\}^+ = (\eta n(x),\zeta n(x))^\top, \quad x\in\partial\Omega^+ \end{gathered} \tag{2.54}$$

with η and ζ being material parameters involved in the field equations (2.1) *and* (2.2)*; here $\widetilde{L}(\partial,0)$ and $T(\partial,n)$ are the operators defined by* (2.14) *and* (2.22) *arising in the hemitropic elasticity without taking into consideration of thermal effects.*

Proof. Let $U^{(j)} = (u^{(j)},\omega^{(j)},\vartheta^{(j)})^\top \in [W_2^1(\Omega^+)]^7$, $j=1,2$, be two solutions to the BVP of statics $(K^{(0)})^+$, $K = I, II, III$. Denote $U := U^{(1)} - U^{(2)}$. Evidently, $U = (u,\omega,\vartheta)^\top \in [W_2^1(\Omega^+)]^7$ solves the homogeneous BVP of statics $(K^{(0)})^+$, $K = I, II, III$. Then by the last equation in (2.4) and formula (2.18) we see that $\vartheta \in W_2^1(\Omega^+)$ is a harmonic function in Ω^+

satisfying the homogeneous Dirichlet, mixed or Neumann type boundary condition. Therefore, in the cases of the BVPs $(I^{(0)})^+$ and $(III^{(0)})^+$ we easily derive that $\vartheta = 0$ in Ω^+ since the support of the Dirichlet condition is not empty, while for the Neumann BVP $(II^{(0)})^+$ we have $\vartheta = \vartheta_0$ in Ω^+ with arbitrary real constant ϑ_0.

Thus, an arbitrary solution to the homogeneous BVPs $(I^{(0)})^+$ and $(III^{(0)})^+$ has the structure $U = (u, \omega, 0)^\top$, where the vector $\widetilde{U} = (u, \omega)^\top$ solves the differential equation $\widetilde{L}(\partial, 0)\widetilde{U} = 0$ in Ω^+ with $\widetilde{L}(\partial, 0)$ defined by (2.14). Further, since $\mathcal{T}(\partial, n)U = T(\partial, n)\widetilde{U}$ for $U = (u, \omega, 0)^\top$, from Green's formula (2.23) with $\widetilde{U}' = \widetilde{U}$ we get $E(\widetilde{U}, \widetilde{U}) = 0$ since the surface integral in the right hand side in (2.23) vanishes in view of the homogeneous boundary conditions. Now, by Lemma 2.1 we easily conclude $u(x) = [a \times x] + b$ and $\omega(x) = a$ in Ω^+ where a and b are arbitrary three-dimensional real constant vectors. The homogeneous Dirichlet conditions for u and ω on $\partial\Omega^+$ in the case of the BVP $(I^{(0)})^+$ or on S_D in the case of the BVP $(III^{(0)})^+$ then imply that u and ω vanish identically in Ω^+. This proves the first part of the theorem.

Now we investigate the homogeneous BVP $(II^{(0)})^+$. As we have established, any solution of the problem has the structure $U = (u, \omega, \vartheta_0)^\top = (\widetilde{U}, \vartheta_0)^\top$, where ϑ_0 is an arbitrary real constant and $\widetilde{U} = (u, \omega)^\top$. From the formulas (2.17), (2.18), (2.22), and from the homogeneous differential equation $L(\partial, 0)U = 0$ in Ω^+ and the homogeneous Neumann condition $\{\mathcal{P}(\partial, n)U\}^+ = 0$ on $\partial\Omega^+$ it follows that

$$\begin{aligned} &\widetilde{L}(\partial, 0)\widetilde{U} = 0, \quad x \in \Omega^+, \\ &\{T(\partial, n)\widetilde{U}\}^+ = \widetilde{F}_0, \quad x \in \partial\Omega^+, \end{aligned} \tag{2.55}$$

where, and

$$\widetilde{F}_0(x) = \vartheta_0(\eta n(x), \zeta n(x))^\top, \quad x \in \partial\Omega^+. \tag{2.56}$$

Here $n(x)$ is the exterior unit normal vector to the boundary $\partial\Omega^+$ at the point $x \in \partial\Omega^+$, while η and ζ are material parameters, and ϑ_0 is a constant temperature. Thus, $\widetilde{U}$ solves the nonhomogeneous Neumann problem of the theory of hemitropic elasticity when thermal effects are not taken into consideration in the governing equations. It is shown in [44] that the necessary and sufficient condition for the problem (2.55)–(2.56) to be solvable reads as

$$\int\limits_{\partial\Omega_+} \widetilde{F}_0(x) \cdot \widetilde{\Psi}(x)\, dS = 0, \tag{2.57}$$

where $\widetilde{\Psi}$ is given by (2.53) with arbitrary three-dimensional real constant vectors a and b.

With the help of the relation $[a \times x] \cdot n = [x \times n] \cdot a$ and the equalities

$$\int\limits_{\partial\Omega_+} n_k(x)\, dS = 0, \quad \int\limits_{\partial\Omega_+} [x_j n_k(x) - x_k n_j(x)]\, dS = 0, \quad k, j = 1, 2, 3,$$

we easily derive that the necessary condition (2.57) is satisfied. Consequently, the BVP (2.55) is solvable for arbitrary ϑ_0 and solutions are defined modulo generalized rigid displacement vector $\widetilde{\Psi}$. Now let us chose a particular solution of the problem (2.54) (which coincides with (2.55) for $\vartheta_0 = 1$) and denote it by $\widetilde{V}_0 = (u_0, \omega_0)^\top$ with $u_0 = (u_{01}, u_{02}, u_{03})^\top$ and $\omega_0 = (\omega_{01}, \omega_{02}, \omega_{03})^\top$. Then clearly $\vartheta_0 \widetilde{V}_0$ is a particular solution of the problem (2.55) and the general solution to the same problem reads as $\vartheta_0 \widetilde{V}_0 + \widetilde{\Psi}$.

Therefore an arbitrary solution of the homogeneous BVP $(II^{(0)})^+$ is representable in the form $U = (u, \omega, \vartheta_0)^\top = (\widetilde{U}, \vartheta_0)^\top$ where $\widetilde{U} = \vartheta_0 \widetilde{V}_0 + \widetilde{\Psi}$. In turn, this leads to the representation $U = \vartheta_0 (u_0, \omega_0, 1)^\top + (\widetilde{\Psi}, 0)^\top$ which completes the proof. □

Remark 2.4. Unfortunately, in contrast to the classical thermoelasticity case, to find the explicit expression for the particular solution vector $\widetilde{V}_0$ for arbitrary domain Ω^+ is problematic in the theory of thermo-hemitropic elasticity. However, $\widetilde{V}_0$ can be constructed explicitly in some particular cases. For example, if the material parameters satisfy the following condition

$$\frac{\eta}{2\mu + 3\lambda} = \frac{\zeta}{2\varkappa + 3\delta}\,, \tag{2.58}$$

then

$$\widetilde{V}_0 = \frac{\eta}{2\mu + 3\lambda}\,(x, 0)^\top.$$

Indeed, one can easily check that in this case

$$\begin{gathered} \widetilde{L}(\partial, 0)\widetilde{V}_0 = 0 \quad \text{in} \quad \Omega^+, \\ \{T(\partial, n)\widetilde{V}_0\}^+ = \frac{\eta}{2\mu + 3\lambda}\,((2\mu + 3\lambda)n, (2\varkappa + 3\delta)n)^\top = \\ = (\eta n(x), \zeta n(x))^\top \quad \text{on} \quad \partial\Omega^+, \end{gathered}$$

for arbitrary domain Ω^+.

Remark 2.5. For some domains with particular geometry it is possible to construct explicitly the particular solution vector $\widetilde{V}_0$ of the problem (2.54) without the restriction (2.58). For example, let Ω^+ be a ball $B(O, R)$ centered at the origin and radius R. Let us look for a particular solution $\widetilde{V}_0 = (u_0, \omega_0)^\top$ of the problem (2.54) in the form (cf., [20])

$$\begin{aligned} u_0(x) &= A_1 x^\top - A_2(\delta + 2\varkappa)\,\frac{dg_0(r)}{dr}\,\widetilde{n}(x), \\ \omega_0(x) &= A_2(\lambda + 2\mu)\,\frac{dg_0(r)}{dr}\widetilde{n}(x), \end{aligned} \tag{2.59}$$

where A_1 and A_2 are unknown scalar constants, $x = (x_1, x_2, x_3)$, $r = |x|$ and

$$\widetilde{n}(x) = \frac{x^\top}{r}\,, \quad g_0(r) = \frac{J_{1/2}(i\lambda_1 r)}{\sqrt{r}} \quad \text{with} \quad \lambda_1^2 = \frac{4\alpha(\lambda + 2\mu)}{d_2}\,;$$

here $J_{1/2}(i\lambda_1 r)$ is the Bessel function of the first kind and d_2 is defined in (2.28). Note that the vector $\widetilde{n}(x)$ for $x \in \partial B(O,R) =: \Sigma(O,R)$ coincides with the exterior unit normal vector, i.e., $\widetilde{n}(x) = n(x) = x^\top/R$ for $x \in \Sigma(O,R)$.

One can easily verify the following identities

$$\Delta[f(r)\widetilde{n}(x)] = \Big[\frac{d}{dr}\Big(\frac{df(r)}{dr} + \frac{2f(r)}{r}\Big)\Big]\widetilde{n}(x), \quad \operatorname{curl}[f(r)\widetilde{n}(x)] = 0,$$
$$\operatorname{grad}\operatorname{div}[f(r)\widetilde{n}(x)] = \Big[\frac{d}{dr}\Big(\frac{df(r)}{dr} + \frac{2f(r)}{r}\Big)\Big]\widetilde{n}(x),$$
$$\frac{d^2 g_0(r)}{dr^2} = -\frac{2}{r}\,\frac{dg_0(r)}{dr} + \lambda_1^2 g_0(r),$$

where $f(\cdot)$ is an arbitrary C^2-smooth function. With the help of these relations we can show that the vector $\widetilde{V}_0 = (u_0, \omega_0)^\top$ with u_0 and ω_0 given by (2.59), solves the differential equation

$$\widetilde{L}(\partial,0)\widetilde{V}_0 = 0 \quad \text{in} \quad B(O,R)$$

for arbitrary constants A_1 and A_2. Further we show that these unknown constants can be chosen so that the boundary condition in (2.54) is satisfied. In view of (2.22) we have

$$T(\partial,n)\widetilde{V}_0 = \Big(T^{(1)}(\partial,n)u_0 + T^{(2)}(\partial,n)\omega_0,\, T^{(3)}(\partial,n)u_0 + T^{(4)}(\partial,n)\omega_0\Big)^\top.$$

Taking into account the equalities (2.19) and

$$\operatorname{div}[f(r)\,\widetilde{n}(x)] = \frac{df(r)}{dr} + \frac{2\,f(r)}{r}\,, \quad \frac{\partial}{\partial n} = \frac{\partial}{\partial r}\,,$$

we can easily show that the traces of the force stress and couple stress vectors on $\Sigma(O,R)$ read as

$$\big\{T^{(1)}(\partial,n)u_0 + T^{(2)}(\partial,n)\omega_0\big\}^+ =$$
$$= \Big[(3\lambda+2\mu)A_1 + 4(\mu\delta - \lambda\varkappa)\,\frac{1}{R}\,\frac{dg_0(R)}{dR}\,A_2\Big]n(x),$$
$$\big\{T^{(3)}(\partial,n)u_0 + T^{(4)}(\partial,n)\omega_0\big\}^+ =$$
$$= \Big\{(3\delta+2\varkappa)A_1 + \Big[\big[\varkappa(\delta+2\varkappa) - \gamma(\lambda+2\mu)\big]\,\frac{4}{R}\,\frac{dg_0(R)}{dR} +$$
$$+ 4\alpha(\lambda+2\mu)g_0(R)\Big]A_2\Big\}n(x).$$

Now the boundary condition in (2.54) leads to the following system of linear algebraic equations

$$\begin{gathered}(3\lambda+2\mu)\,A_1 + 4(\mu\delta-\lambda\varkappa)\,\frac{1}{R}\,\frac{dg_0(R)}{dR}\,A_2 = \eta,\\ (3\delta+2\varkappa)\,A_1 +\\ +\Big[\big[\varkappa(\delta+2\varkappa)-\gamma(\lambda+2\mu)\big]\,\frac{4}{R}\,\frac{dg_0(R)}{dR} + 4\alpha(\lambda+2\mu)g_0(R)\Big]A_2 = \zeta.\end{gathered} \tag{2.60}$$

Whence

$$A_1 = \frac{4\eta}{RD}\Big\{[\varkappa(\delta+2\varkappa)-\gamma(\lambda+2\mu)]\frac{dg_0(R)}{dR}+\alpha(\lambda+2\mu)R\,g_0(R)\Big\} - \\ -\frac{4\zeta(\mu\delta-\lambda\varkappa)}{RD}\,\frac{dg_0(R)}{dR}\,, \tag{2.61}$$

$$A_2 = \frac{\zeta(3\lambda+2\mu)-\eta(3\delta+2\varkappa)}{D}\,, \tag{2.62}$$

where

$$D=\Big\{(3\lambda+2\mu)\big[\varkappa(\delta+2\varkappa)-\gamma(\lambda+2\mu)\big]+ \\ +(3\delta+2\varkappa)(\lambda\varkappa-\mu\delta)\Big\}\,\frac{4}{R}\,\frac{dg_0(R)}{dR}+4\alpha(\lambda+2\mu)(3\lambda+2\mu)g_0(R) \tag{2.63}$$

is the determinant of the above system. By standard arguments we can show that this determinant is different from zero. Otherwise, if $D=0$, the homogeneous version of the system (2.60) will possess a nontrivial solution, A_1' and A_2'. Construct the vectors u_0' and ω_0' by formulas (2.59) with A_1' and A_2' for A_1 and A_2 respectively. Evidently, the vector $\widetilde{V}_0'=(u_0',\omega_0')^\top$ solves then the homogeneous Neumann problem

$$\begin{aligned} &\widetilde{L}(\partial,0)\widetilde{V}_0'=0, \quad x\in B(O,R),\\ &\{T(\partial,n)\widetilde{V}_0'\}^+=0, \quad x\in\Sigma(O,R). \end{aligned} \tag{2.64}$$

On the one hand, by Green's formula (2.23) and Lemma 2.1 it follows that $\widetilde{V}_0'$, as a solution of the problem (2.64), is a rigid displacement vector, i.e., $\widetilde{V}_0'=(a'\times x+b',a')^\top$, where a' and b' are arbitrary three-dimensional constant vectors. On the other hand, in accordance with the representation (2.59) with A_1' and A_2' for A_1 and A_2, it is clear that the vector $\widetilde{V}_0'=(u_0',\omega_0')^\top$ does not belong to the lineal of rigid displacement vectors if $|A_1'|+|A_2'|\neq 0$. This contradiction proves that $D\neq 0$ and consequently the system (2.60) is uniquely solvable. Therefore, the vector $\widetilde{V}_0=(u_0,\omega_0)^\top$, where u_0 and ω_0 are defined by formulas (2.59) with the constants A_1 and A_2 given by (2.61) and (2.62), solves the boundary value problem (2.54) for arbitrary values of the material parameters.

3. Fundamental Matrices of Solutions

‘ Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwarz space $\mathcal{S}'(\mathbb{R}^3)$) which for regular summable functions f and $\widehat{f}$ read as follows

$$\begin{aligned} &\mathcal{F}_{x\to\xi}[f]=\int\limits_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}\,dx=\widehat{f}(\xi),\\ &\mathcal{F}^{-1}_{\xi\to x}[\widehat{f}]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\widehat{f}(\xi)e^{-ix\cdot\xi}\,d\xi=f(x), \end{aligned} \tag{3.1}$$

where $x=(x_1,x_2,x_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$. Note that for arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $f\in\mathcal{S}'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f]=(-i\xi)^\alpha\mathcal{F}[f],\quad \mathcal{F}^{-1}[\xi^\alpha\widehat{f}]=(i\partial)^\alpha\mathcal{F}^{-1}[\widehat{f}], \tag{3.2}$$

where $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$.

Denote by $\Gamma(x,\sigma)=[\Gamma_{kj}(x,\sigma)]_{7\times7}$ the matrix of fundamental solutions of the operator $L(\partial,\sigma)$ (see (2.5)–(2.6))

$$L(\partial,\sigma)\Gamma(x,\sigma)=\delta(x)I_7. \tag{3.3}$$

Here $\delta(\,\cdot\,)$ is the Dirac's delta distribution. We assume that the frequency parameter σ is complex, in general:

$$\sigma=\sigma_1+i\sigma_2,\quad \sigma_1,\sigma_2\in\mathbb{R}. \tag{3.4}$$

We represent the matrix $\Gamma(x,\sigma)$ in the block wise form

$$\begin{aligned}
\Gamma(x,\sigma)&=\begin{bmatrix}\Gamma^{(1)}(x,\sigma) & \Gamma^{(2)}(x,\sigma) & \Gamma^{(5)}(x,\sigma)\\ \Gamma^{(3)}(x,\sigma) & \Gamma^{(4)}(x,\sigma) & \Gamma^{(6)}(x,\sigma)\\ \Gamma^{(7)}(x,\sigma) & \Gamma^{(8)}(x,\sigma) & \Gamma^{(9)}(x,\sigma)\end{bmatrix}_{7\times7},\\
\Gamma^{(j)}(x,\sigma)&=\big[\Gamma^{(j)}_{pq}(x,\sigma)\big]_{3\times3},\ \ j=\overline{1,4},\\
\Gamma^{(l)}(x,\sigma)&=\big[\Gamma^{(l)}_{pq}(x,\sigma)\big]_{3\times1},\ \ l=5,6,\\
\Gamma^{(m)}(x,\sigma)&=\big[\Gamma^{(m)}_{pq}(x,\sigma)\big]_{1\times3},\ \ m=7,8.
\end{aligned}$$

Here $\Gamma^{(9)}(x,\sigma)$ is a scalar function.

By $\widehat{\Gamma}(\xi,\sigma)$ and $\widehat{\Gamma}^{(k)}(\xi,\sigma)$ we denote the Fourier transforms of the matrices $\Gamma(x,\sigma)$ and $\Gamma^{(k)}(x,\sigma)$, $k=\overline{1,9}$.

Applying the Fourier transform to the equation (3.3), and taking into consideration (3.2) and the equality $\mathcal{F}[\delta(\cdot)]=1$, we get

$$L(-i\xi,\sigma)\widehat{\Gamma}(\xi,\sigma)=I_7. \tag{3.5}$$

We have to determine $\widehat{\Gamma}(\xi,\sigma)$ from (3.5) and afterwards with the help of the inverse Fourier transform construct the fundamental matrix $\Gamma(x,\sigma)$ explicitly in terms of standard elementary functions. Evidently, first of all we have to represent the matrix $\widehat{\Gamma}(\xi,\sigma)=[L(-i\xi,\sigma)]^{-1}$ in such form which is convenient for calculation of the inverse Fourier transform.
To this end, we proceed as follows. We set $r:=|\xi|=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$ and introduce the notation

$$\begin{gathered}
A(\xi):=L^{(1)}(-i\xi,\sigma)=\big[-(\mu+\alpha)r^2+\rho\sigma^2\big]I_3-(\lambda+\mu-\alpha)Q(\xi),\\
B(\xi):=L^{(2)}(-i\xi,\sigma)=L^{(3)}(-i\xi,\sigma)=\\
=-(\varkappa+\nu)r^2I_3-(\delta+\varkappa-\nu)Q(\xi)-i2\alpha R(\xi),\\
D(\xi):=L^{(4)}(-i\xi,\sigma)=\\
=\big[\mathcal{I}\sigma^2-4\alpha-(\gamma+\varepsilon)r^2\big]I_3-(\beta+\gamma-\varepsilon)Q(\xi)-i4\nu R(\xi),
\end{gathered} \tag{3.6}$$

where $R(\cdot)$ and $Q(\cdot)$ are defined by (2.7). In view of (2.5)–(2.8) from (3.5) we easily derive

$$\begin{gathered}
A(\xi)\widehat{\Gamma}^{(1)}(\xi,\sigma)+B(\xi)\widehat{\Gamma}^{(3)}(\xi,\sigma)+i\eta\xi^\top\widehat{\Gamma}^{(7)}(\xi,\sigma)=I_3,\\
B(\xi)\widehat{\Gamma}^{(1)}(\xi,\sigma)+D(\xi)\widehat{\Gamma}^{(3)}(\xi,\sigma)+i\zeta\xi^\top\widehat{\Gamma}^{(7)}(\xi,\sigma)=[0]_{3\times3},\\
\eta\sigma\xi\widehat{\Gamma}^{(1)}(\xi,\sigma)+\zeta\sigma\xi\widehat{\Gamma}^{(3)}(\xi,\sigma)+(i\sigma\kappa''-\kappa' r^2)\widehat{\Gamma}^{(7)}(\xi,\sigma)=[0]_{1\times3};
\end{gathered} \tag{3.7}$$

$$\begin{gathered}
A(\xi)\widehat{\Gamma}^{(2)}(\xi,\sigma)+B(\xi)\widehat{\Gamma}^{(4)}(\xi,\sigma)+i\eta\xi^\top\widehat{\Gamma}^{(8)}(\xi,\sigma)=[0]_{3\times3},\\
B(\xi)\widehat{\Gamma}^{(2)}(\xi,\sigma)+D(\xi)\widehat{\Gamma}^{(4)}(\xi,\sigma)+i\zeta\xi^\top\widehat{\Gamma}^{(8)}(\xi,\sigma)=I_3,\\
\eta\sigma\xi\widehat{\Gamma}^{(2)}(\xi,\sigma)+\zeta\sigma\xi\widehat{\Gamma}^{(4)}(\xi,\sigma)+(i\sigma\kappa''-\kappa' r^2)\widehat{\Gamma}^{(8)}(\xi,\sigma)=[0]_{1\times3};
\end{gathered} \tag{3.8}$$

$$\begin{gathered}
A(\xi)\widehat{\Gamma}^{(5)}(\xi,\sigma)+B(\xi)\widehat{\Gamma}^{(6)}(\xi,\sigma)+i\eta\xi^\top\widehat{\Gamma}^{(9)}(\xi,\sigma)=[0]_{3\times1},\\
B(\xi)\widehat{\Gamma}^{(5)}(\xi,\sigma)+D(\xi)\widehat{\Gamma}^{(6)}(\xi,\sigma)+i\zeta\xi^\top\widehat{\Gamma}^{(9)}(\xi,\sigma)=[0]_{3\times1},\\
\eta\sigma\xi\widehat{\Gamma}^{(5)}(\xi,\sigma)+\zeta\sigma\xi\widehat{\Gamma}^{(6)}(\xi,\sigma)+(i\sigma\kappa''-\kappa' r^2)\widehat{\Gamma}^{(9)}(\xi,\sigma)=1.
\end{gathered} \tag{3.9}$$

Applying the relations (see (2.5)–(2.9))

$$\begin{gathered}
A(\xi)=A(-\xi)=A^\top(\xi),\ \ B(\xi)=B^\top(-\xi),\ \ D(\xi)=D^\top(-\xi),\\
Q(\xi)=[Q(\xi)]^\top,\quad [R(\xi)]^\top=-R(\xi)=R(-\xi),\\
Q(\xi)R(\xi)=R(\xi)Q(\xi)=[0]_{3\times3},\\
[Q(\xi)]^2=r^2Q(\xi),\quad [R(\xi)]^2=Q(\xi)-r^2I_3,
\end{gathered} \tag{3.10}$$

we can easily show that the matrices A, B, and D commute to each other. Therefore from the first and the second equations in (3.7) we obtain

$$\begin{gathered}
\big[A(\xi)D(\xi)-B^2(\xi)\big]\widehat{\Gamma}^{(1)}(\xi,\sigma)=i[\zeta B(\xi)-\eta D(\xi)]\xi^\top\widehat{\Gamma}^{(7)}(\xi,\sigma)+D(\xi),\\
\big[A(\xi)D(\xi)-B^2(\xi)\big]\widehat{\Gamma}^{(3)}(\xi,\sigma)=i[\eta B(\xi)-\zeta A(\xi)]\xi^\top\widehat{\Gamma}^{(7)}(\xi,\sigma)-B(\xi).
\end{gathered} \tag{3.11}$$

It can be shown that

$$M(\xi):=A(\xi)D(\xi)-B^2(\xi)=aI_3+bQ(\xi)+icR(\xi), \tag{3.12}$$

where

$$a(\xi):=d_1r^4-d_3r^2+\varrho\sigma^2(\mathcal{I}\sigma^2-4\alpha), \tag{3.13}$$

$$\begin{aligned}
b(\xi):=&\,(d_2-d_1)r^2-\\
&-\big[(\beta+\gamma-\varepsilon)\varrho\sigma^2+(\lambda+\mu-\alpha)(\mathcal{I}\sigma^2-4\alpha)-4\alpha^2\big],
\end{aligned} \tag{3.14}$$

$$c(\xi):=4(\mu\nu-\alpha\varkappa)r^2-4\nu\varrho\sigma^2, \tag{3.15}$$

with

$$\begin{aligned}
d_1&:=(\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2,\\
d_2&:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2,\\
d_3&:=(\mu+\alpha)(\mathcal{I}\sigma^2-4\alpha)+(\gamma+\varepsilon)\varrho\sigma^2+4\alpha^2.
\end{aligned}$$

Moreover, in view of (2.9) by direct calculations we arrive at the following formula for the inverse of the matrix (3.12)

$$M^{-1}(\xi)=\frac{1}{\det M(\xi)}M^*(\xi)=\frac{1}{(a+br^2)(a^2-c^2r^2)}\times \\ \times\Big[a(a+br^2)I_3-(ab+c^2)Q(\xi)-ic(a+br^2)R(\xi)\Big], \quad (3.16)$$

where $M^*(\xi)=[M^*_{kj}(\xi)]$ is the adjoint to the matrix $M(\xi)$,

$$M^*_{11}(\xi)=\begin{vmatrix} a+b\xi_2^2 & b\xi_2\xi_3-ic\xi_1\\ b\xi_2\xi_3+ic\xi_1 & a+b\xi_3^2\end{vmatrix}=a(a+br^2)-(ab+c^2)\xi_1^2,$$

$$M^*_{21}(\xi)=-\begin{vmatrix} b\xi_1\xi_2+ic\xi_3 & b\xi_2\xi_3-ic\xi_1\\ b\xi_1\xi_3-ic\xi_2 & a+b\xi_3^2\end{vmatrix}=-i(a+br^2)c\xi_3-(ab+c^2)\xi_1\xi_2,$$

$$M^*_{31}(\xi)=\begin{vmatrix} b\xi_1\xi_2+ic\xi_3 & a+b\xi_2^2\\ b\xi_1\xi_3-ic\xi_2 & b\xi_2\xi_3+ic\xi_1\end{vmatrix}=-(ab+c^2)\xi_1\xi_3+i(a+br^2)c\xi_2,$$

$$M^*_{12}(\xi)=-\begin{vmatrix} b\xi_1\xi_2-ic\xi_3 & b\xi_1\xi_3+ic\xi_2\\ b\xi_2\xi_3+ic\xi_1 & a+b\xi_3^2\end{vmatrix}=i(a+br^2)c\xi_3-(ab+c^2)\xi_1\xi_2,$$

$$M^*_{22}(\xi)=\begin{vmatrix} a+b\xi_1^2 & b\xi_1\xi_3+ic\xi_2\\ b\xi_1\xi_3-ic\xi_2 & a+b\xi_3^2\end{vmatrix}=a(a+br^2)-(ab+c^2)\xi_2^2,$$

$$M^*_{32}(\xi)=-\begin{vmatrix} a+b\xi_1^2 & b\xi_1\xi_2-ic\xi_3\\ b\xi_1\xi_3-ic\xi_2 & b\xi_2\xi_3+ic\xi_1\end{vmatrix}=-(ab+c^2)\xi_2\xi_3-i(a+br^2)c\xi_1,$$

$$M^*_{13}(\xi)=\begin{vmatrix} b\xi_1\xi_2-ic\xi_3 & b\xi_1\xi_3+ic\xi_2\\ a+b\xi_2^2 & b\xi_2\xi_3-ic\xi_1\end{vmatrix}=-(ab+c^2)\xi_1\xi_3-i(a+br^2)c\xi_2,$$

$$M^*_{23}(\xi)=-\begin{vmatrix} a+b\xi_1^2 & b\xi_1\xi_3+ic\xi_2\\ b\xi_1\xi_2+ic\xi_3 & b\xi_2\xi_3-ic\xi_1\end{vmatrix}=-(ab+c^2)\xi_2\xi_3+i(a+br^2)c\xi_1,$$

$$M^*_{33}(\xi)=\begin{vmatrix} a+b\xi_1^2 & b\xi_1\xi_2-ic\xi_3\\ b\xi_1\xi_3+ic\xi_3 & a+b\xi_2^2\end{vmatrix}=a(a+br^2)-(ab+c^2)\xi_3^2.$$

These formulae imply

$$M^*(\xi)=[M^*_{kj}(\xi)]=a(a+br^2)I_3-(ab+c^2)Q(\xi)-ic(a+br^2)R(\xi).$$

From (3.11) we get

$$\begin{aligned}\widehat\Gamma^{(1)}(\xi,\sigma)&=iM^{-1}(\xi)[\zeta B(\xi)-\eta D(\xi)]\xi^\top\widehat\Gamma^{(7)}(\xi,\sigma)+M^{-1}(\xi)D(\xi),\\ \widehat\Gamma^{(3)}(\xi,\sigma)&=iM^{-1}(\xi)[\eta B(\xi)-\zeta A(\xi)]\xi^\top\widehat\Gamma^{(7)}(\xi,\sigma)-M^{-1}(\xi)B(\xi).\end{aligned}\quad (3.17)$$

With the help of the relations (3.10) and

$$\begin{gathered}M^{-1}(\xi)A(\xi)=A(\xi)M^{-1}(\xi),\quad M^{-1}(\xi)B(\xi)=B(\xi)M^{-1}(\xi),\\ M^{-1}(\xi)D(\xi)=D(\xi)M^{-1}(\xi),\quad R(\xi)\xi^\top=0,\quad \xi R(\xi)=0,\\ M^{-1}(\xi)\xi^\top=\frac{1}{a+br^2}\xi^\top,\quad \xi M^{-1}(\xi)=\frac{1}{a+br^2}\xi,\end{gathered}\quad (3.18)$$

we can rewrite (3.17) as

$$\begin{aligned}\widehat\Gamma^{(1)}(\xi,\sigma)&=\frac{i}{a+br^2}\,[\zeta B(\xi)-\eta D(\xi)]\xi^\top\widehat\Gamma^{(7)}(\xi,\sigma)+M^{-1}(\xi)D(\xi),\\ \widehat\Gamma^{(3)}(\xi,\sigma)&=\frac{i}{a+br^2}\,[\eta B(\xi)-\zeta A(\xi)]\xi^\top\widehat\Gamma^{(7)}(\xi,\sigma)-M^{-1}(\xi)B(\xi).\end{aligned}\quad (3.19)$$

Substitute these expressions into the third equation in (3.7) and apply the formulas

$$\begin{aligned}\xi A(\xi) &= \big[\varrho\sigma^2 - (\lambda+2\mu)r^2\big]\xi, \quad \xi B(\xi) = -(\delta+2\varkappa)r^2\xi,\\ \xi D(\xi) &= \big[\mathcal{I}\sigma^2 - 4\alpha - (\beta+2\gamma)r^2\big]\xi, \quad \xi\xi^\top = r^2, \quad \xi^\top\xi = Q(\xi),\end{aligned} \tag{3.20}$$

to obtain

$$\widehat{\Gamma}^{(7)}(\xi,\sigma) = \frac{c_7(\xi)}{(a^2-c^2r^2)\Lambda_1(\xi)}\xi, \tag{3.21}$$

where

$$c_7(\xi) := \sigma\Big\{\zeta(\delta+2\varkappa)r^2 + \eta\big[\mathcal{I}\sigma^2 - 4\alpha - (\beta+2\gamma)r^2\big]\Big\}(a^2-c^2r^2), \tag{3.22}$$

$$\begin{aligned}\Lambda_1(\xi) &:= \kappa' d_2 r^6 + r^4\Big\{i2\sigma\eta\zeta(\delta+2\varkappa) - i\sigma\eta^2(\beta+2\gamma) -\\ &-i\sigma\zeta^2(\lambda+2\mu) - i\sigma\kappa'' d_2 - \kappa'\big[\rho\sigma^2(\beta+2\gamma) + (\mathcal{I}\sigma^2-4\alpha)(\lambda+2\mu)\big]\Big\} +\\ &+r^2\Big\{i\sigma\eta^2(\mathcal{I}\sigma^2-4\alpha) + i\sigma\kappa''\big[\rho\sigma^2(\beta+2\gamma) + (\mathcal{I}\sigma^2-4\alpha)(\lambda+2\mu)\big] +\\ &+i\zeta^2\rho\sigma^3 + \kappa'\rho\sigma^2(\mathcal{I}\sigma^2-4\alpha)\Big\} - i\kappa''\rho\sigma^3(\mathcal{I}\sigma^2-4\alpha),\end{aligned} \tag{3.23}$$

$$\begin{aligned}a^2 - c^2r^2 &= d_1^2 r^8 - r^6\Big\{2d_1\big[(\mu+\alpha)(\mathcal{I}\sigma^2-4\alpha) + (\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2\big] +\\ &+\big[4\alpha(\kappa+\nu) - 4\nu(\mu+\alpha)\big]^2\Big\} +\\ &+r^4\Big\{\big[(\mu+\alpha)(\mathcal{I}\sigma^2-4\alpha) + (\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2\big]^2 +\\ &+2\rho\sigma^2(\mathcal{I}\sigma^2-4\alpha)d_1 + 32\nu\rho\sigma^2(\nu\mu - \alpha\varkappa)\Big\} -\\ &-r^2\Big\{2\rho\sigma^2(\mathcal{I}\sigma^2-4\alpha)\big[(\mu+\alpha)(\mathcal{I}\sigma^2-4\alpha) +\\ &+(\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2\big] + 16\nu^2\rho^2\sigma^4\Big\} + \rho^2\sigma^4(\mathcal{I}\sigma^2-4\alpha)^2.\end{aligned} \tag{3.24}$$

By (3.21), (3.10), (3.20) and

$$\begin{aligned}A(\xi)Q(\xi) &= \big[\varrho\sigma^2 - (\lambda+2\mu)r^2\big]Q(\xi),\\ B(\xi)Q(\xi) &= -(\delta+2\varkappa)r^2Q(\xi),\\ D(\xi)Q(\xi) &= \big[\mathcal{I}\sigma^2 - 4\alpha - (\beta+2\gamma)r^2\big]Q(\xi),\end{aligned} \tag{3.25}$$

finally we arrive at the equalities

$$\begin{aligned}&\widehat{\Gamma}^{(j)}(\xi,\sigma) =\\ &= \frac{1}{(a^2-c^2r^2)\Lambda_1(\xi)}\big[a_j(\xi)I_3 + b_j(\xi)Q(\xi) + c_j(\xi)R(\xi)\big], \quad j=1,3,\end{aligned} \tag{3.26}$$

where

$$
\begin{aligned}
a_1(\xi) &:= \Lambda_1(\xi)\Big\{a\big[\mathcal{I}\sigma^2-4\alpha-(\gamma+\varepsilon)r^2\big]+4c\nu r^2\Big\},\\
b_1(\xi) &:= -\big[\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)r^2\big]\Big\{i\sigma\eta^2 a\big[\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)r^2\big]+\\
&\quad +i2\sigma a\eta\zeta(\delta+2\varkappa)r^2+(\kappa' r^2-i\sigma\kappa'')(ab+c^2)\Big\}-\\
&\quad -\big[4c\nu+a(\beta+\gamma-\varepsilon)\big]\Lambda_1(\xi)-i\sigma\zeta^2\big[ab+c^2+a(\delta+2\varkappa)^2r^2\big]r^2,\\
c_1(\xi) &:= -i\Lambda_1(\xi)\Big\{c\big[\mathcal{I}\sigma^2-4\alpha-(\gamma+\varepsilon)r^2\big]+4a\nu\Big\},\\
a_3(\xi) &:= \Lambda_1(\xi)\big[a(\varkappa+\nu)-2\alpha c\big]r^2,\\
b_3(\xi) &:= (\delta+2\varkappa)r^2\Big\{(ab+c^2)(i\sigma\kappa''-\kappa' r^2)-\\
&\quad -i\sigma\Big[2a\eta\zeta(\delta+2\varkappa)r^2+a\zeta^2(\varrho\sigma^2-(\lambda+2\mu)r^2)+\\
&\quad +a\eta^2(\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)r^2)\Big]\Big\}-\\
&\quad -i\sigma\eta\zeta(a^2-c^2r^2)+\Lambda_1(\xi)\big[2\alpha c+a(\delta+\varkappa-\nu)\big],\\
c_3(\xi) &:= i\Lambda_1(\xi)\big[2\alpha a-c(\varkappa+\nu)r^2\big].
\end{aligned}
\tag{3.27}
$$

Applying the word for word arguments to the systems (3.8) and (3.9) we get

$$
\begin{aligned}
\widehat{\Gamma}^{(k)}(\xi,\sigma) &= \frac{1}{(a^2-c^2r^2)\Lambda_1(\xi)}\times\\
&\quad\times\ \Big[a_k(\xi)I_3+b_k(\xi)Q(\xi)+c_k(\xi)R(\xi)\Big],\ \ k=2,4,\\
\widehat{\Gamma}^{(l)}(\xi,\sigma) &= \frac{c_l(\xi)}{(a^2-c^2r^2)\Lambda_1(\xi)}\,\xi^\top,\ \ l=5,6,\\
\widehat{\Gamma}^{(8)}(\xi,\sigma) &= \frac{c_8(\xi)}{(a^2-c^2r^2)\Lambda_1(\xi)}\,\xi,\\
\widehat{\Gamma}^{(9)}(\xi,\sigma) &= \frac{c_9(\xi)}{(a^2-c^2r^2)\Lambda_1(\xi)}\,,
\end{aligned}
\tag{3.28}
$$

where

$$
\begin{aligned}
a_2(\xi) &= a_3(\xi), \quad b_2(\xi) = b_3(\xi), \quad c_2(\xi) = c_3(\xi),\\
a_4(\xi) &:= a\big[\varrho\sigma^2-(\mu+\alpha)r^2\big]\Lambda_1(\xi),\\
b_4(\xi) &:= -i\sigma\Big\{2a\eta\zeta(\delta+2\varkappa)\big[\varrho\sigma^2-(\lambda+2\mu)r^2\big]r^2+\\
&\qquad + a\zeta^2\big[\varrho\sigma^2-(\lambda+2\mu)r^2\big]^2+\eta^2r^2\big[ab+c^2+\\
&\qquad + a(\delta+2\varkappa)^2r^2\big]\Big\}-a(\lambda+\mu-\alpha)\Lambda_1(\xi)-\\
&\qquad -(ab+c^2)\big[\varrho\sigma^2-(\lambda+2\mu)r^2\big](\kappa'r^2-i\sigma\kappa''), \qquad (3.29)\\
c_4(\xi) &:= -ic\big[\varrho\sigma^2-(\mu+\alpha)r^2\big]\Lambda_1(\xi),\\
c_5(\xi) &:= i\Big\{\zeta(\delta+2\varkappa)r^2+\eta\big[\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)r^2\big]\Big\}(a^2-c^2r^2),\\
c_6(\xi) &:= i\Big\{\eta(\delta+2\varkappa)r^2+\zeta\big[\varrho\sigma^2-(\lambda+2\mu)r^2\big]\Big\}(a^2-c^2r^2),\\
c_8(\xi) &:= \sigma\Big\{\eta(\delta+2\varkappa)r^2+\zeta\big[\varrho\sigma^2-(\lambda+2\mu)r^2\big]\Big\}(a^2-c^2r^2),\\
c_9(\xi) &:= -(a+br^2)(a^2-c^2r^2).
\end{aligned}
$$

Therefore we can represent the matrix $\widehat{\Gamma}(\xi,\sigma)$ in the form

$$
\widehat{\Gamma}(\xi,\sigma)=[L(-i\xi,\sigma)]^{-1}=\frac{1}{(a^2-c^2r^2)\Lambda_1(\xi)}\,\mathcal{M}(\xi,\sigma), \qquad (3.30)
$$

where

$$
\begin{aligned}
\mathcal{M}(\xi,\sigma) := &\begin{bmatrix} a_1(\xi)I_3 & a_2(\xi)I_3 & [0]_{3\times1}\\ a_3(\xi)I_3 & a_4(\xi)I_3 & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & c_9(\xi)\end{bmatrix}+\\
&+\begin{bmatrix} b_1(\xi)Q(\xi) & b_2(\xi)Q(\xi) & [0]_{3\times1}\\ b_3(\xi)Q(\xi) & b_4(\xi)Q(\xi) & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & 0\end{bmatrix}+\\
&+\begin{bmatrix} c_1(\xi)R(\xi) & c_2(\xi)R(\xi) & c_5(\xi)\xi^\top\\ c_3(\xi)R(\xi) & c_4(\xi)R(\xi) & c_6(\xi)\xi^\top\\ c_7(\xi)\xi & c_8(\xi)\xi & 0\end{bmatrix} \qquad (3.31)
\end{aligned}
$$

It is easy to see that the entries of the 7×7 matrix $\mathcal{M}(\xi,\sigma)$ are polynomials in ξ. Therefore to invert the Fourier transform and find an explicit form for the fundamental matrix $\Gamma(x,\sigma)$ we need the roots of the equation

$$
\Xi(r):=\det L(-i\xi,\sigma)\equiv(a^2-c^2r^2)\Lambda_1(\xi)=0. \qquad (3.32)
$$

Due to the evenness of the functions Λ_1 and $a^2-c^2r^2$ with respect to r, it is clear that if $r=r_0$ is a root of either the equation $a^2-c^2r^2=0$ or $\Lambda_1(\xi)=0$, then so is $r=-r_0$. Denote the roots of the equation

$$
\Lambda_1(\xi)\equiv\kappa'd_2r^6+r^4\Big\{i2\sigma\eta\zeta(\delta+2\varkappa)-i\sigma\eta^2(\beta+2\gamma)-
$$

$$- i\sigma\zeta^2(\lambda+2\mu) - i\sigma\kappa'' d_2 - \kappa'\big[\rho\sigma^2(\beta+2\gamma) + (\mathcal{I}\sigma^2 - 4\alpha)(\lambda+2\mu)\big]\Big\} +$$
$$+ r^2\Big\{ i\sigma\eta^2(\mathcal{I}\sigma^2 - 4\alpha) + i\sigma\kappa''\big[\rho\sigma^2(\beta+2\gamma) + (\mathcal{I}\sigma^2 - 4\alpha)(\lambda+2\mu)\big] +$$
$$+ i\zeta^2\rho\sigma^3 + \kappa'\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)\Big\} - i\kappa''\rho\sigma^3(\mathcal{I}\sigma^2 - 4\alpha) = 0 \quad (3.33)$$

by $\pm k_1$, $\pm k_2$ and $\pm k_3$, and the roots of the equation

$$a^2 - c^2 r^2 \equiv d_1^2 r^8 - r^6\Big\{ 2d_1\Big[(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2\Big] +$$
$$+ \big[4\alpha(\kappa+\nu) - 4\nu(\mu+\alpha)\big]^2\Big\} + r^4\Big\{\Big[(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2\Big]^2 +$$
$$+ 2\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)d_1 + 32\nu\rho\sigma^2(\nu\mu - \alpha\varkappa)\Big\} -$$
$$- r^2\Big\{ 2\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)\big[(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma+\varepsilon)\rho\sigma^2 + 4\alpha^2\big] + 16\nu^2\rho^2\sigma^4\Big\} +$$
$$+ \rho^2\sigma^4(\mathcal{I}\sigma^2 - 4\alpha)^2 = 0 \quad (3.34)$$

by $\pm k_4$, $\pm k_5$, $\pm k_6$ and $\pm k_7$. For simplicity we assume that (see the Appendix A)

$$k_j \neq k_p \quad \text{for} \quad j \neq p, \quad \Im k_j \geq 0, \quad \text{and if} \quad \Im k_j = 0, \quad \text{then} \quad k_j > 0. \quad (3.35)$$

Then we have the relations

$$\Lambda_1(\xi) = \kappa' d_2 \prod_{j=1}^{3}(r^2 - k_j^2), \quad a^2 - c^2 r^2 = d_1^2 \prod_{j=4}^{7}(r^2 - k_j^2). \quad (3.36)$$

Therefore in view of (3.30) we can represent the fundamental solution as

$$\Gamma(x,\sigma) = \mathcal{F}^{-1}_{\xi\to x}[\widehat{\Gamma}(\xi,\sigma)] = \frac{1}{\kappa' d_1^2 d_2}\,\mathcal{F}^{-1}_{\xi\to x}\Big[\mathcal{M}(\xi,\sigma)\,\frac{1}{\Phi(r)}\Big] =$$
$$= \frac{1}{\kappa' d_1^2 d_2}\,\mathcal{M}(i\partial,\sigma)\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{\Phi(r)}\Big], \quad (3.37)$$

where

$$\Phi(r) := \prod_{j=1}^{7}(r^2 - k_j^2). \quad (3.38)$$

Note that

$$\frac{1}{\Phi(r)} = \sum_{j=1}^{7}\frac{p_j}{r^2 - k_j^2}\,, \quad (3.39)$$

where the parameters $p_1,\dots,p_7$ solve the system of linear algebraic equations

$$\begin{aligned} &k_1^{2m}p_1 + k_2^{2m}p_2 + \cdots + k_7^{2m}p_7 = 0, \quad m = 0,1,\dots,5,\\ &k_1^{12}p_1 + k_2^{12}p_2 + \cdots + k_7^{12}p_7 = 1. \end{aligned} \quad (3.40)$$

They can be represented as follows

$$p_j = \Big[\prod_{l=1, l\neq j}^{7} (k_l^2 - k_j^2) \Big]^{-1}. \tag{3.41}$$

Note that, if $\Im k_j \geq 0$, then (for details see the Appendix B)

$$\mathcal{F}^{-1}\Big[\frac{1}{r^2 - k_j^2}\Big] = \frac{e^{ik_j|x|}}{4\pi|x|}\,. \tag{3.42}$$

Therefore

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{\Phi(r)}\Big] = \frac{1}{4\pi}\sum_{j=1}^{7} p_j \frac{e^{ik_j|x|}}{|x|}\,. \tag{3.43}$$

Now from (3.37) and (3.43) it follows that

$$\Gamma(x,\sigma) = \frac{1}{4\pi\kappa' d_1^2 d_2}\,\mathcal{M}(i\partial,\sigma)\sum_{j=1}^{7} p_j \frac{e^{ik_j|x|}}{|x|}\,, \tag{3.44}$$

or

$$\Gamma(x,\sigma) = \frac{1}{4\pi\kappa' d_1^2 d_2}\,\mathcal{M}(i\partial,\sigma)\Psi(x,\sigma), \tag{3.45}$$

where the differential operator $\mathcal{M}(i\partial,\sigma)$ is given by (3.31) with $i\partial$ for ξ and

$$\Psi(x,\sigma) = \sum_{j=1}^{7} p_j \frac{e^{ik_j|x|}}{|x|}\,. \tag{3.46}$$

We can calculate the expression $\mathcal{M}(i\partial,\sigma)\Psi(x,\sigma)$ and rewrite the fundamental solution in a more explicit form. To this end let us note that

$$\Delta\frac{e^{ik_j|x|}}{|x|} = -k_j^2\frac{e^{ik_j|x|}}{|x|}\,,\quad |x|\neq 0,$$

and apply the formulas (3.13)–(3.15), (3.23), (3.27) and (3.29) to obtain

$$\begin{aligned}
a(i\partial)\Psi(x,\sigma) &= \sum_{j=1}^{7} p_j a_j^* \frac{e^{ik_j|x|}}{|x|}\,,\\
b(i\partial)\Psi(x,\sigma) &= \sum_{j=1}^{7} p_j b_j^* \frac{e^{ik_j|x|}}{|x|}\,,\\
c(i\partial)\Psi(x,\sigma) &= \sum_{j=1}^{7} p_j c_j^* \frac{e^{ik_j|x|}}{|x|}\,,\\
a_l(i\partial)\Psi(x,\sigma) &= \sum_{j=4}^{7} a_{lj}^* \frac{e^{ik_j|x|}}{|x|}\,,\quad l=1,2,3,4,\\
b_l(i\partial)\Psi(x,\sigma) &= \sum_{j=1}^{7} b_{lj}^* \frac{e^{ik_j|x|}}{|x|}\,,\quad l=1,2,3,4,
\end{aligned}$$

$$c_l(i\partial)\Psi(x,\sigma)=\sum_{j=4}^{7}c_{lj}^*\frac{e^{ik_j|x|}}{|x|}\,,\ \ l=1,2,3,4,$$
$$c_m(i\partial)\Psi(x,\sigma)=\sum_{j=1}^{3}c_{mj}^*\frac{e^{ik_j|x|}}{|x|}\,,\ \ m=5,6,7,8,9,$$

where

$$a_j^*:=d_1k_j^4-d_3k_j^2+\varrho\sigma^2(\mathcal{I}\sigma^2-4\alpha),$$
$$b_j^*:=(d_2-d_1)k_j^2-\Big[(\beta+\gamma-\varepsilon)\varrho\sigma^2+(\lambda+\mu-\alpha)(\mathcal{I}\sigma^2-4\alpha)-4\alpha^2\Big],$$
$$c_j^*=4(\mu\nu-\alpha\varkappa)k_j^2-4\nu\varrho\sigma^2,$$
$$a_{1j}^*:=\kappa'p_j''d_2\Big\{a_j^*\big[\mathcal{I}\sigma^2-4\alpha-(\gamma+\varepsilon)k_j^2\big]+4\nu k_j^2c_j^*\Big\},$$
$$a_{2j}^*=a_{3j}^*:=\kappa'p_j''d_2\big[(\varkappa+\nu)a_j^*-2\alpha c_j^*\big]k_j^2,$$
$$a_{4j}^*:=\kappa'p_j''d_2\big[\varrho\sigma^2-(\mu+\alpha)k_j^2\big]a_j^*,$$
$$b_{1j}^*:=-p_j\Big\{\big[\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)k_j^2\big]\Big[i\sigma\eta^2(\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)k_j^2)a_j^*+$$
$$+i2\sigma(\delta+2\varkappa)\zeta\eta k_j^2a_j^*+(\kappa'k_j^2-i\sigma\kappa'')(a_j^*b_j^*+(c_j^*)^2)\Big]+$$
$$+i\sigma\zeta^2k_j^2\Big[a_j^*b_j^*+(c_j^*)^2+(\delta+2\varkappa)^2k_j^2a_j^*\Big]+$$
$$+\big[(\beta+\gamma-\varepsilon)a_j^*+4\nu c_j^*\big]\Lambda_{1j}^*\Big\},$$
$$b_{2j}^*=b_{3j}^*:=p_j\big[2\alpha c_j^*+(\delta+\varkappa-\nu)a_j^*\big]\Lambda_{1j}^*+$$
$$+p_j(\delta+2\varkappa)k_j^2\Big\{(i\sigma\kappa''-\kappa'k_j^2)\big[a_j^*b_j^*+(c_j^*)^2\big]-$$
$$-i\sigma\Big[2(\delta+2\varkappa)\zeta\eta k_j^2a_j^*+\zeta^2(\varrho\sigma^2-(\lambda+2\mu)k_j^2)a_j^*+$$
$$+\eta^2(\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)k_j^2)a_j^*\Big]\Big\}-ip_j\sigma\eta\zeta\big[(a_j^*)^2-k_j^2(c_j^*)^2\big],$$
$$b_{4j}^*:=-p_j\big[\varrho\sigma^2-(\lambda+2\mu)k_j^2\big]\Big\{i\sigma\zeta^2a_j^*\big[\varrho\sigma^2-(\lambda+2\mu)k_j^2\big]+$$
$$+i2\sigma(\delta+2\varkappa)\zeta\eta k_j^2a_j^*+\big[a_j^*b_j^*+(c_j^*)^2\big](\kappa'k_j^2-i\sigma\kappa'')\Big\}-$$
$$-ip_j\sigma\eta^2k_j^2\big[a_j^*b_j^*+(c_j^*)^2+(\delta+2\varkappa)^2k_j^2a_j^*\big]-p_j(\lambda+\mu-\alpha)a_j^*\Lambda_{1j}^*,$$
$$c_{1j}^*:=-i\kappa'p_j''d_2\Big\{\big[\mathcal{I}\sigma^2-4\alpha-(\gamma+\varepsilon)k_j^2\big]c_j^*+4\nu a_j^*\Big\},$$
$$c_{2j}^*=c_{3j}^*:=i\kappa'p_j''d_2\big[2\alpha a_j^*-(\varkappa+\nu)k_j^2c_j^*\big],$$
$$c_{4j}^*:=-i\kappa'p_j''d_2\big[\varrho\sigma^2-(\mu+\alpha)k_j^2\big]c_j^*,$$
$$c_{5j}^*:=ip_j'd_1^2\Big\{\zeta(\delta+2\varkappa)k_j^2+\eta\big[\mathcal{I}\sigma^2-4\alpha-(\beta+2\gamma)k_j^2\big]\Big\},$$
$$c_{6j}^*:=ip_j'd_1^2\Big\{\eta(\delta+2\varkappa)k_j^2+\zeta\big[\varrho\sigma^2-(\lambda+2\mu)k_j^2\big]\Big\},$$

$$c^*_{7j} := \sigma p'_j d_1^2 \Big\{ \zeta(\delta + 2\varkappa)k_j^2 + \eta\big[\mathcal{I}\sigma^2 - 4\alpha - (\beta + 2\gamma)k_j^2\big] \Big\},$$

$$c^*_{8j} := \sigma p'_j d_1^2 \Big\{ \eta(\delta + 2\varkappa)k_j^2 + \zeta\big[\varrho\sigma^2 - (\lambda + 2\mu)k_j^2\big] \Big\},$$

$$c^*_{9j} := -p'_j d_1^2 \big[a^*_j + k_j^2 b^*_j\big],$$

$$\Lambda^*_{1j} := \kappa' d_2 \prod_{q=1}^{3} (k_j^2 - k_q^2),$$

$$p'_j := \prod_{q=1, q\neq j}^{3} (k_j^2 - k_q^2)^{-1}, \quad p''_j := \prod_{q=4, q\neq j}^{7} (k_j^2 - k_q^2)^{-1}.$$

From (3.31) and (3.45) with the help of the above relations we get the following representation of the fundamental matrix

$$\Gamma(x,\sigma) = \frac{1}{4\pi\kappa' d_1^2 d_2} \left\{ \begin{bmatrix} \Psi_1(x,\sigma)I_3 & \Psi_2(x,\sigma)I_3 & [0]_{3\times1} \\ \Psi_3(x,\sigma)I_3 & \Psi_4(x,\sigma)I_3 & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & \Psi_5(x,\sigma) \end{bmatrix} + \right.$$

$$+ \begin{bmatrix} Q(\partial)\Psi_6(x,\sigma) & Q(\partial)\Psi_7(x,\sigma) & [0]_{3\times1} \\ Q(\partial)\Psi_8(x,\sigma) & Q(\partial)\Psi_9(x,\sigma) & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & 0 \end{bmatrix} +$$

$$\left. + \begin{bmatrix} R(\partial)\Psi_{10}(x,\sigma) & R(\partial)\Psi_{11}(x,\sigma) & \nabla^\top\Psi_{14}(x,\sigma) \\ R(\partial)\Psi_{12}(x,\sigma) & R(\partial)\Psi_{13}(x,\sigma) & \nabla^\top\Psi_{15}(x,\sigma) \\ \nabla\Psi_{16}(x,\sigma) & \nabla\Psi_{17}(x,\sigma) & 0 \end{bmatrix} \right\}, \quad (3.47)$$

where

$$\Psi_l(x,\sigma) = \sum_{j=4}^{7} a^*_{lj} \frac{e^{ik_j|x|}}{|x|}, \quad \Psi_5(x,\sigma) = \sum_{j=1}^{3} c^*_{9j} \frac{e^{ik_j|x|}}{|x|},$$

$$\Psi_{5+l}(x,\sigma) = -\sum_{j=1}^{7} b^*_{lj} \frac{e^{ik_j|x|}}{|x|}, \quad \Psi_{9+l}(x,\sigma) = i\sum_{j=4}^{7} c^*_{lj} \frac{e^{ik_j|x|}}{|x|},$$

$$\Psi_{13+l}(x,\sigma) = i\sum_{j=1}^{7} c^*_{4+lj} \frac{e^{ik_j|x|}}{|x|}, \quad l = 1,2,3,4.$$

Remark 3.1. Note that (3.44) can be rewritten in the form

$$\Gamma(x,\sigma) = \sum_{j=1}^{7} \Phi^{(j)}(x,\sigma), \quad (3.48)$$

where

$$\Phi^{(j)}(x,\sigma) = \frac{p_j}{4\pi\kappa' d_1^2 d_2}\, \mathcal{M}(i\partial,\sigma)\, \frac{e^{ik_j|x|}}{|x|} \quad (3.49)$$

and $\mathcal{M}(i\partial,\sigma)$ is defined by (3.31). Since $\mathcal{M}(i\partial,\sigma)$ is a matrix differential operator with constant coefficients from the representation (3.49) it follows that the entries of the matrix $\Phi^{(j)}(x,\sigma) = [\Phi^{(j)}_{pq}(x,\sigma)]_{7\times 7}$ are metaharmonic functions corresponding to the wave number k_j, i.e., solutions of the Helmholtz equation

$$(\Delta + k_j^2)\Phi^{(j)}_{pq}(x,\sigma) = 0, \quad |x| \neq 0,$$

and satisfy the Sommerfeld radiation conditions at infinity:

$$\frac{\partial}{\partial|x|}\Phi^{(j)}_{pq}(x,\sigma) - ik_j\Phi^{(j)}_{pq}(x,\sigma) = \\ = \exp\{-\Im k_j|x|\}\mathcal{O}(|x|^{-2}), \;\; p,q,j=\overline{1,7}, \quad (3.50)$$

as $|x| \to +\infty$.

The entries of the matrix $\Phi^{(j)}(x,\sigma)$ and its derivatives satisfy also the following Sommerfeld type radiation conditions at infinity (cf. [58]):

$$\begin{gathered}\Phi^{(j)}_{pq}(x,\sigma) = \exp\big\{-\Im k_j|x|\big\}\mathcal{O}(|x|^{-1}),\\ \frac{\partial}{\partial x_l}\,\Phi^{(j)}_{pq}(x,\sigma) - ik_j\,\frac{x_l}{|x|}\,\Phi^{(j)}_{pq}(x,\sigma) = \\ = \exp\big\{-\Im k_j|x|\big\}\mathcal{O}(|x|^{-2}), \;\; l=1,2,3.\end{gathered} \quad (3.51)$$

These asymptotic equalities can be differentiated any times with respect to the variable x.

In accordance with formulas (3.48), (3.49) and Corollary A.2 (see the Appendix A) we see that for $\Im\sigma = \sigma_2 > 0$ the entries of the matrix $\Gamma(x,\sigma)$ decay exponentially as $|x| \to \infty$ since $\Im k_j > 0$, $j=\overline{1,7}$.

Remark 3.2. Note that the matrix $\Gamma^*(x,\sigma) := [\Gamma(-x,\sigma)]^\top$ represents a fundamental solution to the formally adjoint differential operator $L^*(\partial,\sigma) \equiv [L(-\partial,\sigma)]^\top$,

$$L^*(\partial,\sigma)[\Gamma(-x,\sigma)]^\top = I_7\delta(x). \quad (3.52)$$

In the case of repeated roots (i.e., when (3.35) is violated) the fundamental solution can be obtained from (3.44) by the standard limiting procedure.

3.1. **Fundamental matrix of the operator of statics.** Here we construct the fundamental matrix for the equilibrium equations, i.e., for the operator of statics $L(\partial)$ defined by (2.11) (see simultaneous equations (2.4) with $\sigma = 0$). Denote this fundamental matrix by $\Gamma(x)$. We apply again the approach based on the Fourier transform technique for the equations of statics. For the Fourier transform of the fundamental matrix then we obtain (cf. (3.30) and (3.31))

$$\widehat{\Gamma}(\xi) = [L(-i\xi,0)]^{-1} = \begin{bmatrix}\widehat{\Gamma}^{(1)}(\xi) & \widehat{\Gamma}^{(2)}(\xi) & \widehat{\Gamma}^{(5)}(\xi)\\ \widehat{\Gamma}^{(3)}(\xi) & \widehat{\Gamma}^{4)}(\xi) & \widehat{\Gamma}^{(6)}(\xi)\\ \widehat{\Gamma}^{(7)}(\xi) & \widehat{\Gamma}^{(8)}(\xi) & \widehat{\Gamma}^{(9)}(\xi)\end{bmatrix}_{7\times 7}, \quad (3.53)$$

where

$$\begin{aligned}
\widehat{\Gamma}^{(j)}(\xi) &= [\widehat{\Gamma}^{(j)}_{pq}(\xi)]_{3\times 3} = \widetilde{a}_j(\xi)I_3 + \widetilde{b}_j(\xi)Q(\xi) + \widetilde{c}_j(\xi)R(\xi), \quad j=\overline{1,4},\\
\widehat{\Gamma}^{(l)}(\xi) &= [\widehat{\Gamma}^{(l)}_{pq}(\xi)]_{3\times 1} = \widetilde{c}_l(\xi)\xi^\top, \quad l=5,6,\\
\widehat{\Gamma}^{(m)}(\xi) &= [0]_{3\times 1}, \quad m=7,8,\\
\widehat{\Gamma}^{(9)}(\xi) &= -\frac{1}{\kappa' r^2}, \quad r=|\xi|,
\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}
\widetilde{a}_1(\xi) &= \frac{1}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\Big\{-d_1(\gamma+\varepsilon)r^2 -\\
&\quad -4\big[\alpha d_1 + \alpha\mu(\gamma+\varepsilon) + 4\nu(\alpha\varkappa - \mu\nu)\big] - \frac{16\alpha^2\mu}{r^2}\Big\},\\
\widetilde{a}_2(\xi) = \widetilde{a}_3(\xi) &= \frac{1}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\times\\
&\quad \times\Big\{d_1(\varkappa+\nu)r^2 + 4\alpha\big[\mu(\varkappa+\nu) - 2(\mu\nu-\alpha\varkappa)\big]\Big\},\\
\widetilde{a}_4(\xi) &= -\frac{\mu+\alpha}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\,(d_1r^2+4\alpha\mu),\\
\widetilde{b}_1(\xi) &= \frac{1}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\Big\{d_1(\gamma+\varepsilon) +\\
&\quad +4\big[\alpha d_1 + \alpha\mu(\gamma+\varepsilon) - 4\nu(\mu\nu-\alpha\varkappa)\big]\,\frac{1}{r^2} + \frac{16\alpha^2\mu}{r^4}\Big\} -\\
&\quad -\frac{1}{d_2r^2(r^2+\lambda_1^2)}\Big(\beta+2\gamma+\frac{4\alpha}{r^2}\Big),\\
\widetilde{b}_2(\xi) = \widetilde{b}_3(\xi) &= -\frac{1}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\times\\
&\quad \times\Big\{d_1(\varkappa+\nu) + 4\alpha\big[\mu(\varkappa+\nu) - 2(\mu\nu-\alpha\varkappa)\big]\,\frac{1}{r^2}\Big\} +\\
&\quad +\frac{\delta+2\varkappa}{d_2r^2(r^2+\lambda_1^2)}\,,\\
\widetilde{b}_4(\xi) &= \frac{\mu+\alpha}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\Big(d_1+\frac{4\alpha\mu}{r^2}\Big) - \frac{\lambda+2\mu}{d_2r^2(r^2+\lambda_1^2)}\,,\\
\widetilde{c}_1(\xi) &= \frac{4i}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\Big[-\nu d_1 + (\gamma+\varepsilon)(\mu\nu-\alpha\varkappa) - \frac{4\alpha^2\varkappa}{r^2}\Big],\\
\widetilde{c}_2(\xi) = \widetilde{c}_3(\xi) &= \frac{2i}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\Big[\alpha d_1 - 2(\varkappa+\nu)(\mu\nu-\alpha\varkappa) + \frac{4\alpha^2\mu}{r^2}\Big],\\
\widetilde{c}_4(\xi) &= \frac{4i(\mu+\alpha)(\mu\nu-\alpha\varkappa)}{d_1^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}\,,\\
\widetilde{c}_5(\xi) &= \frac{i}{\kappa' d_2r^2(r^2+\lambda_1^2)}\Big[\zeta(\delta+2\varkappa) - \eta(\beta+2\gamma) - \frac{4\alpha\eta}{r^2}\Big],
\end{aligned}$$

$$\widetilde{c}_6(\xi)=\frac{i}{\kappa' d_2 r^2(r^2+\lambda_1^2)}\,\big[\eta(\delta+2\varkappa)-\zeta(\lambda+2\mu)\big].$$

Here

$$\begin{aligned}
\lambda_1^2&=\frac{4\alpha(\lambda+2\mu)}{d_2}>0,\\
\lambda_{2,3}^2&=\frac{4}{d_1^2}\Big\{2(\mu\nu-\alpha\varkappa)^2-\alpha\mu d_1\pm\\
&\qquad\pm i2(\mu\nu-\alpha\varkappa)\sqrt{(\mu+\alpha)\big[\alpha(\mu\gamma-\varkappa^2)+\mu(\alpha\varepsilon-\nu^2)\big]}\Big\}.
\end{aligned}\tag{3.55}$$

We assume that $\lambda_1>0$, $\Im\lambda_2>0$ and $\Im\lambda_3>0$. Note that due to (2.26) the expression under the square root is positive. To find the inverse Fourier transform of the matrix $\widehat{\Gamma}(\xi)$ we apply the results collected in the Appendix B and the following decompositions:

$$\begin{aligned}
\frac{1}{(r^2-\lambda_2^2)(r^2-\lambda_3^2)}&=\frac{1}{\lambda_2^2-\lambda_3^2}\Big(\frac{1}{r^2-\lambda_2^2}-\frac{1}{r^2-\lambda_3^2}\Big),\\
\frac{r^2}{(r^2-\lambda_2^2)(r^2-\lambda_3^2)}&=\frac{1}{\lambda_2^2-\lambda_3^2}\Big(\frac{\lambda_2^2}{r^2-\lambda_2^2}-\frac{\lambda_3^2}{r^2-\lambda_3^2}\Big),\\
\frac{1}{r^2(r^2-\lambda_2^2)(r^2-\lambda_3^2)}&=\frac{1}{\lambda_2^2\lambda_3^2r^2}+\frac{1}{\lambda_2^2(\lambda_2^2-\lambda_3^2)(r^2-\lambda_2^2)}-\\
&\qquad-\frac{1}{\lambda_3^2(\lambda_2^2-\lambda_3^2)(r^2-\lambda_3^2)}\,,\\
\frac{1}{r^4(r^2-\lambda_2^2)(r^2-\lambda_3^2)}&=\frac{\lambda_2^2+\lambda_3^2}{\lambda_2^4\lambda_3^4r^2}+\frac{1}{\lambda_2^2\lambda_3^2r^4}+\frac{1}{\lambda_2^4(\lambda_2^2-\lambda_3^2)(r^2-\lambda_2^2)}-\\
&\qquad-\frac{1}{\lambda_3^4(\lambda_2^2-\lambda_3^2)(r^2-\lambda_3^2)}\,,\\
\frac{1}{r^2(r^2+\lambda_1^2)}&=\frac{1}{\lambda_1^2}\Big(\frac{1}{r^2}-\frac{1}{r^2+\lambda_1^2}\Big),\\
\frac{1}{r^4(r^2+\lambda_1^2)}&=-\frac{1}{\lambda_1^4r^2}+\frac{1}{\lambda_1^2r^4}+\frac{1}{\lambda_1^4(r^2+\lambda_1^2)}\,.
\end{aligned}$$

Finally we arrive at the following explicit representation for the fundamental matrix

$$\begin{aligned}
\Gamma(x)=\mathcal{F}^{-1}_{\xi\to x}[\widehat{\Gamma}(\xi)]&=\begin{bmatrix}[\Gamma^{(1)}_{pq}(x)]_{3\times3} & [\Gamma^{(2)}_{pq}(x)]_{3\times3} & [\Gamma^{(5)}_{pq}(x)]_{3\times1}\\ [\Gamma^{(3)}_{pq}(x)]_{3\times3} & [\Gamma^{(4)}_{pq}(x)]_{3\times3} & [\Gamma^{(6)}_{pq}(x)]_{3\times1}\\ [\Gamma^{(7)}_{pq}(x)]_{1\times3} & [\Gamma^{(8)}_{pq}(x)]_{1\times3} & \Gamma^{(9)}(x)\end{bmatrix}_{7\times7}=\\
&=\frac{1}{4\pi}\begin{bmatrix}\widetilde{\Psi}_1(x)I_3 & \widetilde{\Psi}_2(x)I_3 & [0]_{3\times1}\\ \widetilde{\Psi}_3(x)I_3 & \widetilde{\Psi}_4(x)I_3 & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & \widetilde{\Psi}_5(x)\end{bmatrix}_{7\times7}-
\end{aligned}$$

$$
-\frac{1}{4\pi}\begin{bmatrix} Q(\partial)\widetilde{\Psi}_6(x) & Q(\partial)\widetilde{\Psi}_7(x) & [0]_{3\times1} \\ Q(\partial)\widetilde{\Psi}_8(x) & Q(\partial)\widetilde{\Psi}_9(x) & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & 0 \end{bmatrix}_{7\times7} +
$$

$$
+\frac{1}{4\pi}\begin{bmatrix} R(\partial)\widetilde{\Psi}_{10}(x) & R(\partial)\widetilde{\Psi}_{11}(x) & \nabla^\top\widetilde{\Psi}_{14}(x) \\ R(\partial)\widetilde{\Psi}_{12}(x) & R(\partial)\widetilde{\Psi}_{13}(x) & \nabla^\top\widetilde{\Psi}_{15}(x) \\ [0]_{1\times3} & [0]_{1\times3} & 0 \end{bmatrix}_{7\times7}, \tag{3.56}
$$

where the operators $Q(\partial)$, $R(\partial)$ and ∇ are defined by (2.7) and

$$
\widetilde{\Psi}_1(x) = -\frac{\gamma+\varepsilon}{d_1|x|} - \frac{1}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big\{d_1(\gamma+\varepsilon)\lambda_j^2 +
$$

$$
+4\big[\alpha d_1+\alpha\mu(\gamma+\varepsilon)+4\nu(\alpha\varkappa-\mu\nu)\big]+\frac{16\alpha^2\mu}{\lambda_j^2}\Big\}\,\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$
\widetilde{\Psi}_2(x) = \widetilde{\Psi}_3(x) = \frac{\varkappa+\nu}{d_1|x|} + \frac{1}{d_1^2(\lambda_2^2-\lambda_3^2)}\times
$$

$$
\times\sum_{j=2}^{3}(-1)^j\Big\{d_1(\varkappa+\nu)\lambda_j^2+4\alpha\big[\mu(\varkappa+\nu)+2(\alpha\varkappa-\mu\nu)\big]\Big\}\,\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$
\widetilde{\Psi}_4(x) = -\frac{\mu+\alpha}{d_1|x|} - \frac{\mu+\alpha}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j(d_1\lambda_j^2+4\alpha\mu)\,\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$
\widetilde{\Psi}_5(x) = -\frac{1}{\kappa'|x|}\,,
$$

$$
\widetilde{\Psi}_6(x) = -\frac{(\lambda+\mu)|x|}{2\mu(\lambda+2\mu)} + \frac{(\delta+2\varkappa)^2 d_2}{4\alpha(\lambda+2\mu)^2}\,\frac{e^{-\lambda_1|x|}-1}{|x|} +
$$

$$
+\frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\Big\{\frac{\gamma+\varepsilon}{d_1}+\frac{4}{d_1^2\lambda_j^2}\,\big[\alpha d_1+\alpha\mu(\gamma+\varepsilon)+4\nu(\alpha\varkappa-\mu\nu)\big]+
$$

$$
+\frac{16\alpha^2\mu}{d_1^2\lambda_j^4}\Big\}\,\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$
\widetilde{\Psi}_7(x) = \widetilde{\Psi}_8(x) = -\frac{\delta+2\varkappa}{4\alpha(\lambda+2\mu)}\,\frac{e^{-\lambda_1|x|}-1}{|x|} -
$$

$$
-\frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\Big\{\frac{\varkappa+\nu}{d_1}+\frac{4\alpha}{d_1^2\lambda_j^2}\,\big[\mu(\varkappa+\nu)+2(\alpha\varkappa-\mu\nu)\big]\Big\}\,\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$
\widetilde{\Psi}_9(x) = \frac{1}{4\alpha}\,\frac{e^{-\lambda_1|x|}-1}{|x|} + \frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\,\frac{\mu+\alpha}{d_1^2}\,\Big(d_1+\frac{4\alpha\mu}{\lambda_j^2}\Big)\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$
\widetilde{\Psi}_{10}(x) = \frac{4}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big[\nu d_1+(\gamma+\varepsilon)(\alpha\varkappa-\mu\nu)+\frac{4\alpha^2\varkappa}{\lambda_j^2}\Big]\,\frac{e^{i\lambda_j|x|}-1}{|x|}\,,
$$

$$\widetilde{\Psi}_{11}(x)=\widetilde{\Psi}_{12}(x)=\frac{2}{d_1^2(\lambda_2^2-\lambda_3^2)}\times$$
$$\times\sum_{j=2}^{3}(-1)^j\Big[2(\varkappa+\nu)(\mu\nu-\alpha\varkappa)-\alpha d_1-\frac{4\alpha^2\mu}{\lambda_j^2}\Big]\frac{e^{i\lambda_j|x|}-1}{|x|},$$
$$\widetilde{\Psi}_{13}(x)=\frac{4(\mu+\alpha)(\alpha\varkappa-\mu\nu)}{d_1^2(\lambda_2^2-\lambda_3^2)}\,\frac{e^{i\lambda_2|x|}-e^{i\lambda_3|x|}}{|x|},$$
$$\widetilde{\Psi}_{14}(x)=-\frac{\eta|x|}{2\kappa'(\lambda+2\mu)}+\big[\zeta(\lambda+2\mu)-\eta(\delta+2\varkappa)\big]\frac{\delta+2\varkappa}{4\alpha\kappa'(\lambda+2\mu)^2}\,\frac{e^{-\lambda_1|x|}-1}{|x|},$$
$$\widetilde{\Psi}_{15}(x)=\frac{\eta(\delta+2\varkappa)-\zeta(\lambda+2\mu)}{4\alpha\kappa'(\lambda+2\mu)}\,\frac{e^{-\lambda_1|x|}-1}{|x|}.$$

One can easily verify that in a vicinity of the origin and at infinity the fundamental matrix has the following asymptotic behaviour:

$$\Gamma(x)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(1)]_{3\times1}\\ [\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(1)]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{7\times7}\quad\text{as }|x|\to0, \tag{3.57}$$

$$\Gamma(x)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(|x|^{-2})]_{3\times3} & [\varkappa_0\frac{x_j}{|x|}+\mathcal{O}(|x|^{-2})]_{3\times1}\\ [\mathcal{O}(|x|^{-2})]_{3\times3} & [\mathcal{O}(|x|^{-2})]_{3\times3} & [\mathcal{O}(|x|^{-2})]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{7\times7} \tag{3.58}$$
$$\text{as }|x|\to\infty,$$

with $\varkappa_0=-\frac{\eta}{2(\lambda+2\mu)}$.

Remark 3.3. Note that from the above results we can obtain the explicit expression for the fundamental matrix $\widetilde{\Gamma}(x)$ of the operator of statics $\widetilde{L}(\partial,0)$ of the hemitropic elasticity when the thermal effects are not taken into consideration (see (2.14)). We have to set $\eta=\zeta=0$ and delete in (3.56) the seventh column and the seventh row. We arrive at the formula

$$\widetilde{\Gamma}(x)=\mathcal{F}_{\xi\to x}\begin{bmatrix}\widehat{\Gamma}^{(1)}(\xi) & \widehat{\Gamma}^{(2)}(\xi)\\ \widehat{\Gamma}^{(3)}(\xi) & \widehat{\Gamma}^{4)}(\xi)\end{bmatrix}_{6\times6}=\begin{bmatrix}[\Gamma^{(1)}_{pq}(x)]_{3\times3} & [\Gamma^{(2)}_{pq}(x)]_{3\times3}\\ [\Gamma^{(3)}_{pq}(x)]_{3\times3} & [\Gamma^{(4)}_{pq}(x)]_{3\times3}\end{bmatrix}_{6\times6}=$$
$$=\frac{1}{4\pi}\begin{bmatrix}\widetilde{\Psi}_1(x)I_3 & \widetilde{\Psi}_2(x)I_3\\ \widetilde{\Psi}_2(x)I_3 & \widetilde{\Psi}_4(x)I_3\end{bmatrix}_{6\times6}-\frac{1}{4\pi}\begin{bmatrix}Q(\partial)\widetilde{\Psi}_6(x) & Q(\partial)\widetilde{\Psi}_7(x)\\ Q(\partial)\widetilde{\Psi}_7(x) & Q(\partial)\widetilde{\Psi}_9(x)\end{bmatrix}_{6\times6}+$$
$$+\frac{1}{4\pi}\begin{bmatrix}R(\partial)\widetilde{\Psi}_{10}(x) & R(\partial)\widetilde{\Psi}_{11}(x)\\ R(\partial)\widetilde{\Psi}_{11}(x) & R(\partial)\widetilde{\Psi}_{13}(x)\end{bmatrix}_{6\times6}, \tag{3.59}$$

where the functions $\widetilde{\Psi}_j$ are as above. From the explicit form of the functions $\widetilde{\Psi}_j$ and formula (3.59) it follows that for sufficiently large $|x|$ (i.e., as $|x|\to\infty$) we have the relations

$$\Gamma^{(1)}_{pq}(x)=\mathcal{O}(|x|^{-1}),\ \ \Gamma^{(j)}_{pq}(x)=\mathcal{O}(|x|^{-2}),\ \ j=2,3,4,\ \ p,q=1,2,3, \tag{3.60}$$

while $\widetilde{\Gamma}(x) = [\mathcal{O}(|x|^{-1})]_{6\times 6}$ as $|x| \to 0$. These asymptotic relations can be differentiated any times with respect to the variables x_1, x_2, x_3.

3.2. **Principal singular parts of the fundamental matrices.** In this subsection we will write down explicitly the principal singular part of the fundamental matrices (3.47) and (3.56). This principal part $\Gamma_0(x)$ represents a 7×7 fundamental matrix of the operator $L_0(\partial)$ defined by (2.12) and solves the equation:

$$L_0(\partial)\Gamma_0(x) = \delta(x) I_7. \tag{3.61}$$

It is clear that

$$\Gamma_0(x) = \begin{bmatrix} \widetilde{\Gamma}_0(x) & [0]_{6\times 1} \\ [0]_{1\times 6} & \Gamma_0^{(9)}(x) \end{bmatrix}_{7\times 7}, \tag{3.62}$$

where

$$\Gamma_0^{(9)}(x) = -\frac{1}{4\pi\kappa'|x|},$$

and $\widetilde{\Gamma}_0(x) = [\widetilde{\Gamma}_{0pq}(x)]_{6\times 6}$ is a homogeneous of order -1 fundamental matrix of the operator $\widetilde{L}_0(\partial)$ defined by (2.14). This matrix is constructed in [44] explicitly and has the form

$$\widetilde{\Gamma}_0(x) = -\frac{1}{8\pi d_1 d_2 |x|} \left\{ \begin{bmatrix} d_1^* I_3 & d_2^* I_3 \\ d_2^* I_3 & d_3^* I_3 \end{bmatrix} - \frac{1}{|x|^2} \begin{bmatrix} d_4^* Q(x) & d_5^* Q(x) \\ d_5^* Q(x) & d_6^* Q(x) \end{bmatrix} \right\}, \tag{3.63}$$

where

$$d_1^* := d_2(\gamma+\varepsilon) + d_1(\beta + 2\gamma), \quad d_2^* := -\big[d_2(\varkappa+\nu) + d_1(\delta + 2\varkappa)\big],$$
$$d_3^* := d_2(\mu+\alpha) + d_1(\lambda+2\mu), \quad d_4^* := d_1(\beta+2\gamma) - d_2(\gamma+\varepsilon),$$
$$d_5^* := -\big[d_1(\delta+2\varkappa) - d_2(\varkappa+\nu)\big], \quad d_6^* := d_1(\lambda+2\mu) - d_2(\mu+\alpha).$$

Note that $\Gamma_0(x) = \Gamma_0^\top(x) = \Gamma_0(-x)$ and the entries of the matrix $\Gamma_0(x)$ are homogeneous functions of order -1. For an arbitrary multi–index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and an arbitrary complex number σ it can easily be shown that in a neighbourhood of the origin (i.e., for small $|x|$)

$$\partial^\alpha \big[\Gamma(x,\sigma) - \Gamma_0(x)\big] = \mathcal{O}(|x|^{-|\alpha|}), \ \ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \tag{3.64}$$

which shows that $\Gamma_0(x)$ is a principal singular part of the matrices $\Gamma(x,\sigma)$ and $\Gamma(x)$.

3.3. **Special representation of the principal singular part.** In this subsection we derive some formulae which will help us to calculate the principal symbol matrices of the boundary integral (pseudodifferential) operators generated by the boundary layer potentials.

Due to the evenness of the entries of the matrix $L_0(\xi)$ we have

$$\Gamma_0(x) = -\frac{1}{8\pi^3} \int\limits_{\mathbb{R}^3} [L_0(\xi)]^{-1} e^{-ix\cdot\xi}\, d\xi = -\frac{1}{8\pi^3} \int\limits_{\mathbb{R}^3} [L_0(\xi)]^{-1} e^{ix\cdot\xi}\, d\xi, \tag{3.65}$$

where the above formal integrals are understood as generalized Fourier transforms, i.e.,

$$\Gamma_0(x) = -\mathcal{F}^{-1}[L_0^{-1}(\xi)] = -\frac{1}{8\pi^3}\,\mathcal{F}[L_0^{-1}(\xi)].$$

One can show that $L_0(\xi)$ is a positive definite matrix for $\xi \in \mathbb{R}^3 \setminus \{0\}$, since $\widetilde{L}_0(\xi)$ is positive definite (see [44]).

Let $E = [e_{kj}]_{3\times 3} : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal matrix with $\det E = 1$,

$$EE^\top = E^\top E = I_3. \tag{3.66}$$

Then

$$\begin{aligned}\Gamma_0(Ex) &= -\frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3}[L_0(\xi)]^{-1}e^{-iEx\cdot\xi}\,d\xi = -\frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3}\big[L_0(E\xi)\big]^{-1}e^{-ix\cdot\xi}\,d\xi = \\ &= \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2}e^{-i\widetilde{x}\cdot\widetilde{\xi}}\bigg\{-\frac{1}{2\pi}\int\limits_{\mathbb{R}^1}\big[L_0(E\xi)\big]^{-1}e^{-ix_3\xi_3}\,d\xi_3\bigg\}\,d\widetilde{\xi},\end{aligned} \tag{3.67}$$

where $\widetilde{x} = (x_1, x_2)$, $\widetilde{\xi} = (\xi_1, \xi_2)$, i.e.,

$$\Gamma_0(Ex) = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\bigg[-\frac{1}{2\pi}\int\limits_{\mathbb{R}^1}\big[L_0(E\xi)\big]^{-1}e^{-ix_3\xi_3}\,d\xi_3\bigg]. \tag{3.68}$$

This implies (due to the Cauchy integral theorem for analytic functions)

$$\begin{aligned}\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}\big[\Gamma_0(Ex)\big] &= -\frac{1}{2\pi}\int\limits_{\mathbb{R}^1}\big[L_0(E\xi)\big]^{-1}e^{-ix_3\xi_3}\,d\xi_3 = \\ &= \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int\limits_{\ell^+}\big[L_0(E\xi)\big]^{-1}e^{-ix_3\xi_3}\,d\xi_3 & \text{for } x_3 \le 0, \\ +\dfrac{1}{2\pi}\displaystyle\int\limits_{\ell^-}\big[L_0(E\xi)\big]^{-1}e^{-ix_3\xi_3}\,d\xi_3 & \text{for } x_3 \ge 0,\end{cases}\end{aligned} \tag{3.69}$$

where ℓ^+ [resp. ℓ^-] is a closed simple curve of positive, counter clockwise orientation in the upper [resp. lower] half-plane of the complex ξ_3-plane ($\xi_3 = \xi_3' + i\xi_3''$) enveloping all the roots (with respect to ξ_3) of the equation $\det L_0(E\xi) = 0$ with positive [resp. negative] imaginary parts. Clearly, (3.69) does not depend on the shape of ℓ^+ [resp. ℓ^-].

The integration in (3.69) is performed counter clockwise. It can easily be shown that the entries of the matrix (3.69) with $x_3 = 0$ are homogeneous functions in $\widetilde{\xi}$ of order -1. Moreover, from (3.69) it follows that the matrix $\big[-\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\Gamma_0(Ex)]\big|_{x_3=0}\big]_{7\times 7}$ is positive definite for arbitrary $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$ due to the positive definiteness of the matrix $L_0(E\xi)$. As we will see below this matrix with opposite sign represents the principal homogeneous symbol of the single-layer potential associated with the matrix $\Gamma(\cdot, \sigma)$.

3.4. **Integral representation formulae of solutions.** Let us introduce the generalized single and double layer potentials, and the Newton type volume potential

$$V(\varphi)(x) = \int\limits_S \Gamma(x-y,\sigma)\varphi(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{3.70}$$

$$W(\varphi)(x) = \int\limits_S \big[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\big]^\top \varphi(y)\,dS_y, \ x \in \mathbb{R}^3 \setminus S, \tag{3.71}$$

$$N_{\Omega^\pm}(\psi)(x) = \int\limits_{\Omega^\pm} \Gamma(x-y,\sigma)\psi(y)\,dy, \quad x \in \mathbb{R}^3, \tag{3.72}$$

where $\mathcal{P}^*(\partial, n)$ is the boundary differential operator defined by (2.21), $\Gamma(\cdot,\sigma)$ is the fundamental matrix given by (3.44) or (3.47), $\varphi = (\varphi_1, \dots, \varphi_7)^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1, \dots, \psi_7)^\top$ is defined on $\Omega^\pm$ and we assume that in the case of Ω^- the support of the density vector-function ψ of the Newtonian potential is a compact set.

Due to the equality

$$\sum_{j=1}^{7} L_{kj}(\partial_x,\sigma)\Big(\big[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\big]^\top\Big)_{jp} =$$

$$= \sum_{j,q=1}^{7} L_{kj}(\partial_x,\sigma)\mathcal{P}^*_{pq}(\partial_y, n(y))\Gamma_{jq}(x-y,\sigma) =$$

$$= \sum_{j,q=1}^{7} \mathcal{P}^*_{pq}(\partial_y, n(y))L_{kj}(\partial_x,\sigma)\Gamma_{jq}(x-y,\sigma) = 0, \quad x \neq y, \quad k,p = \overline{1,7},$$

it can easily be checked that the potentials defined by (3.70) and (3.71) are C^∞–smooth in $\mathbb{R}^3 \setminus S$ and solve the homogeneous equation $L(\partial,\sigma)U(x) = 0$ in $\mathbb{R}^3 \setminus S$ for an arbitrary L_p-summable vector function φ. The volume potential solves the nonhomogeneous equation

$$L(\partial,\sigma)N_{\Omega^\pm}(\psi) = \psi \quad \text{in} \quad \Omega^\pm \quad \text{for} \quad \psi \in [C^{0,\kappa}(\overline{\Omega^\pm})]^7. \tag{3.73}$$

The relation (3.73) holds true for an arbitrary $\psi \in [L_p(\Omega^\pm)]^7$ with $1<p<\infty$.

With the help of Green's formula (2.38) and Remark 3.2 by standard arguments we can prove the following assertions (cf., e.g., [36], [7], [37], [43], Ch. I, Lemma 2.1; Ch. II, Lemma 8.2).

Theorem 3.4. *Let $S = \partial\Omega^+$ be $C^{1,\kappa}$-smooth with $0 < \kappa \leq 1$, either $\sigma = 0$ or $\sigma = \sigma_1 + i\sigma_2$ with $\sigma_2 > 0$, and let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^7$. Then there holds the integral representation formula*

$$W(\{U\}^+)(x) - V(\{\mathcal{P}U\}^+)(x) + N_{\Omega^+}(L(\partial,\sigma)U)(x) =$$

$$= \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases} \tag{3.74}$$

This formula can be extended to Lipschitz domains and to vector functions satisfying the conditions $U \in [W^1_p(\Omega^+)]^7$ *and* $L(\partial,\sigma)U \in [L_p(\Omega^+)]^7$ *with* $1 < p < \infty$.

Proof. For the smooth case it easily follows from Green's formula (2.38) with the domain of integration $\Omega^+ \setminus B(x,\varepsilon')$, where $x \in \Omega^+$ is treated as a fixed parameter, $B(x,\varepsilon')$ is a ball centered at the point x and radius $\varepsilon' > 0$ and $\overline{B(x,\varepsilon')} \subset \Omega^+$. One needs to take the j-th column of the fundamental matrix $\Gamma^*(y-x,\sigma)$ for $V(y)$, calculate the surface integrals over the sphere $\Sigma(x,\varepsilon') := \partial B(x,\varepsilon')$ and pass to the limit as $\varepsilon' \to 0$.

The second part of the theorem can be shown by standard limiting procedure. □

Similar representation formula holds in the exterior domain Ω^- if a vector U and its derivatives possess some asymptotic properties at infinity. In particular, the following assertion holds.

Theorem 3.5. *Let* $S = \partial\Omega^-$ *be* $C^{1,\kappa}$*-smooth with* $0 < \kappa \le 1$ *and let* U *be a regular vector of the class* $[C^2(\overline{\Omega^-})]^7$*, such that for any multi-index* $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ *with* $0 \le |\alpha| = \alpha_1+\alpha_2+\alpha_3 \le 2$*, the function* $\partial^\alpha U_j$ *is polynomially bounded at infinity, i.e., for sufficiently large* $|x|$

$$\big|\partial^\alpha U_j(x)\big| \le C_0|x|^m, \ \ j = \overline{1,7}, \tag{3.75}$$

with some constants m *and* $C_0 > 0$*. Then there holds the integral representation formula*

$$-W(\{U\}^-)(x) + V(\{\mathcal{P}U\}^-)(x) + N_{\Omega^-}(L(\partial,\sigma)U)(x) =$$
$$= \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-, \end{cases} \tag{3.76}$$

with $\sigma = \sigma_1 + i\sigma_2$*, where* $\sigma_2 > 0$.

This formula can be extended to Lipschitz domains and to vector functions satisfying the conditions: $U \in [W^1_{p,loc}(\Omega^-)]^7$, $L(\partial,\sigma)U \in [L_{p,loc}(\Omega^-)]^7$ *with* $1 < p < \infty$ *and* $L(\partial,\sigma)U(x)$ *is polynomially bounded at infinity.*

Proof. The proof immediately follows from Theorem 3.4 and Remark 3.1. Indeed, one needs to write the integral representation formula (3.74) for bounded domain $\Omega^- \cap B(0,R)$, send then R to $+\infty$ and take into consideration that the surface integral over $\Sigma(0,R)$ tends to zero due to the conditions (3.75) and the exponential decay of the fundamental matrix at infinity.

The second part of the theorem again can be shown by standard limiting procedure. □

Corollary 3.6. *Let $\sigma = \sigma_1 + i\sigma_2$ with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$, and U be a solution to the homogeneous equations $L(\partial,\sigma)U = 0$ in $\Omega^\pm$ satisfying the condition* (3.75) *and $U \in [C^{1,\kappa}(\overline{\Omega^\pm})]^7$ with some $0 < \kappa \leq 1$. Then the following representation formula holds*

$$U(x) = W([U]_S)(x) - V([\mathcal{P}U]_S)(x), \quad x \in \Omega^\pm, \tag{3.77}$$

where $[U]_S = \{U\}^+ - \{U\}^-$ and $[\mathcal{P}U]_S = \{\mathcal{P}U\}^+ - \{\mathcal{P}U\}^-$ on S.

Proof. It immediately follows from Theorems 3.4 and 3.5. □

4. Uniqueness Theorems for Unbounded Domains

4.1. **Uniqueness results for pseudo-oscillation problems.** Here we prove the counterpart of Theorem 2.2 for the exterior BVPs.

Theorem 4.1. *Let $\sigma = \sigma_1 + i\sigma_2$ with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$, and $\Phi^{(-)} \in [L_{2,comp}(\Omega^-)]^7$. Then the boundary value problems $(I^{(\sigma)})^-$, $(II^{(\sigma)})^-$ and $(III^{(\sigma)})^-$ have at most one solution in the class of vector functions which are polynomially bounded at infinity and belong to the space $[W^1_{2,loc}(\Omega^-)]^7$.*

Proof. Let $U^{(1)} = (u^{(1)}, \omega^{(1)}, \vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \omega^{(2)}, \vartheta^{(2)})^\top$ be two solutions of the BVP $(K^{(\sigma)})^-$, $K = I, II, III$. Denote $U := U^{(1)} - U^{(2)}$. The vector $U \in [W^1_{2,loc}(\Omega^-)]^7$ is polynomially bounded at infinity and solves the corresponding homogeneous BVP. By Theorem 3.5, it follows that $U = (\widetilde{U}, \vartheta)^\top$ with $\widetilde{U} = (u, \omega)^\top$ actually decays exponentially at infinity. Therefore we have the following Green's formula (cf. (2.39))

$$\begin{aligned} - \big\langle \{U'\}^-, \{\mathcal{P}(\partial,n)U\}^- \big\rangle_{\partial\Omega^-} &= \int\limits_{\Omega^-} U' \cdot L(\partial,\sigma)U \, dx + \\ + \int\limits_{\Omega^-} \Big[E(\widetilde{U}', \widetilde{U}) - \varrho\sigma^2 u' \cdot u - \mathcal{I}\sigma^2 \omega' \cdot \omega - \eta\vartheta \operatorname{div} u' - \zeta\vartheta \operatorname{div} \omega' - i\eta\sigma\vartheta' \operatorname{div} u - \\ - i\zeta\sigma\vartheta' \operatorname{div} \omega - i\sigma\kappa''\vartheta'\vartheta + \kappa' \operatorname{grad}\vartheta' \cdot \operatorname{grad}\vartheta \Big] \, dx. \end{aligned} \tag{4.1}$$

where $\langle \cdot\,, \cdot \rangle_{\partial\Omega^-}$ denotes the duality between the spaces $[H_2^{1/2}(\partial\Omega^-)]^7$ and $[H_2^{-1/2}(\partial\Omega^-)]^7$. Now, by the same approach as in the proof of Theorem 2.2, we arrive at the relation

$$\int\limits_{\Omega^-} \Big[E(\widetilde{U}, \overline{\widetilde{U}}) - \varrho\sigma^2 |u|^2 - \mathcal{I}\sigma^2 |\omega|^2 + \kappa' C_0 |\nabla\vartheta|^2 + \kappa'' |\vartheta|^2 \Big] \, dx = 0, \tag{4.2}$$

where $E(U, \overline{\widetilde{U}})$ and C_0 are given by (2.30) and (2.49) respectively. Whence, by the word for word arguments applied in the proof of Theorem 2.2, we derive that $U = 0$ in Ω^-. □

4.2. **Uniqueness results for static problems.** For the readers convenience let us formulate here the exterior BVP of statics.

Problem $(I^{(0)})^-$**.** Find a solution vector $U=(u,\omega,\vartheta)^\top \in [W^1_{2,loc}(\Omega^-)]^7$ to the equilibrium equation

$$L(\partial,0)U=\Phi \quad \text{in} \quad \Omega^-, \tag{4.3}$$

satisfying the Dirichlet type boundary condition

$$\{U\}^-=f \quad \text{on} \quad S=\partial\Omega^-. \tag{4.4}$$

Problem $(II^{(0)})^-$**.** Find a solution vector $U=(u,\omega,\vartheta)^\top \in [W^1_{2,loc}(\Omega^-)]^7$ to the equation (4.3) satisfying the Neumann type boundary condition

$$\{\mathcal{P}(\partial,n)U\}^-=F \quad \text{on} \quad S. \tag{4.5}$$

Problem $(III^{(0)})^-$**.** Find a solution vector $U=(u,\omega,\vartheta)^\top \in [W^1_{2,loc}(\Omega^-)]^7$ to the equation (4.3) satisfying mixed type boundary conditions

$$\{U\}^-=f^{(D)} \quad \text{on} \quad S_D, \tag{4.6}$$

$$\{\mathcal{P}(\partial,n)U\}^-=F^{(N)} \quad \text{on} \quad S_N. \tag{4.7}$$

As above, we assume that $\Phi=(\widetilde{\Phi},\Phi_7)^\top$ with $\widetilde{\Phi}=(\Phi_1,\dots,\Phi_6)^\top$ has a compact support and the boundary data are as in (2.48) with $p=2$. The equation (4.3) is again understood in the distributional sense, and the Dirichlet and Neumann type boundary conditions in the usual trace and generalized trace sense.

It is easy to see that the BVPs for the temperature function $\vartheta\in W^1_{2,loc}(\Omega^-)$ are separated as independent BVPs for the Laplace equation

$$\kappa'\Delta\vartheta=\Phi_7 \quad \text{in} \quad \Omega^-, \tag{4.8}$$

with the Dirichlet boundary condition

$$\{\vartheta\}^-=f_7 \quad \text{on} \quad S, \tag{4.9}$$

or with the Neumann boundary condition

$$\kappa'\{\partial_n\vartheta\}^-=F_7 \quad \text{on} \quad S \tag{4.10}$$

or with the mixed boundary conditions

$$\{\vartheta\}^-=f_7^{(D)} \quad \text{on} \quad S_D, \quad \kappa'\{\partial_n\vartheta\}^-=F_7^{(N)} \quad \text{on} \quad S_N. \tag{4.11}$$

If we require that ϑ vanishes at infinity, $\vartheta=o(1)$ as $|x|\to\infty$, then these problems are uniquely solvable and, since Φ_7 has a compact support, we have the following asymptotic for sufficiently large $|x|$ (see, e.g., [19])

$$\partial^\alpha\vartheta(x)=\mathcal{O}(|x|^{-1-|\alpha|}), \quad |x|\to\infty, \tag{4.12}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is an arbitrary multi-index. More precisely,

$$\vartheta(x) = \frac{\theta_0}{|x|} + \mathcal{O}(|x|^{-2}), \tag{4.13}$$

$$\operatorname{grad} \vartheta(x) = -\frac{\theta_0}{|x|^3}\, x + \mathcal{O}(|x|^{-3}), \quad \vartheta_0 = const, \tag{4.14}$$

where

$$\theta_0 = \lim_{|x|\to\infty} |x|\vartheta(x) =$$

$$= -\frac{1}{4\pi} \int\limits_S \{\partial_n \vartheta(x)\}^- dS - \frac{1}{4\pi\kappa'} \int\limits_{\Omega^*} \Phi_7(x)\, dx, \quad \Omega^* = \Omega^- \cap \operatorname{supp} \Phi_7.$$

Assuming that ϑ is known, the above formulated BVPs can be reformulated as follows.

Problem $(\widetilde{I}^{(0)})^-$. Find a solution vector $\widetilde{U} = (u, \omega)^\top \in [W^1_{2,loc}(\Omega^-)]^6$ to the equation

$$\widetilde{L}(\partial, 0)\widetilde{U} = \widetilde{\Phi} + \widetilde{\Psi} \quad \text{in} \quad \Omega^-, \tag{4.15}$$

satisfying the Dirichlet type boundary condition

$$\{\widetilde{U}\}^- = \widetilde{f} \ \text{ on } \ S. \tag{4.16}$$

Problem $(\widetilde{II}^{(0)})^-$. Find a solution vector $\widetilde{U} = (u, \omega)^\top \in [W^1_{2,loc}(\Omega^-)]^6$ to the equation (4.15) satisfying the Neumann type boundary condition

$$\{T(\partial, n)\widetilde{U}\}^- = \widetilde{F} + \widetilde{G} \ \text{ on } \ S. \tag{4.17}$$

Problem $(\widetilde{III}^{(0)})^-$. Find a solution vector $\widetilde{U} = (u, \omega)^\top \in [W^1_{2,loc}(\Omega^-)]^6$ to the equation (4.15) satisfying the mixed boundary conditions

$$\{\widetilde{U}\}^- = \widetilde{f}^{(D)} \ \text{ on } \ S_D, \tag{4.18}$$

$$\{T(\partial, n)\widetilde{U}\}^- = \widetilde{F}^{(N)} + \widetilde{G} \ \text{ on } \ S_N. \tag{4.19}$$

Here $\widetilde{L}(\partial, 0)$ is defined by (2.14) with $\sigma = 0$, while $T(\partial, n)$ is given by (2.22), and

$$\widetilde{\Psi} = (\eta \operatorname{grad} \vartheta, \zeta \operatorname{grad} \vartheta)^\top \quad \text{in} \quad \Omega^-, \tag{4.20}$$

$$\widetilde{G} = (\eta\vartheta n, \zeta\vartheta n)^\top \quad \text{on} \quad S. \tag{4.21}$$

We see that the right hand side vector in equation (4.15) has not a compact support and it decays at infinity as $\mathcal{O}(|x|^{-2})$ due to (4.14), since $\widetilde{\Phi}$ has a compact support. Therefore, solutions to equation (4.15) do not vanish at infinity, in general.

To establish the asymptotic of solutions at infinity, we rewrite equation (4.15) in the form

$$\widetilde{L}(\partial,0)\widetilde{U} = -\frac{\theta_0}{|x|^3}\begin{bmatrix}\eta x\\ \zeta x\end{bmatrix}_{6\times 1} + \widetilde{\Psi}^{(1)} + \widetilde{\Phi}, \quad x \in \Omega^-, \tag{4.22}$$

where

$$\widetilde{\Psi}^{(1)}(x) = (\eta\psi(x), \zeta\psi(x))^\top, \quad \psi(x) = \operatorname{grad}\vartheta(x) + \frac{\theta_0}{|x|^3}\,x. \tag{4.23}$$

In view of (4.14), we have $\widetilde{\Psi}^{(1)}(x) = \mathcal{O}(|x|^{-3})$ as $|x| \to \infty$.

Now, we prove several technical lemmas. In what follows, without loss of generality, we assume that the origin lies in Ω^+.

Lemma 4.2. *A particular solution to the differential equation*

$$\widetilde{L}(\partial,0)\widetilde{U}(x) = -\frac{\theta_0}{|x|^3}\begin{bmatrix}\eta x\\ \zeta x\end{bmatrix}_{6\times 1}, \quad x \in \mathbb{R}^3 \setminus \{0\}, \tag{4.24}$$

reads as

$$\widetilde{U}^{(0)}(x) = (u^{(0)}(x), \omega^{(0)}(x))^\top := \theta_0\Big(\alpha_1\frac{x}{|x|}, \alpha_2\frac{x}{|x|^3}\Big)^\top, \tag{4.25}$$

where

$$\alpha_1 = \frac{\eta}{2(\lambda+2\mu)}, \quad \alpha_2 = \frac{\zeta(\lambda+2\mu) - \eta(\delta+2\kappa)}{4\alpha(\lambda+2\mu)}, \tag{4.26}$$

and $u^{(0)}(x) = \mathcal{O}(1)$ *and* $\omega^{(0)}(x) = \mathcal{O}(|x|^{-2})$ *as* $|x| \to \infty$.

Lemma 4.3. *Let*

$$\widetilde{U}^{(1)}(x) = (u^{(1)}(x), \omega^{(1)}(x))^\top := \\ := \int\limits_{\Omega^-} \widetilde{\Gamma}(x-y)\big[\widetilde{\Psi}^{(1)}(y) + \widetilde{\Phi}(y)\big]\,dy, \quad x \in \Omega^-, \tag{4.27}$$

where $\widetilde{\Psi}^{(1)}$ *and* $\widetilde{\Phi}$ *are as in* (4.22) *and* $\widetilde{\Gamma}$ *is the fundamental matrix of the operator* $\widetilde{L}(\partial,0)$ *given by* (3.59)*. Then the vector* (4.27) *belongs to the space* $[W^2_{2,loc}(\Omega^-)]^6$*, solves the differential equation*

$$\widetilde{L}(\partial,0)\widetilde{U}^{(1)} = \widetilde{\Psi}^{(1)} + \widetilde{\Phi} \quad \text{in} \quad \Omega^-, \tag{4.28}$$

and possesses the following asymptotic

$$\widetilde{U}^{(1)}(x) = (u^{(1)}(x), \omega^{(1)}(x))^\top = \begin{bmatrix}[\mathcal{O}(|x|^{-1}\ln|x|)]_{3\times 1}\\ [\mathcal{O}(|x|^{-2}\ln|x|)]_{3\times 1}\end{bmatrix} \quad \text{as} \quad |x| \to \infty. \tag{4.29}$$

Proof. Since $\widetilde{\Phi}$ has a compact support, by Remark 3.3 we conclude that

$$\int\limits_{\Omega^-} \widetilde{\Gamma}(x-y)\widetilde{\Phi}(y)\,dy = \begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{3\times 1}\\ [\mathcal{O}(|x|^{-2})]_{3\times 1}\end{bmatrix} \quad \text{as} \quad |x| \to \infty. \tag{4.30}$$

Further, we use the representation

$$\int\limits_{\Omega^-} \widetilde{\Gamma}(x-y)\widetilde{\Psi}^{(1)}(y)\,dy = \sum_{l=1}^{4}\int\limits_{\Omega_l} \widetilde{\Gamma}(x-y)\widetilde{\Psi}^{(1)}(y)\,dy, \tag{4.31}$$

where

$$\Omega^- = \bigcup_{l=1}^{4}\overline{\Omega_l}, \quad \Omega_1 = B\Big(O,\frac{1}{2}\,|x|\Big)\setminus\Omega^+, \quad \Omega_2 = B\Big(x,\frac{1}{2}\,|x|\Big)\cap\Omega^-,$$

$$\Omega_3 = \Big\{B\Big(O,\frac{3}{2}\,|x|\Big)\cap\Omega^-\Big\}\setminus\Big\{B\Big(O,\frac{1}{2}\,|x|\Big)\cup B\Big(x,\frac{1}{2}\,|x|\Big)\Big\},$$

$$\Omega_4 = \Omega^-\setminus B\Big(O,\frac{3}{2}\,|x|\Big).$$

We recall that $B(z,R)$ stands for a ball centered at z and radius R.

Applying the asymptotic relation $\widetilde{\Psi}^{(1)}(x) = \mathcal{O}(|x|^{-3})$ as $|x|\to\infty$ and properties of the fundamental matrix $\widetilde{\Gamma}$ exposed in Remark 3.3, one can easily derive that

$$\int\limits_{\Omega^-} \widetilde{\Gamma}(x-y)\widetilde{\Psi}^{(1)}(y)\,dy = \begin{bmatrix}\big[\mathcal{O}(|x|^{-1}\ln|x|)\big]_{3\times1}\\ \big[\mathcal{O}(|x|^{-2}\ln|x|)\big]_{3\times1}\end{bmatrix} \quad \text{as} \quad |x|\to\infty. \tag{4.32}$$

Thus, the relation (4.29) holds.

Equality (4.28) can be shown by standard arguments, since $\widetilde{\Psi}^{(1)}+\widetilde{\Phi}\in L_2(\Omega^-)$ (see, e.g., [37]). □

Lemma 4.4. *Any solution of the homogeneous equation*

$$\widetilde{L}(\partial,0)\widetilde{V} = 0 \quad in \quad \Omega^-, \tag{4.33}$$

which is bounded at infinity, has the following asymptotic

$$\widetilde{V}(x) = C + \begin{bmatrix}\big[\mathcal{O}(|x|^{-1})\big]_{3\times1}\\ \big[\mathcal{O}(|x|^{-2})\big]_{3\times1}\end{bmatrix} \quad as \quad |x|\to\infty, \tag{4.34}$$

where $C=(C_1,C_2,C_3,0,0,0)^\top$ with arbitrary constants C_j, $j=1,2,3$.

Proof. Let $\widetilde{V}$ be a bounded at infinity solution of equation (4.33). Due to the ellipticity of the operator $\widetilde{L}(\partial,0)$, we have the imbedding $\widetilde{V}\in[C^\infty(\Omega^-)]^6$.

Choose a number R such that $\overline{\Omega^+}\subset B(O,R)$ and extend the vector function $\widetilde{V}$ inside the ball $B(O,R)$ preserving the C^∞-smoothness. Denote the extended vector function by $\widetilde{W}$. Evidently, $\widetilde{W}\in[C^\infty(\mathbb{R}^3)]^6$ and

$$\widetilde{W}(x) = \widetilde{V}(x) \quad \text{for} \quad x\in\mathbb{R}^3\backslash\overline{B(O,R)}. \tag{4.35}$$

Moreover, in accordance with (4.33),

$$\widetilde{L}(\partial,0)\widetilde{W}(x) = \widetilde{H}(x) \quad \text{for} \quad x\in\mathbb{R}^3, \tag{4.36}$$

where $\widetilde{H}\in[C^\infty(\mathbb{R}^3)]^6$ and $\operatorname{supp}\widetilde{H}\subset B(O,R)$.

Applying the generalized Fourier transform to (4.36) leads to the following equation

$$\widetilde{L}(-i\xi,0)\widehat{\widetilde{W}}(x)=\widehat{\widetilde{H}},\ \ \xi\in\mathbb{R}^3, \tag{4.37}$$

which is understood in the sense of the Schwartz space of tempered distributions $\mathcal{S}'(\mathbb{R}^3)$. For the determinant of the matrix $\widetilde{L}(-i\xi,0)$ we have

$$\det\widetilde{L}(-i\xi,0)=d_1^2d_2|\xi|^6(|\xi|^2+\lambda_1^2)(|\xi|^2-\lambda_2^2)(|\xi|^2-\lambda_3^2),$$

where d_1 and d_2 are positive constants defined by (2.28), and $\lambda_1^2>0$, λ_2^2 and λ_3^2 are mutually conjugate complex constants given by (3.55). Therefore we see that $\det\widetilde{L}(-i\xi,0)=0$ only for $\xi=0$. Note that $\widetilde{L}^{-1}(-i\xi,0)$ is the Fourier transform of the fundamental matrix $\widetilde{\Gamma}(x)$ (see Remark 3.3)

$$\widetilde{L}^{-1}(-i\xi,0)=\widehat{\widetilde{\Gamma}}(\xi)=\begin{bmatrix}\widehat{\Gamma}^{(1)}(\xi) & \widehat{\Gamma}^{(2)}(\xi)\\ \widehat{\Gamma}^{(3)}(\xi) & \widehat{\Gamma}^{(4)}(\xi)\end{bmatrix}_{6\times 6}, \tag{4.38}$$

where $\widehat{\Gamma}^k(\xi)$, $k=\overline{1,4}$, are defined by (3.54). The entries of this matrix have the following weak singularities at the origin

$$\widehat{\widetilde{\Gamma}}(\xi)=\widetilde{L}^{-1}(-i\xi,0)=\begin{bmatrix}\big[\mathcal{O}(|\xi|^{-2})\big]_{3\times3} & \big[\mathcal{O}(|\xi|^{-1})\big]_{3\times3}\\ \big[\mathcal{O}(|\xi|^{-1})\big]_{3\times3} & \big[\mathcal{O}(1)\big]_{3\times3}\end{bmatrix}. \tag{4.39}$$

Therefore, form (4.37) we deduce

$$\widehat{\widetilde{W}}(\xi)=\widehat{\widetilde{\Gamma}}(\xi)\widehat{\widetilde{H}}(\xi)+\sum_{|\alpha|\le m}C_\alpha\delta^{(\alpha)}(\xi), \tag{4.40}$$

where $\delta(\cdot)$ is the Dirac distribution, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index, $\delta^{(\alpha)}=\partial^\alpha\delta$, C_α are arbitrary constant vectors and m is some nonnegative integer.

Since $\widetilde{H}$ has a compact support, its Fourier transform $\widehat{\widetilde{H}}$ is analytic and by the inverse Fourier transform we get from (4.40)

$$\widetilde{W}(x)=\mathcal{F}^{-1}_{\xi\to x}\big[\widehat{\widetilde{\Gamma}}(\xi)\widehat{\widetilde{H}}(\xi)\big]+\sum_{|\alpha|\le m}C_\alpha x^\alpha=$$

$$=\int\limits_{\Omega^{(1)}}\widetilde{\Gamma}(x-y)\widetilde{H}(y)dy+\sum_{|\alpha|\le m}C_\alpha x^\alpha,\ \ x\in\mathbb{R}^3, \tag{4.41}$$

where $\Omega^{(1)}=\operatorname{supp}\widetilde{H}$.

With the help of the asymptotic behaviour of the fundamental matrix $\widetilde{\Gamma}(x)$ at infinity, we derive

$$\int\limits_{\Omega^{(1)}}\widetilde{\Gamma}(x-y)\widetilde{H}(y)\,dy=\begin{bmatrix}\big[\mathcal{O}(|x|^{-1})\big]_{3\times1}\\ \big[\mathcal{O}(|x|^{-2})\big]_{3\times1}\end{bmatrix}\quad\text{as}\quad|x|\to\infty. \tag{4.42}$$

Therefore, the boundedness of the vector function $\widetilde{W}(x)$ at infinity implies $C_\alpha = 0$ for all α with $|\alpha| \geq 1$. From (4.41) we then get

$$\widetilde{W}(x) = \int\limits_{\Omega^{(1)}} \widetilde{\Gamma}(x-y)\widetilde{H}(y)\,dy + C, \quad x \in \mathbb{R}^3, \tag{4.43}$$

where $C = (C_1, \dots, C_6)^\top$ is an arbitrary constant vector.

Taking into account that $\widetilde{W}$ solves the homogeneous equation in the exterior of $B(O,R)$,

$$\widetilde{L}(\partial, 0)\widetilde{W}(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \overline{B(O,R)}. \tag{4.44}$$

Since the first summand in (4.43) solves the homogeneous equation (4.44), the constant vector C must satisfy the same homogeneous equation. This leads to the equalities $C_4 = C_5 = C_6 = 0$, which along with (4.42) completes the proof. □

Lemma 4.5. *Any solution* $\widetilde{U} = (u, \omega)^\top$ *of equation* (4.22)*, which is bounded at infinity and satisfies the condition*

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(O,R)} u(x)\, d\Sigma(O,R) = 0, \tag{4.45}$$

possesses the following asymptotic behaviour at infinity

$$\widetilde{U}(x) = \begin{bmatrix} \big[\theta_0 \alpha_1 x|x|^{-1} + \mathcal{O}(|x|^{-1}\ln|x|)\big]_{3\times1} \\ \big[\mathcal{O}(|x|^{-2}\ln|x|)\big]_{3\times1} \end{bmatrix} \quad \textit{as} \quad |x| \to \infty, \tag{4.46}$$

where α_1 *is given by* (4.26)*.*

Proof. Let $\widetilde{U}$ be a bounded at infinity solution of equation (4.22) and satisfy the condition (4.45). Put

$$\widetilde{U}^*(x) := \widetilde{U}(x) - \widetilde{U}^{(0)}(x) - \widetilde{U}^{(1)}(x), \quad x \in \Omega^-, \tag{4.47}$$

where $\widetilde{U}^{(0)}$ and $\widetilde{U}^{(1)}$ are given by (4.25) and (4.27) respectively. It is clear that $\widetilde{U}^*$ is bounded at infinity and solves the homogeneous equation

$$\widetilde{L}(\partial, 0)\widetilde{U}^*(x) = 0, \quad x \in \Omega^-.$$

Therefore, by Lemma 4.4

$$\widetilde{U}^*(x) = C + \begin{bmatrix} \big[\mathcal{O}(|x|^{-1})\big]_{3\times1} \\ \big[\mathcal{O}(|x|^{-2})\big]_{3\times1} \end{bmatrix} \quad \text{as} \quad |x| \to \infty, \tag{4.48}$$

where $C = (C_1, C_2, C_3, 0, 0, 0)^\top$ and C_j, $j = \overline{1,3}$, are arbitrary constants. Consequently, for the vector $\widetilde{U}(x) = \widetilde{U}^*(x) + \widetilde{U}^{(0)}(x) + \widetilde{U}^{(1)}(x)$ we get

$$\widetilde{U}(x) = C + \begin{bmatrix} \big[\theta_0 \alpha_1 x|x|^{-1} + \mathcal{O}(|x|^{-1}\ln|x|)\big]_{3\times1} \\ \big[\mathcal{O}(|x|^{-2}\ln|x|)\big]_{3\times1} \end{bmatrix} \quad \text{as} \quad |x| \to \infty. \tag{4.49}$$

In view of the equality

$$\int\limits_{\Sigma(O,R)} x\, d\Sigma(O,R) = 0,$$

the condition (4.45) implies $C_1 = C_2 = C_3 = 0$, which completes the proof. □

Now, having in hand the above results, we can formulate the following uniqueness theorem.

Theorem 4.6. *The exterior BVPs of statics* $(I^{(0)})^-$, $(II^{(0)})^-$ *and* $(III^{(0)})^-$ *have at most one solution vector* $U = (u,\omega,\vartheta)^\top = (\widetilde{U},\vartheta)^\top \in [W^1_{2,loc}(\Omega^-)]^7$ *satisfying the following conditions at infinity:*

$$\widetilde{U}(x) = \mathcal{O}(1) \quad \textit{and} \quad \vartheta(x) = o(1) \quad \textit{as} \quad |x| \to \infty, \tag{4.50}$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(O,R)} u(x)\, d\Sigma(O,R) = 0. \tag{4.51}$$

Proof. It suffices to show that the homogeneous BVPs have only the trivial solution in the class of vector functions satisfying the conditions (4.50) and (4.51). Let $U = (\widetilde{U},\vartheta)^\top \in [W^1_{2,loc}(\Omega^-)]^7$ be such solution. Since the homogeneous BVPs for the temperature function are separated, we get $\vartheta = 0$ in Ω^- (see (4.8)–(4.11)). Consequently, $\theta_0 = \lim_{|x|\to\infty} |x|\vartheta(x) = 0$. Therefore, $\widetilde{U} = (u,\omega)^\top$ solves the homogeneous equation

$$\widetilde{L}(\partial,0)\widetilde{U} = 0 \quad \text{in} \quad \Omega^-, \tag{4.52}$$

is bounded at infinity and satisfies the condition (4.51). Therefore, by Lemma 4.5 we have

$$\widetilde{U}(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1}\ln|x|0]_{3\times1} \\ [\mathcal{O}(|x|^{-2}\ln|x|)]_{3\times1} \end{bmatrix} \quad \text{as} \quad |x| \to \infty. \tag{4.53}$$

For vector functions with the asymptotic (4.53) at infinity, there holds Green's identity

$$\int\limits_{\Omega^-} \big[\widetilde{U}\cdot\widetilde{L}(\partial,0)\widetilde{U} + E(\widetilde{U},\widetilde{U})\big]\, dx = -\int\limits_{\partial\Omega^-} \{\widetilde{U}\}^- \cdot \{T(\partial,n)\widetilde{U}\}^-\, dS \tag{4.54}$$

where the bilinear form $E(\widetilde{U},\widetilde{U})$ is given by (2.30). Due to (4.52) and since $\widetilde{U}$ satisfies the homogeneous boundary conditions (see (4.16)–(4.19)), from (4.54) we get

$$\int\limits_{\Omega^-} E(\widetilde{U},\widetilde{U})\, dx = 0. \tag{4.55}$$

By Lemma 2.1,

$$u(x) = [a\times x] + b, \ \ \omega(x) = a, \ \ x \in \Omega^-, \tag{4.56}$$

and in view of (4.53) we conclude $a = b = 0$, i.e., $\widetilde{U} = (u,\omega)^\top = 0$ in Ω^-. □

5. Properties of Layer Potentials

Here we establish the mapping and regularity properties of the single and double layer potentials and the boundary pseudodifferential operators generated by them in the Hölder ($C^{m,\kappa}$), Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces. They can be established by standard methods (see, e.g., [7], [8], [14], [9], [10], [19], [27], [37], [41], [42], [43], and [44]). We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter σ (σ' and σ'' say) have the same smoothness properties and possess the same jump relations, since the entries of the difference of the fundamental matrices $\Gamma(x,\sigma') - \Gamma(x,\sigma'')$ are bounded functions in $\mathbb{R}^3$ and their derivatives of order m have a singularity of type $\mathcal{O}(|x|^{-m})$ in a vicinity of the origin. Moreover, the boundary integral operators generated by the single layer potentials (respectively, by the double layer potentials) constructed by the kernels $\Gamma(x,\sigma')$ and $\Gamma(x,\sigma'')$ differ by a compact perturbations. Therefore, using the word for word arguments given in [8], [27], [37], and [44] we can prove the following theorems concerning the above introduced layer potentials.

For simplicity, henceforward we assume (if not otherwise stated) that

$$\begin{gathered} S = \partial\Omega^{\pm} \in C^{m,\kappa} \quad \text{with integer} \quad m \geq 2 \quad \text{and} \quad 0 < \kappa \leq 1; \\ \sigma = \sigma_1 + i\sigma_2, \;\; \sigma_1 \in \mathbb{R}, \;\; \Im\sigma = \sigma_2 > 0. \end{gathered} \tag{5.1}$$

Theorem 5.1. *Let S, m, and κ be as in* (5.1)*, $0 < \kappa' < \kappa$, and let $k \leq m-1$ be integer. Then the operators*

$$V: \; [C^{k,\kappa'}(S)]^7 \to [C^{k+1,\kappa'}(\overline{\Omega^{\pm}})]^7, \;\; W: \; [C^{k,\kappa'}(S)]^7 \to [C^{k,\kappa'}(\overline{\Omega^{\pm}})]^7 \tag{5.2}$$

are continuous.

For any $g \in [C^{0,\kappa'}(S)]^7$, $h \in [C^{1,\kappa'}(S)]^7$, and any $x \in S$

$$[V(g)(x)]^{\pm} = V(g)(x) = \mathcal{H}g(x), \tag{5.3}$$

$$\big[\mathcal{P}(\partial_x, n(x))V(g)(x)\big]^{\pm} = \big[\mp 2^{-1} I_7 + \mathcal{K}\big]g(x), \tag{5.4}$$

$$[W(g)(x)]^{\pm} = \big[\pm 2^{-1} I_7 + \mathcal{N}\big]g(x), \tag{5.5}$$

$$\big[\mathcal{P}(\partial_x, n(x))W(h)(x)\big]^{+} = \big[\mathcal{P}(\partial_x, n(x))W(h)(x)\big]^{-} = \mathcal{L}h(x), \tag{5.6}$$

where

$$\mathcal{H}g(x) := \int\limits_S \Gamma(x-y,\sigma)g(y)\,dS_y, \tag{5.7}$$

$$\mathcal{K}g(x) := \int\limits_S \big[\mathcal{P}(\partial_x, n(x))\Gamma(x-y,\sigma)\big]g(y)\,dS_y, \tag{5.8}$$

$$\mathcal{N}g(x) := \int\limits_S \big[\mathcal{P}^*(\partial_y, n(y))\Gamma^{\top}(x-y,\sigma)\big]^{\top}g(y)\,dS_y, \tag{5.9}$$

$$\mathcal{L}h(x) := \lim_{\Omega^{\pm}\ni z\to x\in S} \mathcal{P}(\partial_z, n(x)) \int_S \big[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(z-y,\sigma)\big]^\top h(y)\, dS_y. \tag{5.10}$$

Proof. The proof of the relations (5.2)–(5.5) can be performed by standard arguments (see, e.g., [27], Ch.5, and [44]). We demonstrate here only a simplified proof of the relation (5.6), the so called *Liapunov-Tauber type theorem.* Let $h \in [C^{1,\kappa'}(S)]^7$ and consider the double layer potential $U := W(h) \in [C^{1,\kappa'}(\overline{\Omega^{\pm}})]^7$. Then by Corollary 3.6 and the jump relations (5.5), we have

$$U(x) = W([U]_S)(x) - V([\mathcal{P}U]_S)(x), \quad x \in \Omega^{\pm},$$

i.e.,

$$W(h)(x) = W(h)(x) - V([\mathcal{P}W(h)]_S)(x), \quad x \in \Omega^{\pm},$$

since $[U]_S = \{W(h)\}^+ - \{W(h)\}^- = h$ on S due to (5.5). Therefore $V([\mathcal{P}W(h)]_S) = 0$ in $\Omega^{\pm}$ and in view of (5.4) we conclude

$$\big\{\mathcal{P}V([\mathcal{P}W(h)]_S)\big\}^- - \big\{\mathcal{P}V([\mathcal{P}W(h)]_S)\big\}^+ = \\ = [\mathcal{P}W(h)]_S = \{\mathcal{P}W(h)\}^+ - \{\mathcal{P}W(h)\}^- = 0$$

on S, which completes the proof. □

With the help of the explicit form of the fundamental matrix $\Gamma(x-y,\sigma)$ it can easily be shown that the operators $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator. For a C^∞-smooth surfaces S all these operators can be treated as pseudodifferential operators on S (cf., [1], [19]).

Theorem 5.2. *Let S be a Lipschitz surface. Then the operators* (5.2) *can be extended to the continuous mappings*

$$V:\ [H_2^{-\frac12}(S)]^7 \to [H_2^1(\Omega^{\pm})]^7, \quad W:\ [H_2^{\frac12}(S)]^7 \to [H_2^1(\Omega^{\pm})]^7.$$

The jump relations (5.3)–(5.6) *on S remain valid for the extended operators in the corresponding function spaces.*

Proof. It is word for word of the proofs of the similar theorems in [7], [19] and [37]. □

Theorem 5.3. *Let S, m, κ, κ' and k be as in Theorem* 5.1. *Then the operators*

$$\mathcal{H}:\ [C^{k,\kappa'}(S)]^7 \to [C^{k+1,\kappa'}(S)]^7, \tag{5.11}$$

$$:\ [H_2^{-\frac12}(S)]^7 \to [H_2^{\frac12}(S)]^7, \tag{5.12}$$

$$\mathcal{K}:\ [C^{k,\kappa'}(S)]^7 \to [C^{k,\kappa'}(S)]^7, \tag{5.13}$$

$$:\ [H_2^{-\frac12}(S)]^7 \to [H_2^{-\frac12}(S)]^7, \tag{5.14}$$

$$\mathcal{N}:\ [C^{k,\kappa'}(S)]^7 \to [C^{k,\kappa'}(S)]^7, \tag{5.15}$$

$$: \ [H_2^{\frac{1}{2}}(S)]^7 \to [H_2^{\frac{1}{2}}(S)]^7, \tag{5.16}$$

$$\mathcal{L}: \ [C^{k,\kappa'}(S)]^7 \to [C^{k-1,\kappa'}(S)]^7, \tag{5.17}$$

$$: \ [H_2^{\frac{1}{2}}(S)]^7 \to [H_2^{-\frac{1}{2}}(S)]^7 \tag{5.18}$$

are continuous. Moreover,

(i) *the principal homogeneous symbol matrices of the operators* $\pm 2^{-1}I_7 + \mathcal{K}$ *and* $\pm 2^{-1}I_7 + \mathcal{N}$ *are non-degenerate, while the principal homogeneous symbol matrices of the operators* $-\mathcal{H}$ *and* $\mathcal{L}$ *are positive definite;*

(ii) *the operators* $\mathcal{H}$, $\pm 2^{-1}I_7 + \mathcal{K}$, $\pm 2^{-1}I_7 + \mathcal{N}$, *and* $\mathcal{L}$ *are elliptic pseudodifferential operators* (*of order* -1, 0, 0, *and* 1, *respectively*) *with zero index;*

(iii) *the following equalities hold in appropriate function spaces:*

$$\begin{gathered} \mathcal{N}\mathcal{H} = \mathcal{H}\mathcal{K}, \quad \mathcal{L}\mathcal{N} = \mathcal{K}\mathcal{L}, \\ \mathcal{H}\mathcal{L} = -4^{-1}I_7 + \mathcal{N}^2, \quad \mathcal{L}\mathcal{H} = -4^{-1}I_7 + \mathcal{K}^2. \end{gathered} \tag{5.19}$$

(iv) *The operators* (5.12), (5.14), (5.16), *and* (5.18) *are bounded if* S *is a Lipschitz surface.*

Proof. Proof of the mapping properties (5.11)–(5.18) are standard and can be performed as in [7], [8], [19], [27], [37], [43], and [44].

The item (iii) follows from the jump relations for the layer potentials and the general integral representation formulas of solutions to the homogeneous equation $L(\partial, \sigma)U = 0$.

Proof of items (i) and (ii) is based on the positive definiteness of the potential energy functional. Indeed, it can be shown that under the restrictions (2.26) the matrix $L_0(\xi)$ is positive definite for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \setminus \{0\}$ (see (2.12)–(2.13)). On the other hand, due to the results obtained in Subsection 3.4 we can derive that the principal homogeneous symbol matrix of the operator $\mathcal{H}$, generated by the single-layer potential associated with the matrix $\Gamma(\cdot, \sigma)$, in a local coordinate system reads as (see (3.69))

$$\mathfrak{S}(\widetilde{\xi}, x; \mathcal{H}) = -\frac{1}{2\pi} \int\limits_{\mathbb{R}^1} \big[L_0(E\xi)\big]^{-1} d\xi_3 = \mp \frac{1}{2\pi} \int\limits_{\ell^{\pm}} \big[L_0(E\xi)\big]^{-1} d\xi_3, \tag{5.20}$$

where $\widetilde{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, $\ell^{\pm}$ is as in (3.69) and $E = E(x)$ is an orthogonal matrix associated with a local coordinate system at the point $x \in S$,

$$E(x) = \begin{bmatrix} l_1(x) & m_1(x) & n_1(x) \\ l_2(x) & m_2(x) & n_2(x) \\ l_3(x) & m_3(x) & n_3(x) \end{bmatrix}. \tag{5.21}$$

Here $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outward unit normal vector to the surface S, and $l(x) = (l_1(x), l_2(x), l_3(x))$ and $m(x) = (m_1(x), m_2(x), m_3(x))$ are orthogonal unit vectors in the tangential plane at the point $x \in S$.

From (5.20) we conclude that $\mathfrak{S}(\widetilde{\xi}, x; -\mathcal{H})$ is a homogeneous matrix of order -1 in $\widetilde{\xi}$ and is positive definite for all $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$.

The principal homogeneous symbol matrices of the operators $\pm 2^{-1}I_7+\mathcal{K}$, $\pm 2^{-1}I_7+\mathcal{N}$ and $\mathcal{L}$ read as

$$\mathfrak{S}(\widetilde{\xi},x;\mp 2^{-1}I_7+\mathcal{K})=\pm\frac{i}{2\pi}\int\limits_{\ell^\pm}\mathcal{P}_0(E\xi,n(x))[L_0(E\xi)]^{-1}\,d\xi_3, \quad (5.22)$$

$$\mathfrak{S}(\widetilde{\xi},x;\pm 2^{-1}I_7+\mathcal{N})=\mp\frac{i}{2\pi}\int\limits_{\ell^\pm}[L_0(E\xi)]^{-1}\mathcal{P}_0^\top(E\xi,n(x))\,d\xi_3, \quad (5.23)$$

$$\mathfrak{S}(\widetilde{\xi},x;\mathcal{L})=\mp\frac{1}{2\pi}\int\limits_{\ell^\pm}\mathcal{P}_0(E\xi,n(x))[L_0(E\xi)]^{-1}\mathcal{P}_0^\top(E\xi,n(x))\,d\xi_3, \quad (5.24)$$

where $E=E(x)$ is given by (5.21) and $\mathcal{P}_0(\partial,n)$ is the principal homogeneous part of the operator (2.18), i.e.,

$$\mathcal{P}_0(\partial,n)=\begin{bmatrix}T_0(\partial,n) & [0]_{6\times 1}\\ [0]_{1\times 6} & \kappa'\partial_n\end{bmatrix}_{7\times 7} \quad (5.25)$$

with $T_0(\partial,n)$ the principal homogeneous part of the operator $T(\partial,n)$ (see (2.19) and (2.22))

$$\begin{gathered}T_0(\partial,n)=\begin{bmatrix}T_0^{(1)}(\partial,n) & T_0^{(2)}(\partial,n)\\ T_0^{(3)}(\partial,n) & T_0^{(4)}(\partial,n)\end{bmatrix}_{6\times 6},\\ T_0^{(j)}=[T_{0pq}^{(j)}]_{3\times 3},\quad j=\overline{1,4},\\ T_{0pq}^{(1)}(\partial,n)=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q,\\ T_{0pq}^{(2)}(\partial,n)=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q,\\ T_{pq}^{(3)}(\partial,n)=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q,\\ T_{pq}^{(4)}(\partial,n)=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\beta n_p\partial_q.\end{gathered} \quad (5.26)$$

Evidently, $\mathcal{P}_0(\partial,n)$ is the principal homogeneous part of the operator (2.21) as well.

The entries of the matrices (5.22) and (5.23) are homogeneous functions of order 0 in $\widetilde{\xi}$, while the entries of the matrix (5.24) are homogeneous functions of order +1 in $\widetilde{\xi}$.

Applying the equalities $E\xi\cdot n=\xi_3$ and $([L_0(E\xi)]^{-1})_{77}=|E\xi|^{-2}=|\xi|^{-2}$, we easily derive that

$$\begin{gathered}[\mathfrak{S}(\widetilde{\xi},x;\mathcal{H})]_{77}=-2^{-1}|\xi|^{-1},\quad [\mathfrak{S}(\widetilde{\xi},x;\mathcal{L})]_{77}=2^{-1}|\xi|,\\ \big[\mathfrak{S}(\widetilde{\xi},x;\mp 2^{-1}I_7+\mathcal{K})\big]_{77}=\mp 2^{-1},\quad \big[\mathfrak{S}(\widetilde{\xi},x;\mp 2^{-1}I_7+\mathcal{N})\big]_{77}=\mp 2^{-1}.\end{gathered} \quad (5.27)$$

The explicit expressions for the symbol matrices (5.22)–(5.24) yield

$$\begin{gathered}\mathfrak{S}(\widetilde{\xi},x;-2^{-1}I_7+\mathcal{K})=\overline{\mathfrak{S}^\top(\widetilde{\xi},x;-2^{-1}I_7+\mathcal{N})},\\ \mathfrak{S}(\widetilde{\xi},x;2^{-1}I_7+\mathcal{K})=\overline{\mathfrak{S}^\top(\widetilde{\xi},x;2^{-1}I_7+\mathcal{N})}.\end{gathered} \quad (5.28)$$

Furthermore, the matrices (5.22) and (5.23) are non-degenerate, which can be shown by standard arguments. In fact, let us consider the Dirichlet BVP and the Neumann BVP (associated with the operator $\mathcal{P}_0(\partial_x, n)$) for the differential equation $L_0(\partial_x)U(x) = 0$ in the half spaces $\mathbb{R}^3_+(n) := n \cdot x > 0$ and $\mathbb{R}^3_-(n) := n \cdot x < 0$, where $n = (n_1, n_2, n_3)$ is an arbitrary constant unit vector. Denote by $l = (l_1, l_2, l_3)$ and $m = (m_1, m_2, m_3)$ orthogonal unit vectors lying in the plane orthogonal to n, such that $\det E = 1$, where E is the orthogonal matrix having the structure (5.21).

With the help of the change of variables $x = E\zeta$, where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, the domains $\mathbb{R}^3_+(n)$ and $\mathbb{R}^3_-(n)$ are transformed into the half spaces $\zeta_3 > 0$ and $\zeta_3 < 0$, and $L_0(\partial_x)$ and $\mathcal{P}_0(\partial_x, n)$ into the operators $L_0(E\partial_\zeta)$ and $\mathcal{P}_0(E\partial_\zeta, n)$ respectively. Applying the partial Fourier transform $\mathcal{F}_{\widetilde{\zeta}\to\widetilde{\xi}}$ with $\widetilde{\zeta} = (\zeta_1, \zeta_2)$, the BVPs will be transformed into the Dirichlet and Neumann type BVPs for the system of ordinary differential equations in ζ_3 either in the interval $(-\infty, 0)$ or $(0, +\infty)$ with the operators $L_0(E\Lambda)$ and $\mathcal{P}_0(E\Lambda, n)$, where

$$\Lambda := \Big(-i\xi_1, -i\xi_2, \frac{d}{d\zeta_3}\Big)^\top.$$

Evidently, these problems depend on the parameters $\widetilde{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and their homogeneous versions read as follows: Find a solution vector $U(\zeta_3)$ to the equation

$$L_0(E\Lambda)U(\zeta_3) = 0, \tag{5.29}$$

either in $(-\infty, 0)$, satisfying one of the following boundary conditions:

$$\begin{aligned} (I)^+ :& \lim_{\zeta_3\to 0-} U(\zeta_3) = 0, \\ (II)^+ :& \lim_{\zeta_3\to 0-} \mathcal{P}_0(E\Lambda, n)U(\zeta_3) = 0, \end{aligned} \tag{5.30}$$

or in $(0, +\infty)$, satisfying one of the following boundary conditions:

$$\begin{aligned} (I)^- :& \lim_{\zeta_3\to 0+} U(\zeta_3) = 0, \\ (II)^- :& \lim_{\zeta_3\to 0+} \mathcal{P}_0(E\Lambda, n)U(\zeta_3) = 0. \end{aligned} \tag{5.31}$$

With the help of the positive definiteness of the potential energy quadratic form and inequalities (2.26), it can be easily verified that the homogeneous boundary value problems $(I)^\pm$ and $(II)^\pm$ possesses only the trivial solution in the space of vector functions decaying at infinity.

Further, we can check that the columns of the matrices (cf. (3.69))

$$\mathfrak{U}^{(-)}(\zeta_3) := -\frac{1}{2\pi}\int\limits_{\ell^+} [L_0(E\xi)]^{-1} e^{-i\zeta_3\xi_3}\, d\xi_3, \tag{5.32}$$

$$\mathfrak{U}^{(+)}(\zeta_3) := \frac{1}{2\pi}\int\limits_{\ell^-} [L_0(E\xi)]^{-1} e^{-i\zeta_3\xi_3}\, d\xi_3, \tag{5.33}$$

represent a complete system of solutions to the homogeneous differential equation (5.29). Moreover, the entries of (5.32) decay exponentially as $\zeta_3 \to -\infty$ and grow exponentially as $\zeta_3 \to +\infty$, while the entries of (5.33) decay exponentially as $\zeta_3 \to +\infty$ and grow exponentially as $\zeta_3 \to -\infty$. Due to the above mentioned uniqueness results, it follows immediately that the four linear systems of algebraic equations with unknown $C \in \mathbb{C}^7$,

$$\mathfrak{U}^{(\pm)}(0)C = 0, \tag{5.34}$$

$$\mathcal{P}_0(E\Lambda, n)\mathfrak{U}^{(\pm)}(0)C = 0, \tag{5.35}$$

have only the trivial solution, that is the corresponding determinants are different from zero. Since the matrices $\mathcal{P}_0(E\Lambda, n)\mathfrak{U}^{(\pm)}(0)$ coincide with the (5.22), it follows that the symbol matrices (5.22) are non-degenerate for $\widetilde{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$. By (5.28), we conclude that the matrices (5.23) are non-degenerate as well.

Moreover, from the last equalities in (5.19) and the non-degeneracy of the matrices (5.20) and (5.22)–(5.23) it follows that the symbol matrix $\mathfrak{S}(\widetilde{\xi}, x; \mathcal{L})$ is non-degenerate.

Now, we show that the symbol matrix $\mathfrak{S}(\widetilde{\xi}, x; \mathcal{L})$ is positive definite. To this end, let us consider the double layer potential $W_0 = W_0(g)$ in domains $\mathbb{R}^3_+(n)$ and $\mathbb{R}^3_-(n)$

$$W_0(g)(x) = \int\limits_{S(n)} \big[\mathcal{P}_0(\partial_y, n)\Gamma_0(x-y)\big]^\top g(y)\, dS(n), \tag{5.36}$$

where $\mathcal{P}_0(\partial, n)$ is defined by (5.25)–(5.26) and is the principal homogeneous part of the operator (2.21), $\Gamma_0(\cdot)$ is the principal singular part of the matrix $\Gamma(\cdot, \sigma)$ (see Subsections 3.3 and 3.4), $S(n) = \partial\mathbb{R}^3_\pm(n)$, $n = (n_1, n_2, n_3)$ is an arbitrary constant vector which is the unit normal vector to the plane $S(n)$ directed into the half space $\mathbb{R}^3_+(n)$, and $g = (g_1, \dots, g_7)^\top$ is an arbitrary complex valued vector function from the Schwartz space $[\mathcal{S}(S(n))]^7$ of rapidly decaying vector functions. Since $W_0(g)$ solves the homogeneous equation $L_0(\partial)W_0(g) = 0$ in $\mathbb{R}^3_\pm(n)$ and at infinity has the following decay properties $W_0(g)(x) = \mathcal{O}(|x|^{-1})$ and $\partial^\alpha W_0(g)(x) = \mathcal{O}(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, there hold Green's formulas

$$\int\limits_{\mathbb{R}^3_+(n)} E_0(W_0, \overline{W}_0) = -\int\limits_{S(n)} \{\mathcal{P}_0(\partial, n)W_0\}^- \cdot \{\overline{W}_0\}^-\, dS(n), \tag{5.37}$$

$$\int\limits_{\mathbb{R}^3_-(n)} E_0(W_0, \overline{W}_0) = \int\limits_{S(n)} \{\mathcal{P}_0(\partial, n)W_0\}^+ \cdot \{\overline{W}_0\}^+\, dS(n), \tag{5.38}$$

where for $U = (u, \omega, \vartheta)^\top$

$$E_0(U, \overline{U}) =$$

$$= \frac{3\lambda+2\mu}{3}\Big|\operatorname{div}u+\frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega\Big|^2+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\Big)|\operatorname{div}\omega|^2+$$
$$+\frac{\mu}{2}\sum_{k,j=1,k\neq j}^{3}\Big|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\Big)\Big|^2+$$
$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\Big)\Big|^2+$$
$$+\Big(\gamma-\frac{\kappa^2}{\mu}\Big)\sum_{k,j=1,k\neq j}^{3}\Big[\frac{1}{2}\Big|\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\Big|^2+\frac{1}{3}\Big|\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\Big|^2\Big]+$$
$$+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)|\operatorname{curl}\omega|^2+\alpha\Big|\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega\Big|^2+\sum_{j=1,k\neq j}^{3}\Big|\frac{\partial\vartheta}{\partial x_j}\Big|^2. \tag{5.39}$$

In view of Liapunov–Tauber type theorem (see (5.6))

$$\big\{\mathcal{P}_0(\partial,n)W_0(g)\big\}^- = \big\{\mathcal{P}_0(\partial,n)W_0(g)\big\}^+ =: \mathcal{L}_0 g. \tag{5.40}$$

The operator $\mathcal{L}_0$ is a compact perturbation of the operator $\mathcal{L}$ and therefore their principal homogeneous symbols coincide (see (5.24))

$$\mathfrak{S}(\widetilde{\xi},z;\mathcal{L}) = \mathfrak{S}(\widetilde{\xi};\mathcal{L}_0), \quad z\in S, \tag{5.41}$$

provided $n=n(z)$ in (5.40).

Due to the inequalities (2.26), we have $E_0(U,\overline{U})\geq 0$ and $E_0(U,U)=0$ implies

$$U(x) = (b_1,\dots,b_7)^\top, \tag{5.42}$$

where b_j, $j=\overline{1,7}$, are arbitrary constants.

With the help of the jump relation $\{W_0(g)\}^+ - \{W_0(g)\}^- = g$ and equality (5.40) we get from (5.37)

$$\int\limits_{\mathbb{R}^3_+(n)} E_0(W_0,\overline{W}_0) + \int\limits_{\mathbb{R}^3_-(n)} E_0(W_0,\overline{W}_0) = \int\limits_{S(n)} \mathcal{L}_0 g(x)\cdot\overline{g(x)}\,dS(n) \geq 0. \tag{5.43}$$

Since $W_0(g)$ decays at infinity, it can easily be shown that in (5.43) we have strong inequality if g does not vanish identically.

Under the change of variables $x=E\zeta$ in (5.43) and denoting $h(\zeta)=g(E\zeta)$, we arrive at the inequality

$$\int\limits_{S(n)} \mathcal{L}_0 g(x)\cdot\overline{g(x)}\,dS(n) = \int\limits_{\mathbb{R}^2} \mathcal{L}_0 h(\widetilde{\zeta})\cdot\overline{h(\widetilde{\zeta})}\,d\widetilde{\zeta} > 0 \tag{5.44}$$

for all $h\in[\mathcal{S}(\mathbb{R}^2)]^7$ and $h\not\equiv 0$.

Applying the Parseval–Plancherel formula and the Fourier transform formula of convolution,

$$\int\limits_{\mathbb{R}^2} f(\widetilde{x})\,\overline{g(\widetilde{x})}\,d\widetilde{x} = \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2} \widehat{f}(\widetilde{\xi})\,\overline{\widehat{g}(\widetilde{\xi})}\,d\widetilde{\xi}, \quad \mathcal{F}[f*g](\widetilde{\xi}) = \widehat{f}(\widetilde{\xi})\,\overline{\widehat{g}(\widetilde{\xi})},$$

from (5.44) we get

$$\int\limits_{S(n)} \mathcal{L}_0 g(x)\cdot\overline{g(x)}\,dS(n)=\int\limits_{\mathbb{R}^2} \mathcal{L}_0 h(\widetilde{\zeta})\cdot\overline{h(\widetilde{\zeta})}\,d\widetilde{\zeta}=$$
$$=\frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2} \mathfrak{S}(\widetilde{\xi};\mathcal{L}_0)\widehat{h}(\widetilde{\xi})\cdot\overline{\widehat{h}(\widetilde{\xi})}\,d\widetilde{\xi}>0, \quad (5.45)$$

Since, $h\in[\mathcal{S}(\mathbb{R}^2)]^7$ is an arbitrary nonzero vector, form (5.45) we deduce that the matrix $\mathfrak{S}(\widetilde{\xi};\mathcal{L}_0)$ is positive definite for all $\widetilde{\xi}\in\mathbb{R}^2\setminus\{0\}$, which proves positive definiteness of the principal homogeneous symbol matrix of the operator $\mathcal{L}$ due to (5.41).

From positive definiteness of the matrices $\mathfrak{S}(\widetilde{\xi},x;-\mathcal{H})$ and $\mathfrak{S}(\widetilde{\xi},x;\mathcal{L})$ it follows that the index of the operators $\mathcal{H}$ and $\mathcal{L}$ equal to zero.

Now we show that the operators $\pm 2^{-1}I_7+\mathcal{K}$ and $\pm 2^{-1}I_7+\mathcal{N}$ have zero index as well. We demonstrate the proof for the operator $2^{-1}I_7+\mathcal{K}$. For the other operators the proof is word for word.

First of all, let us note that the operator

$$\mathcal{H}:\ [H_2^{-\frac{1}{2}}(S)]^7\to[H_2^{\frac{1}{2}}(S)]^7 \quad (5.46)$$

is injective due to the uniqueness theorems for the Dirichlet interior and exterior BVPs (see Theorems 2.2 and 4.1). Consequently, (5.46) is invertible. Evidently, the adjoint operator

$$\mathcal{H}^*:[H_2^{-\frac{1}{2}}(S)]^7\to[H_2^{\frac{1}{2}}(S)]^7 \quad (5.47)$$

is invertible as well.

Further, in view of the first equality in (5.19), we get

$$\mathcal{H}(2^{-1}I_7+\mathcal{K})=(2^{-1}I_7+\mathcal{N})\mathcal{H}. \quad (5.48)$$

On the other hand, taking into consideration that

$$\mathfrak{S}(\widetilde{\xi},x;-2^{-1}I_7+\mathcal{K}^*)=\overline{\mathfrak{S}^\top(\widetilde{\xi},x;-2^{-1}I_7+\mathcal{K})},$$

from the second equality in (5.28), it follows that the principal homogenous symbol matrices of the operators $2^{-1}I_7+\mathcal{N}$ and $2^{-1}I_7+\mathcal{K}^*$ coincide and therefore $\mathcal{N}-\mathcal{K}^*$ is a compact operator, i.e.,

$$\operatorname{ind}(2^{-1}I_7+\mathcal{N})=\operatorname{ind}(2^{-1}I_7+\mathcal{K}^*), \quad (5.49)$$

where $\mathcal{K}^*$ is adjoint to the operator $\mathcal{K}$.

From (5.48) we have

$$\begin{aligned}(2^{-1}I_7+\mathcal{K})&=\mathcal{H}^{-1}(2^{-1}I_7+\mathcal{N})\mathcal{H},\\(2^{-1}I_7+\mathcal{K}^*)&=\mathcal{H}^*(2^{-1}I_7+\mathcal{N}^*)[\mathcal{H}^*]^{-1},\end{aligned} \quad (5.50)$$

which implies that

$$\operatorname{dimker}(2^{-1}I_7+\mathcal{K})=\operatorname{dimker}(2^{-1}I_7+\mathcal{N}),$$
$$\operatorname{dimker}(2^{-1}I_7+\mathcal{K}^*)=\operatorname{dimker}(2^{-1}I_7+\mathcal{N}^*).$$

Therefore,

$$\mathrm{ind}(2^{-1}I_7 + \mathcal{K}) = \mathrm{ind}(2^{-1}I_7 + \mathcal{N}) = \mathrm{ind}(2^{-1}I_7 + \mathcal{K}^*)$$

due to (5.49), and since $\mathrm{ind}(2^{-1}I_7 + \mathcal{K}) = -\mathrm{ind}(2^{-1}I_7 + \mathcal{K}^*)$, it follows that $\mathrm{ind}(2^{-1}I_7 + \mathcal{K}) = 0$.

Finally, the item (iv) follows from Green's formulas and Theorem 5.2 (see similar theorems in [37]). □

The next proposition is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [21], [14], [53], [54], and the references therein).

Theorem 5.4. *Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$, and $\mathcal{L}$ be as in Theorems* 5.1 *and* 5.3 *and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The layer potential operators* (3.70), (3.71) *and the boundary integral* (*pseudodifferential*) *operators* (5.7)–(5.10) *can be extended to the following continuous operators*

$$V:\ [B^s_{p,p}(S)]^7 \to [H_p^{s+1+\frac{1}{p}}(\Omega^\pm)]^7 \quad \Big[\, [B^s_{p,q}(S)]^7 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^\pm)]^7 \,\Big],$$

$$W:\ [B^s_{p,p}(S)]^7 \to [H_p^{s+\frac{1}{p}}(\Omega^\pm)]^7 \quad \Big[\, [B^s_{p,q}(S)]^7 \to [B_{p,q}^{s+\frac{1}{p}}(\Omega^\pm)]^7 \,\Big],$$

$$\mathcal{H}:\ [H^s_p(S)]^7 \to [H^{s+1}_p(S)]^7 \quad \Big[\, [B^s_{p,q}(S)]^7 \to [B^{s+1}_{p,q}(S)]^7 \,\Big], \tag{5.51}$$

$$\mathcal{K}:\ [H^s_p(S)]^7 \to [H^s_p(S)]^7 \quad \Big[\, [B^s_{p,q}(S)]^7 \to [B^s_{p,q}(S)]^7 \,\Big], \tag{5.52}$$

$$\mathcal{N}:\ [H^s_p(S)]^7 \to [H^s_p(S)]^7 \quad \Big[\, [B^s_{p,q}(S)]^7 \to [B^s_{p,q}(S)]^7 \,\Big], \tag{5.53}$$

$$\mathcal{L}:\ [H^{s+1}_p(S)]^7 \to [H^s_p(S)]^7 \quad \Big[\, [B^{s+1}_{p,q}(S)]^7 \to [B^s_{p,q}(S)]^7 \,\Big]. \tag{5.54}$$

The jump relations (5.3)–(5.6) *remain valid for arbitrary $g \in [B^s_{p,q}(S)]^7$ with $s \in \mathbb{R}$ if the limiting values* (*traces*) *on S are understood in the sense described in* [53].

The operators (5.51)–(5.54) *are elliptic pseudodifferential operators with zero index. The null-spaces of the operators* (5.51)–(5.54) *are invariant with respect to p, q, and s.*

Proof. The proof follows from Theorem 5.3 by duality and interpolation arguments (see similar theorems in [10] and [8]. □

6. Existence Results for Pseudo-Oscillation Problems

Here we apply the potential method and prove existence theorems for the Dirichlet and Neumann type BVPs for pseudo-oscillation equations (see Subsection 2.4). We reduce the original BVPs to the equivalent integral equations on the boundary of the elastic body under consideration and investigate their Fredholm properties. In particular, we show that the corresponding integral (pseudodifferential) operators are invertible. Without loss of generality we consider the BVPs for the homogeneous differential equation $L(\partial, \sigma)U(x) = 0$, since a particular solution to the nonhomogeneous equation (2.43) can be written explicitly in the form of volume potential $N_{\Omega^{\pm}}(\Phi)$ (see (3.73)).

Moreover, throughout this section we assume that the conditions (5.1) are fulfilled if not otherwise stated.

6.1. **Investigation of the Dirichlet type interior and exterior BVPs.** These problems are formulated in Subsection 2.4 as problems $(I^{(\sigma)})^{+}$ and $(I^{(\sigma)})^{-}$ (see (2.43)–(2.44)). We assume that $\Phi^{(\pm)}=0$ and look for solutions in $\Omega^{\pm}$ in the form of double layer potential $U=W(h)$ (see (3.71)). Applying the jump relations for the double layer potential (see Theorem 5.1) and taking into consideration the boundary conditions (2.44), for the unknown density vector function $h=(h_1,\dots,h_7)^{\top}$ we get the boundary integral equations

$$[2^{-1}I_7+\mathcal{N}]h=f \quad \text{on} \quad S, \tag{6.1}$$

in the case of Problem $(I^{(\sigma)})^{+}$, and

$$[-2^{-1}I_7+\mathcal{N}]h=f \quad \text{on} \quad S, \tag{6.2}$$

in the case of Problem $(I^{(\sigma)})^{-}$.

Here the operator $\mathcal{N}$ is given by (5.9). Due to Theorem 5.3, the operators $\pm 2^{-1}I_7+\mathcal{N}$ are singular integral operators of normal type with index zero. This leads to the following existence theorems.

Theorem 6.1. *Let $S\in C^{2,\alpha}$ and $f\in[C^{1,\beta}(S)]^7$ with $0<\beta<\alpha\leq 1$. Then the BVP $(I^{(\sigma)})^{+}$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^{+}})]^7$ and the solution is represented by the double layer potential $W(h)$ defined by* (3.71)*, where $h\in[C^{1,\beta}(S)]^7$ is a unique solution of the integral equation* (6.1)*.*

Proof. The uniqueness follows from Theorems 5.1 and 2.2. It remains to show that the singular integral operator

$$2^{-1}I_7+\mathcal{N}:\ [C^{1,\beta}(S)]^7\to[C^{1,\beta}(S)]^7 \tag{6.3}$$

is invertible.

Due to Theorem 5.3, we conclude that (6.3) is a Fredholm operator with zero index. Further, we show that $\ker[2^{-1}I_7+\mathcal{N}]$ is trivial. Indeed, let h_0 solve the homogeneous equation

$$[2^{-1}I_7+\mathcal{N}]h_0=0 \quad \text{on} \quad S. \tag{6.4}$$

Construct the double layer potential $W(h_0)$. Since $h_0\in[C^{1,\beta}(S)]^7$, we have $W(h_0)\in[C^{1,\beta}(\overline{\Omega^{\pm}})]^7$. In view of equation (6.4), we see that then $[W(h_0)(x)]^{+}=0$ for $x\in S$ and by the uniqueness Theorem 2.2 we get $W(h_0)(x)=0$ for $x\in\Omega^{+}$. Consequently, $[\mathcal{P}(\partial,n)W(h_0)(x)]^{+}=0$ for $x\in S$. By the Liapunov-Tauber theorem (see Theorem 5.1)

$$\big[\mathcal{P}(\partial,n)W(h_0)(x)\big]^{+}=\big[\mathcal{P}(\partial,n))W(h_0)(x)\big]^{-}=0,\ \ x\in S,$$

i.e., $W(h_0)$ solves the exterior Neumann type boundary value problem $(II^{(\sigma)})^{-}$ and decays at infinity exponentially. Therefore, $W(h_0)(x)=0$

in Ω^- by Theorem 2.2. Since

$$[W(h_0)(x)]^+ - [W(h_0)(x)]^- = 2h_0(x), \quad x \in S,$$

we conclude that $h_0 = 0$ on S, which shows that $\ker[2^{-1}I_7 + \mathcal{N}]$ is trivial. Therefore, (6.3) is invertible. □

Quite similarly, by the word for word arguments and with the help of Theorem 4.1, we can show that the operator

$$-2^{-1}I_7 + \mathcal{N}: \ [C^{1,\beta}(S)]^7 \to [C^{1,\beta}(S)]^7 \tag{6.5}$$

is invertible, which leads to the existence theorem for the Dirichlet type exterior BVP.

Theorem 6.2. *Let $S \in C^{2,\alpha}$ and $f \in [C^{1,\beta}(S)]^7$ with $0 < \beta < \alpha \leq 1$. Then the BVP $(I^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $[C^{1,\beta}(\overline{\Omega^-})]^7$ and decaying at infinity, and the solution is represented by the double layer potential $W(h)$ defined by* (3.71), *where $h \in [C^{1,\beta}(S)]^7$ is a unique solution of the integral equation* (6.2).

6.2. **Investigation of the Neumann type interior and exterior BVPs.** These problems are formulated in Subsection 2.4 as problems $(II^{(\sigma)})^+$ and $(II^{(\sigma)})^-$ (see (2.43), (2.45)). As above, we assume that $\Phi^{(\pm)} = 0$ and look for solutions in $\Omega^\pm$ in the form of the single layer potential $U = V(g)$ (see (3.70)). Applying the jump relations for the single layer potential (see Theorem 5.1) and taking into consideration the boundary conditions (2.45), for the unknown density vector function $g = (g_1, \dots, g_7)^\top$ we get the boundary integral equations

$$[-2^{-1}I_7 + \mathcal{K}]g = F \quad \text{on} \quad S, \tag{6.6}$$

in the case of Problem $(II^{(\sigma)})^+$, and

$$[2^{-1}I_7 + \mathcal{K}]g = F \quad \text{on} \quad S, \tag{6.7}$$

in the case of Problem $(II^{(\sigma)})^-$.

Here the operator $\mathcal{K}$ is given by (5.8). Due to Theorem 5.3, the operators $\pm 2^{-1}I_7 + \mathcal{K}$ are singular integral operators of normal type with index zero. This yields the following existence theorems.

Theorem 6.3. *Let $S \in C^{1,\alpha}$ and $F \in [C^{0,\beta}(S)]^7$ with $0 < \beta < \alpha \leq 1$. Then the BVP $(II^{(\sigma)})^+$ is uniquely solvable in the space $[C^{1,\beta}(\overline{\Omega^+})]^7$ and the solution is represented by the single layer potential $V(g)$ defined by* (3.70), *where $g \in [C^{0,\beta}(S)]^7$ is a unique solution of the integral equation* (6.6).

Proof. The uniqueness is a consequence of Theorems 5.1 and 2.2. Now, we show that the operator

$$-2^{-1}I_7 + \mathcal{K}: \ [C^{0,\beta}(S)]^7 \to [C^{0,\beta}(S)]^7 \tag{6.8}$$

is invertible.

Due to Theorem 5.3, the operator (6.3) is a Fredholm operator with zero index. Therefore, it remains to show that $\ker[-2^{-1}I_7+\mathcal{K}]$ is trivial. Let g_0 solve the homogeneous equation

$$[-2^{-1}I_7+\mathcal{K}]g_0=0, \quad \text{on} \quad S. \tag{6.9}$$

Construct the single layer potential $V(g_0)$. Evidently, $V(g_0)\in[C^{1,\beta}(\overline{\Omega^+})]^7$, since $g_0\in[C^{0,\beta}(S)]^7$. Moreover, $V(g_0)$ solves the homogeneous Problem $(II^{(\sigma)})^+$ and therefore it vanishes identically in Ω^+, due to Theorem 2.2. Further, by Theorem 5.1 we have $[V(g_0)(x)]^+=[V(g_0)(x)]^-=0$ for $x\in S$, and since it decays at infinity, by the uniqueness theorem for the Dirichlet exterior BVP, we conclude $V(g_0)(x)=0$ for $x\in\Omega^-$. Finally, with the help of equality

$$[\mathcal{P}(\partial,n))V(g_0)(x)]^- - [\mathcal{P}(\partial,n))V(g_0)(x)]^+ = 2g_0(x), \;\; x\in S,$$

we derive $g_0=0$ on S, which proves that $\ker[-2^{-1}I_7+\mathcal{K}]$ is trivial. Thus, the operator (6.8) is invertible. □

By the word for word arguments we can prove that the operator

$$2^{-1}I_7+\mathcal{K}:\; [C^{0,\beta}(S)]^7\to[C^{0,\beta}(S)]^7 \tag{6.10}$$

is invertible, which leads to the existence theorem for the Neumann type exterior BVP.

Theorem 6.4. *Let $S\in C^{1,\alpha}$ and $F\in[C^{0,\beta}(S)]^7$ with $0<\beta<\alpha\le 1$. Then the BVP $(II^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $[C^{1,\beta}(\overline{\Omega^-})]^7$ and decaying at infinity, and the solution is represented by the single layer potential $V(g)$ defined by* (3.70)*, where $g\in[C^{0,\beta}(S)]^7$ is a unique solution of the integral equation* (6.7)*.*

Remark 6.5. Note that, if $S\in C^{2,\alpha}$, the operators

$$\pm 2^{-1}I_7+\mathcal{K}:\; [C^{1,\beta}(S)]^7\to[C^{1,\beta}(S)]^7 \tag{6.11}$$

are invertible as well.

6.3. **Investigation of the basic BVPs by the first kind integral equations.** Here we apply an alternative approach and reduce the basic interior and exterior BVPs, considered in the previous subsections, to the first kind integral (pseudodifferential) equations. We shall essentially apply the results obtained in this subsection in the study of mixed BVPs

6.3.1. *Investigation of the Dirichlet problem with the help of the first kind integral equations.* We look for a solution to the problems $(I^{(\sigma)})^+$ and $(I^{(\sigma)})^-$ (see (2.43)–(2.44) with $\Phi^{(\pm)}=0$) in the form of the single layer potential $U=V(g)$ (see (3.70)). In both cases, for the interior and exterior BVPs, we arrive at the equation

$$\mathcal{H}g=f \quad \text{on} \quad S, \tag{6.12}$$

where $\mathcal{H}$ is defined by (5.7).

We have the following existence theorem.

Theorem 6.6. *Let $S \in C^{2,\alpha}$ and $f \in [C^{1,\beta}(S)]^7$ with $0 < \beta < \alpha \le 1$. Then the BVPs $(I^{(\sigma)})^{\pm}$ are uniquely solvable in the class of vector functions belonging to the space $[C^{1,\beta}(\overline{\Omega^{\pm}})]^7$ and decaying at infinity, and the solution is represented by the single layer potential $V(g)$ defined by* (3.70)*, where $g \in [C^{0,\beta}(S)]^7$ is a unique solution of the integral equation* (6.12)*.*

Proof. The uniqueness follows from Theorems 5.1, 2.2 and 4.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{H}:\ [C^{0,\beta}(S)]^7 \to [C^{1,\beta}(S)]^7. \tag{6.13}$$

To this end, we apply the operator $\mathcal{L}$ (see (5.10)) to both sides of equation (6.12) and take into consideration the operator equalities (5.19),

$$\mathcal{L}\mathcal{H}g \equiv [-4^{-1}I_7 + \mathcal{K}^2]g = \mathcal{L}f \quad \text{on} \quad S. \tag{6.14}$$

Clearly, $\mathcal{L}f \in [C^{0,\beta}(S)]^7$ due to Theorem 5.3. Since the operators (6.8) and (6.10) are invertible, we conclude that the singular integral operator

$$\mathcal{L}\mathcal{H} = [-2^{-1}I_7 + \mathcal{K}][2^{-1}I_7 + \mathcal{K}]:\ [C^{0,\alpha}(S)]^7 \to [C^{0,\alpha}(S)]^7 \tag{6.15}$$

is invertible as well. Therefore, from (6.14) we get the following representation of a solution of equation (6.12)

$$g = [-4^{-1}I_7 + \mathcal{K}^2]^{-1}\mathcal{L}f \in [C^{0,\beta}(S)]^7. \tag{6.16}$$

With the help of the uniqueness Theorem 2.2, one can easily show that the operators

$$\mathcal{H}:\ [C^{0,\beta}(S)]^7 \to [C^{1,\beta}(S)]^7, \quad \mathcal{L}:\ [C^{1,\beta}(S)]^7 \to [C^{0,\beta}(S)]^7, \tag{6.17}$$

are injective. Therefore, the equations (6.12) and (6.14) are equivalent and the operator (6.13) is invertible, which completes the proof. □

Corollary 6.7. *A solution $U \in [C^{1,\beta}(\overline{\Omega^{\pm}})]^7$ of the BVP $(I^{(\sigma)})^{\pm}$ with $\Phi^{(\pm)} = 0$ is uniquely representable in the form*

$$U(x) = V(\mathcal{H}^{-1}f)(x), \quad x \in \Omega^{\pm}, \tag{6.18}$$

where $f = \{U\}^{\pm}$ on S and

$$\mathcal{H}^{-1}:\ [C^{1,\beta}(S)]^7 \to [C^{0,\beta}(S)]^7 \tag{6.19}$$

is the inverse to the operator (6.13)*.*

6.3.2. *Investigation of the Neumann problem with the help of the first kind integral equations.* We look for a solution to the problems $(II^{(\sigma)})^{+}$ and $(II^{(\sigma)})^{-}$ (see (2.43), (2.45) with $\Phi^{(\pm)} = 0$) in the form of the double layer potential $U = W(h)$ (see (3.71)). In both cases, for the interior and exterior BVPs, we arrive then at the equation

$$\mathcal{L}h = F \quad \text{on} \quad S, \tag{6.20}$$

where $\mathcal{L}$ is defined by (5.10).

We have the following existence theorem.

Theorem 6.8. *Let $S \in C^{2,\alpha}$ and $F \in [C^{0,\beta}(S)]^7$ with $0 < \beta < \alpha \le 1$. Then the BVPs $(II^{(\sigma)})^{\pm}$ are uniquely solvable in the class of vector functions belonging to the space $[C^{1,\beta}(\overline{\Omega^{\pm}})]^7$ and decaying at infinity, and the solution is represented by the double layer potential $W(h)$ defined by* (3.71), *where $h \in [C^{1,\beta}(S)]^7$ is a unique solution of the integral equation* (6.20).

Proof. The uniqueness follows from Theorems 5.1, 2.2 and 4.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{L}: \ [C^{1,\beta}(S)]^7 \to [C^{0,\beta}(S)]^7. \tag{6.21}$$

To this end, we apply the operator $\mathcal{H}$ (see (5.7)) to both sides of equation (6.20) and take into consideration the operator equalities (5.19),

$$\mathcal{H}\mathcal{L}h \equiv [-4^{-1}I_7 + \mathcal{N}^2]h = \mathcal{H}F \quad \text{on} \quad S. \tag{6.22}$$

Clearly, $\mathcal{H}F \in [C^{1,\beta}(S)]^7$ due to Theorem 5.3. Since the operators (6.3) and (6.5) are invertible, we conclude that the singular integral operator

$$\mathcal{H}\mathcal{L} = [-2^{-1}I_7 + \mathcal{N}][2^{-1}I_7 + \mathcal{N}]: \ [C^{1,\alpha}(S)]^7 \to [C^{1,\alpha}(S)]^7 \tag{6.23}$$

is invertible as well. Therefore, from (6.22) we get the following representation of a solution of equation (6.20)

$$h = [-4^{-1}I_7 + \mathcal{N}^2]^{-1}\mathcal{H}F \in [C^{1,\beta}(S)]^7. \tag{6.24}$$

Since the operators (6.17) are injective, we conclude that the equations (6.20) and (6.22) are equivalent and the operator (6.21) is invertible, which completes the proof. □

Corollary 6.9. *A solution $U \in [C^{1,\beta}(\overline{\Omega^{\pm}})]^7$ of the BVP $(II^{(\sigma)})^{\pm}$ with $\Phi^{(\pm)} = 0$ is uniquely representable in the form*

$$U(x) = W(\mathcal{L}^{-1}F)(x), \ \ x \in \Omega^{\pm}, \tag{6.25}$$

where $F = \{\mathcal{P}(\partial, n)U\}^{\pm}$ on S and

$$\mathcal{L}^{-1}: \ [C^{0,\beta}(S)]^7 \to [C^{1,\beta}(S)]^7 \tag{6.26}$$

is the inverse to the operator (6.21).

6.4. **Existence results for the data from the Bessel potential and Besov spaces.** Here we extend the existence results to that case when the data of the BVPs are from the Bessel potential and Besov spaces and solutions are sought in the space $[W_p^1(\Omega^{\pm})]^7$ (see problem setting in Subsection 2.4 and inclusions (2.48)). First we formulate the following auxiliary lemma which directly follows from Theorems 5.4 and 6.1–6.4.

Lemma 6.10. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and $S \in C^{\infty}$. Then the operators*

$$\mathcal{H}: \ [H_p^s(S)]^7 \to [H_p^{s+1}(S)]^7 \ \ \Big[[B_{p,q}^s(S)]^7 \to [B_{p,q}^{s+1}(S)]^7\Big], \tag{6.27}$$

$$\pm 2^{-1} + \mathcal{K}: \ [H_p^s(S)]^7 \to [H_p^s(S)]^7 \ \ \Big[[B_{p,q}^s(S)]^7 \to [B_{p,q}^s(S)]^7\Big], \tag{6.28}$$

$$\pm 2^{-1}+\mathcal{N}: \ [H_p^s(S)]^7 \to [H_p^s(S)]^7 \ \ \Big[[B_{p,q}^s(S)]^7 \to [B_{p,q}^s(S)]^7\Big], \tag{6.29}$$

$$\mathcal{L}: \ [H_p^{s+1}(S)]^7 \to [H_p^s(S)]^7 \ \ \Big[[B_{p,q}^{s+1}(S)]^7 \to [B_{p,q}^s(S)]^7\Big], \tag{6.30}$$

are invertible.

This lemma implies the following existence result.

Theorem 6.11. *Let $S \in C^\infty$ and $f \in [B_{p,p}^{1-\frac{1}{p}}(S)]^7$. Then the BVPs $(I^{(\sigma)})^\pm$ are uniquely solvable in the space $[W_p^1(\Omega^\pm)]^7$ and the solutions are represented by the double layer potential $W(h)$ defined by* (3.71)*, where $h \in [B_{p,p}^{1-\frac{1}{p}}(S)]^7$ is a unique solution of the pseudodifferential equation*

$$[2^{-1}I_7 + \mathcal{N}]h = f \quad on \quad S, \tag{6.31}$$

in the case of problem $(I^{(\sigma)})^+$ and of the pseudodifferential equation

$$[-2^{-1}I_7 + \mathcal{N}]h = f \quad on \quad S, \tag{6.32}$$

in the case of problem $(I^{(\sigma)})^-$.

Theorem 6.12. *Let $S \in C^\infty$ and $F \in [B_{p,p}^{-\frac{1}{p}}(S)]^7$. Then the BVPs $(II^{(\sigma)})^\pm$ are uniquely solvable in the space $[W_p^1(\Omega^\pm)]^7$ and the solutions are represented by the single layer potential $V(g)$ defined by* (3.70)*, where $g \in [B_{p,p}^{-\frac{1}{p}}(S)]^7$ is a unique solution of the pseudodifferential equation*

$$[-2^{-1}I_7 + \mathcal{K}]g = F \quad on \quad S, \tag{6.33}$$

in the case of problem $(II^{(\sigma)})^+$ and of the pseudodifferential equation

$$[2^{-1}I_7 + \mathcal{K}]g = F \quad on \quad S, \tag{6.34}$$

in the case of problem $(II^{(\sigma)})^-$.

Theorem 6.13. *Let $S \in C^\infty$ and $f \in [B_{p,p}^{1-\frac{1}{p}}(S)]^7$. Then the BVPs $(I^{(\sigma)})^\pm$ are uniquely solvable in the space $[W_p^1(\Omega^\pm)]^7$ and the solutions are represented by the single layer potential $V(g)$, where $g \in [B_{p,p}^{-\frac{1}{p}}(S)]^7$ is a unique solution of the pseudodifferential equation*

$$\mathcal{H}g = f \quad on \quad S. \tag{6.35}$$

A solution $U \in [W_p^1(\Omega^\pm)]^7$ of the homogeneous equation $L(\partial,\sigma)U = 0$ in $\Omega^\pm$ is uniquely representable in the form

$$U(x) = V\left(\mathcal{H}^{-1}\{U\}^+\right)(x), \ \ x \in \Omega^\pm, \tag{6.36}$$

where $\mathcal{H}^{-1}$ is inverse to the operator

$$\mathcal{H} : [B_{p,p}^{-\frac{1}{p}}(S)]^7 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^7. \tag{6.37}$$

Theorem 6.14. *Let $S \in C^\infty$ and $F \in [B_{p,p}^{-\frac{1}{p}}(S)]^7$. Then the BVPs $(II^{(\sigma)})^\pm$ are uniquely solvable in the space $[W_p^1(\Omega^\pm)]^7$ and the solutions are represented by the double layer potential $W(h)$, where $h \in [B_{p,p}^{1-\frac{1}{p}}(S)]^7$ is a unique solution of the pseudodifferential equation*

$$\mathcal{L}h = F \quad on \quad S. \tag{6.38}$$

Remark 6.15. Lemma 6.10 and Theorems 6.11 − 6.14 with $p = 2$ remain valid for Lipschitz domains due to Theorems 5.2, 5.3.(iv) and the uniqueness Theorem 2.2.

6.5. **Investigation of the mixed type BVPs.** Having in hand the results obtained in the previous subsections, we can investigate the mixed type BVPs. In general, solutions to the mixed type BVPs are not regular at the lines, where the boundary conditions change their type (e.g., Dirichlet to Neumann). Therefore we have to look for solutions in the space $[W_p^1(\Omega^\pm)]^7$.

First, we consider the interior mixed type BVP $(III^{(\sigma)})^+$. We have to find a solution $U = (u, \omega, \vartheta)^\top \in [W_p^1(\Omega^+)]^7$ to the homogeneous equation $L(\partial, \sigma)U = 0$ in Ω^+, which satisfies the boundary conditions

$$\{U\}^+ = f^{(D)} \quad \text{on} \quad S_D, \tag{6.39}$$

$$\{\mathcal{P}(\partial, n)U\}^+ = F^{(N)} \quad \text{on} \quad S_N, \tag{6.40}$$

where

$$f^{(D)} \in [B_{p,p}^{1-\frac{1}{p}}(S_D)]^7, \quad F^{(N)} \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^7. \tag{6.41}$$

For simplicity, throughout this subsection we assume that S and $\partial S_D = \partial S_N$ are C^∞-smooth.

Denote by $f^{(e)}$ a fixed extension of the vector-function $f^{(D)}$ from S_D onto S preserving the functional space:

$$f^{(e)} \in [B_{p,p}^{1-\frac{1}{p}}(S)]^7, \quad r_{S_D} f^{(e)} = f^{(D)} \quad \text{on} \quad S_D. \tag{6.42}$$

Recall that $r_{\mathcal{M}}$ denotes the restriction operator to $\mathcal{M}$.

Evidently, an arbitrary extension f of $f^{(D)}$ onto the whole of S, which preserves the functional space, can be then represented as

$$f = f^{(e)} + \varphi \ \text{ with } \ \varphi \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^7. \tag{6.43}$$

In accordance with Lemma 6.10 and Theorem 6.13, we can look for a solution in the form

$$U = V\big(\mathcal{H}^{-1}(f^{(e)} + \varphi)\big), \tag{6.44}$$

where $\varphi \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^7$ is an unknown vector function.

In view of (6.42), it is easy to check that the Dirichlet condition (6.39) on S_D is satisfied automatically. It remains only to satisfy the Neumann condition (6.40) on S_N, which leads to the pseudodifferential equation

$$[-2^{-1}I_7 + \mathcal{K}]\mathcal{H}^{-1}(f^{(e)} + \varphi) = F^{(N)} \tag{6.45}$$

on the open subsurface S_N for the unknown vector function φ.

Let

$$\mathcal{A} := [-2^{-1}I_7 + \mathcal{K}]\mathcal{H}^{-1}, \tag{6.46}$$

$$F^{(0)} := F^{(N)} - r_{S_N}\mathcal{A}f^{(e)} \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^7. \tag{6.47}$$

The operator $\mathcal{A}$ is known as the Steklov–Poincaré type operator. Equation (6.45) can be rewritten then in the form

$$r_{S_N}\mathcal{A}\varphi = F^{(0)} \quad \text{on} \quad S_N, \tag{6.48}$$

which is a pseudodifferential equation on the submanifold S_N with boundary ∂S_N. Due to Theorem 5.4 and Lemma 6.10, the operator $\mathcal{A}$ has the following mapping property

$$\mathcal{A}\ : [B_{p,p}^{1-\frac{1}{p}}(S)]^7 \to [B_{p,p}^{-\frac{1}{p}}(S)]^7, \tag{6.49}$$

i.e., $\mathcal{A}$ is a pseudodifferential operator of order 1.

Lemma 6.16. *The principal homogeneous symbol matrix of the operator $\mathcal{A}$ is positive definite.*

Proof. It is clear that the principal homogeneous symbol matrices of the operators $\mathcal{A}$ and its main singular part $\mathcal{A}_0 := [-2^{-1}I_7 + \mathcal{K}_0]\mathcal{H}_0^{-1}$ are the same. Note that the operators with subscript 0 are generated by the potentials with the kernel matrix $\Gamma_0(\cdot)$, which is the fundamental matrix of the operator $L_0(\partial)$ constructed in Subsection 3.3 (see (2.12)) and represents a principal singular part of the matrix $\Gamma(\cdot,\sigma)$ (see (3.64)).

We write Green's formula in Ω^+ for real-valued vector functions $U = U' = V_0(\mathcal{H}_0^{-1}g)$ with arbitrary $g \in [H_2^{\frac{1}{2}}(S)]^7$ to obtain (cf. (5.38))

$$\Big\langle \big\{\mathcal{P}_0(\partial,n)V_0(\mathcal{H}_0^{-1}g)\big\}^+, \big\{V_0(\mathcal{H}_0^{-1}g)\big\}^+ \Big\rangle_S = \\ = \int\limits_{\Omega^+} E_0\big(V_0(\mathcal{H}_0^{-1}g), V_0(\mathcal{H}_0^{-1}g)\big)\,dx \ge 0,$$

where $E_0(\cdot,\cdot)$ is defined by (5) and n is the outward unit normal vector to S. Whence

$$\big\langle [-2^{-1}I_7 + \mathcal{K}_0]\mathcal{H}_0^{-1}g, g \big\rangle_S \ge 0, \tag{6.50}$$

where we have a strict inequality if g is not a constant vector (see the arguments concerning the formula (5.42) in the proof of Theorem 5.3). Since the principal homogeneous symbol matrices of the operators $-2^{-1}I_7 + \mathcal{K}_0$ and $\mathcal{H}_0^{-1}$ are nondegenerate (see Theorem 5.3) and g is an arbitrary vector function of the space $[H_2^{\frac{1}{2}}(S)]^7$, it follows from (6.50) that the principal homogeneous symbol matrix of the composition of these operators (i.e., of the operator $\mathcal{A}$) is positive definite. □

Now we are in a position to prove the following main lemma.

Lemma 6.17. *The operators*

$$r_{S_N}\mathcal{A}:\ [\tilde{H}_p^s(S_N)]^7 \to [H_p^{s-1}(S_N)]^7, \tag{6.51}$$

$$: \ [\tilde{B}^s_{p,q}(S_N)]^7 \to [B^{s-1}_{p,q}(S_N)]^7, \tag{6.52}$$

are invertible if

$$1/p - 1/2 < s < 1/p + 1/2. \tag{6.53}$$

Proof. The mapping properties (6.51) and (6.52) follow from Theorem 5.4, since $\mathcal{A}$ is a pseudodifferential operator of order 1.

To prove the invertibility of the operators (6.51) and (6.52), we first consider the case $p = 2$, $s = 1/2$, and $q = 2$, and show that the null space of the operator

$$r_{S_N}\mathcal{A}: \ [\widetilde{H}^{\frac{1}{2}}_2(S_N)]^7 \to [H^{-\frac{1}{2}}_2(S_N)]^7$$

is trivial, i.e., the equation

$$r_{S_N}\mathcal{A}\varphi = 0 \ \text{ on } \ S_N \tag{6.54}$$

admits only the trivial solution in the space $[\widetilde{H}^{\frac{1}{2}}_2(S_N)]^7$. Recall that $\widetilde{H}^s_2(S_N) = \widetilde{B}^s_{2,2}(S_N)$ and $H^s_2(S_N) = B^s_{2,2}(S_N)$ for $s \in \mathbb{R}$.

Let $\varphi \in [\widetilde{H}^{\frac{1}{2}}(S_N)]^7$ be a solution of the homogeneous equation (6.54). It is clear that the vector

$$U = V(\mathcal{H}^{-1}\varphi)$$

belongs to the space $[H^1_2(\Omega^+)]^7 = [W^1_2(\Omega^+)]^7$ and solves the homogeneous mixed Problem $(III^{(\sigma)})^+$. Therefore, $U(x) = V(\mathcal{H}^{-1}\varphi)(x) = 0$ for $x \in \Omega^+$, due to Theorem 2.2 and, consequently, $\{U\}^+ = \varphi = 0$ on S. Since the principal singular part of the operator $\mathcal{A}$ is self-adjoint (due to the positive definiteness of the principal homogeneous symbol matrix of $\mathcal{A}$), we conclude that the index of $\mathcal{A}$ equals to zero and thus, by Theorem C.1 (see the Appendix C) the operator

$$r_{S_N}\mathcal{A}: \ [\widetilde{H}^{\frac{1}{2}}_2(S_N)]^7 \to [H^{-\frac{1}{2}}_2(S_N)]^7$$

is invertible.

Since the principal homogeneous symbol matrix of the operator $\mathcal{A}$ is positive definite, Theorem C.1 with $\nu = 1$ completes the proof. □

With the help of this lemma we can prove the following main existence result.

Theorem 6.18. *Let $4/3 < p < 4$ and the conditions* (6.41) *be fulfilled. Then Problem $(III^{(\sigma)})^+$ has a unique solution $U \in [W^1_p(\Omega^+)]^7$ which is representable in the form of single layer potential* (6.44),

$$U = V\big(\mathcal{H}^{-1}(f^{(e)} + \varphi)\big), \tag{6.55}$$

where $f^{(e)} \in [B^{1-1/p}_{p,p}(S)]^7$ is a fixed extension of the vector function $f^{(D)} \in [B^{1-1/p}_{p,p}(S_D)]^7$ from S_D onto S preserving the functional space and $\varphi \in [\widetilde{B}^{1-1/p}_{p,p}(S_N)]^7$ is defined by the uniquely solvable pseudodifferential equation

$$r_{S_N}\mathcal{A}\varphi = F^{(0)} \ \text{ on } \ S_N \tag{6.56}$$

with

$$F^{(0)} := F^{(N)} - r_{_{S_N}} \mathcal{A} f^{(e)} \in [B^{-1/p}_{p,p}(S_N)]^7.$$

Proof. First we note that in accordance with Lemma 6.17, equation (6.56) is uniquely solvable for $s = 1 - 1/p$ and $4/3 < p < 4$, where the last inequality follows from the inequality (6.53). This implies the solvability of Problem $(III^{(\sigma)})^+$ in the space $[W^1_p(\Omega^+)]^7$ with $p \in (4/3, 4)$.

Next, we show the uniqueness of solution in the space $[W^1_p(\Omega^+)]^7$ for arbitrary $p \in (4/3, 4)$ (for $p = 2$ it has been proved in Theorem 2.2). Let $U \in [W^1_p(\Omega^+)]^7$ be some solution of the homogeneous mixed Problem $(III^{(\sigma)})^+$. Clearly, then

$$\{U\}^+ \in [\widetilde{B}^{1-1/p}_{p,p}(S_N)]^7. \tag{6.57}$$

By Theorem 6.13, we have the representation

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^+\big)(x), \ \ x \in \Omega^+.$$

Since U satisfies the homogeneous Neumann condition (6.40) on S_N, we arrive at the equation

$$r_{_{S_N}} \mathcal{A}\{U\}^+ = 0 \ \text{ on } \ S_N,$$

whence $\{U\}^+ = 0$ on S follows due to the inclusion (6.57), Lemma 6.17, and the inequality $4/3 < p < 4$. Therefore, $U = 0$ in Ω^+. □

Further, we prove almost the best regularity results for solutions to the mixed type boundary value problem $(III^{(\sigma)})^+$.

Theorem 6.19. *Let the conditions* (6.41) *and the inequalities*

$$4/3 < p < 4, \ \ 1 < t < \infty, \ \ 1 \le q \le \infty, \ \ 1/t - 1/2 < s < 1/t + 1/2, \tag{6.58}$$

be fulfilled, and let $U \in [W^1_p(\Omega^+)]^7$ *be the unique solution to the mixed problem* $(III^{(\sigma)})^+$.

In addition,

(i) *if*

$$f^{(D)} \in [B^s_{t,t}(S_D)]^7, \ \ F^{(N)} \in [B^{s-1}_{t,t}(S_N)]^7, \tag{6.59}$$

then

$$U \in [H^{s+1/t}_t(\Omega^+)]^7; \tag{6.60}$$

(ii) *if*

$$f^{(D)} \in [B^s_{t,q}(S_D)]^7, \ \ F^{(N)} \in [B^{s-1}_{t,q}(S_N)]^7, \tag{6.61}$$

then

$$U \in [B^{s+1/t}_{t,q}(\Omega^+)]^7; \tag{6.62}$$

(iii) *if*

$$f^{(D)} \in [C^{\alpha_0}(S_D)]^7, \ \ F^{(N)} \in [B^{\alpha_0-1}_{\infty,\infty}(S_N)]^7, \ \ \alpha_0 > 0, \tag{6.63}$$

then

$$U \in [C^{\beta_0}(\overline{\Omega^+})]^7 \ \text{ with any } \ \beta_0 \in (0, \alpha_1), \ \ \alpha_1 := \min\{\alpha_0, 1/2\}. \tag{6.64}$$

Proof. Applying Lemma 6.17, Theorem 6.18, the inclusions (6.41) and (6.59) (resp. (6.61)) along with the inequalities (6.58), we conclude that the solution vector U is represented by (6.55) with $\varphi \in [\tilde{B}^s_{t,t}(S_N)]^7$ (resp. $\varphi \in [\tilde{B}^s_{t,q}(S_N)]^7$) in view of (6.56) and $F^{(0)} \in [B^{s-1}_{t,t}(S_N)]^7$ (resp. $F^{(0)} \in [B^{s-1}_{t,q}(S_N)]^7$).

Note that $f^{(e)} \in [B^s_{t,t}(S)]^7$ (resp. $f^{(e)} \in [B^s_{t,q}(S)]^7$) is some extension of the vector $f^{(D)}$ onto the whole of S. Therefore, by Theorem 6.18 and the representation formula (6.55) the inclusion (6.60) (resp. (6.62)) follows.

To prove (iii) we use the following embeddings (see, e.g., [56], [57])

$$C^{\alpha_0}(\mathcal{S}) = B^{\alpha_0}_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha_0-\varepsilon_0}_{\infty,1}(\mathcal{S}) \subset \\ \subset B^{\alpha_0-\varepsilon_0}_{\infty,q}(\mathcal{S}) \subset B^{\alpha_0-\varepsilon_0}_{t,q}(\mathcal{S}) \subset C^{\alpha_0-\varepsilon_0-k/t}(\mathcal{S}), \quad (6.65)$$

where ε_0 is an arbitrary small positive number, $\mathcal{S} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < t < \infty$, $\alpha_0 - \varepsilon_0 - k/t > 0$, α_0 and $\alpha_0 - \varepsilon_0 - k/t$ are not integers. From (6.63) and the embeddings (6.65) the condition (6.62) follows with any $s \le \alpha_0 - \varepsilon_0$.

Bearing in mind (6.58) and taking t sufficiently large and ε_0 sufficiently small, we may put $s = \alpha_0 - \varepsilon_0$ if

$$1/t - 1/2 < \alpha_0 - \varepsilon_0 < 1/t + 1/2, \quad (6.66)$$

and $s \in (1/t - 1/2, 1/t + 1/2)$ if

$$1/t + 1/2 < \alpha_0 - \varepsilon_0. \quad (6.67)$$

By (6.62) the solution U belongs then to $[B^{s+1/t}_{t,q}(\Omega^+)]^7$ with $s + 1/t = \alpha_0 - \varepsilon_0 + 1/t$ if (6.66) holds, and with $s+1/t \in (2/t-1/2, 2/t+1/2)$ if (6.67) holds. In the last case we can take $s+1/t = 2/t+1/2-\varepsilon_0$. Therefore, we have either $U \in [B^{\alpha_0-\varepsilon_0+1/t}_{t,q}(\Omega^+)]^7$, or $U \in [B^{1/2+2/t-\varepsilon_0}_{t,q}(\Omega^+)]^7$ in accordance with the inequalities (6.66) and (6.67). The last embedding in (6.65) (with $k = 3$) yields that either $U \in [C^{\alpha_0-\varepsilon_0-2/t}(\overline{\Omega^+})]^7$, or $U \in [C^{1/2-\varepsilon_0-1/t}(\overline{\Omega^+})]^7$ which lead to the inclusion

$$U \in [C^{\alpha_1-\varepsilon_0-2/t}(\overline{\Omega^+})]^7, \quad (6.68)$$

where $\alpha_1 := \min\{\alpha_0, 1/2\}$. Since t is sufficiently large and ε_0 is sufficiently small, the embedding (6.68) completes the proof. □

Remark 6.20. By the same arguments, it can be shown that the uniqueness, existence and regularity results, similar to the above ones, hold also true for the exterior boundary value problem $(III^{(\sigma)})^-$. We note only that the solution is representable again in the form of the single layer potential (6.55), where $f^{(e)}$ is the same as above, and φ is the unique solution of the pseudodifferential equation

$$r_{S_N} \widetilde{\mathcal{A}}\varphi = \widetilde{F}^{(0)} \ \text{ on } \ S_N, \quad (6.69)$$

where

$$\widetilde{\mathcal{A}} := [2^{-1} I_7 + \mathcal{K}]\mathcal{H}^{-1}, \tag{6.70}$$

$$\widetilde{F}^{(0)} := F^{(N)} - r_{_{S_N}} \widetilde{\mathcal{A}} f^{(e)}. \tag{6.71}$$

The operator $\widetilde{\mathcal{A}}$ has the same properties as $\mathcal{A}$ described above in Lemmas 6.16 and 6.17.

Remark 6.21. Lemma 6.17 and Theorems 6.18 with $p = 2$ remain valid for Lipschitz domains due to Theorems 5.2, 5.3.(iv) and the uniqueness Theorem 2.2.

7. Appendix A: Properties of the Characteristic Roots and Wave Numbers

Here we investigate the properties of roots of the equation (3.32) with respect to r. In particular we prove the following assertion.

Lemma A.1. *Let $\sigma = \sigma_1 + i\sigma_2$ be a complex parameter with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then*

$$\Xi(r) := \det L(-i\xi, \sigma) \equiv (a^2 - c^2 r^2)\Lambda_1(\xi) \neq 0$$

for arbitrary $\xi \in \mathbb{R}^3$; here Λ_1 and $a^2 - c^2r^2$ are given by formulas (3.23) *and* (3.24).

Proof. We prove the lemma by contradiction. To this end, let σ be as in the lemma and assume that $\Xi(r) := \det L(-i\xi, \sigma) = 0$ for some $r = |\xi|$ with $\xi \in \mathbb{R}^3$ (cf. (3.32)). Then the system of linear equations $L(-i\xi, \sigma)X = 0$ has a nontrivial solution $X \in \mathbb{C}^7 \setminus \{0\}$. Denote $X = (X^{(1)}, X^{(2)}, X^{(3)})^\top$ with $X^{(j)} = (X_1^{(j)}, X_2^{(j)}, X_3^{(j)})^\top \in \mathbb{C}^3$, $j = 1, 2$, and $X^{(3)} \in \mathbb{C}$. In view of (2.5) and (2.6), the linear system in question can be written as

$$L^{(1)}(-i\xi,\sigma)X^{(1)} + L^{(2)}(-i\xi,\sigma)X^{(2)} + L^{(5)}(-i\xi,\sigma)X^{(3)} = 0,$$

$$L^{(3)}(-i\xi,\sigma)X^{(1)} + L^{(4)}(-i\xi,\sigma)X^{(2)} + L^{(6)}(-i\xi,\sigma)X^{(3)} = 0,$$

$$L^{(7)}(-i\xi,\sigma)X^{(1)} + L^{(8)}(-i\xi,\sigma)X^{(2)} + L^{(9)}(-i\xi,\sigma)X^{(3)} = 0,$$

i.e.,

$$\begin{aligned} &\Big\{\big[-(\mu+\alpha)|\xi|^2 + \varrho\sigma^2\big] I_3 - (\lambda+\mu-\alpha)Q(\xi)\Big\} X^{(1)} + \\ &+\Big\{-(\varkappa+\nu)|\xi|^2 I_3 - (\delta+\varkappa-\nu)Q(\xi) - i2\alpha R(\xi)\Big\} X^{(2)} + \\ &+ i\eta\xi^\top X^{(3)} = 0, \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\Big\{-(\varkappa+\nu)|\xi|^2 I_3 - (\delta+\varkappa-\nu)Q(\xi) - i2\alpha R(\xi)\Big\} X^{(1)} + \\ &+\Big\{\big[-(\gamma+\varepsilon)|\xi|^2 + \mathcal{I}\sigma^2 - 4\alpha\big] I_3 - (\beta+\gamma-\varepsilon)Q(\xi) - i4\nu R(\xi)\Big\} X^{(2)} + \\ &+ i\zeta\xi^\top X^{(3)} = 0, \end{aligned} \tag{A.2}$$

$$\eta\sigma(\xi\cdot X^{(1)}) + \zeta\sigma(\xi\cdot X^{(2)}) + (-\kappa'|\xi|^2 + i\sigma\kappa'')X^{(3)} = 0. \tag{A.3}$$

From (3.23) and (3.24) it follows that $\Xi(0)\neq 0$. Therefore in what follows we assume $|\xi|\neq 0$.

We recall that the central dot denotes the real scalar product, $a\cdot b=\sum\limits_{j=1}^{3}a_jb_j$ for $a,b\in\mathbb{C}^3$, and $c\times d$ denotes the vector product of two vectors $c,d\in\mathbb{C}^3$. Note that

$$\begin{aligned}&(a\times b)\cdot c=-(a\times c)\cdot b,\quad a\times(b\times c)=(a\cdot c)b-(a\cdot b)c,\\ &Q(\xi)X^{(j)}=\xi^\top(\xi\cdot X^{(j)}),\quad R(\xi)X^{(j)}=\xi\times X^{(j)},\ \ j=1,2.\end{aligned}$$

Multiply equation (A.1) by $\overline{X^{(1)}}$, equation (A.2) by $\overline{X^{(2)}}$ and the complex conjugate of equation (A.3) by $C_0X^{(3)}$ with

$$C_0=-\frac{i}{\overline{\sigma}} \tag{A.4}$$

and sum the results to obtain

$$\begin{aligned}&\big[-(\mu+\alpha)|\xi|^2+\varrho\sigma^2\big]|X^{(1)}|^2-(\lambda+\mu-\alpha)|\xi\cdot X^{(1)}|^2-\\ &\quad-(\varkappa+\nu)|\xi|^2(X^{(2)}\cdot\overline{X^{(1)}})-(\delta+\varkappa-\nu)(\xi\cdot X^{(2)})(\xi\cdot\overline{X^{(1)}})-\\ &\quad-i2\alpha(\xi\times X^{(2)})\cdot\overline{X^{(1)}}-(\varkappa+\nu)|\xi|^2(X^{(1)}\cdot\overline{X^{(2)}})-\\ &\quad-(\delta+\varkappa-\nu)(\xi\cdot X^{(1)})(\xi\cdot\overline{X^{(2)}})-i2\alpha(\xi\times X^{(1)})\cdot\overline{X^{(2)}}+\\ &\quad+\big[-(\gamma+\varepsilon)|\xi|^2+\mathcal{I}\sigma^2-4\alpha\big]|X^{(2)}|^2-(\beta+\gamma-\varepsilon)|\xi\cdot X^{(2)}|^2-\\ &\quad-i4\nu(\xi\times X^{(2)})\cdot\overline{X^{(2)}}+\frac{i}{\overline{\sigma}}\big(\kappa'|\xi|^2+i\overline{\sigma}\kappa''\big)|X^{(3)}|^2=0.\end{aligned}\tag{A.5}$$

Keep in mind that the real parts of the expressions $(\xi\times X^{(2)})\cdot\overline{X^{(1)}}+(\xi\times X^{(1)})\cdot\overline{X^{(2)}}$ and $(\xi\times X^{(2)})\cdot\overline{X^{(2)}}$ vanish and separate the imaginary part of equation (A.5)

$$\sigma_1\Big[2\varrho\sigma_2|X^{(1)}|^2+2\mathcal{I}\sigma_2|X^{(2)}|^2+\frac{\kappa'|\xi|^2}{|\sigma|^2}\,|X^{(3)}|^2\Big]=0.$$

Whence $X^{(1)}=0$, $X^{(2)}=0$, $X^{(3)}=0$ follow for $\sigma_1\neq 0$ since $\sigma_2>0$ and $|\xi|\neq 0$.

Further, let $\sigma_1=0$ and $\sigma=i\sigma_2$. Then by multiplication of (A.1) and (A.2) by ξ, and (A.3) by $-|\xi|^2\sigma_2^{-1}$ we get

$$\big[(\lambda+2\mu)|\xi|^2+\varrho\sigma_2^2\big](\xi\cdot X^{(1)})+(\delta+2\varkappa)|\xi|^2(\xi\cdot X^{(2)})-i\eta|\xi|^2X^{(3)}=0, \tag{A.6}$$

$$\begin{aligned}&(\delta+2\varkappa)|\xi|^2(\xi\cdot X^{(1)})+\big[(\beta+2\gamma)|\xi|^2+\mathcal{I}\sigma_2^2+4\alpha\big](\xi\cdot X^{(2)})-\\ &\quad-i\zeta|\xi|^2X^{(3)}=0,\end{aligned}\tag{A.7}$$

$$-i\eta|\xi|^2(\xi\cdot X^{(1)})-i\zeta|\xi|^2(\xi\cdot X^{(2)})+\frac{|\xi|^2}{\sigma_2}\big(\kappa'|\xi|^2+\sigma_2\kappa''\big)X^{(3)}=0. \tag{A.8}$$

Introduce the notation $z_1=\xi\cdot X^{(1)}$, $z_2=\xi\cdot X^{(2)}$ and $z_3=X^{(3)}$. Multiply (A.6) by $\overline{z_1}$, (A.7) by $\overline{z_2}$, and the conjugate of (A.8) by z_3 and sum to obtain

$$\big[(\lambda+2\mu)|\xi|^2+\varrho\sigma_2^2\big]|z_1|^2+(\delta+2\varkappa)|\xi|^2 z_2\overline{z_1}+(\delta+2\varkappa)|\xi|^2 z_1\overline{z_2}+$$
$$+\big[(\beta+2\gamma)|\xi|^2+\mathcal{I}\sigma_2^2+4\alpha\big]|z_2|^2+\frac{|\xi|^2}{\sigma_2}\big(\kappa'|\xi|^2+\sigma_2\kappa''\big)|z_3|^2=0. \quad (A.9)$$

With the help of the inequality (see (2.26)) $(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0$ we easily conclude from (A.9)

$$z_1=\xi\cdot X^{(1)}=0,\quad z_2=\xi\cdot X^{(2)},\quad z_3=X^{(3)}=0. \quad (A.10)$$

Therefore, we can rewrite (A.1)–(A.3) in the equivalent form

$$\big[(\mu+\alpha)|\xi|^2+\varrho\sigma_2^2\big]X^{(1)}+(\varkappa+\nu)|\xi|^2X^{(2)}+i2\alpha(\xi\times X^{(2)})=0, \quad (A.11)$$
$$(\varkappa+\nu)|\xi|^2X^{(1)}+i2\alpha(\xi\times X^{(1)})+\big[(\gamma+\varepsilon)|\xi|^2+\mathcal{I}\sigma_2^2+4\alpha\big]X^{(2)}+$$
$$+i4\nu(\xi\times X^{(2)})=0. \quad (A.12)$$

Applying the standard formulas of vector analysis to the vectors $X^{(j)}$, $j=1,2$, satisfying the condition (A.10) we get:

$$\xi\times(\xi\times X^{(j)})=-|\xi|^2X^{(j)},$$
$$\xi\times(\xi\times X^{(j)})\cdot X^{(k)}=-|\xi|^2(X^{(j)}\cdot X^{(k)}), \quad (A.13)$$
$$(\xi\times X^{(j)})\cdot(\xi\times X^{(k)})=|\xi|^2(X^{(j)}\cdot X^{(k)}),$$

Multiply equation (A.11) by $\overline{X^{(1)}}$, (A.12) by $\overline{X^{(2)}}$ and sum

$$\Psi(X^{(1)},X^{(2)}):=\big[(\mu+\alpha)|\xi|^2+\varrho\sigma_2^2\big]|X^{(1)}|^2+(\varkappa+\nu)|\xi|^2(X^{(2)}\cdot\overline{X^{(1)}})+$$
$$+i2\alpha(\xi\times X^{(2)})\cdot\overline{X^{(1)}}+(\varkappa+\nu)|\xi|^2(X^{(1)}\cdot\overline{X^{(2)}})+$$
$$+i2\alpha(\xi\times X^{(1)})\cdot\overline{X^{(2)}}+\big[(\gamma+\varepsilon)|\xi|^2+\mathcal{I}\sigma_2^2+4\alpha\big]|X^{(2)}|^2+i4\nu(\xi\times X^{(2)})\cdot\overline{X^{(2)}}=0.$$

Using the relations (A.13) we can rewrite the function Ψ in the form:

$$\Psi(X^{(1)},X^{(2)})=\frac{\alpha\varepsilon-\nu^2}{\alpha}\,|\xi\times X^{(2)}|^2+$$
$$+\frac{1}{\alpha}\,\big|\alpha\xi\times X^{(1)}+\nu\xi\times X^{(2)}-i2\alpha X^{(2)}\big|^2+\big(\mu|\xi|^2+\varrho\sigma_2^2\big)|X^{(1)}|^2+$$
$$+\big(\gamma|\xi|^2+\mathcal{I}\sigma_2^2\big)|X^{(2)}|^2+\varkappa|\xi|^2\big(X^{(2)}\cdot\overline{X^{(1)}}+X^{(1)}\cdot\overline{X^{(2)}}\big)=0\,.$$

With the help of the inequalities $\alpha>0$, $\alpha\varepsilon-\nu^2>0$ and $\mu\gamma-\varkappa^2>0$ (see (2.26)) we easily derive from the last equality that $X^{(1)}=X^{(2)}=0$.

Thus we have shown that the system $L(-i\xi,\sigma)X=0$ possesses only the trivial solution. This contradiction proves the lemma. □

Corollary A.2. *Let $\sigma=\sigma_1+i\sigma_2$ be a complex parameter with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Consider the equation*

$$\Xi(r)\equiv(a^2-c^2r^2)\Lambda_1(\xi)=0 \quad (A.14)$$

with respect to r, where Λ_1 and $a^2-c^2r^2$ are given by formulas (3.23) *and* (3.24). *The roots $\pm k_j$, $j=\overline{1,7}$, of equation* (A.14) *are complex with $\Im k_j>0$, $j=\overline{1,7}$.*

8. Appendix B: Fourier Transform of Some Standard Functions

8.1. **Fourier transform of standard homogeneous functions.** Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the generalized direct and inverse Fourier transforms in the space of tempered distributions defined as in (3.1). The following formulas are true

$$\begin{gathered}\mathcal{F}_{x\to\xi}[\partial_p f]=-i\xi_p\mathcal{F}_{x\to\xi}[f],\quad \mathcal{F}_{x\to\xi}[x_p f]=-i\partial_p\mathcal{F}_{x\to\xi}[f],\\ \mathcal{F}^{-1}_{\xi\to x}[\partial_p g]=ix_p\mathcal{F}^{-1}_{\xi\to x}[g],\quad \mathcal{F}^{-1}_{\xi\to x}[\xi_p g]=i\partial_p\mathcal{F}^{-1}_{\xi\to x}[g],\ \ p=1,2,3.\end{gathered}\tag{B.1}$$

For the regular functionals $|\xi|^{-k}$ with $k=1,2$, and $\xi_p\xi_q|\xi|^{-4}$ with $p,q=1,2,3$, and the singular functional $\xi_p|\xi|^{-4}$ which are understood in the Value Principal sense, we have the following formulas (see, e.g., [18], [14])

$$\mathcal{F}^{-1}_{\xi\to x}[|\xi|^{-1}]=\frac{1}{2\pi^2|x|^2},\quad \mathcal{F}^{-1}_{\xi\to x}[|\xi|^{-2}]=\frac{1}{4\pi|x|},\tag{B.2}$$

$$\mathcal{F}^{-1}_{\xi\to x}[\xi_p|\xi|^{-4}]=-\frac{ix_p}{8\pi|x|},\ \ p=1,2,3,\tag{B.3}$$

$$\begin{gathered}\mathcal{F}^{-1}_{\xi\to x}\big[\xi_p\xi_q|\xi|^{-4}\big]=\\ =\partial_q\mathcal{F}^{-1}_{\xi\to x}[\xi_p|\xi|^{-4}]=\frac{1}{8\pi}\Big[\frac{\delta_{pq}}{|x|}-\frac{x_px_q}{|x|^3}\Big],\ \ p,q=1,2,3.\end{gathered}\tag{B.4}$$

8.2. **Fourier transform of functions related to the Helmholtz operator.** Here we calculate the inverse Fourier transform of the regular functional $(|\xi|^2-\tau^2)^{-1}$, where $\tau=\omega+i\varepsilon$ with $\omega\in\mathbb{R}$ and $\varepsilon>0$. Since the function under consideration is square integrable, we can write (see, e.g., [18])

$$\begin{aligned}H(x,\tau)&:=\mathcal{F}^{-1}_{\xi\to x}\big[(|\xi|^2-\tau^2)^{-1}\big]=\mathcal{F}^{-1}_{\xi\to x}\big[\big(|\xi|^2+(\varepsilon-i\omega)^2\big)^{-1}\big]\\ &=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}\frac{e^{-ix\cdot\xi}}{|\xi|^2+(\varepsilon-i\omega)^2}\,d\xi.\end{aligned}\tag{B.5}$$

Let $\Lambda(\widetilde{x})=[\Lambda_{kj}(\widetilde{x})]_{3\times3}$ with $\widetilde{x}=x/|x|$ be an orthogonal matrix with properties:

$$\det\Lambda(\widetilde{x})=1,\quad \Lambda^\top(\widetilde{x})x=(0,0,|x|)^\top.$$

Perform the transform of variables in (B.5): $\xi=\Lambda(\widetilde{x})\eta$. Evidently, $x\cdot\Lambda(\widetilde{x})\eta=|x|\eta_3$, $|\xi|=|\eta|$ and $d\xi=d\eta$. Therefore we get from (B.5)

$$H(x,\tau)=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\eta|<R}\frac{e^{-i|x|\eta_3}}{|\eta|^2+(\varepsilon-i\omega)^2}\,d\eta.\tag{B.6}$$

Introduce the spherical co-ordinates

$$\begin{gathered}\eta_1=\varrho\cos\varphi\sin\vartheta,\ \ \eta_2=\varrho\sin\varphi\sin\vartheta,\ \ \eta_3=\varrho\cos\vartheta,\\ \varrho=|\eta|,\ \ \varphi\in[0,2\pi],\ \ \vartheta\in[0,\pi],\end{gathered}$$

and rewrite (B.6) as follows

$$H(x,\tau)=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_0^R\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{e^{-i|x|\varrho\cos\vartheta}}{\varrho^2+(\varepsilon-i\omega)^2}\,\varrho^2\sin\vartheta\,d\vartheta\,d\varphi\,d\varrho=$$

$$=\frac{1}{4\pi^2}\lim_{R\to\infty}\int\limits_0^R\frac{\varrho^2d\varrho}{\varrho^2+(\varepsilon-i\omega)^2}\int\limits_0^{\pi}\frac{1}{i|x|\varrho}\Big(\frac{\partial}{\partial\vartheta}e^{-i|x|\varrho\cos\vartheta}\Big)\,d\vartheta=$$

$$=-\frac{i}{4\pi^2|x|}\lim_{R\to\infty}\int\limits_0^R\frac{\varrho}{\varrho^2+(\varepsilon-i\omega)^2}\big[e^{i|x|\varrho}-e^{-i|x|\varrho}\big]\,d\varrho=$$

$$=\frac{1}{2\pi^2|x|}\lim_{R\to\infty}\int\limits_0^R\frac{\varrho\sin(|x|\varrho)}{\varrho^2+(\varepsilon-i\omega)^2}\,d\varrho=$$

$$=\frac{1}{2\pi^2|x|}\int\limits_0^{\infty}\frac{\varrho\sin(|x|\varrho)}{\varrho^2+(\varepsilon-i\omega)^2}\,d\varrho. \tag{B.7}$$

With the help of the formula

$$\int\limits_0^{\infty}\frac{t\sin(at)}{t^2+\beta^2}dt=\frac{\pi}{2}e^{-a\beta}\quad\text{for}\quad a>0,\Re\beta>0,$$

finally we get

$$H(x,\tau)=\frac{e^{-|x|(\varepsilon-i\omega)}}{4\pi|x|}=\frac{e^{i\tau|x|}}{4\pi|x|}\,, \tag{B.8}$$

i.e.,

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-\tau^2}\Big]=\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-(\omega+i\varepsilon)^2}\Big]=\frac{e^{i\tau|x|}}{4\pi|x|}\,, \tag{B.9}$$

where $\tau=\omega+i\varepsilon$ with $\varepsilon>0$ and $\omega\in\mathbb{R}$.
Quite analogously we can derive the similar formula

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-\tau^2}\Big]=\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-(\omega-i\varepsilon)^2}\Big]=\frac{e^{-i\tau|x|}}{4\pi|x|} \tag{B.10}$$

for $\tau=\omega-i\varepsilon$ with $\varepsilon>0$ and $\omega\in\mathbb{R}$.
Passing to the limit as $\varepsilon\to0$, we obtain from (B.9) and (B.10):

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-(\omega\pm i0)^2}\Big]=\frac{e^{\pm i\omega|x|}}{4\pi|x|}\,. \tag{B.11}$$

By these formulas we can construct fundamental solutions for the Helmholtz equation

$$(\Delta+\omega^2)\gamma^{(\pm)}(x,\omega)=\delta(x),\ \ \omega\in\mathbb{R}^3,$$

corresponding to maximally decreasing out going and incoming waves,

$$\gamma^{(\pm)}(x,\omega):=-\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-(\omega\pm i0)^2}\Big]=-\frac{e^{\pm i\omega|x|}}{4\pi|x|}\,. \tag{B.12}$$

It is clear that these fundamental solutions satisfy the Sommerfeld radiation conditions at infinity

$$\frac{\partial}{\partial |x|}\gamma^{(\pm)}(x,\omega) \mp i\omega\gamma^{(\pm)}(x,\omega) = \mathcal{O}(|x|^{-2}).$$

The above described procedure is the so called *limiting absorption principle.*

9. Appendix C: Some Results from the Theory of Pseudodifferential Equations on Manifolds with Boundary

Here we recall some results from the theory of strongly elliptic pseudo-differential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission and crack problems by the potential methods.

They can be found in [14], [21], [54].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(x,\xi;\mathcal{A})$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}}, \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x),\dots,\lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\mathfrak{S}(x,0,\dots,0,+1;\mathcal{A})\big]^{-1}\big[\mathfrak{S}(x,0,\dots,0,-1;\mathcal{A}))\big], \ \ x \in \partial\overline{\mathcal{M}},$$

and introduce the notation

$$\delta_j(x) = \Re\big[(2\pi i)^{-1}\ln\lambda_j(x)\big], \ \ j = 1,\dots,N.$$

Here the branch in the logarithmic function $\ln\zeta$ is chosen with regard to the inequality $-\pi < \arg\zeta \le \pi$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$, $j = \overline{1,N}$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. Remark that in the particular case, when $\mathfrak{S}(x,\xi;\mathcal{A})$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $\delta_j(x) = 0$ for $j = 1,\dots,N$, since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1,N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem.

Theorem C.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\Re\mathfrak{S}(x,\xi;\mathcal{A})\eta\cdot\eta \ge c_0|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\mathcal{A}: \ \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N \ \ \Big[\big[\widetilde{B}^s_{p,t}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,t}(\mathcal{M})\big]^N\Big], \quad \text{(C.1)}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x \in \partial\mathcal{M}, 1 \le j \le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x \in \partial\mathcal{M}, 1 \le j \le N} \delta_j(x). \tag{C.2}$$

Moreover, the null-spaces and indices of the operators (C.1) *are the same* (*for all values of the parameter* $t \in [1, +\infty]$) *provided* p *and* s *satisfy the inequality* (C.2).

In particular, if $\mathfrak{S}(x, \xi; \mathcal{A})$ *is a positive definite matrix for every* $x \in \overline{\mathcal{M}}$ *and* $\xi \in \mathbb{R}^n \setminus \{0\}$, *the inequalities* (C.2) *read as*

$$\frac{1}{p} - 1 < s - \frac{\nu}{2} < \frac{1}{p}. \tag{C.3}$$

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