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THE WELL-POSEDNESS OF A SEMILINEAR WAVE EQUATION ASSOCIATED WITH A LINEAR INTEGRAL EQUATION AT THE BOUNDARY

Abstract. In this paper, we prove the well-posedness for a mixed nonhomogeneous problem for a semilinear wave equation associated with a linear integral equation at the boundary.

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[illegible]

1. Introduction

We investigate the following problem: find a pair (u, Q) of functions satisfying

$$u_{tt} - \mu(t)u_{xx} + F(u, u_t) = f(x,t), \ \ 0 < x < 1, \ \ 0 < t < T, \tag{1.1}$$

$$u(0,t) = 0, \tag{1.2}$$

$$-\mu(t)u_x(1,t) = Q(t), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.4}$$

where $F(u, u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$ with $p, q \geq 2$, K, λ given constants, u_0, u_1, f, μ are given functions satisfying conditions specified later, and the unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfy the following integral equation

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)\,ds \tag{1.5}$$

with g, k, K_1, λ_1 given functions.

This problem is a mathematical model describing the shock of a rigid body and a viscoelastic bar (see [1], [2], [8], [9], [10], [11]) considered by several authors.

In [1], with $F(u, u_t) = Ku + \lambda u_t$, $\mu(t) \equiv a^2$, $f(x,t) = 0$, An and Trieu studied the equation $(1.1)_1$ in the domain $[0,l] \times [0,T]$ when the initial data are homogeneous, namely $u(x,0) = u_t(x,0) = 0$ and the boundary conditions are given by

$$\begin{cases} Eu_x(0,t) = -f(t), \\ u(l,t) = 0, \end{cases} \tag{1.6}$$

where E is a constant.

In [6], Long and Dinh considered the problem (1.1)–(1.4) with $\lambda_1(t) \equiv 0$, $K_1(t) = h \geq 0$, $\mu(t) = 1$, the unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfying the following integral equation

$$Q(t) = hu(1,t) - g(t) - \int_0^t k(t-s)u(1,s)\,ds. \tag{1.7}$$

We note that Eq. (1.7) is deduced from a Cauchy problem for an ordinary differential equation at the boundary $x = 1$.

In [2], Bergounioux, Long and Dinh proved the unique solvability for the problem (1.1), (1.4), where $\mu(t) \equiv 1$, $F(u, u_t)$ is linear and the mixed boundary conditions (1.2), (1.3) replaced by

$$u_x(0,t) = hu(0,t) + g(t) - \int_0^t k(t-s)u(0,s)\,ds, \tag{1.8}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0. \tag{1.9}$$

In [12], Santos studied the asymptotic behavior of the solution of the problem (1.1), (1.2), (1.4) in the case where $F(u, u_t) = 0$ associated with a boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)ds = 0, \quad t > 0. \tag{1.10}$$

In [8], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the solution of the problem (1.1)–(1.4) in the case where $\mu(t) = 1$, $Q(t) = K_1 u(1,t) + \lambda_1 u_t(1,t)$, $u_x(0,t) = P(t)$, where $P(t)$ satisfies (1.7) instead of $Q(t)$.

In [9]–[11], Long, Lê and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of the problem (1.1)–(1.5) when $F(u, u_t) = Ku + \lambda u_t$.

The present paper consists of two main parts. In Part 1, we prove a theorem on existence and uniqueness of a weak solution (u, Q) of the problem (1.1)–(1.5). The proof is based on a Galerkin type approximation associated with various energy estimates type bounds, weak convergence and compactness arguments. The main difficulties encountered here are the boundary condition at $x = 1$ and the presence of the nonlinear term $F(u, u_t)$. In order to overcome these particular difficulties, stronger assumptions on the initial conditions u_0, u_1 and parameters K, λ will be imposed. It is remarkable that the linearization method from the papers [3], [7] can not be used in [2], [5], [6]. In the second part we show the stability of the solution of the problem (1.1)–(1.5) in suitable spaces. The results obtained here may be considered as generalizations of those in An and Trieu [1] and in Long, Dinh, Lê, Truc and Santos ([2], [3], [5]–[12]).

2. The Existence and Uniqueness of the Solution

First we introduce some preliminary results and notation used in this paper. Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p = L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \le p \le \infty$, $m = 0, 1, \ldots$.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle \cdot\,, \cdot \rangle$ the scalar product in L^2 or the dual scalar product of a continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space to X. We denote by $L^p(0,T;X)$, $1 \le p \le \infty$, the Banach space of the real measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \ \text{ if } \ 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \ \text{ if } \ p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

We put

$$V = \{v \in H^1 : \ v(0) = 0\}, \tag{2.1}$$

$$a(u,v) = \Big\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \Big\rangle = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx. \tag{2.2}$$

Here V is a closed subspace of H^1 and $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)}$ are two equivalent norms on V.

Then we have the following lemma.

Lemma 1. *The imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \le \|v\|_V \tag{2.3}$$

for all $v \in V$.

We omit the detailed proof because of its obviousness.

The process is continued by making the following essential assumptions:

(H_1) $K, \lambda \ge 0$;

(H_2) $u_0 \in V \cap H^2$, and $u_1 \in H^1$;

(H_3) g, K_1, $\lambda_1 \in H^1(0,T)$, $\lambda_1(t) \ge \lambda_0 > 0$, $K_1(t) \ge 0$;

(H_4) $k \in H^1(0,T)$;

(H_5) $\mu \in H^2(0,T)$, $\mu(t) \ge \mu_0 > 0$;

(H_6) f, $f_t \in L^2(Q_T)$.

Then we have the following theorem.

Theorem 1. *Let (H_1)–(H_6) hold. Then for every $T > 0$ there exists a unique weak solution (u,Q) of the problem (1.1)–(1.5) such that*

$$\begin{cases} u \in L^\infty\big(0,T; V \cap H^2\big) \cap L^p(Q_T), \\ u_t \in L^\infty\big(0,T;V\big) \cap L^q(Q_T), \ \ u_{tt} \in L^\infty\big(0,T;L^2\big), \\ u(1,\cdot\,) \in H^2(0,T), \ \ Q \in H^1(0,T). \end{cases} \tag{2.4}$$

Remark 1. By $L^\infty(0,T;V) \subset L^p(Q_T)$ $\forall p$, $1 \le p < \infty$, it follows from (2.4) that the component u in the weak solution (u,Q) of the problem (1.1)–(1.5) satisfies

$$\begin{cases} u \in C^0\big(0,T;V\big) \cap C^1\big(0,T;L^2\big) \cap L^\infty\big(0,T;V \cap H^2\big), \\ u_t \in L^\infty(0,T;V). \end{cases} \tag{2.5}$$

Proof. The proof consists of Steps 1–4.

Step 1. *The Galerkin approximation.* Let $\{\omega_j\}$ be a denumerable base of $V \cap H^2$. Look for the approximate solution of the problem (1.1)–(1.5) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)\omega_j, \tag{2.6}$$

where the coefficient functions c_{mj} satisfy the following system of ordinary differential equations

$$\langle u''_m(t), \omega_j\rangle + \mu(t)\langle u_{mx}(t), \omega_{jx}\rangle + Q_m(t)\omega_j(1) + \big\langle F\big(u_m(t), u'_m(t)\big), \omega_j\big\rangle = \\ = \langle f(t), \omega_j\rangle, \quad 1 \le j \le m, \tag{2.7}$$

$$Q_m(t) = K_1(t)u_m(1,t) + \lambda_1(t)u'_m(1,t) - g(t) - \int_0^t k(t-s)u_m(1,s)\,ds, \tag{2.8}$$

$$\begin{cases} u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj}\omega_j \to u_0 \quad \text{strongly in } V \cap H^2, \\ u'_m(0) = u_{1m} = \sum_{j=1}^{m} \beta_{mj}\omega_j \to u_1 \quad \text{strongly in } H^1. \end{cases} \tag{2.9}$$

From the assumptions of Theorem 1, the system (2.7)–(2.9) has a solution (u_m, Q_m) on some interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. *A priori estimates. A priori estimates* I. Substituting (2.8) into (2.7), then multiplying the j^{th} equation of (2.7) by $c'_{mj}(t)$ and summing up with respect to j, we get

$$\frac{1}{2}\frac{d}{dt}\|u'_m(t)\|^2 + \frac{1}{2}\mu(t)\frac{d}{dt}\|u_{mx}(t)\|^2 + \\ + \Big[K_1(t)u_m(1,t) + \lambda_1(t)u'_m(1,t) - g(t) - \int_0^t k(t-s)u_m(1,s)\,ds\Big]u'_m(1,t) + \\ + \big\langle F\big(u_m, u'_m\big), u'_m(t)\big\rangle = \langle f(t), u'_m(t)\rangle. \tag{2.10}$$

Integrating (2.10) with respect to t, we get after some rearrangements

$$S_m(t) = S_m(0) + \int_0^t \mu'(s)\|u_{mx}(s)\|^2\,ds + \int_0^t K'_1(s)u_m^2(1,s)\,ds + \\ + 2\int_0^t g(s)u'_m(1,s)\,ds + 2\int_0^t \langle f(s), u'_m(s)\rangle\,ds + \\ + 2\int_0^t u'_m(1,s)\Big(\int_0^s k(s-\tau)u_m(1,\tau)\,d\tau\Big)\,ds, \tag{2.11}$$

where

$$S_m(t) = \|u_m'(t)\|^2 + \mu(t)\|u_{mx}(t)\|^2 + K_1(t)u_m^2(1,t) + \frac{2K}{p}\|u_m(t)\|_{L^p}^p + \\ + 2\lambda\int_0^t \|u_m'(s)\|_{L^q}^q\, ds + 2\int_0^t \lambda_1(s)|u_m'(1,s)|^2\, ds. \tag{2.12}$$

Using the inequality

$$2ab \le \beta a^2 + \frac{1}{\beta}b^2,\ \ \forall\, a,\, b \in \mathbb{R},\ \ \beta > 0, \tag{2.13}$$

and the inequalities

$$S_m(t) \ge \|u_m'(t)\|^2 + \mu_0\|u_{mx}(t)\|^2 + 2\lambda_0\int_0^t |u_m'(1,s)|^2\, ds, \tag{2.14}$$

$$|u_m(1,t)| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{\frac{S_m(t)}{\mu_0}}, \tag{2.15}$$

we will estimate respectively the terms on the right-hand side of (2.11) as follows

$$\int_0^t \mu'(s)\|u_{mx}(s)\|^2\, ds \le \frac{1}{\mu_0}\int_0^t |\mu'(s)|S_m(s)\, ds, \tag{2.16}$$

$$\int_0^t K_1'(s)u_m^2(1,s)\, ds \le \frac{1}{\mu_0}\int_0^t |K_1'(s)|S_m(s)\, ds, \tag{2.17}$$

$$2\int_0^t g(s)u_m'(1,s)\, ds \le \frac{1}{\beta}\|g\|_{L^2(0,T)}^2 + \frac{\beta}{2\lambda_0}\, S_m(t), \tag{2.18}$$

$$2\int_0^t u_m'(1,s)\left(\int_0^s k(s-\tau)u_m(1,\tau)\, d\tau\right) ds \le \\ \le \frac{\beta}{2\lambda_0}\, S_m(t) + \frac{1}{\beta\mu_0}T\|k\|_{L^2(0,T)}^2 \int_0^t S_m(s)\, ds, \tag{2.19}$$

$$2\int_0^t \langle f(s), u_m'(s)\rangle\, ds \le \|f\|_{L^2(Q_T)}^2 + \int_0^t S_m(s)\, ds. \tag{2.20}$$

In addition, from the assumptions (H_1), (H_2), (H_5) and the imbedding $H^1 \hookrightarrow L^p(0,1)$, $p \ge 1$, there exists a positive constant C_1 such that

$$S_m(0) = \|u_{1m}\|^2 + \mu(0)\|u_{0mx}\|^2 +$$

$$+K_1(0)u_{0m}^2(1)+\frac{2K}{p}\|u_{0m}\|_{L^p}^p \le C_1 \text{ for all } m. \tag{2.21}$$

Combining (2.11), (2.12), (2.16)–(2.21), we obtain

$$S_m(t) \le C_1 + \frac{1}{\beta}\|g\|_{L^2(0,T)}^2 + \|f\|_{L^2(Q_T)}^2 + \frac{\beta}{\lambda_0}S_m(t) +$$

$$+\int_0^t \left[1+\frac{1}{\beta\mu_0}T\|k\|_{L^2(0,T)}^2+\frac{1}{\mu_0}\left(|\mu'(s)|+|K_1'(s)|\right)\right]S_m(s)\,ds. \tag{2.22}$$

By choosing $\beta = \frac{\lambda_0}{2}$, we deduce from (2.22) that

$$S_m(t) \le M_T^{(1)} + \int_0^t N_T^{(1)}(s)S_m(s)\,ds, \tag{2.23}$$

where

$$\begin{cases} M_T^{(1)} = 2C_1 + \dfrac{4}{\lambda_0}\|g\|_{L^2(0,T)}^2 + 2\|f\|_{L^2(Q_T)}^2, \\ N_T^{(1)}(s) = 2\left[1+\dfrac{2}{\lambda_0\mu_0}T\|k\|_{L^2(0,T)}^2 + \dfrac{1}{\mu_0}\left(|\mu'(s)|+|K_1'(s)|\right)\right], \\ N_T^{(1)} \in L^1(0,T). \end{cases} \tag{2.24}$$

By Gronwall's lemma, we deduce from (2.23), (2.24) that

$$S_m(t) \le M_T^{(1)} exp\left(\int_0^t N_T^{(1)}(s)ds\right) \le C_T, \text{ for all } t \in [0,T]. \tag{2.25}$$

A priori estimate II. Now differentiating (2.7) with respect to t, we have

$$\langle u_m'''(t),\omega_j\rangle + \mu(t)\langle u_{mx}'(t),\omega_{jx}\rangle + \mu'(t)\langle u_{mx}(t),\omega_{jx}\rangle + Q_m'(t)\omega_j(1) +$$
$$+\left\langle K(p-1)|u_m|^{p-2}u_m' + \lambda(q-1)|u_m'|^{q-2}u_m'',\omega_j\right\rangle = \langle f'(t),\omega_j\rangle \tag{2.26}$$

for all $1 \le j \le m$.

Multiplying the j^{th} equation of (2.28) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we have after some persistent rearrangements

$$X_m(t) = X_m(0) + 2\mu'(0)\langle u_{0mx},u_{1mx}\rangle - 2\mu'(t)\langle u_{mx}(t),u_{mx}'(t)\rangle +$$
$$+3\int_0^t \mu'(s)\|u_{mx}'(s)\|^2\,ds + 2\int_0^t \mu''(s)\langle u_{mx}(s),u_{mx}'(s)\rangle\,ds -$$
$$-2\int_0^t \left[K_1'(s)-k(0)\right]u_m(1,s)u_m''(1,s)\,ds -$$
$$-2\int_0^t \left[K_1(s)+\lambda_1'(s)\right]u_m'(1,s)u_m''(1,s)\,ds +$$

$$+\,2\int_0^t u_m''(1,s)\Big(g'(s)+\int_0^s k'(s-\tau)u_m(1,\tau)\,d\tau\Big)\,ds-$$

$$-\,2\int_0^t \big\langle K(p-1)|u_m(s)|^{p-2}u_m'(s),u_m''(s)\big\rangle\,ds+$$

$$+\,2\int_0^t \langle f'(s),u_m''(s)\rangle\,ds, \tag{2.27}$$

where

$$X_m(t)=\|u_m''(t)\|^2+\mu(t)\|u_{mx}'(t)\|^2+2\int_0^t \lambda_1(s)|u_m''(1,s)|^2\,ds+$$

$$+\,\frac{8}{q^2}(q-1)\lambda\int_0^t \Big\|\frac{\partial}{\partial t}\big(|u_m'(s)|^{\frac{q-2}{2}}u_m'(s)\big)\Big\|^2\,ds. \tag{2.28}$$

From the assumptions (H_1), (H_2), (H_5), (H_6) and the imbedding $H^1(0,1)\hookrightarrow L^p(0,1)$, $p\geq 1$, there exist positive constants D_1, D_2 depending on $\mu(0)$, u_0, u_1, K, λ, f such that

$$\begin{cases} X_m(0)=\|u_m''(0)\|^2+\mu(0)\|u_{1mx}\|^2\leq \\ \qquad\leq \mu(0)\|u_{0mxx}\|+K\|u_{0m}\|_{L^{2p-2}}^{p-1}+\lambda\|u_{1m}\|_{L^{2q-2}}^{q-1}+ \\ \qquad+\|f(0)\|+\mu(0)\|u_{1mx}\|^2\leq D_1, \\ 2\mu'(0)\langle u_{0mx},u_{1mx}\rangle\leq 2|\mu'(0)|\|u_{0mx}\|\|u_{1mx}\|\leq D_2 \end{cases} \tag{2.29}$$

for all m.

Taking into account the inequality (2.13) with β replaced by β_1 and the following inequalities

$$X_m(t)\geq \|u_m''(t)\|^2+\mu_0\|u_{mx}'(t)\|^2+2\lambda_0\int_0^t |u_m''(1,s)|^2\,ds, \tag{2.30}$$

$$|u_m(1,t)|\leq \|u_m(t)\|_{C^0(\overline{\Omega})}\leq \|u_{mx}(t)\|\leq \sqrt{\frac{S_m(t)}{\mu_0}}\leq \sqrt{\frac{C_T}{\mu_0}}, \tag{2.31}$$

$$|u_m'(1,t)|\leq \|u_m'(t)\|_{C^0(\overline{\Omega})}\leq \|u_{mx}'(t)\|\leq \sqrt{\frac{X_m(t)}{\mu_0}}, \tag{2.32}$$

we estimate, without any difficulties, the terms in the right-hand side of (2.27) as follows

$$-2\mu'(t)\langle u_{mx}(t),u_{mx}'(t)\rangle\leq \beta_1 X_m(t)+\frac{1}{\beta_1\mu_0^2}\,C_T|\mu'(t)|^2, \tag{2.33}$$

$$2\int_0^t \mu''(s)\langle u_{mx}(s), u'_{mx}(s)\rangle\, ds \le \frac{C_T}{\beta_1\mu_0^2}\,\|\mu''\|^2_{L^2(0,T)} + \beta_1\int_0^t X_m(s)\, ds, \tag{2.34}$$

$$3\int_0^t \mu'(s)\|u'_{mx}(s)\|^2\, ds \le \frac{3}{\mu_0}\int_0^t |\mu'(s)| X_m(s)\, ds, \tag{2.35}$$

$$-2\int_0^t \big[K_1'(s) - k(0)\big] u_m(1,s) u''_m(1,s)\, ds \le$$

$$\le \frac{C_T}{\mu_0\beta_1}\|K_1' - k(0)\|^2_{L^2(0,T)} + \frac{\beta_1}{2\lambda_0}\, X_m(t), \tag{2.36}$$

$$-2\int_0^t \big[K_1(s) + \lambda_1'(s)\big] u'_m(1,s) u''_m(1,s)\, ds \le$$

$$\le \frac{2}{\mu_0\beta_1}\int_0^t \big[K_1^2(s) + |\lambda_1'(s)|^2\big] X_m(s)\, ds + \frac{\beta_1}{2\lambda_0} X_m(t), \tag{2.37}$$

$$2\int_0^t u''_m(1,s)\Big(g'(s) + \int_0^s k'(s-\tau) u_m(1,\tau)\, d\tau\Big)\, ds \le$$

$$\le \frac{\beta_1}{2\lambda_0}\, X_m(t) + \frac{2}{\beta_1}\Big[\|g'\|^2_{L^2(0,T)} + \frac{C_T}{\mu_0} T\|k'\|^2_{L^1(0,T)}\Big], \tag{2.38}$$

$$-2K(p-1)\int_0^t \big\langle |u_m(s)|^{p-2} u'_m(s), u''_m(s)\big\rangle\, ds \le$$

$$\le 2\frac{p-1}{\sqrt{\mu_0}}\, K\Big(\frac{C_T}{\mu_0}\Big)^{\frac{p-2}{2}}\int_0^t X_m(s)\, ds, \tag{2.39}$$

$$2\int_0^t \langle f'(s), u''_m(s)\rangle ds \le \beta_1\int_0^t X_m(s)\, ds + \frac{1}{\beta_1}\|f'\|^2_{L^2(Q_T)}. \tag{2.40}$$

In terms of (2.27), (2.29), (2.33)–(2.40) we obtain that

$$\begin{aligned} X_m(t) \le\ & D_1 + D_2 + \frac{C_T}{\beta_1\mu_0^2}\,|\mu'(t)|^2 + \frac{C_T}{\beta_1\mu_0^2}\,\|\mu''\|^2_{L^2(0,T)} + \\ & + \frac{C_T}{\beta_1\mu_0}\,\|K_1' - k(0)\|^2_{L^2(0,T)} + \frac{1}{\beta_1}\,\|f'\|^2_{L^2(Q_T)} \\ & + \beta_1\Big(1 + \frac{1}{2\lambda_0}\Big) X_m(t) + \frac{2}{\beta_1}\Big[\|g'\|^2_{L^2(0,T)} + \frac{C_T}{\mu_0} T\|k'\|^2_{L^1(0,T)}\Big] + \end{aligned}$$

$$+ 2\int_0^t \Big[\beta_1 + \frac{3}{2\mu_0}\,|\mu'(s)| + \frac{1}{\beta_1\mu_0}\,\big(K_1^2(s) + |\lambda_1'(s)|^2\big) + \\ + \frac{p-1}{\sqrt{\mu_0}}\,K\Big(\frac{C_T}{\mu_0}\Big)^{\frac{p-2}{2}}\Big]\int_0^t X_m(s)\,ds. \qquad (2.41)$$

By the choice of $\beta_1 > 0$ such that

$$\beta_1\Big(1 + \frac{3}{2\lambda_0}\Big) \le \frac{1}{2}\,, \qquad (2.42)$$

we obtain

$$X_m(t) \le \widetilde{M}_T^{(2)}(t) + \int_0^t N_T^{(2)}(s)X_m(s)\,ds, \qquad (2.43)$$

where

$$\begin{cases} \widetilde{M}_T^{(2)}(t) = 2D_1 + 2D_2 + \dfrac{2C_T}{\beta_1\mu_0^2}\,|\mu'(t)|^2 + \dfrac{2C_T}{\beta_1\mu_0^2}\,\|\mu''\|^2_{L^2(0,T)} + \\ \qquad + \dfrac{2C_T}{\beta_1\mu_0}\,\|K_1' - k(0)\|^2_{L^2(0,T)} + \dfrac{2}{\beta_1}\,\|f'\|^2_{L^2(Q_T)} + \\ \qquad + \dfrac{4}{\beta_1}\Big[\|g'\|^2_{L^2(0,T)} + \dfrac{C_T}{\mu_0}T\|k'\|^2_{L^1(0,T)}\Big], \\ N_T^{(2)}(s) = 4\Big[\beta_1 + \dfrac{3}{2\mu_0}\,|\mu'(s)| + \dfrac{1}{\beta_1\mu_0}\,\big(K_1^2(s) + |\lambda_1'(s)|^2\big) + \\ \qquad + \dfrac{p-1}{\sqrt{\mu_0}}\,K\Big(\dfrac{C_T}{\mu_0}\Big)^{\frac{p-2}{2}}\Big], \\ N_T^{(2)} \in L^1(0,T). \end{cases} \qquad (2.44)$$

From the assumptions (H_3)–(H_6) and the embedding $H^1(0,T) \hookrightarrow C^0([0,T])$ we deduce that

$$\widetilde{M}_T^{(2)}(t) \le M_T^{(2)} \ \text{ for all } \ t \in [0,T], \qquad (2.45)$$

where $M_T^{(2)}$ is a positive constant depending on T, D_1, D_2, C_T, μ, β_1, g, f, K_1, λ_1. From (2.43)–(2.45) and Gronwall's inequality we derive that

$$X_m(t) \le M_T^{(2)}\exp\Big(\int_0^t N_T^{(2)}(s)\,ds\Big) < D_T \ \text{ for all } \ t \in [0,T]. \qquad (2.46)$$

On the other hand, we deduce from (2.8), (2.12), (2.25), (2.28), (2.46) that

$$\|Q_m'\|^2_{L^2(0,T)} \le \frac{5D_T}{2\lambda_0}\|\lambda_1\|^2_\infty + \frac{5T^2C_T}{\mu_0}\|k'\|^2_{L^2(0,T)} + 5\|g'\|^2_{L^2(0,T)} + \\ + \frac{5D_T}{\mu_0}\Big(\|K_1 + \lambda_1'\|^2_{L^2(0,T)}\|K_1' - k(0)\|^2_{L^2(0,T)}\Big), \qquad (2.47)$$

where $\|\lambda_1\|_\infty = \|\lambda_1\|_{L^\infty(0,T)}$.

Taking into account the assumptions (H_3), (H_4), we deduce from (2.47) that

$$\|Q_m\|_{H^1(0,T)} \leq C_T \quad \text{for all } m, \tag{2.48}$$

where C_T is a positive constant depending only on T.

Step 3. *Limiting process.* In view of (2.12), (2.25), (2.28), (2.46) and (2.48), we conclude the existence of a subsequence of (u_m, Q_m), also denoted by (u_m, Q_m), such that

$$\begin{cases} u_m \to u \ \text{in} & L^\infty(0,T;V) \ \text{weakly}^\star, \\ u_m \to u \ \text{in} & L^\infty(0,T;L^p) \ \text{weakly}^\star, \\ u'_m \to u' \ \text{in} & L^\infty(0,T;V) \ \text{weakly}^\star, \\ u'_m \to u' \ \text{in} & L^\infty(0,T;L^q) \ \text{weakly}^\star, \\ u''_m \to u'' \ \text{in} & L^\infty(0,T;L^2) \ \text{weakly}^\star, \\ u_m(1,\cdot) \to u(1,\cdot) \ \text{in} & H^2(0,T) \ \text{weakly}, \\ |u_m|^{p-2}u_m \to \chi_1 \ \text{in} & L^\infty(0,T;L^{p/p-1}) \ \text{weakly}^\star, \\ |u'_m|^{q-2}u'_m \to \chi_2 \ \text{in} & L^\infty(0,T;L^{q/q-1}) \ \text{weakly}^\star, \\ Q_m \to \widetilde{Q} \ \text{in} & H^1(0,T) \ \text{weakly}. \end{cases} \tag{2.49}$$

With the help of the compactness lemma of J.L. Lions ([4, p. 57]) and the embeddings $H^2(0,T) \hookrightarrow H^1(0,T), H^1(0,T) \hookrightarrow C^0([0,T])$, we can deduce from $(2.49)_{1,3,6,7}$ the existence of a subsequence, still denoted by (u_m, Q_m), such that

$$\begin{cases} u_m \to u \ \text{strongly in} & L^2(Q_T), \\ u'_m \to u' \ \text{strongly in} & L^2(Q_T), \\ u_m(1,\cdot) \to u(1,\cdot) \ \text{strongly in} & H^1(0,T), \\ u'_m(1,\cdot) \to u'(1,\cdot) \ \text{strongly in} & C^0[0,T], \\ Q_m \to \widetilde{Q} \ \text{strongly in} & C^0[0,T]. \end{cases} \tag{2.50}$$

The remarkable results of (2.8) and $(2.50)_{3-4}$ help us to affirm that

$$Q_m(t) \to K_1(t)u(1,t) + \lambda_1(t)u'(1,t) - g(t) - \int_0^t k(t-s)u(1,s)\,ds \equiv$$
$$\equiv Q(t) \ \text{strongly in} \ C^0[0,T]. \tag{2.51}$$

On account of $(2.50)_5$ and (2.51), we conclude that

$$Q(t) = \widetilde{Q}(t). \tag{2.52}$$

By means of the inequality

$$\big||x|^\alpha x - |y|^\alpha y\big| \leq (\alpha+1)R^\alpha |x-y|,$$
$$\forall\, x,\, y \in [-R,R] \ \text{for all} \ R > 0, \ \ \alpha \geq 0, \tag{2.53}$$

it follows from (2.31) that

$$\big||u_m|^{p-2}u_m - |u|^{p-2}u\big| \le (p-1)R^{p-2}|u_m - u|, \quad R = \sqrt{\frac{C_T}{\mu_0}}. \tag{2.54}$$

Hence, it follows from (2.54), $(2.50)_1$ that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \ \text{ strongly in } \ L^2(Q_T). \tag{2.55}$$

By the same way, we are able to get from (2.53) with $R = \sqrt{\frac{D_T}{\mu_0}}$, $(2.49)_3$ and $(2.50)_2$ that

$$|u'_m|^{p-2}u'_m \to |u_t|^{p-2}u_t \ \text{ strongly in } \ L^2(Q_T). \tag{2.56}$$

As a result, we deduce from (2.55), (2.56) that

$$F(u_m, u'_m) \to F(u, u_t) \ \text{ strongly in } \ L^2(Q_T). \tag{2.57}$$

Passing to limit in (2.7)–(2.9), by $(2.49)_{1,5}$, (2.51)–(2.52) and (2.57) we have (u, Q) satisfying the problem

$$\begin{gathered} \langle u''(t), v\rangle + \mu(t)\langle u_x(t), v_x\rangle + Q(t)v(1)+ \\ +\langle K|u(t)|^{p-2}u(t) + \lambda|u_t(t)|^{q-2}u_t(t), v\rangle = \langle f(t), v\rangle, \ \ \forall\, v \in V, \end{gathered} \tag{2.58}$$

$$u(0) = u_0, \ \ u'(0) = u_1, \tag{2.59}$$

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)\,ds, \tag{2.60}$$

in $L^2(0,T)$ weakly. Nevertheless, we obtain from $(2.42)_5$, (2.57) and the assumptions (H_5)–(H_6), that

$$u_{xx} = \frac{1}{\mu(t)}\left[u'' + F(u, u_t) - f\right] \in L^\infty\big(0, T; L^2\big). \tag{2.61}$$

Thus $u \in L^\infty\big(0, T; V \cap H^2\big)$ and the existence result of the theorem is proved completely.

Step 4. Uniqueness of the solution. We start this part by letting (u_1, Q_1) and (u_2, Q_2) be two weak solutions of the problem (1.1)–(1.5) such that

$$\begin{cases} u_i \in L^\infty\big(0, T; V \cap H^2\big) \cap L^p(Q_T), \\ u'_i \in L^\infty\big(0, T; V\big) \cap L^q(Q_T), \ \ u''_i \in L^\infty\big(0, T; L^2\big), \\ u_i(1, \cdot\,) \in H^2(0,T), \ \ Q_i \in H^1(0,T), \ \ i = 1, 2. \end{cases} \tag{2.62}$$

As a result, (u, Q) with $u = u_1 - u_2$ and $Q = Q_1 - Q_2$ satisfies the following variational problem

$$\begin{cases} \langle u''(t), v\rangle + \mu(t)\langle u_x(t), v_x\rangle + Q(t)v(1)+ \\ \qquad + K\big\langle |u_1(t)|^{p-2}u_1(t) - |u_2(t)|^{p-2}u_2(t), v\big\rangle+ \\ \qquad + \lambda\big\langle |u'_1(t)|^{q-2}u'_1(t) - |u'_2(t)|^{q-2}u'_2(t), v\big\rangle = 0 \ \ \forall\, v \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{2.63}$$

and

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u'(1,t) - \int_0^t k(t-s)u(1,s)\,ds. \tag{2.64}$$

Choosing $v = u'$ in $(2.63)_1$ and integrating with respect to t, we arrive at

$$\begin{aligned} S(t) \le & \int_0^t \mu'(s)\|u_x(s)\|^2\,ds + \int_0^t K_1'(s)u^2(1,s)\,ds \\ & + 2\int_0^t u'(1,s)\Big(\int_0^s k(s-\tau)u(1,\tau)d\tau\Big)\,ds \\ & - 2K\int_0^t \big\langle |u_1(s)|^{p-2}u_1(s) - |u_2(s)|^{p-2}u_2(s), u'(s)\big\rangle\,ds, \end{aligned} \tag{2.65}$$

where

$$\begin{aligned} S(t) = & \|u'(t)\|^2 + \mu(t)\|u_x(t)\|^2 + K_1(t)u^2(1,t) + \\ & + 2\int_0^t \lambda_1(s)|u'(1,s)|^2\,ds. \end{aligned} \tag{2.66}$$

Note that

$$S(t) \ge \|u'(t)\|^2 + \mu_0\|u_x(t)\|^2 + 2\lambda_0\int_0^t |u'(1,s)|^2\,ds, \tag{2.67}$$

$$|u(1,t)| \le \|u(t)\|_{C^0(\overline{\Omega})} \le \|u_x(t)\| \le \sqrt{\frac{S(t)}{\mu(t)}} \le \sqrt{\frac{S(t)}{\mu_0}}\,. \tag{2.68}$$

We again use the inequalities (2.13) and (2.53) with $\alpha = p-2$, $R = max_{i=1,2}$ $\|u_i\|_{L^\infty(0,T;V)}$. Then it follows from (2.65)–(2.68) that

$$\begin{aligned} S(t) \le & \frac{1}{\mu_0}\int_0^t \big(\|\mu'\|_\infty + |K_1'(s)|\big)S(s)\,ds + \frac{\beta}{2\lambda_0}\,S(t) + \\ & + \frac{T}{\beta\mu_0}\,\|k\|^2_{L^2(0,T)}\int_0^t S(\tau)\,d\tau + \frac{1}{\sqrt{\mu_0}}\,(p-1)KR^{p-2}\int_0^t S(s)\,ds. \end{aligned} \tag{2.69}$$

Choosing $\beta > 0$ such that $\beta\frac{1}{2\lambda_0} \le \frac{1}{2}$, we obtain from (2.69) that

$$S(t) \le \int_0^t q_1(s)S(s)ds, \tag{2.70}$$

where

$$\begin{cases} q_1(s) = \dfrac{1}{\mu_0}\left(\|\mu'\|_\infty + |K_1'(s)|\right) + \dfrac{2T}{\beta\mu_0}\,\|k\|^2_{L^2(0,T)} + \\ \qquad + \dfrac{2}{\sqrt{\mu_0}}\,(p-1)KR^{p-2}, \\ q_1 \in L^2(0,T). \end{cases} \tag{2.71}$$

By Gronwall's lemma, we deduce that $S \equiv 0$ and Theorem 1 is proved completely. □

Remark 2. In the case where $p, q > 2$ and $K, \lambda < 0$, the question about the existence of a solution of the problem (1.1)–(1.5) is still open. However, we have received the answer when $p = q = 2$ and $K, \lambda \in \mathbb{R}$ published in [11].

3. The Stability of the Solution

In this section we assume that the functions u_0, u_1 satisfy (H_2). By Theorem 1, the problem (1.1)–(1.5) has a unique weak solution (u, Q) depending on μ, K, λ, f, K_1, λ_1, g, k. So we have

$$u = u(\mu, K, \lambda, f, K_1, \lambda_1, g, k), \quad Q = Q(\mu, K, \lambda, f, K_1, \lambda_1, g, k), \tag{3.1}$$

where $(\mu, K, \lambda, f, K_1, \lambda_1, g, k)$ satisfy the assumptions (H_1), (H_3)–(H_6) and u_0, u_1 are fixed functions satisfying (H_2).

We put

$$\Im(\mu_0, \lambda_0) = \Big\{(\mu, K, \lambda, f, K_1, \lambda_1, g, k) : (\mu, K, \lambda, f, K_1, \lambda_1, g, k) \\ \text{satisfy the assumptions } (H_1),\ (H_3)\text{–}(H_6)\Big\},$$

where $\mu_0 > 0$, $\lambda_0 > 0$ are given constants.

Then the following theorem is valid.

Theorem 2. *For every $T > 0$, let (H_1)–(H_6) hold. Then the solutions of the problem* (1.1)–(1.5) *are stable with respect to the data* $(\mu, K, \lambda, f, K_1, \lambda_1, g, k)$, *i.e., if*

$$(\mu, K, \lambda, f, K_1, \lambda_1, g, k),\ (\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j) \in \Im(\mu_0, \lambda_0),$$

are such that

$$\begin{cases} \|\mu^j - \mu\|_{H^2(0,T)} \to 0, \quad |K^j - K| + |\lambda^j - \lambda| \to 0, \\ \|f^j - f\|_{L^2(Q_T)} + \|f_t^j - f_t\|_{L^2(Q_T)} \to 0, \\ \|K_1^j - K_1\|_{H^1(0,T)} \to 0, \quad \|\lambda_1^j - \lambda_1\|_{H^1(0,T)} \to 0, \\ \|g^j - g\|_{H^1(0,T)} \to 0, \quad \|k^j - k\|_{H^1(0,T)} \to 0 \end{cases} \tag{3.2}$$

as $j \to +\infty$, then

$$\big(u_j, u_j', u_j(1,\cdot), Q_j\big) \to \big(u, u', u(1,\cdot), Q\big) \tag{3.3}$$

in $L^\infty(0,T;V) \times L^\infty(0,T;L^2) \times H^1(0,T) \times L^2(0,T)$ *strongly as* $j \to +\infty$, *where*

$$u_j = u\big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big),$$
$$Q_j = Q\big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big).$$

Proof. First of all, we have that the data $\big(\mu, K, \lambda, f, K_1, \lambda_1, g, k\big)$ satisfy

$$\begin{cases} \|\mu\|_{H^2(0,T)} \le \mu^\star, \;\; 0 \le K \le K^\star, \;\; 0 \le \lambda \le \lambda^\star, \\ \|f\|_{L^2(Q_T)} + \|f_t\|_{L^2(Q_T)} \le f^\star, \\ \|K_1\|_{H^1(0,T)} \le K_1^\star, \;\; \|\lambda_1\|_{H^1(0,T)} \le \lambda_1^\star, \\ \|g\|_{H^1(0,T)} \le g^\star, \;\; \|k\|_{H^1(0,T)} \le k^\star, \end{cases} \tag{3.4}$$

where $\mu^\star$, $K^\star$, $\lambda^\star$, $f^\star$, $K_1^\star$, $\lambda_1^\star$, $g^\star$, $k^\star$ are fixed positive constants. Therefore, the a priori estimates of the sequences $\{u_m\}$ and $\{Q_m\}$ in the proof of Theorem 1 satisfy

$$\|u_m'(t)\|^2 + \mu_0\|u_{mx}(t)\|^2 + 2\lambda_0 \int_0^t |u_m'(1,s)|^2\,ds \le M_T, \;\; \forall\, t \in [0,T], \tag{3.5}$$

$$\|u_m''(t)\|^2 + \mu_0\|u_{mx}'(t)\|^2 + 2\lambda_0 \int_0^t |u_m''(1,s)|^2\,ds \le M_T, \;\; \forall\, t \in [0,T], \tag{3.6}$$

$$\|Q_m\|_{H^1(0,T)} \le M_T, \tag{3.7}$$

where M_T is a positive constant depending on T, u_0, u_1, μ_0, λ_0, $\mu^\star$, $K^\star$, $\lambda^\star$, $f^\star$ (independent of μ, K, λ, f, K_1, λ_1, g, k).

Hence the limit (u,Q) of the sequence $\{(u_m,Q_m)\}$ defined by (2.6)–(2.8) in suitable spaces is a weak solution of the problem (1.1)–(1.5) satisfying the estimates (3.5)–(3.7).

Now by (3.2) we can assume that there exist positive constants $\mu^\star$, $K^\star$, $\lambda^\star$, $f^\star$, $K_1^\star$, $\lambda_1^\star$, $g^\star$, $k^\star$ such that the data $\big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$ satisfy (3.4) with $\big(\mu, K, \lambda, f, K_1, \lambda_1, g, k\big) = \big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$. Then, by the above remark, we have that the solution (u_j, Q_j) of the problem (1.1)–(1.5) corresponding to

$$\big(\mu, K, \lambda, f, K_1, \lambda_1, g, k\big) = \big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$$

satisfies

$$\|u_j'(t)\|^2 + \mu_0\|u_{jx}(t)\|^2 + 2\lambda_0 \int_0^t |u_j'(1,s)|^2\,ds \le M_T, \;\; \forall\, t \in [0,T], \tag{3.8}$$

$$\|u_j''(t)\|^2 + \mu_0\|u_{jx}'(t)\|^2 + 2\lambda_0 \int_0^t |u_j''(1,s)|^2\,ds \le M_T, \;\; \forall\, t \in [0,T], \tag{3.9}$$

$$\|Q_j\|_{H^1(0,T)} \le M_T. \tag{3.10}$$

Put

$$\begin{cases} \widetilde{\mu}_j = \mu^j - \mu, \;\; \widetilde{K}_j = K^j - K, \;\; \widetilde{\lambda}_j = \lambda^j - \lambda, \\ \widetilde{f}_j = f^j - f, \;\; \widetilde{K}_{1j} = K_1^j - K_1, \;\; \widetilde{\lambda}_1^j = \lambda_1^j - \lambda_1, \\ \widetilde{g}_j = g^j - g, \;\; \widetilde{k}_j = k^j - k. \end{cases} \tag{3.11}$$

Consequently, $v_j = u_j - u$, $P_j = Q_j - Q$ satisfy the following variational problem

$$\begin{cases} \langle v_j''(t), v\rangle + \mu(t)\langle v_{jx}(t), v_x\rangle + P_j(t)v(1)+ \\ \qquad + K_j\big\langle |u_j|^{p-2}u_j - |u|^{p-2}u, v\big\rangle + \\ \qquad + \lambda_j\big\langle |u_j'|^{q-2}u_j' - |u'|^{q-2}u', v\big\rangle \\ = \langle \widetilde{f}_j, v\rangle - \widetilde{\mu_j}(t)\langle u_{jx}(t), v_x\rangle - \\ \qquad - \widetilde{K_j}\langle |u|^{p-2}u, v\rangle - \widetilde{\lambda}_j\langle |u'|^{q-2}u', v\rangle \;\; \forall\, v \in V, \\ \quad v_j(0) = v_j'(0) = 0, \end{cases} \tag{3.12}$$

where

$$P_j(t) = Q_j(t) - Q(t) =$$

$$= K_1(t)v_j(1,t) + \lambda_1(t)v_{jt}(1,t) - \int_0^t k(t-s)v_j(1,s)\,ds - \widehat{g}_j(t), \tag{3.13}$$

$$\widehat{g}_j(t) = \widetilde{g}_j(t) - \widetilde{K_1}_j(t)u_j(1,t) - \widetilde{\lambda_1}_j(t)u_{jt}(1,t) + $$

$$+ \int_0^t \widetilde{k}_j(t-s)u_j(1,s)\,ds. \tag{3.14}$$

Substituting $P_j(t)$ into (3.12), then taking $v = v_j'$ in $(3.12)_1$ and integrating in t, we obtain

$$S_j(t) \le \int_0^t \mu_j'(s)\|v_{jx}(x)\|^2\,ds + \int_0^t K_1'(s)v_j^2(1,s)\,ds +$$

$$+ 2\int_0^t v_j'(1,\tau)\,d\tau \int_0^\tau k(\tau - s)v_j(1,s)\,ds + 2\int_0^t \big\langle \widetilde{f}_j, v_j'(s)\big\rangle\,ds -$$

$$- 2\widetilde{K}_j\int_0^t \big\langle |u|^{p-2}u, v_j'(s)\big\rangle\,ds - 2\widetilde{\lambda}_j\int_0^t \big\langle |u'|^{q-2}u', v_j'(s)\big\rangle\,ds +$$

$$+ 2\int_0^t \widehat{g}_j(s)v_j'(1,s)\,ds - 2\int_0^t \widetilde{\mu}_j(s)\big\langle u_{jx}(s), v_{jx}'(s)\big\rangle\,ds -$$

$$- 2K_j\int_0^t \big\langle |u_j|^{p-2}u_j - |u|^{p-2}u, v_j'(s)\big\rangle\,ds, \tag{3.15}$$

where

$$S_j(t) = \|v_j'(t)\|^2 + \mu(t)\|v_{jx}(t)\|^2 + K_1(t)|v_j(1,t)|^2 + \\ + 2\int_0^t \lambda_1(s)|v_j'(1,s)|^2\,ds. \quad (3.16)$$

Using the inequalities (2.12), (3.8), (3.9) and

$$S_j(t) \geq \|v_j'(t)\|^2 + \mu_0\|v_{jx}(t)\|^2 + 2\lambda_0\int_0^t |v_j'(1,s)|^2\,ds, \quad (3.17)$$

we can prove the following inequality in a similar manner

$$S_j(t) \leq \frac{\beta}{\lambda_0}\,S_j(t) + \frac{1}{\beta}\,\|\widehat{g}_j\|^2_{L^2(0,T)} + \|\widetilde{f}_j\|^2_{L^2(Q_T)} + \frac{1}{\mu_0}\,TM_T\|\widetilde{\mu}_j\|^2_\infty + \\ + T\Big(\frac{M_T}{\mu_0}\Big)^{p-1}\big|\widetilde{K}_j\big|^2 + T\Big(\frac{M_T}{\mu_0}\Big)^{q-1}\big|\widetilde{\lambda}_j\big|^2 + \\ + \int_0^t \bigg[4 + \|\mu'\|^2_\infty + \frac{1}{\beta\mu_0}\,T\|k\|^2_{L^2(0,T)} + \\ + \frac{2K^\star}{\sqrt{\mu_0}}\,(p-1)R^{p-2} + |K_1'(s)|\bigg]S_j(s)\,ds \quad (3.18)$$

for all $\beta > 0$ and $t \in [0,T]$.

Choose $\beta > 0$ such that $\frac{\beta}{\lambda_0} \leq 1/2$ and denote

$$\widetilde{R}_j = \frac{2}{\beta}\|\widehat{g}_j\|^2_{L^2(0,T)} + 2\big\|\widetilde{f}_j\big\|^2_{L^2(Q_T)} + \frac{2}{\mu_0}\,TM_T\|\widetilde{\mu}_j\|^2_\infty + \\ + 2T\Big(\frac{M_T}{\mu_0}\Big)^{p-1}|\widetilde{K}_j|^2 + 2T\Big(\frac{M_T}{\mu_0}\Big)^{q-1}|\widetilde{\lambda}_j|^2, \quad (3.19)$$

$$\phi(s) = 2\bigg[4 + \|\mu'\|^2_\infty + \frac{1}{\beta\mu_0}\,T\|k\|^2_{L^2(0,T)} + \frac{2K^\star}{\sqrt{\mu_0}}\,(p-1)R^{p-2} + |K_1'(s)|\bigg]. \quad (3.20)$$

Then from (3.18)–(3.20) we have

$$S_j(t) \leq \widetilde{R}_j + \int_0^t \phi(s)S_j(s)\,ds. \quad (3.21)$$

By Gronwall's lemma, we obtain from (3.21) that

$$S_j(t) \leq \widetilde{R}_j \exp\bigg(\int_0^t \phi(s)\,ds\bigg) \leq D_T^{(1)}\widetilde{R}_j, \quad \forall\, t \in [0,T], \quad (3.22)$$

where $D_T^{(1)}$ is a positive constant.

On the other hand, using the imbedding $H^1(0,T) \hookrightarrow C^0\big([0,T]\big)$, it follows from (3.13), (3.14), (3.17), (3.19) and (3.22) that

$$\|P_j\|_{L^2(0,T)} \leq$$
$$\leq \left(\sqrt{\frac{T}{\mu_0}}\,\|K_1\|_\infty + \frac{1}{\sqrt{2\lambda_0}}\,\|\lambda_1\|_\infty + \sqrt{\frac{T}{\mu_0}}\,\|k\|_{L^2(0,T)}\right)\sqrt{D_T^{(1)}\widetilde{R}_j}+$$
$$+\|\widehat{g}_j\|_{L^2(0,T)}, \tag{3.23}$$

$$\widetilde{R}_j \leq \frac{2}{\beta}\,\|\widehat{g}_j\|^2_{L^2(0,T)} + 2\big\|\widetilde{f}_j\big\|^2_{L^2(Q_T)} + \frac{2}{\mu_0}\,TM_T\big\|\widetilde{\mu}_j\big\|^2_{H^1(0,T)}+$$
$$+2T\Big(\frac{M_T}{\mu_0}\Big)^{p-1}\big|\widetilde{K}_j\big|^2 + 2T\Big(\frac{M_T}{\mu_0}\Big)^{q-1}\big|\widetilde{\lambda}_j\big|^2 \leq$$
$$\leq D_T^{(2)}\Big(\big\|\widehat{g}_j\big\|^2_{L^2(0,T)} + \big\|\widetilde{f}_j\big\|^2_{L^2(Q_T)} + \big\|\widetilde{\mu}_j\big\|^2_{H^1(0,T)} + \big|\widetilde{K}_j\big|^2 + \big|\widetilde{\lambda}_j\big|^2\Big), \tag{3.24}$$

$$\|\widehat{g}_j\|_{L^2(0,T)} \leq \big\|\widetilde{g}_j\big\|_{H^1(0,T)} + \sqrt{\frac{TM_T}{\mu_0}}\,\big\|\widetilde{K}_{1j}\big\|_{H^1(0,T)}+$$
$$+\sqrt{\frac{M_T}{2\lambda_0}}\,\big\|\widetilde{\lambda}_{1j}\big\|_{H^1(0,T)} + \sqrt{\frac{TM_T}{\mu_0}}\big\|\widetilde{k}_j\big\|_{H^1(0,T)} \leq$$
$$\leq D_T^{(3)}\Big(\|\widetilde{g}_j\|_{H^1(0,T)} + \big\|\widetilde{K_{1j}}\big\|_{H^1(0,T)} + \big\|\widetilde{\lambda_{1j}}\big\|_{H^1(0,T)} + \|\widetilde{k}_j\|_{H^1(0,T)}\Big). \tag{3.25}$$

Finally, by (3.2), (3.11) and the estimates (3.22)–(3.25), we deduce that (3.3) holds. Hence, Theorem 2 is proved completely. □

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