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# ON THE GLOBAL SOLVABILITY OF THE CAUCHY CHARACTERISTIC PROBLEM FOR A NONLINEAR WAVE EQUATION IN A LIGHT CONE OF THE FUTURE

**Abstract.** For a nonlinear wave equation occurring in relativistic quantum mechanics we investigate the problem on the global solvability of the Cauchy characteristic problem in a light cone of the future.

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**რეზიუმე.** ერთი არაწრფივი განტოლებისათვის, რომელიც წარმოიშობა რელატივისტურ კვანტურ მექანიკაში, გამოკვლეულია კოშის მახასიათებლური ამოცანის გლობალური ამოხსნადობის საკითხი კონუსურ არეში.

**1. Statement of the Problem.** Consider a nonlinear wave equation of the type

$$(\Box + m^2)u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} + m^2 u = -\lambda u^3 + F, \tag{1}$$

where $\lambda \neq 0$ and $m \geq 0$ are given real constants, $F$ is a given and $u$ is an unknown real function. As is known, the equation (1) arises in the relativistic quantum mechanics ([1]–[5]).

For the equation (1) we consider the Cauchy characteristic problem on finding in a frustum of the light cone of the future $D_T : |x| < t < T$, $x = (x_1, x_2, x_3)$, $T = const > 0$, a solution $u(x,t)$ of that equation by the boundary condition

$$u\big|_{S_T} = g, \tag{2}$$

where $g$ is a given real function on the characteristic conic surface $S_T$ : $t = |x|$, $t \leq T$. When considering the case $T = +\infty$, we assume that $D_\infty : t > |x|$ and $S_\infty = \partial D_\infty : t = |x|$.

Note that the Cauchy characteristic problem for the equation (1) with the boundary conditions $u|_{t=0} = u_0$, $u_t|_{t=0} = u_1$ has been investigated in [1]–[5]. For more general semi-linear equations of type (1) the issues of existence or nonexistence of a global solution of that problem have been considered and studied in [6]–[21].

In the linear case, that is for $\lambda = 0$, the problem (1), (2) is posed correctly and its global solvability takes place in the corresponding function spaces ([22]–[26]). In the nonlinear case, for the equation $\Box u + m^2 u = -\lambda |u|^p u + F$, $p = const \neq 0$, coinciding with (1) for $p = 2$, the global solvability of the Cauchy characteristic problem for $\lambda > 0$, $m = 0$ and $0 < p < 1$ in the Sobolev space $W_2^1(D_T)$ has been proved in [27]. In the present paper we prove the global solvability of the problem (1), (2) for $\lambda > 0$ in the Sobolev space $W_2^2(D_T)$. The uniqueness of a solution of that problem will also be proved, and a result on the absence of the global solvability of the problem (1), (2) will be given for the case $\lambda < 0$. In this direction the works [28]–[30] are noteworthy.

**2. A Priori Estimate of a Solution of the Problem (1), (2) for $\lambda > 0$.** For the sake of simplicity of our exposition, the boundary condition (2) will be assumed to be homogeneous, i.e.,

$$u\big|_{S_T} = 0. \tag{3}$$

Suppose $\overset{\circ}{W}{}_2^k(D_T, S_T) = \{u \in W_2^k(D_T) : u|_{S_T} = 0\}$, where $W_2^k(D_T)$ is the well-known Sobolev space, and the condition $u|_{S_T} = 0$ will be understood in the sense of the trace theory ([31, pp. 56, 70]).

**Definition 1.** Let $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$. The function $u = u(x,t)$ is said to be a solution of the problem (1), (3) of the class $W_2^2$, if $u \in W_2^2(D_T)$, it satisfies both the equation (1) almost everywhere in the domain $D_T$ and the boundary condition (3) in the sense of the trace theory (and hence $\overset{\circ}{W}{}_2^2(D_T, S_T)$).

**Definition 2.** Let $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$. The function $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ is said to be a strong generalized solution of the problem $(1), (3)$ of the class $W_2^2$, if there exists a sequence of functions $u_n \in C^\infty(\overline{D}_T)$ satisfying the boundary condition (3) and

$$\lim_{n\to\infty} \|u_n - u\|_{W_2^2(D_T)} = 0, \quad \lim_{n\to\infty} \|F_n - F\|_{W_2^1(D_T)} = 0, \tag{4}$$

where

$$F_n = \Box u_n + m^2 u_n + \lambda u_n^3 \quad \text{and} \quad \operatorname{supp} F_n \cap S_T = \varnothing. \tag{5}$$

*Remark* 1. It can be easily seen that a strong generalized solution of the problem $(1), (3)$ of the class $W_2^2$ in the sense of Definition 2 is also a solution of the problem $(1), (3)$ of the class $W_2^2$ in the sense of Definition 1, since, as it will be noted below, the first equality of (4) implies $\lim\limits_{n\to\infty} \|u_n^3 - u^3\|_{L_2(D_T)} = 0$. On the other hand, in the next section we will prove that the problem $(1), (3)$ is solvable in the sense of Definition 2, whereas the uniqueness of a solution of that problem in the sense of Definition 1 will be proved in Section 4. This obviously implies that a solution of the problem is unique in the sense of Definition 2, and the above definitions are equivalent.

**Definition 3.** Let $F \in L_{2,loc}(D_\infty)$ and $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ for any $T > 0$. We say that the problem $(1), (3)$ is globally solvable, if for any $T > 0$ this problem has a solution of the class $W_2^2$ in the domain $D_T$ in the sense of Definition 1.

**Lemma 1.** *Let $\lambda \geq 0$ and $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$. Then for any strong generalized solution $u$ of the problem $(1), (3)$ of the class $W_2^2$ the following a priori estimate*

$$\begin{aligned} \|u\|_{W_2^2(D_T)} \leq \sigma_1 \|F\|_{L_2(D_T)} + \lambda \sigma_2 \|F\|^3_{L(D_T)} + \\ + \sigma_3 \|F\|_{W_2^1(D_T)} \exp\Big[\sigma_4 + \lambda\sigma_5 \|F\|^2_{L_2(D_T)}\Big] \end{aligned} \tag{6}$$

*is valid with positive constants $\sigma_i = \sigma_i(m, T)$, $i = 1, \dots, 5$, not depending on $u$ and $F$.*

*Proof.* By Definition 2 of strong generalized solution $u$ of the problem $(1), (3)$ of the class $W_2^2$, there exists a sequence of functions $u_n \in C^\infty(\overline{D}_T)$ satisfying the conditions $(3), (4)$ and (5), and hence

$$\Box u_n + m^2 u_n = -\lambda u_n^3 + F_n, \quad u_n \in C^\infty(\overline{D}_T), \tag{7}$$

$$u_n\big|_{S_T} = 0. \tag{8}$$

The proof of the lemma runs in several steps.

**1⁰.** Assuming $\Omega_\tau := D_\infty \cap \{t = \tau\}$, we first show that the a priori estimate

$$\int\limits_{\Omega_t} \left[ u_n^2 + \left(\frac{\partial u_n}{\partial t}\right)^2 + \sum_{i=1}^{3} \left(\frac{\partial u_n}{\partial x_i}\right)^2 \right] dx \le c \int\limits_{D_t} F_n^2 \, dx \, dt, \quad 0 < t < T, \tag{9}$$

is valid with a constant $c$ not depending on $u_n$ and $F_n$. Indeed, multiplying both sides of the equation (7) by $\frac{\partial u_n}{\partial t}$ and integrating over the domain $D_\tau$, $0 < \tau \le T$, we obtain

$$\frac{1}{2} \int\limits_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_n}{\partial t}\right)^2 dx \, dt - \int\limits_{D_\tau} \Delta u_n \frac{\partial u_n}{\partial t} \, dx \, dt + \frac{m^2}{2} \int\limits_{D_\tau} \frac{\partial}{\partial t} (u_n)^2 \, dx \, dt +$$
$$+ \frac{\lambda}{4} \int\limits_{D_\tau} \frac{\partial}{\partial t} (u_n)^4 \, dx \, dt = \int\limits_{D_\tau} F_n \frac{\partial u_n}{\partial t} \, dx \, dt. \tag{10}$$

By $\nu = (\nu_1, \nu_2, \nu_3, \nu_0)$ we denote the unit vector of the outer normal to $S_T \setminus \{(0,0,0,0)\}$. The integration by parts with regard for the equality (8) and $\nu|_{\Omega_\tau} = (0,0,0,1)$ results in

$$\int\limits_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_n}{\partial t}\right)^2 dx \, dt = \int\limits_{D_\tau} \left(\frac{\partial u_n}{\partial t}\right)^2 \nu_0 \, ds = \int\limits_{\Omega_\tau} \left(\frac{\partial u_n}{\partial t}\right)^2 dx + \int\limits_{S_\tau} \left(\frac{\partial u_n}{\partial t}\right)^2 \nu_0 \, ds,$$

$$\int\limits_{D_\tau} \frac{\partial}{\partial t} (u_n)^{2k} \, dx \, dt = \int\limits_{\partial D_\tau} (u_n)^{2k} \nu_0 \, ds = \int\limits_{\Omega_\tau} (u_n)^{2k} \, dx, \quad k = 1, 2,$$

$$\int\limits_{D_\tau} \frac{\partial^2 u_n}{\partial x_i^2} \frac{\partial u_n}{\partial t} \, dx \, dt = \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial t} \nu_i \, ds - \frac{1}{2} \int\limits_{D\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_n}{\partial x_i}\right)^2 dx \, dt =$$
$$= \int\limits_{\partial D_\tau} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial t} \nu_i \, ds - \frac{1}{2} \int\limits_{\partial D_\tau} \left(\frac{\partial u_n}{\partial x_i}\right)^2 \nu_0 \, ds = \int\limits_{\partial D_\tau} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial t} \nu_i \, ds -$$
$$- \frac{1}{2} \int\limits_{S_\tau} \left(\frac{\partial u_n}{\partial x_i}\right)^2 \nu_0 \, ds - \frac{1}{2} \int\limits_{\Omega_\tau} \left(\frac{\partial u_n}{\partial x_i}\right)^2 dx, \quad i = 1, 2, 3,$$

whence by virtue of (10) we obtain

$$\int\limits_{D_\tau} F_n \frac{\partial u_n}{\partial t} \, dx \, dt = \int\limits_{S_\tau} \frac{1}{2\nu_0} \left[ \sum_{i=1}^{3} \left( \frac{\partial u_n}{\partial x_i} \nu_0 - \frac{\partial u_n}{\partial t} \nu_i \right)^2 + \right.$$
$$\left. + \left(\frac{\partial u_n}{\partial t}\right)^2 \left( \nu_0^2 - \sum_{j=1}^{3} \nu_j^2 \right) \right] ds + \frac{1}{2} \int\limits_{\Omega_\tau} \left[ \left(\frac{\partial u_n}{\partial t}\right)^2 + \sum_{i=1}^{3} \left(\frac{\partial u_n}{\partial x_i}\right)^2 \right] dx +$$
$$+ \frac{\lambda}{4} \int\limits_{\Omega_\tau} u_n^4 \, dx + \frac{m^2}{2} \int\limits_{\Omega_\tau} u_n^2 \, dx. \tag{11}$$

Since $S_\tau$ is a characteristic surface, we have

$$\left(\nu_0^2-\sum_{j=1}^{3}\nu_j^2\right)\Big|_{S_\tau}=0. \tag{12}$$

Taking into account that $(\nu_0\,\frac{\partial}{\partial x_i}-\nu_i\,\frac{\partial}{\partial t})$, $i=1,2,3$, is the interior differential operator on $S_\tau$, owing to (8) we find

$$\left(\frac{\partial u_n}{\partial x_i}\,\nu_0-\frac{\partial u_n}{\partial t}\,\nu_i\right)\Big|_{S_\tau}=0,\quad i=1,2,3. \tag{13}$$

With regard for (12) and (13), the equation (11) yields

$$\int\limits_{\Omega_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx+\frac{\lambda}{2}\int\limits_{\Omega_\tau}u_n^4\,dx+m^2\int\limits_{\Omega_\tau}u_n^2\,dx=$$
$$=2\int\limits_{D_\tau}F_n\,\frac{\partial u_n}{\partial t}\,dx\,dt\,. \tag{14}$$

Since $\lambda\geq 0$, $m^2\geq 0$, by the Cauchy inequality $2F_n\,\frac{\partial u_n}{\partial t}\leq F_n^2+\left(\frac{\partial u_n}{\partial t}\right)^2$ it follows from (14) that

$$\int\limits_{\Omega_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx\leq\int\limits_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2dx\,dt+\int\limits_{D_\tau}F_n^2\,dx\,dt\,. \tag{15}$$

Reasoning in a standard way, from the equalities $u_n|_{S_\tau}=0$ and $u_n(x,\tau)=\int\limits_{|x|}^{\tau}\frac{\partial u_n(x,t)}{\partial t}\,dt$, $x\in\Omega_\tau$, $0<\tau\leq T$, we arrive at the inequality ([31, p. 63])

$$\int\limits_{\Omega_\tau}u_n^2\,dx\leq T\int\limits_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2dx\,dt. \tag{16}$$

Adding the inequalities (15) and (16), we obtain

$$\int\limits_{\Omega_\tau}\left[u_n^2+\left(\frac{\partial u_n}{\partial t}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx\leq$$
$$\leq(1+T)\int\limits_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2dx\,dt+\int\limits_{D_\tau}F_n^2\,dx\,dt\,. \tag{17}$$

Introduce the notation

$$w(\delta)=\int\limits_{\Omega_\delta}\left[u_n^2+\left(\frac{\partial u_n}{\partial t}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx.$$

Then by virtue of (17) we have

$$w(\delta) = (1+T)\int\limits_{D_\tau}\left[u_n^2 + \left(\frac{\partial u_n}{\partial t}\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx\,dt + \int\limits_{D_\delta} F_n^2\,dx\,dt =$$

$$= (1+T)\int\limits_0^\delta w(\sigma)\,d\sigma + \|F_n\|^2_{L_2^{(}D_\delta)}, \quad 0<\delta<T. \tag{18}$$

From (18), taking into account the fact that the expression $\|F_n\|^2_{L:_2(D_\delta)}$ as a function of $\delta$ is nondecreasing, by the Gronwall lemma ([32, p. 13]) we find that

$$w(\delta) \le \|F_n\|^2_{L_2(D_\delta)} \exp(1+T)\delta \le \|F_n\|^2_{L_2(D_\delta)} \exp(1+T)T,$$

whence for $\delta = t$ we obtain the inequality (9) with the constant $c = \exp(1+T)T$.

**2$^0$.** Since owing to (5) we have $\operatorname{supp} F_n \cap S_T = \varnothing$, there exists a positive number $\delta_n < T$ such that

$$\operatorname{supp} F_n \subset D_{T,\delta_n} = \{(x,t)\in D_T : t > |x| + \delta_n\}. \tag{19}$$

In this section we will show that

$$u_n\big|_{D_T\setminus \overline{D}_{T,\delta_n}} = 0. \tag{20}$$

Indeed, let $(x^0,t^0)\in D_T\setminus \overline{D}_{t,\delta_n}$. Introduce the domain $D_{x^0,t^0} = \{(x,t)\in R^4 : |x| < t < t^0 - |x-x^0|\}$ which is bounded from below by the surface $S_T$ and from above by the boundary $S^-_{x^0,t^0} = \{(x,t)\in R^4 : t = t^0 - |x-x^0|\}$ of the light cone of the past $G^-_{x^0,t^0} = \{(x,t)\in R^4 : t < t^0 - |x-x^0|\}$ with the vertex at the point $(x^0,t^0)$. By virtue of (19), we have

$$F_n\big|_{D_{x^0,t^0}} = 0, \quad (x^0,t^0)\in D_T\setminus \overline{D}_{T,\delta_n}\,. \tag{21}$$

Assume $D_{x^0,t^0,\tau} := D_{x^0,t^0}\cap\{t<\tau\}$, $\Omega_{x^0,t^0,\tau} := D_{x^0,t^0}\cap\{t=\tau\}$, $0<\tau<t^0$. Then $\partial D_{x^0,t^0,\tau} = S_{1,\tau}\cup S_{2,\tau}\cup S_{3,\tau}$, where $S_{1,\tau} = \partial D_{x^0,t^0,\tau}\cap S_\infty$, $S_{2,\tau} = \partial D_{x^0,t^0,\tau}\cap S^-_{x^0,t^0}$, $S_{3,\tau} = \partial D_{x^0,t^0,\tau}\cap \overline{\Omega}_{x^0,t^0,\tau}$. Just in the same way as when obtaining the equality (11), we multiply both sides of the equation (7) by $\frac{\partial u_n}{\partial t}$, integrate over the domain $D_{x^0,t^0,\tau}$, $0<\tau<t^0$, and taking into account (8) and (21), we obtain

$$0 = \int\limits_{S_{1,\tau}\cup S_{2,\tau}} \frac{1}{2\nu_0}\left[\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\,\nu_0 - \frac{\partial u_n}{\partial t}\,\nu_i\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2\left(\nu_0^2 - \sum_{j=1}^{3}\nu_j^2\right)\right]ds +$$

$$+ \int\limits_{S_{2,\tau}\cup S_{3,\tau}} \left[\frac{\lambda}{4}\,u_n^4 + \frac{m^2}{2}\,u_n^2\right]\nu_0\,ds +$$

$$+\frac{1}{2}\int\limits_{S_{3,\tau}}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx. \tag{22}$$

By (8) and (12), taking into account the fact that $S_{2,\tau}$ is, just like $S_{1,\tau}$, a characteristic surface and hence $\left(\nu_0^2-\sum\limits_{j=1}^{3}\nu_j^2\right)\Big|_{S_{1,\tau}\cup S_{2,\tau}}=0$ and also that

$$\nu_0\big|_{S_{1,\tau}}=-\frac{1}{\sqrt{2}}<0,\quad \nu_0\big|_{S_{2,\tau}}=\frac{1}{\sqrt{2}}>0,\quad \nu_0\big|_{S_{3,\tau}}=1,$$

$$\left(\frac{\partial u}{\partial x_i}\,\nu_0-\frac{\partial u}{\partial t}\,\nu_i\right)\bigg|_{S_{1,\tau}}=0,$$

$$\left(\frac{\partial u}{\partial x_i}\,\nu_0-\frac{\partial u}{\partial t}\,\nu_i\right)^2\bigg|_{S_{2,\tau}}\geq 0,\quad i=1,2,3,$$

we have

$$\int\limits_{S_{1,\tau}\cup S_{2,\tau}}\frac{1}{2\nu_0}\left[\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\,\nu_0-\frac{\partial u_n}{\partial t}\,\nu_i\right)^2+\left(\frac{\partial u_n}{\partial t}\right)^2\left(\nu_0^2-\sum_{j=1}^{3}\nu_j^2\right)\right]ds\geq 0. \tag{23}$$

Taking into account (23), from (22) we get

$$\int\limits_{S_{3,\tau}}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx\leq M\int\limits_{S_{2,\tau}\cup S_{3,\tau}}u_n^2\,ds,\quad 0<\tau<t^0. \tag{24}$$

Here, since $u_n\in C^\infty(\overline{D}_T)$ and $|\nu_0|\leq 1$, we can take

$$M=m^2+\frac{\lambda}{2}\,\|u_n\|^2_{C(\overline{D}_T)}<+\infty \tag{25}$$

in the capacity of a nonnegative constant $M$ independent of the parameter $\tau$. As far as $u_n|_{S_T}=0$ and $S_T: t=|x|$, $t\leq T$, we will have

$$u_n(x,t)=\int\limits_{|x|}^{t}\frac{\partial u_n(x,\sigma)}{\partial t}\,d\sigma,\quad (x,t)\in S_{2,\tau}\cup S_{3,\tau}. \tag{26}$$

Reasoning in a standard way ([31, p. 63]), from the equality (26) we get that

$$\int\limits_{S_{2,\tau}\cup S_{3,\tau}}u_n^2\,ds\leq 2t^0\int\limits_{D_{x^0,t^0,\tau}}\left(\frac{\partial u_n}{\partial t}\right)^2dx,\quad 0<\tau<t^0. \tag{27}$$

Assuming $v(\tau)=\int\limits_{S_{3,\tau}}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\sum\limits_{i=1}^{3}\left(\frac{\partial u_n}{\partial x_i}\right)^2\right]dx$, from (24) and (27) we easily obtain

$$v(\tau)\leq 2t^0M\int\limits_{0}^{\tau}v(\delta)\,d\delta,\quad 0<\tau<t^0,$$

whence by virtue of (25) and the Gronwall lemma it immediately follows that $v(\tau)=0$, $0<\tau<t^0$, and hence $\frac{\partial u_n}{\partial t}=\frac{\partial u_n}{\partial x_1}=\frac{\partial u_n}{\partial x_2}=\frac{\partial u_n}{\partial x_3}=0$ in the domain $D_{x^0,t^0}$. Therefore $u_n|_{D_{x^0,t^0}}=const$, and taking into account the homogeneous boundary condition (8), we find that $u_n|_{D_{x^0,t^0}}=0\ \forall(x^0,t^0)\in D_T\setminus\overline{D}_{T,\delta_n}$. Thus the equality (20) is proved.

**$3^0$.** We will now pass to proving the a priori estimate (6). By (20), extending $u_n$ from the domain $D_T$ to the layer $\Sigma_T=\{(x,t)\in R^4:0<t<T\}$ as zero and preserving for it the same designation, we obtain

$$u_n\in C^\infty\Big(\overline{\Sigma}_T\Big),\quad u_n\Big|_{\overline{\Sigma}_T\setminus\overline{D}_{T,\delta_n}}=0. \tag{28}$$

In particular, from (28) it follows that $u_n=0$ for $|x|\ge T$.

Differentiating the equality (7) with respect to the variable $x_i$, we obtain

$$\Box u_{n,x_i}+m^2u_{n,x_i}=-3\lambda u_n^2u_{n,x_i}+F_{n,x_i},\quad i=1,2,3, \tag{29}$$

where $u_{n,x_i}=\frac{\partial u_n}{\partial x_i}$, $F_{n,x_i}=\frac{\partial F_n}{\partial x_i}$. Suppose

$$E(\tau)=\frac{1}{2}\sum_{i=1}^{3}\int\limits_{\Omega_\tau}\bigg(u^2_{n,x_it}+\sum_{k=1}^{3}u^2_{n,x_ix_k}\bigg)\,dx,\quad \Omega_\tau=D_\infty\cap\{t=\tau\}. \tag{30}$$

By virtue of (28), in the right-hand side of (30) we can take instead of $\Omega_\tau$ a three-dimensional ball $B(0,T):|x|<T$ in the plane $t=\tau$.

Differentiating the equality (30) with respect to the variable $\tau$ and then integrating by parts, with regard for (7), (28) and (29) we find that

$$\begin{aligned}
E'(\tau)&=\sum_{i=1}^{3}\int\limits_{\Omega_\tau}\bigg(u_{n,x_it}u_{n,x_itt}+\sum_{k=1}^{3}u_{n,x_ix_k}u_{n,x_ix_kt}\bigg)\,dx=\\
&=\sum_{i=1}^{3}\int\limits_{\Omega_\tau}\bigg(u_{n,x_itt}u_{n,x_it}-\sum_{k=1}^{3}u_{n,x_ix_kx_k}u_{n,x_it}\bigg)\,dx=\\
&=\sum_{i=1}^{3}\int\limits_{\Omega_\tau}(\Box u_{n,x_i})u_{n,x_it}\,dx=\\
&=\sum_{i=1}^{3}\int\limits_{\Omega_t}\big[-m^2u_{n,x_i}-3\lambda u_n^2u_{n,x_i}+F_{n,x_i}\big]u_{n,x_it}\,dx=\\
&=\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}\big[-m^2u_{n,x_i}-3\lambda u_n^2u_{n,x_i}+F_{n,x_i}\big]u_{n,x_it}\,dx.
\end{aligned} \tag{31}$$

By Hölder's inequality ([33, p. 134])

$$\Big|\int f_1f_2f_3\,dx\Big|\le\|f_1\|_{L_{p_1}}\|f_2\|_{L_{p_2}}\|f_3\|_{L_{p_3}}$$

$$\text{for } p_1=3,\ p_2=6,\ p_3=2,\ \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1,$$

and by the Cauchy inequality, for the right-hand side of (31) we have the estimate

$$I=\Bigg|\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}\big[-m^2u_{n,x_i}-3\lambda u_n^2u_{n,x_i}+F_{n,x_i}\big]u_{n,x_it}\,dx\Bigg|\le$$
$$\le\frac{m^2}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}u_{n,x_i}^2\,dx+\frac{m^2}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}u_{n,x_it}^2\,dx+\frac{1}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}F_{n,x_i}^2\,dx+$$
$$+\frac{1}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}u_{n,x_it}^2\,dx+3\lambda\Bigg|\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}u_n^2u_{n,x_i}u_{n,x_it}\,dx\Bigg|\le$$
$$\le\frac{m^2+1}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}u_{n,x_i}^2\,dx+\frac{m^2}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}u_{n,x_it}^2\,dx+$$
$$+\frac{1}{2}\sum_{i=1}^{3}\int\limits_{\substack{|x|<T,\\ t=\tau}}F_{n,x_i}^2\,dx+3\lambda\sum_{i=1}^{3}\|u_n^2\|_{L_3((|x|<T,\,t=\tau))}\times$$
$$\times\|u_{n,x_i}\|_{L_6((|x|<T,\,t=\tau))}\|u_{n,x_it}\|_{L_2((|x|<T,\,t=\tau))}. \tag{32}$$

By the well-known theorem of imbedding of the space $W_m^l(\Omega)$ into $L_p(\Omega)$ for $m=2$, $l=1$, $p=6$ ([31, p. 84]; [34, p. 111]), there takes place the inequality

$$\|v\|_{L_6((|x|<T))}\le c_1\|v\|_{\overset{\circ}{W}{}_2^1((|x|<t))}\quad\forall v\in\overset{\circ}{W}{}_2^1((|x|<T)) \tag{33}$$

with a positive constant $c_1$ not depending on $v$.

We also have ([31, p. 117])

$$\sum_{i=1}^{3}\int\limits_{|x|<T}v_{x_i}^2\,dx\le c_2\sum_{i,j=1}^{n}\int\limits_{|x|<T}v_{x_ix_k}^2\,dx\quad\forall v\in\overset{\circ}{W}{}_2^2((|x|<T)) \tag{34}$$

with a positive constant $c_2$ not depending on $v$.

Applying the inequality (33) to the functions $u_n$ and $u_{n,x_i}$ which by virtue of (28) belong to the space $\overset{\circ}{W}{}_2^1((|x|<T))$ for the fixed $t=\tau$, we obtain

$$\begin{aligned}\|u_n\|_{L_6((\|x\|<T,\,t=\tau))}&\le c_1\|u_n\|_{\overset{\circ}{W}{}_2^1((\|x\|<T,\,t=\tau))},\\ \|u_{n,x_i}\|_{L_6((\|x\|<T,\,t=\tau))}&\le c_1\|u_{n,x_i}\|_{\overset{\circ}{W}{}_2^2((\|x\|<T,\,t=\tau))}.\end{aligned} \tag{35}$$

By (9), (30), (33) and (35), we have

$$\|u_n^2\|_{L_3((\|x\|<T,\,t=\tau))}\|u_{n,x_i}\|_{L_6((\|x\|<T,\,t=\tau))}\|u_{n,x_it}\|_{L_2((\|x\|<T,\,t=\tau))}\le$$

$$\le \|u_n\|^2_{L_6((\|x\|<T,\, t=\tau))} c_1 \|u_n\|_{\overset{\circ}{W}{}^2_2((\|x\|<T,\, t=\tau))} [2E(\tau)]^{\frac{1}{2}} \le$$

$$\le c_1 c \|F_n\|^2_{L_2(D_\tau)} c_1 [2E(\tau)]^{\frac{1}{2}} [2E(\tau)]^{\frac{1}{2}} = 2cc_1^2 \|F_n\|^2_{L_2(D_\tau)} E(\tau). \tag{36}$$

By (9), (32), (34) and (36), we have

$$I \le \frac{m^2+1}{2}\, c_2\, 2E(\tau) + m^2 E(\tau) + \frac{1}{2}\, \|F_n\|^2_{W_2^1((\|x\|<T,\, t=\tau))} + \\ + 6\lambda cc_1^2 \|F_n\|^2_{L_2(D_\tau)} E(\tau). \tag{37}$$

By (31) and (37), we find that

$$E'(\tau) \le \alpha(\tau)E(\tau) + \beta(\tau) \le \alpha(T)E(\tau) + \beta(\tau), \quad \tau \le T. \tag{38}$$

Here

$$\begin{aligned} \alpha(\tau) &= (m^2+1)c_2 + m^2 + 6\lambda cc_1^2 \|F_n\|^2_{L_2(D_\tau)}, \\ \beta(\tau) &= \frac{1}{2}\, \|F_n\|^2_{W_2^1((\|x\|<T,\, t=\tau))}. \end{aligned} \tag{39}$$

Since according to (28) we have $E(0) = 0$, multiplying both sides of the inequality (38) by $\exp[-\alpha(T)\tau]$ and integrating in a standard way, we obtain

$$E(\tau) \le e^{\alpha(T)\tau} \int\limits_0^\tau e^{-\alpha(T)\sigma} \beta(\sigma)\, d\sigma \le e^{\alpha(T)\tau} \int\limits_0^\tau \beta(\sigma)\, d\sigma =$$

$$= \frac{1}{2}\, e^{\alpha(T)\tau} \int\limits_0^\tau \|F_n\|^2_{W_2^1((\|x\|<T,\, t=\sigma))}\, d\sigma = \frac{1}{2}\, e^{\alpha(T)\tau} \|F_n\|^2_{W_2^1(D_\tau)} \le$$

$$\le \frac{1}{2}\, e^{\alpha(T)T} \|F_n\|^2_{W_2^1(D_T)}, \quad 0 \le \tau \le T. \tag{40}$$

By the equality (7),

$$u_{n,tt} = \Delta u_n - m^2 u_n - \lambda u_n^3 + F. \tag{41}$$

Squaring both parts of the equality (41) and using the inequality $\Big(\sum\limits_{i=1}^n a_i\Big)^2 \le n \sum\limits_{i=1}^n a_i^2$ for $n = 4$, we will have

$$\int\limits_{\Omega_\tau} u^2_{n,tt}\, dx \le 4\lambda^2 \int\limits_{\substack{|x|<T,\\ t=\tau}} u_n^6\, dx + 4 \int\limits_{\substack{|x|<T,\\ t=\tau}} \big[(\Delta u_n)^2 + m^4 u_n^2 + F_n^2\big]\, dx,$$

whence by virtue of (9), (35) and the fact that $(\Delta u_n)^2 \le 3 \sum\limits_{i=1}^3 u^2_{n,x_i x_i}$, we find that

$$\int\limits_{\Omega_\tau} u^2_{n,tt}\, dx \le 4\lambda^2 c_1^6 \|u_n\|^2_{\overset{\circ}{W}{}^1_2((\|x\|<T,\, t=\tau))} + 24E(\tau) + 4m^2 c \|F_n\|^2_{L_2(D_\tau)} +$$

$$+ 4\|F_n\|^2_{L_2((\|x\|<T,\, t=\tau))} \le 4\lambda^2 c_1^6 \|F_n\|^6_{L_2(D_\tau)} + 4m^4 c \|F_n\|^2_{L_2(D_\tau)} +$$

$$+\,4\|F_n\|^2_{L_2((\|x\|<T,\,t=\tau))}+24E(\tau). \tag{42}$$

Due to (40), it follows from (42) that

$$\int\limits_{D_T} u^2_{n,tt}\,dx\,dt=\int\limits_0^T d\tau\int\limits_{\Omega_\tau} u^2_{n,tt}\,dx\le 4\lambda^2 c_1^6 T\|F_n\|^6_{L_2(D_T)}+$$
$$+\,4m^4cT\|F_n\|^2_{L_2(D_T)}+4\|F_n\|^2_{L_2(D_T)}+12e^{\alpha(T)T}\|F_n\|^2_{W_2^1(D_T)}. \tag{43}$$

From (9), (30), (40) and (43) we now find that

$$\|u_n\|^2_{W_2^2(D_T)}=\int\limits_0^T d\tau\int\limits_{\Omega_\tau}\bigg[u_n^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2+\sum_{i=1}^3\Big(\frac{\partial u_n}{\partial x_i}\Big)^2+u^2_{n,tt}+$$
$$+\sum_{i=1}^3 u^2_{n,x_it}+\sum_{i,k=1}^3 u^2_{n,x_ix_k}\bigg]dx\le$$
$$\le\int\limits_0^T\bigg[c\int\limits_{D_\tau}F_n^2\,dx\,dt\bigg]d\tau+\int\limits_{D_T}u^2_{n,tt}\,dx\,dt+\int\limits_0^T 2E(\tau)\,d\tau\le$$
$$\le cT\|F_n\|^2_{L_2(D_T)}+4\lambda^2c_1^6T\|F_n\|^6_{L_2(D_T)}+4m^4cT\|F_n\|^2_{L_2(D_T)}+$$
$$+\,4\|F_n\|^2_{L_2(D_T)}+12e^{\alpha(T)T}\|F_n\|^2_{W_2^1(D_T)}+Te^{\alpha(T)T}\|F_n\|^2_{W_2^1(D_T)}=$$
$$=(cT+4m^4cT+4)\|F_n\|^2_{L_2(D_T)}+4\lambda^2c_1^6T\|F_n\|^6_{L_2(D_T)}+$$
$$+\,(12+T)e^{\alpha(T)T}\|F_n\|^2_{W_2^1(D_T)}. \tag{44}$$

Taking into account the obvious inequality $\Big(\sum\limits_{i=1}^3|a_i|\Big)^{\frac12}\le\sum\limits_{i=1}^3|a_i|^{\frac12}$ and the equality (39), from (44) we obtain

$$\|u_n\|_{W_2^2(D_T)}\le\sigma_1\|F_n\|_{L_2(D_T)}+\lambda\sigma_2\|F_n\|^3_{L_2(D_T)}+$$
$$+\,\sigma_3\|F_n\|^2_{W_2^1(D_T)}\exp\big[\sigma_4+\lambda\sigma_5\|F_n\|^2_{L_2(D_T)}\big], \tag{45}$$

where $\sigma_1=(cT+4m^4cT+4)^{\frac12}$, $\sigma_2=2c_1^3T^{\frac12}$, $\sigma_3=(12+T)^{\frac12}$, $\sigma_4=\frac12((m^2+1)c_2+m^2)$, $\sigma_5=3cc_1^2$. By (4), passing the in inequality (45) to the limit as $n\to\infty$, we obtain the a priori estimate (6). Thus the proof of Lemma 1 is complete. □

## 3. The Global Solvability of the Problem (1), (3) for $\lambda>0$.

*Remark* 2. Before we proceed to considering the question on the solvability of the nonlinear problem (1), (3), let us consider the same question for the linear case in the form needed for us, when in the equation (1) the parameter $\lambda=0$, i.e., for the problem

$$Lu(x,t)=F(x,t),\quad (x,t)\in D_T\quad (L:=\Box+m^2), \tag{46}$$
$$u(x,t)=0,\quad (x,t)\in S_T. \tag{47}$$

In this case for $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ we analogously introduce the notion of a strong generalized solution $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ of the problem (46), (47) for which there exists a sequence of the functions $u_n \in C^\infty(\overline{D}_T)$ satisfying the boundary condition (47) and

$$\lim_{n\to\infty} \|u_n - u\|_{W_2^2(D_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n - F\|_{W_2^1(D_T)} = 0. \tag{48}$$

As it follows from the a priori estimate (6), for that solution for $\lambda = 0$ we have the estimate

$$\|u\|_{W_2^2(D_T)} \le c_0 \|F\|_{W_2^1(D_T)} \tag{49}$$

with a positive constant $c_0$ not depending on $u$ and $F$.

Since the space $C_0^\infty(\overline{D}_T, S_T) = \{F \in C^\infty(\overline{D}_T) : \operatorname{supp} F \cap S_T = \varnothing\}$ of the infinitely differentiable in $\overline{D}_T$ functions vanishing in some bounded neighborhood (its own for each function) of the set $S_T$ is dense in $\overset{\circ}{W}{}_2^1(D_T, S_T)$, for a given $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ there exists a sequence of functions $F_n \in C_0^\infty(\overline{D}_T, S_T)$ such that $\lim\limits_{n\to\infty} \|F_n - F\|_{W_2^1(D_T)} = 0$. For the fixed $n$, extending the function $F_n$ from the domain $D_T$ to the layer $\Sigma_T = \{(x,t) \in R^4 : 0 < t < T\}$ as zero and preserving the same notation, we will have $F_n \in C^\infty(\overline{\Sigma}_T)$ with the carrier $\operatorname{supp} F_n \subset D_\infty : t > |x|$. Denote by $u_n$ the solution of the Cauchy linear problem: $Lu_n = F_n$, $u_n|_{t=0} = 0$, $\frac{\partial u_n}{\partial t}\big|_{t=0} = 0$ in the layer $\Sigma_T$, which, as is known, exists, is unique and belongs to the space $C^\infty(\overline{\Sigma}_T)$ ([35, p. 192]). Moreover, since $\operatorname{supp} F_n \subset D_\infty$, $u_n|_{t=0} = 0$, $\frac{\partial u_n}{\partial t}\big|_{t=0} = 0$, taking into account the geometry of the domain of dependence of a solution of the linear wave equation, we will have $\operatorname{supp} u_n \subset D_\infty$ ([35, p. 191]). Retaining for the restriction of the function $u_n$ to the domain $D_T$ the same designation, we can easily see that $u_n \in C^\infty(\overline{D}_T)$, $u_n|_{S_T} = 0$, and because of (49),

$$\|u_n - u\|_{W_2^2(D_T)} \le c_0 \|F_n - F\|_{W_2^1(D_T)}. \tag{50}$$

Since the sequence $\{F_n\}$ is fundamental in $\overset{\circ}{W}{}_2^1(D_T, S_T)$, by (50) and the fact that $u_n|_{S_T} = 0$, the sequence $\{u_n\}$ is likewise fundamental in the whole space $\overset{\circ}{W}{}_2^2(D_T, S_T) = \{u \in W_2^2(D_T) : u|_{S_T} = 0\}$. Thus there exists a function $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ such that $\lim\limits_{n\to\infty} \|u_n - u\|_{W_2^2(D_T)} = 0$, and since by the condition $Lu_n = F_n \to F$ in the space $W_2^1(D_T)$, the function $u$ will, by Remark 2, be a strong generalized solution of the problem (46), (47) from the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$. The uniqueness of the solution from the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$ follows from the a priori estimate (49). Consequently, for the solution $u$ of the problem (46), (47) we can write $u = L^{-1}F$, where $L^{-1} : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^2(D_T, S_T)$ is a linear continuous operator, whose norm admits, by virtue of (49), the estimate

$$\|L^{-1}\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)\to\overset{\circ}{W}{}_2^2(D_T,S_T)}\leq c_0. \tag{51}$$

*Remark* 3. Below, we will show that the Nemytski operator $N : \overset{\circ}{W}{}_2^2(D_T,S_T) \to \overset{\circ}{W}{}_2^1(D_T,S_T)$ acting by the formula $Nu = -\lambda u^3$ is continuous and compact.

Indeed, first we note that if $u \in \overset{\circ}{W}{}_2^2(D_T,S_T)$, then since $D_T$ is a bounded domain from $R^4$, therefore $u \in L_q(D_T)$ for any $q \geq 1$, and moreover, the imbedding operator $I_1 : \overset{\circ}{W}{}_2^2(D_T,S_T) \to L_q(D_T)$ is continuous and compact ([31, p. 84]). Note also that the imbedding operator $I_2 : W_2^1(D_T) \to L_p(D_T)$ is a linear continuous one for $1 < p < 4$ ([31, p. 83]). Therefore, if $u \in \overset{\circ}{W}{}_2^2(D_T,S_T)$, then $u^2, u^k \in L_6(D_T)$, and since $\frac{\partial u}{\partial x_i} \in W_2^1(D_T)$, we have $\frac{\partial u}{\partial x_i} \in L_3(D_T)$. As is known, if $f_i \in L_{p_i}(D_T)$, $i=1,2$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{r}$, $p_i > 1$, $r > 1$, then $f_1 f_2 \in L_r(D_T)$ ([4, p. 45]). For $p_1 = 6$, $p_2 = 3$, $r = 2$, $(\frac{1}{6}+\frac{1}{3}=\frac{1}{2})$, $f_1 = u^2$, $f_2 = \frac{\partial u}{\partial x_i}$, we obtain $\frac{\partial Nu}{\partial x_i} = -3\lambda u^2 \frac{\partial u}{\partial x_i} \in L_2(D_T)$, $i = 1,2,3$. Analogously, we have $\frac{\partial Nu}{\partial t} \in L_2(D_T)$.

Let $X$ be a bounded set in $\overset{\circ}{W}{}_2^2(D_T,S_T)$, and let $\{u_n\}$ be a sequence taken arbitrarily in $X$. Since the space $\overset{\circ}{W}{}_2^2(D_T,S_T)$ is compact-imbedded into the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$ ([34, p. 183]), there exist a subsequence $\{u_{n_k}\}$ and a function $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ such that

$$\lim_{k\to\infty}\|u_{n_k}-u\|_{L_2(D_T)} = \lim_{k\to\infty}\left\|\frac{\partial u_{n_k}}{\partial t}-\frac{\partial u}{\partial t}\right\|_{L_2(D_T)} = \\ = \lim_{k\to\infty}\left\|\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right\|_{L_2(D_T)} = 0. \tag{52}$$

On the other hand, according to the above-said, there exists a subsequence of the sequence $\{u_{n_k}\}$, for which we will preserve the same designation, such that

$$\begin{gathered}\lim_{k\to\infty}\|u_{n_k}^2-v_0\|_{L_6(D_T)}=0, \quad \lim_{k\to\infty}\left\|\frac{\partial u_{n_k}}{\partial x_i}-v_i\right\|_{L_3(D_T)}=0, \quad i=1,2,3,\\ \lim_{k\to\infty}\|u_{n_k}^3-v\|_{L_2(D_T)}=0, \quad \lim_{k\to\infty}\left\|\frac{\partial u_{n_k}}{\partial t}-v_4\right\|_{L_3(D_T)}=0,\end{gathered} \tag{53}$$

where $v_0, v, v_i$, $i = 1,2,3,4$, are some functions from the corresponding spaces: $L_6(D_T)$, $L_2(D_T)$ for $v_0, v$, and $L_3(D_T)$ for the remaining $v_i$. Reasoning in a standard manner and using the notion of generalized Sobolev's derivative, from (52) and (53) we find that

$$v_0 = u^2, \quad v = u^3, \quad v_i = \frac{\partial u}{\partial x_i}, \quad i=1,2,3, \quad v_4 = \frac{\partial u}{\partial t}. \tag{54}$$

Let us now show that

$$\begin{aligned} &\lim_{k\to\infty}\left\|\frac{\partial N u_{n_k}}{\partial x_i}-\frac{\partial N u}{\partial x_i}\right\|_{L_2(D_T)}=0, \quad i=1,2,3,\\ &\lim_{k\to\infty}\left\|\frac{\partial N u_{n_k}}{\partial t}-\frac{\partial N u}{\partial t}\right\|_{L_2(D_T)}=0. \end{aligned} \tag{55}$$

Indeed, using Hölder's inequality for $p=3$, $q=\frac{3}{2}$ $(\frac{1}{p}+\frac{1}{q}=1)$, we obtain

$$\begin{aligned} &\left\|\frac{\partial N u_{n_k}}{\partial x_i}-\frac{\partial N u}{\partial x_i}\right\|^2_{L_2(D_T)}=9\lambda^2\int\limits_{D_T}\left(u^2_{n_k}\frac{\partial u_{n_k}}{\partial x_i}-u^2\frac{\partial u}{\partial x_i}\right)^2dx\,dt=\\ &=9\lambda^2\int\limits_{D_T}\left[(u^2_{n_k}-u^2)\frac{\partial u_{n_k}}{\partial x_i}+u^2\left(\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)\right]^2dx\,dt\le\\ &\le 18\lambda^2\int\limits_{D_T}(u^2_{n_k}-u^2)^2\left(\frac{\partial u_{n_k}}{\partial x_i}\right)^2dx\,dt+18\lambda^2\int\limits_{D_T}u^4\left(\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)^2dx\,dt\\ &\le 18\lambda^2\|(u^2_{n_k}-u^2)^2\|_{L_3(D_T)}\left\|\left(\frac{\partial u_{n_k}}{\partial x_i}\right)^2\right\|_{L_{\frac{3}{2}}(D_T)}+\\ &\quad+18\lambda^2\|u^4\|_{L_3(D_T)}\left\|\left(\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)^2\right\|_{L_{\frac{3}{2}}(D_T)}=\\ &=18\lambda^2\|u^2_{n_k}-u^2\|^2_{L_6(D_T)}\left\|\frac{\partial u_{n_k}}{\partial x_i}\right\|^2_{L_3(D_T)}+\\ &\quad+18\lambda^2\|u^2\|^2_{L_6(D_T)}\left\|\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right\|^2_{L_3(D_T)}. \end{aligned} \tag{56}$$

By virtue of (53), the sequence $\left\{\left\|\frac{\partial u_{n_k}}{\partial x_i}\right\|^2_{L_3(D_T)}\right\}$ is bounded. Hence from (56), by virtue of (53) and (54), we obtain the first three equalities from (55) for $i=1,2,3$. The last equality from (55) is proved analogously. Finally, the fact that $Nu_{n_k}\to Nu$ in $L_2(D_T)$ follows directly from (53) and (54). Thus the statement of Remark 3 is proved.

*Remark* 4. When writing the equalities

$$\lim_{k\to\infty}\|u^2_{n_k}-v_0\|_{L_6(D_T)}=0, \quad \lim_{k\to\infty}\|u^3_{n_k}-v\|_{L_2(D_T)}=0$$

from (53), we used the following: (1) the imbedding operator $I_1$ : $\overset{\circ}{W}{}^1_2(D_T,S_T)\to L_q(D_T)$ for any $q\ge 1$ is continuous and compact; (2) if $u\in L_{pr}(D_T)$, $p\ge 1$, $r\ge 1$, then $|u|^p\in L_r(D_T)$, and from the fact that $u_n\to u$ in $L_{pr}(D_T)$ it follows that $|u_n|^p\to|u|^p$ in $L_r(D_T)$ (analogously, if $u_n\to u$ in $L_{3r}(D_T)$, then $u^3_n\to u^3$ in $L_r(D_T)$), which is a consequence of the following theorem ([34, p. 66]): the nonlinear Nemytski operator $\mathcal{H}$, acting by the formula $u \to h(x,u)$, where the function $h = h(x,\xi)$ possesses the Carathéodory property, acts continuously from the space $L_p(D_T)$ to $L_r(D_T)$, $p \geq 1$, $r \geq 1$, if and only if $|h(x,\xi)| \leq d(x) + \alpha|\xi|^{\frac{p}{r}}$ $\forall \xi \in (-\infty,\infty)$, where $d \in L_r(D_T)$ and $\delta = const \geq 0$. In our case, $h(x,\xi) = |\xi|^p$, i.e., $\mathcal{H}u = |u|^p$. Note also that we have for the present proved that the operator $N$ from Remark 3 is compact from the space $\overset{\circ}{W}{}^2_2(D_T,S_T)$ to $W^1_2(D_T)$. In its turn, this implies that the operator is continuous as well, since the above-mentioned spaces, being Hilbert ones, are also reflexive ([34, p. 182]). Finally, the fact that the image $N(\overset{\circ}{W}{}^2_2(D_T,S_T))$ is, in fact, a subspace of the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$ follows from the following reasoning. If $u \in \overset{\circ}{W}{}^2_2(D_T,S_T)$, then there exists a sequence $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T,S_T) = \{u \in C^2(\overline{D}_T) : u|_{S_T} = 0\}$ such that $u_n \to u$ in the space $\overset{\circ}{W}{}^2_2(D_T,S_T)$. But according to the above-said, $Nu_n \to Nu$ in the space $W^1_2(D_T)$, and since $Nu_n = -\lambda u_n^3 \in \overset{\circ}{C}{}^2(\overline{D}_T,S_T) \subset \overset{\circ}{W}{}^1_2(D_T,S_T)$, owing to the completeness of the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$, we finally obtain $N(\overset{\circ}{W}{}^2_2(D_T,S_T)) \subset \overset{\circ}{W}{}^1_2(D_T,S_T)$, and hence the operator $N : \overset{\circ}{W}{}^2_2(D_T,S_T) \to \overset{\circ}{W}{}^1_2(D_T,S_T)$ is continuous and compact.

*Remark* 5. As is said in Remark 1 of Section 2, the first equality of (4) implies that $\lim\limits_{n\to\infty} \|u_n^3 - u^3\|_{L_2(D_T)} = 0$. The latter is a direct consequence of the statement in Remark 3. From the above remarks it immediately follows that if $F \in \overset{\circ}{W}{}^1_2(D_T,S_T)$, then the function $u \in \overset{\circ}{W}{}^2_2(D_T,S_T)$ is a strong generalized solution of the problem (1), (3) of the class $W^2_2$ if and only if this function is, in view of (51), a solution of the following functional equation

$$u = L^{-1}(-\lambda u^3 + F) \tag{57}$$

in the space $\overset{\circ}{W}{}^2_2(D_T,S_T)$.

We rewrite the equation (57) in the form

$$u = Au := L^{-1}(Nu + F), \tag{58}$$

where, according to Remark 3, the operator $N : \overset{\circ}{W}{}^2_2(D_T,S_T) \to \overset{\circ}{W}{}^1_2(D_T,S_T)$ is continuous and compact. Consequently, owing to (51), the operator $A : \overset{\circ}{W}{}^2_2(D_T,S_T) \to \overset{\circ}{W}{}^2_2(D_T,S_T)$ is likewise continuous and compact. At the same time, by Lemma 1, for any parameter $\tau \in [0,1]$ and any solution $u \in \overset{\circ}{W}{}^2_2(D_T,S_T)$ of the equation with the parameter $u = \tau Au$ the following a priori estimate is valid:

$$\|u\|_{W^2_2(D_T)} \leq \sigma_1\tau\|F\|_{L_2(D_T)} + \tau\lambda\sigma_2\tau^3\|F\|^3_{L_2(D_T)} + \\ + \sigma_3\tau\|F\|_{W^1_2(D_T)} \exp\Big[\sigma_4 + \tau\lambda\sigma_5\tau^2\|F\|^2_{L_2(D_T)}\Big] \leq \sigma_1\|F\|_{L_2(D_T)} +$$

$$+\lambda\sigma_2\|F\|^3_{L_2(D_T)}+\sigma_3\|F\|_{W_2^1(D_T)}\exp\left[\sigma_4+\lambda\sigma_5\|F\|^2_{L_2(D_T)}\right]=C_0(\lambda,\sigma_i,F),$$

where $C_0=C_0(\lambda,\sigma_i,F)$ is a positive constant not depending on $u$ and on the parameter $\tau$.

Therefore, by the Leray–Schauder theorem ([36, p. 375]), the equation (58) and hence the problem (1), (3) has at least one strong generalized solution of the class $W_2^2$ in the domain $D_T$. Thus taking into account Remark 1 and Definitions 1, 2 and 3 of Section 2, the following theorem is valid.

**Theorem 1.** *Let $\lambda>0$, $F\in L_{2,loc}(D_\infty)$ and $F\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ for any $T>0$. Then the problem* (1), (3) *is globally solvable, i.e., for any $T>0$ this problem has a solution of the class $W_2^2$ in the domain $D_T$ in the sense of Definition* 1.

Suppose

$$\overset{\circ}{W}{}^k_{2,loc}(D_\infty,S_\infty)=\{v\in L_{2,loc}(D_\infty): v|_{D_T}\in \overset{\circ}{W}{}^k_2(D_T,S_T)\ \ \forall T>0\}.$$

In the next section we will prove the uniqueness of a solution of the problem (1), (3) of the class $W_2^2$ in the sense of Definition 1. This and Theorem 1 allow us to conclude that the theorem below is valid.

**Theorem 2.** *Let $\lambda>0$ and $F\in \overset{\circ}{W}{}^1_{2,loc}(D_\infty,S_\infty)$. Then the problem* (1), (3) *has in the light cone of the future $D_\infty$ a unique global solution $u$ from the space $\overset{\circ}{W}{}^2_{2,loc}(D_\infty,S_\infty)$ which satisfies the equation* (1) *almost everywhere in the domain $D_\infty$ and the boundary condition* (3) *in the sense of the trace theory.*

## 4. The Uniqueness of a Solution of the Problem (1), (3) of the Class $W_2^2$.

**Lemma 2.** *The problem* (1), (3) *fails to have more than one solution of the class $W_2^2$ in the sense of Definition* 1.

*Proof.* Let $u_1$ and $u_2$ be two solutions of the problem (1), (3) of the class $W_2^2$ in the domain $D_T$ in the sense of Definition 1. Then for the difference $u=u_2-u_1$ we have

$$(\Box+m^2)u=-3\lambda u_2^3+3\lambda u_1^3, \tag{59}$$

$$u,u_1,u_2\in \overset{\circ}{W}{}^2_2(D_T,S_T). \tag{60}$$

Multiplying both parts of the equation (59) by $u_t$ and integrating over the domain $D_\tau$ just in the same manner as when writing (14), we obtain

$$\int\limits_{\Omega_\tau}\left[u_t+\sum_{i=1}^3 u_{x_i}^2\right]dx+m^2\int\limits_{\Omega_\tau}u^2\,dx=$$

$$=-6\lambda\int\limits_{D_\tau}(u_2^3-u_1^3)u_t\,dx\,dt,\quad 0<\tau\le T. \tag{61}$$

Estimate the first part of the equality (61). We have

$$\Big| -6\lambda \int\limits_{D_\tau} (u_2^3-u_1^3)u_t\,dx\,dt\Big| = 6|\lambda|\,\Big|\int\limits_0^\tau d\sigma \int\limits_{\Omega_\sigma}(u_2^3-u_1^3)u_t\,dx\,dt\Big| \le$$

$$\le 6|\lambda|\int\limits_0^\tau d\sigma \int\limits_{\Omega_\sigma} |u_2-u_1||u_2^2+u_2u_1+u_1^2|\,|u_t|\,dx\,dt\Big| \le$$

$$\le 9|\lambda|\int\limits_0^\tau d\sigma \int\limits_{\Omega_\sigma}(u_2^2+u_1^2)|u|\,|u_t|\,dx\,dt. \tag{62}$$

Using Hölder's inequality for $p_1=6$, $p_2=3$, $p_3=2$ ($\frac{1}{6}+\frac{1}{3}+\frac{1}{2}=1$), we obtain

$$\int\limits_{\Omega_\sigma}(u_2^2+u_1^2)|u|\,|u_t|\,dx\,dt \le \big(\|u_2^2\|_{L_3(\Omega_\sigma)}+\|u_1^2\|_{L_3(\Omega_\sigma)}\big)\|u\|_{L_6(\Omega_\sigma)}\|u_t\|_{L_2(\Omega_\sigma)} =$$

$$= \big(\|u_2\|^2_{L_6(\Omega_\sigma)}+\|u_1\|^2_{L_6(\Omega_\sigma)}\big)\|u\|_{L_6(\Omega_\sigma)}\|u_t\|_{L_2(\Omega_\sigma)}, \quad 0<\sigma\le T. \tag{63}$$

By the imbedding theorem, we have ([31, pp. 69, 78])

$$\begin{aligned} &\|v|_{\Omega_\sigma}\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)} \le C(T)\|v\|_{W_2^2(D_T)} \quad (\dim\Omega_\sigma=3, \quad \dim D_T=4),\\ &\|v|_{\Omega_\sigma}\|_{L_6(\Omega_\sigma)} \le \beta\|v|_{\Omega_\sigma}\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)} \le \beta C(T)\|v\|_{W_2^2(D_T)}, \end{aligned} \tag{64}$$

where the positive constants $C(T)$ and $\beta$ do not depend on the parameter $\sigma\in(0,T]$ and on the function $v$.

By virtue of (60), it follows from (63) and (64) that

$$\int\limits_{\Omega_\sigma}(u_2^2+u_1^2)|u|\,|u_t|\,dx \le 2M\|u\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}\|u_t\|_{L_2(\Omega_\sigma)} \le$$

$$\le M\big(\|u\|^2_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}+\|u_t\|^2_{L_2(\Omega_\sigma)}\big) = M\int\limits_{\Omega_\tau}\Big[u_t^2+\sum_{i=1}^3 u_{x_i}^2\Big]dx, \tag{65}$$

where

$$M=\beta^3C^2(T)\max\big(\|u_1\|^2_{W_2^2(D_T)}, \|u_2\|^2_{W_2^2(D_T)}\big)<+\infty.$$

Assuming $w(\tau)=\int_{\Omega_\tau}\Big[u_t^2+\sum_{i=1}^3 u_{x_i}^2\Big]dx$, by (62)–(65) we have

$$w(\tau)\le 9|\lambda|M\int\limits_0^\tau w(\sigma)\,d\sigma,$$

whence by the Gronwall lemma we find that $w=0$, i.e., $u_t=u_{x_1}=u_{x_2}=u_{x_3}=0$. Consequently, $u=const$, and since by the condition of the lemma $u|_{S_T}=0$, we have $u=0$ and hence $u_2=u_1$, which proves the lemma. □

**5. The Absence of the Global Solvability of the Problem (1), (3) for $\lambda < 0$.** The following statement is valid: let $\lambda < 0$, $F = \mu F_0$, $F_0 \in C(\overline{D}_\infty)$, $\operatorname{supp} F_0 \cap S_\infty = \varnothing$, $F_0 \geq 0$, $F_0 \not\equiv 0$, $\mu = const > 0$. Then for any $T > 0$ there exists a number $\mu_0 = \mu_0(T) > 0$ such that for $\mu \geq \mu_0$ the problem (1), (3) fails to have a classical solution $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T) = \{u \in C^2(\overline{D}_T) : u|_{S_T} = 0\}$ in the domain $D_T$.

We omit here the proof of that statement because it repeats the proof of an analogous result for the wave equation with the power nonlinearity with two spatial variables [27] and relies essentially on the method of test functions [14].

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