Memoirs on Differential Equations and Mathematical Physics
Volume 41, 2007, 11–26

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ON THE SOLVABILITY OF THE CAUCHY PROBLEM FOR SYSTEMS OF TWO LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. The Cauchy problem for a system of two scalar non-Volterra linear functional differential equations is considered. Necessary and sufficient conditions of the solvability are obtained. The method which is used here can be applied to many other boundary value problems for functional differential equations with monotone operators. Some new results on the solvability are announced in the last part of the article.

2000 Mathematics Subject Classification. 34K10, 34K13, 34B05.
Key words and phrases. Cauchy problem, boundary value problem, functional differential equation, periodic problem.

რეზიუმე. ნაშრომში განხილულია კოშის ამოცანა ორი სკალარული არავოლტერული წრფივი ფუნქციონალურ-დიფერენციალური განტოლების სისტემისათვის. მიღებულია ამოხსნადობის აუცილებელი და საკმარისი პირობები. აქ მოყვანილი მეთოდი მონოტონური ოპერატორებიანი ფუნქციონალურ-დიფერენციალური განტოლებებისათვის ბევრი სხვა სასაზღვრო ამოცანისათვის შეიძლება იქნეს გამოყენებული. ნაშრომის უკანასკნელ ნაწილში დაანონსებულია რამდენიმე ახალი შედეგი ამოხსნადობის შესახებ.

In this article dedicated to the memory of Professor N. V. Azbelev, new results on solvability of the Cauchy problem for a system of two first-order functional differential equations will be obtained. Moreover, in the last part of the article, some new necessary and sufficient conditions of solvability for other problems will be given. Solvability of boundary value problems for functional differential equations was one on the favorite topics of Professor N. V. Azbelev. The method presented below can provide unimprovable conditions of solvability for many boundary value problems for functional differential equations with monotone operators.

1. We use the following notation.

$\mathbf{R}$ is the space of real numbers.

$[a, b] \subset \mathbf{R}$ is the finite closed interval $(-\infty < a < b < +\infty)$.

$\mathbf{C} = \mathbf{C}[a, b]$ is the space of continuous functions $x : [a, b] \mapsto \mathbf{R}$ with the norm $\| x\|_{\mathbf{C}} = \max_{t\in[a,b]} |x(t)|$.

$\mathbf{L} = \mathbf{L}[a, b]$ is the space of Lebesgue integrable functions $z : [a, b] \mapsto \mathbf{R}$ with the norm $\| z\|_{\mathbf{L}} = \int_a^b |z(t)|\, dt$.

All equalities and inequalities with functions from $\mathbf{L}$ are understood almost everywhere on $[a, b]$.

$\mathbf{W}^n = \mathbf{W}^n[a, b]$, $n \geqslant 1$, is the space of all functions $x : [a, b] \mapsto \mathbf{R}$ with absolutely continuous $(n-1)$-th derivative with the norm $\| x\|_{\mathbf{W}^n} = \sum_{i=0}^{n-1} |x^{(i)}(a)| + \int_a^b |x^{(n)}(t)|\, dt$; $\mathbf{W} = \mathbf{W}^1$.

$\mathcal{L}(\mathbf{X}, \mathbf{Y})$ is the linear space of all linear bounded operators acting from the space $\mathbf{X}$ into the space $\mathbf{Y}$. An operator T from $\mathcal{L}(\mathbf{C}, \mathbf{L})$ is called positive if $(Tx)(t) \geqslant 0$, $t \in [a, b]$, for every nonnegative function $x \in \mathbf{C}$. $\mathcal{L}^+ = \mathcal{L}^+(\mathbf{C}, \mathbf{L})$ is the set of all positive operators from $\mathcal{L}(\mathbf{C}, \mathbf{L})$.

If $T \in \mathcal{L}^+$, then

$$\| T\|_{\mathbf{C}\mapsto\mathbf{L}} = \int_a^b (T1)(s)\, ds.$$

2. Consider the boundary value problem for the system of two first-order functional differential equations

$$\begin{cases} \dot{x}(t) = (T_{11}x)(t) + (T_{12}y)(t) + f_1(t), & t \in [a, b], \\ \dot{y}(t) = (T_{21}x)(t) + (T_{22}y)(t) + f_2(t), & t \in [a, b], \\ \ell_1(x, y) = \alpha_1, \quad \ell_2(x, y) = \alpha_2, \end{cases} \tag{1}$$

where f_1, $f_2 \in \mathbf{L}$, α_1, $\alpha_2 \in \mathbf{R}$, $T_{ij} = T_{ij}^+ - T_{ij}^-$, T_{ij}^+, $T_{ij}^- \in \mathcal{L}^+$, $i, j = 1, 2$, $\ell_i \in \mathcal{L}(\mathbf{W} \times \mathbf{W}, \mathbf{R})$, $i = 1, 2$.

By a solution of (1) we mean a pair of functions $(x, y) \in \mathbf{W} \times \mathbf{W}$ such that the boundary conditions $\ell_i(x, y) = \alpha_i$, $i = 1, 2$, hold and the functional differential equations in (1) hold almost everywhere.

Problem (1) is called uniquely solvable if there exists a unique solution of (1) for every $f_i \in \mathbf{L}$, $\alpha_i \in \mathbf{R}$, $i = 1, 2$.

It is well known that the problem (1) has the Fredholm property [1]. Therefore, the following assertion holds.

Lemma 1. *Problem* (1) *is uniquely solvable if and only if the homogeneous problem*

$$\begin{cases} \dot{x} = T_{11}x + T_{12}y, \\ \dot{y} = T_{21}x + T_{22}y, \\ \ell_1(x,y) = 0, \quad \ell_2(x,y) = 0, \end{cases} \tag{2}$$

has only the trivial solution.

Now we present the main lemma. Its modifications can be useful for many other boundary value problems for functional differential equations with monotone operators.

Lemma 2. *If the problem* (2) *has a nontrivial solution, then there exist points* τ_1, τ_2, θ_1, $\theta_2 \in [a,b]$, *and functions* p_{ij}, $q_{ij} \in \mathbf{L}$ *such that*

$$\begin{aligned} -(T_{ij}^{-}1)(t) \leqslant p_{ij}(t), \quad q_{ij}(t) \leqslant (T_{ij}^{+}1)(t), \quad t \in [a,b], \quad i,j = 1,2, \\ p_{ij}(t) + q_{ij}(t) = (T_{ij}^{+}1)(t) - (T_{ij}^{-}1)(t), \quad t \in [a,b], \quad i,j = 1,2, \end{aligned} \tag{3}$$

and the boundary value problem

$$\begin{cases} \dot{x}(t) = p_{11}(t)x(\tau_1) + q_{11}(t)x(\tau_2) + p_{12}(t)y(\theta_1) + q_{12}(t)y(\theta_2), \quad t \in [a,b], \\ \dot{y}(t) = p_{21}(t)x(\tau_1) + q_{21}(t)x(\tau_2) + p_{22}(t)y(\theta_1) + q_{22}(t)y(\theta_2), \quad t \in [a,b], \\ \ell_1(x,y) = 0, \quad \ell_2(x,y) = 0, \end{cases} \tag{4}$$

has a nontrivial solution.

Proof. Suppose the problem (2) has a nontrivial solution (x,y) and

$$\begin{aligned} \min_{t \in [a,b]} x(t) = x(\tau_1), \quad \max_{t \in [a,b]} x(t) = x(\tau_2), \\ \min_{t \in [a,b]} y(t) = y(\theta_1), \quad \max_{t \in [a,b]} y(t) = y(\theta_2). \end{aligned}$$

Using the inequalities

$$x(\tau_1)\,1 \leqslant x(t) \leqslant x(\tau_2)\,1 \quad y(\theta_1)\,1 \leqslant y(t) \leqslant y(\theta_2)\,1, \quad t \in [a,b],$$

for the positive operators T_{ij}^{+}, T_{ij}^{-} we get the inequalities

$$\begin{aligned} T_{11}^{+}1x(\tau_1) - T_{11}^{-}1x(\tau_2) + T_{12}^{+}1y(\theta_1) - T_{12}^{-}1y(\theta_2) \leqslant \\ \leqslant \dot{x} \leqslant T_{11}^{+}1x(\tau_2) - T_{11}^{-}1x(\tau_1) + T_{12}^{+}1y(\theta_2) - T_{12}^{-}1y(\theta_1) \end{aligned}$$

and

$$\begin{aligned} T_{21}^{+}1x(\tau_1) - T_{21}^{-}1x(\tau_2) + T_{22}^{+}1y(\theta_1) - T_{22}^{-}1y(\theta_2) \leqslant \\ \leqslant \dot{y}(t) \leqslant T_{21}^{+}1x(\tau_2) - T_{21}^{-}1x(\tau_1) + T_{22}^{+}1y(\theta_2) - T_{22}^{-}1y(\theta_1). \end{aligned}$$

Then for some function ζ such that $\zeta(t) \in [0,1]$, $t \in [a,b]$, we have the equality

$$\begin{aligned}\dot{x} &= (1-\zeta)\left(T_{11}^{+}1x(\tau_1) - T_{11}^{-}1x(\tau_2) + T_{12}^{+}1y(\theta_1) - T_{12}^{-}1y(\theta_2)\right)+\\ &+\zeta\left(T_{11}^{+}1x(\tau_2) - T_{11}^{-}1x(\tau_1) + T_{12}^{+}1y(\theta_2) - T_{12}^{-}1y(\theta_1)\right) =\\ &= p_{11}x(\tau_1) + q_{11}x(\tau_2) + p_{12}y(\theta_1) + q_{12}y(\theta_2),\end{aligned}$$

where

$$\begin{aligned}p_{11} &= (1-\zeta)T_{11}^{+}1 - \zeta T_{11}^{-}1, \quad q_{11} = \zeta T_{11}^{+}1 - (1-\zeta)T_{11}^{-}1,\\ p_{12} &= (1-\zeta)T_{12}^{+}1 - \zeta T_{12}^{-}1, \quad q_{12} = \zeta T_{12}^{+}1 - (1-\zeta)T_{11}^{-}1.\end{aligned} \tag{5}$$

Besides, for some function ξ such that $\xi(t) \in [0,1]$, $t \in [a,b]$, we have the equality

$$\begin{aligned}\dot{y} &= (1-\xi)\left(T_{21}^{+}1x(\tau_1) - T_{21}^{-}1x(\tau_2) + T_{22}^{+}1y(\theta_1) - T_{22}^{-}1y(\theta_2)\right)+\\ &+\xi\left(T_{21}^{+}1x(\tau_2) - T_{21}^{-}1x(\tau_1) + T_{22}^{+}1y(\theta_2) - T_{22}^{-}1y(\theta_1)\right) =\\ &= p_{21}x(\tau_1) + q_{21}x(\tau_2) + p_{22}y(\theta_1) + q_{22}y(\theta_2),\end{aligned}$$

where

$$\begin{aligned}p_{21} &= (1-\xi)T_{21}^{+}1 - \xi T_{21}^{-}1, \quad q_{21} = \xi T_{21}^{+}1 - (1-\xi)T_{21}^{-}1,\\ p_{22} &= (1-\xi)T_{22}^{+}1 - \xi T_{22}^{-}1, \quad q_{22} = \xi T_{22}^{+}1 - (1-\xi)T_{21}^{-}1.\end{aligned} \tag{6}$$

There exist functions $\zeta(t) \in [0,1]$ and $\xi(t) \in [0,1]$ such that (5) and (6) hold if and only if the conditions (3) are fulfilled.

So, the problem (4) has a nontrivial solution (x,y) and the conditions (3) hold. □

The problem (4) is finite-dimensional. Hence it can be solved explicitly. For the Cauchy problem ($\ell_1(x,y) = x(a)$, $\ell_1(x,y) = y(a)$) we obtain the following necessary and sufficient conditions guaranteeing the solvability.

Theorem 3. *The Cauchy problem*

$$\begin{cases}\dot{x} = T_{11}x + T_{12}y + f_1,\\ \dot{y} = T_{21}x + T_{22}y + f_2,\\ x(a) = \alpha_1, \quad y(a) = \alpha_2,\end{cases} \tag{7}$$

is uniquely solvable for all operators $T_{ij} = T_{ij}^{+} - T_{ij}^{-}$, *where the operators* $T_{ij}^{+}, T_{ij}^{-} \in \mathcal{L}^{+}$, $i,j = 1,2$, *satisfy the inequalities*

$$\begin{aligned}&\| T_{11}^{+}\| \leqslant A^{+}, \quad \| T_{11}^{-}\| \leqslant A^{-}, \quad \| T_{12}^{+}\| \leqslant C^{+}, \quad \| T_{12}^{-}\| \leqslant C^{-},\\ &\| T_{21}^{+}\| \leqslant D^{+}, \quad \| T_{21}^{-}\| \leqslant D^{-}, \quad \| T_{22}^{+}\| \leqslant B^{+}, \quad \| T_{22}^{-}\| \leqslant B^{-},\end{aligned}$$

if and only if

$$\triangle_1 = \begin{vmatrix}1-a_1^{+} & a_1^{-} & -c_1^{+} & c_1^{-}\\ a_2^{-}-1 & 1-a_2^{+} & c_2^{-} & -c_2^{+}\\ -d_1^{+} & d_1^{-} & 1-b_1^{+} & b_1^{-}\\ d_2^{-} & -d_2^{+} & b_2^{-}-1 & 1-b_2^{+}\end{vmatrix} > 0$$

and

$$\triangle_2 = \begin{vmatrix} 1-a_1^+ & a_1^- & -c_1^+ & c_1^- \\ a_2^- - 1 & 1-a_2^+ & c_2^- & -c_2^+ \\ -d_2^+ & d_2^- & 1-b_2^+ & b_2^- - 1 \\ d_1^- & -d_1^+ & b_1^- & 1-b_1^+ \end{vmatrix} > 0$$

for all numbers a_i^+, a_i^-, b_i^+, b_i^-, c_i^+, c_i^-, d_i^+, d_i^-, $i = 1,2$, *such that*

$$\begin{gathered} a_1^+ + a_2^+ \leqslant A^+, \quad a_1^- + a_2^- \leqslant A^-, \\ b_1^+ + b_2^+ \leqslant B^+, \quad b_1^- + b_2^- \leqslant B^-, \\ c_1^+ + c_2^+ \leqslant C^+, \quad c_1^- + c_2^- \leqslant C^-, \\ d_1^+ + d_2^+ \leqslant D^+, \quad d_1^- + d_2^- \leqslant D^-, \\ a_i^+, a_i^-, b_i^+, b_i^-, c_i^+, c_i^-, d_i^+, d_i^- \geqslant 0, \quad i = 1,2. \end{gathered}$$

The proof of Theorem 3 is based on Lemmas 1 and 2.

Remark 4. The problem (7) is uniquely solvable for all $T_{ij} = T_{ij}^+ - T_{ij}^-$, where the operators $T_{ij}^+, T_{ij}^- \in \mathcal{L}^+$, $i,j = 1,2$, satisfy the inequalities

$$\begin{gathered} \|T_{11}^+\| \leqslant A^+, \quad \|T_{11}^-\| \leqslant A^-, \quad \|T_{12}^+\| \leqslant C^+, \quad \|T_{12}^-\| \leqslant C^-, \\ \|T_{21}^+\| \leqslant D^+, \quad \|T_{21}^-\| \leqslant D^-, \quad \|T_{22}^+\| \leqslant B^+, \quad \|T_{22}^-\| \leqslant B^-, \end{gathered}$$

if and only if the problem (7) is uniquely solvable for all $T_{ij} = T_{ij}^+ - T_{ij}^-$, where the operators T_{ij}^+, $T_{ij}^- \in \mathcal{L}^+$, $i,j = 1,2$, satisfy the equalities

$$\begin{gathered} \|T_{11}^+\| = A^+, \quad \|T_{11}^-\| = A^-, \quad \|T_{12}^+\| = C^+, \quad \|T_{12}^-\| = C^-, \\ \|T_{21}^+\| = D^+, \quad \|T_{21}^-\| = D^-, \quad \|T_{22}^+\| = B^+, \quad \|T_{22}^-\| = B^-. \end{gathered}$$

3. Now we consider a particular case of the problem (7):

$$\begin{cases} \dot{x} = T_{11}x + T_{12}y + f_1, \\ \dot{y} = -T_{21}x + T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2, \end{cases} \tag{8}$$

where $T_{ij} \in \mathcal{L}^+$, $i,j = 1,2$.

The conditions guaranteeing the unique solvability of this problem are obtained in [27]. Here we will improve that result and obtain necessary and sufficient conditions for the problem (8) to be uniquely solvable.

As it is shown in [27], the problem (8) is uniquely solvable if and only if the problem

$$\begin{cases} \dot{x} = T_{11}x - T_{12}y + f_1, \\ \dot{y} = T_{21}x + T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2 \end{cases} \tag{9}$$

is uniquely solvable.

The application of Theorem 3 yields necessary and sufficient conditions of the unique solvability.

Lemma 5. *The Cauchy problems* (8) *and* (9) *are uniquely solvable for all operators* $T_{ij} \in \mathcal{L}^+$, $i,j = 1,2$, *such that*

$$\|T_{11}\| \leqslant A, \quad \|T_{12}\| \leqslant C, \quad \|T_{21}\| \leqslant D, \quad \|T_{22}\| \leqslant B,$$

if and only if

$$\triangle_1 = \begin{vmatrix} 1-a_1 & 0 & -c_1 & 0 \\ -1 & 1-a_2 & 0 & c_2 \\ 0 & d_1 & 1-b_1 & 0 \\ d_2 & 0 & -1 & 1-b_2 \end{vmatrix} > 0$$

and

$$\triangle_2 = \begin{vmatrix} 1-a_1 & 0 & -c_1 & 0 \\ -1 & 1-a_2 & 0 & c_2 \\ 0 & d_1 & 1-b_1 & -1 \\ d_2 & 0 & 0 & 1-b_2 \end{vmatrix} > 0$$

for all nonnegative numbers a_i, b_i, c_i, d_i, $i = 1,2$, *satisfying the inequalities*

$$a_1 + a_2 \leqslant A, \quad b_1 + b_2 \leqslant B, \quad c_1 + c_2 \leqslant C, \quad d_1 + d_2 \leqslant D.$$

Theorem 6. *The Cauchy problems* (8), (9) *are uniquely solvable for all positive operators* T_{ij}, $i,j = 1,2$, *such that*

$$\|T_{11}\| \leqslant A, \quad \|T_{12}\| \leqslant C, \quad \|T_{21}\| \leqslant D, \quad \|T_{22}\| \leqslant B,$$

if and only if

$$A < 1, \quad B < 1, \quad CD < \left(\sqrt[3]{1-A} + \sqrt[3]{1-B}\right)^3.$$

Proof. Using Lemma 5, we get

$$\triangle_1 = (1-a_1)(1-a_2)(1-b_1)(1-b_2) + d_1\big(c_2(1-a_2) + c_1(1-b_2)\big) - c_1c_2d_1d_2.$$

If $A \geqslant 1$, then $\triangle_1 \leqslant 0$ for $a_1 = A$, $a_2 = b_1 = b_2 = d_1 = 0$. Therefore, the inequality $A < 1$ (and, similarly, $B < 1$) is necessary for the inequality $\min \triangle_1 > 0$. Further, the coefficients in $\triangle_1$ at b_2, a_2, and d_2 are nonpositive. Hence $\triangle_1$ takes its minimum value for maximum possible values of a_2, b_2, and d_2: $a_2 = A - a_1$, $b_2 = B - b_1$, $d_2 = D - d_1$.

Now, the coefficients at a_1^2 and b_1^2 in the expression

$$\begin{aligned} \triangle_1 &= (1-a_1)(1-A+a_1)(1-b_1)(1-B+b_1) + \\ &\quad + d_1\big(c_2(1-A+a_1) + c_1(1-B+b_1)\big) - c_1c_2d_1(D-d_1) \end{aligned}$$

are negative. So, the function $\triangle_1$ takes its minimum value for a_1, b_1 at the ends of the intervals $[0,A]$, $[0,B]$ correspondently. Choosing the minimum of these four values, we get

$$\triangle_1 = (1-A)(1-B) + d_1\big(c_2(1-A) + c_1(1-B)\big) - c_1c_2d_1(D-d_1).$$

Since $\triangle_1$ is a linear function of c_2, then the minimum value can be taken either for $c_2 = 0$ (then $\triangle_1 > 0$) or for $c_2 = C - c_1$. Let $C > 0$ and $c_2 = C - c_1$. Obviously, $\triangle_1 > 0$ for $d_1 = 0$ or for $d_1 = D$. Let $d_1 \in (0, D)$.

Minimizing $\triangle_1$ over c_1, we get that $\triangle_1 > 0$ for $(D - d_1)C \leqslant |B - A|$, and for $(D - d_1)C > |B - A|$ the expression $\triangle_1$ takes its minimum value at

$$c_1 = \frac{1}{2}\Big(C - \frac{A - B}{D - d_1}\Big).$$

Then

$$\triangle_1 = (1 - A)(1 - B) + d_1\Big(C\frac{2 - A - B}{2} - \frac{(A - B)^2}{2(D - d_1)}\Big) - \\ - \frac{1}{4}\Big(C^2 - \frac{(A - B)^2}{(D - d_1)^2}\Big) d_1(D - d_1).$$

We need conditions under which $\min\triangle_1 > 0$. It is more convenient to find conditions under which $\min(\triangle_1(D - d_1)) > 0$. We have

$$2(D - d_1)\triangle_1 = 2(1 - A)(1 - B)(D - d_1) + \\ + d_1\big(C(2 - A - B)(D - d_1) - (A - B)^2\big) - \frac{d_1}{2}\big(C^2(D - d_1)^2 - (A - B)^2\big).$$

Finding the zeros of the derivative of $(D - d_1)\triangle_1$ by the variable d_1, we obtain that the minimum is taken at

$$d_1 = \frac{1}{3}\Big(D - \frac{2 - A - B}{C}\Big). \tag{10}$$

It is clear that $d_1 \in (0, D)$ if and only if $CD > 2 - A - B$. In this case the condition $(D - d_1)C \geqslant |B - A|$ is fulfilled. If $d_1 < 0$ (that is, if $CD \leqslant 2 - A - B$), then the minimum value of $(D - d_1)\triangle_1$ is positive.

So, let $CD > 2 - A - B$ and d_1 be defined by (10). Then after some transformations we obtain

$$(D - d_1)\triangle_1 = \frac{-1}{27C}\big((CD - (2 - A - B))^3 - 27\,CD(1 - A)(1 - B)\big).$$

The real root of the cubic equation

$$\big(CD - (2 - A - B)\big)^3 - 27\,CD(1 - A)(1 - B) = 0$$

with respect to the product CD can be found explicitly. It is easy to prove that the unique positive root of this equation is

$$CD = M \stackrel{\text{def}}{=} \big(\sqrt[3]{1 - A} + \sqrt[3]{1 - B}\big)^3.$$

Therefore, $\min\triangle_1 > 0$ if and only if $CD < M$. In a similar way we obtain that $\min\triangle_2 > 0$ if and only if $CD < M$, $A < 1$, $B < 1$. This completes the proof. □

Remark 7. The exact value of the upper bound for the product CD in the solvability conditions of the problems (8), (9) confirms the good quality of the approximate value

$$CD < 4\sqrt{(1 - A)(1 - B)} + \big(\sqrt{1 - A} + \sqrt{1 - B}\big)^2,$$

which is found in [27]. Indeed, the difference between the exact value and the approximate one does not exceed 0.18 for all $A, B \in [0, 1)$. Also, as it is shown in [27], this estimate is exact for $A = B$.

4. In the last part of the article we include some results on solvability of various boundary value problems for various functional differential equations (without proofs). All proofs are based on some analogues of Lemma 2 and will be presented in forthcoming articles.

The following theorems 8–15 are dealing with particular cases of the boundary value problem for the second order functional differential equation

$$\begin{cases} x^{(n)}(t) = (T^+x)(t) - (T^-x)(t) + (Q^+\dot{x})(t) - (Q^-\dot{x})(t) + f(t), \ t \in [a, b], \\ \ell_i x = \alpha_i, \ \ i = 1, \dots, n, \end{cases} \tag{11}$$

where $n = 2$ or $n = 3$, T^+, T^-, Q^+, $Q^- \in \mathcal{L}^+$, $\ell_i \in \mathcal{L}(\mathbf{W}^n, \mathbf{R})$, $i = 1, \dots, n$.

Such problems are considered in [17]–[26].

The problem (11) is called uniquely solvable if for every $f \in \mathbf{L}$ and $\alpha_i \in \mathbf{R}$, $i = 1, 2$, there exists a unique function $x \in \mathbf{W}^n$ such that the functional differential equation in (11) is fulfilled almost everywhere and the boundary conditions $\ell_i x = \alpha_i$, $i = 1, 2$, hold.

Theorem 8. *The periodic boundary value problem*

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \ \ t \in [a, b], \\ x(b) - x(a) = \alpha_1, \ \ \ \dot{x}(a) - \dot{x}(b) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T^+, T^- \in \mathcal{L}^+(\mathbf{C}, \mathbf{L})$ *such that* $\| T^+ \| = \mathcal{T}^+$, $\| T^- \| = \mathcal{T}^-$ *if and only if either*

$$\mathcal{T}^- < \mathcal{T}^+, \ \ \ 0 \leqslant \mathcal{T}^- \leqslant \frac{3}{L},$$

$$\frac{\mathcal{T}^-}{1 - \frac{L\mathcal{T}^-}{4}} \leqslant \mathcal{T}^+ \leqslant \frac{1}{L}(8 + 4\sqrt{4 - L\mathcal{T}^-})$$

or

$$\mathcal{T}^+ < \mathcal{T}^-, \ \ \ 0 \leqslant \mathcal{T}^+ \leqslant \frac{3}{L},$$

$$\frac{\mathcal{T}^+}{1 - \frac{L\mathcal{T}^+}{4}} \leqslant \mathcal{T}^- \leqslant \frac{1}{L}(8 + 4\sqrt{4 - L\mathcal{T}^+}),$$

where $L \equiv b - a$.

Theorem 9. *The periodic boundary value problem*

$$\begin{cases} \dddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \ \ t \in [a, b], \\ x(b) - x(a) = \alpha_1, \ \ \ \dot{x}(a) - \dot{x}(b) = \alpha_2, \ \ \ \ddot{x}(a) - \ddot{x}(b) = \alpha_3, \end{cases}$$

is uniquely solvable for all operators T^+, $T^- \in \mathcal{L}^+$ such that $\| T^+\| = \mathcal{T}^+$, $\| T^-\| = \mathcal{T}^-$ if and only if either

$$\mathcal{T}^- < \mathcal{T}^+, \quad 0 \leqslant \mathcal{T}^- \leqslant \frac{24}{L^2}\,,$$
$$\frac{\mathcal{T}^-}{1 - \frac{L^2\,\mathcal{T}^-}{32}} \leqslant \mathcal{T}^+ \leqslant \frac{64}{L^2}\Big(1 + \sqrt{1 - \frac{L^2\,\mathcal{T}^-}{32}}\Big)$$

or

$$\mathcal{T}^+ < \mathcal{T}^-, \quad 0 \leqslant \mathcal{T}^+ \leqslant \frac{24}{L^2}\,,$$
$$\frac{\mathcal{T}^+}{1 - \frac{L^2\,\mathcal{T}^+}{32}} \leqslant \mathcal{T}^- \leqslant \frac{64}{L^2}\Big(1 + \sqrt{1 - \frac{L^2\,\mathcal{T}^+}{32}}\Big),$$

where $L \equiv b - a$.

Theorem 10. *The antiperiodic boundary value problem*

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), & t \in [a, b], \\ x(b) + x(a) = \alpha_1, \quad \dot{x}(a) + \dot{x}(b) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T^+, T^- \in \mathcal{L}^+$ such that $\| T^+\| \leqslant \mathcal{T}^+$, $\| T^-\| \leqslant \mathcal{T}^-$ if and only if

$$\mathcal{T}^- + \mathcal{T}^+ \leqslant \frac{4}{b - a}\,.$$

Theorem 11. *The boundary value problem*

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (Q^-\dot{x})(t) + f(t), & t \in [0, 1], \\ x(0) = \alpha_1, \quad x(1) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T^+, Q^- \in \mathcal{L}^+$ such that

$$\| T^+\| \leqslant \mathcal{T}^+, \quad \| Q^-\| \leqslant \mathcal{Q}^-$$

if and only if

$$\mathcal{Q}^- \leqslant 1, \quad \mathcal{T}^+ \leqslant 8\big(1 + \sqrt{1 - \mathcal{Q}^-}\big).$$

Theorem 12. *The boundary value problem*

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) + (Q^+\dot{x})(t) + f(t), & t \in [a, b], \\ x(a) = \alpha_1, \quad x(b) = \alpha_2, \end{cases} \tag{12}$$

is uniquely solvable for all operators $T^+, Q^+ \in \mathcal{L}^+$ *such that* $\|T^+\| \leqslant \mathcal{T}^+$, $\|Q^+\| \leqslant \mathcal{Q}^+$ *if and only if*

$$\mathcal{Q}^+ \leqslant 1,$$

$$\mathcal{T}^+ \leqslant 1 + \frac{2(1+\sqrt{1+\mathcal{Q}^+})}{\mathcal{Q}^+} \quad \text{for } \mathcal{Q}^+ \geqslant \mathcal{Q}_0^+,$$

$$\mathcal{T}^+ \leqslant \frac{(3+(3+\mathcal{Q}^+)\sqrt{1-2\mathcal{Q}^+})(4+\mathcal{Q}^+)^3}{12+8\mathcal{Q}^+ + 13\mathcal{Q}^{+2} + 8\mathcal{Q}^{+3} + (12+8\mathcal{Q}^+ - \mathcal{Q}^{+2})(1+\mathcal{Q}^+)\sqrt{1-2\mathcal{Q}^+}}\,, \quad \text{for } \mathcal{Q}^+ \leqslant \mathcal{Q}_0^+,$$

where $\mathcal{Q}_0^+ \approx 0.3157$ *is the unique solution of the equation*

$$1 + \frac{2(1+\sqrt{1+q})}{q} = T_0(q)$$

on the interval $q \in (0, 1/2]$.

Corollary 13. *The problem* (12) *is uniquely solvable for all operators* T^+, $Q^+ \in \mathcal{L}^+$ *such that* $\|T^+\| \leqslant \mathcal{T}^+$, $\|Q^+\| \leqslant \mathcal{Q}^+$ *if*

$$\mathcal{T}^+ \leqslant 12 + 4\sqrt{1-2\,\mathcal{Q}^+} \quad \text{for } \mathcal{Q}^+ \leqslant \mathcal{Q}_0^+,$$

where $\mathcal{Q}_0^+ \approx 0.3157$ *is the unique solution of the equation*

$$1 + \frac{2\,(1+\sqrt{1+q})}{q} = T_0(q)$$

on the interval $q \in (0, 1/2]$.

Theorem 14. *The boundary value problem*

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), & t \in [a,b], \\ x(a) = \alpha_1, \quad x(b) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T^+, T^- \in \mathcal{L}^+$ *such that* $\|T^+\| \leqslant \mathcal{T}^+$, $\|T^-\| \leqslant \mathcal{T}^-$ *if and only if*

$$\mathcal{T}^- \leqslant \frac{4}{L}, \quad \mathcal{T}^+ \leqslant \frac{8}{L}\Big(1 + \sqrt{1 - \frac{L\mathcal{T}^-}{4}}\Big),$$

where $L \equiv b - a$.

Theorem 15. *The boundary value problem*

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), & t \in [a,b], \\ x(a) = \alpha_1, \quad \dot{x}(b) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T^+, T^- \in \mathcal{L}^+$ *such that* $\|T^+\| \leqslant \mathcal{T}^+$, $\|T^-\| \leqslant \mathcal{T}^-$ *if and only if*

$$\mathcal{T}^- \leqslant \frac{1}{L}, \quad \mathcal{T}^+ \leqslant \frac{4}{L}\big(1 + \sqrt{1 - L\,\mathcal{T}^-}\big),$$

where $L \equiv b - a$.

In Theorem 16 a boundary value problem for a singular functional differential equation is considered in a special Banach space defined, for example, in [2], [16]. Such problems are investigated in [3], [4].

Theorem 16. *The singular boundary value problem*

$$\begin{cases} (t-a)\,\ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [a,b], \\ x(a) = \alpha_1, \quad \dot{x}(b) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T^+, T^- \in \mathcal{L}^+$ *such that* $\|\,T^+\| \leqslant \mathcal{T}^+$, $\|\,T^-\| \leqslant \mathcal{T}^-$ *if and only if*

$$\mathcal{T}^- \leqslant 1, \qquad \mathcal{T}^+ \leqslant 1 + 2\sqrt{1-\mathcal{T}^-}.$$

In Theorem 17 we obtain a result on solvability of the generalized Cauchy problem for scalar functional differential equations. The Cauchy problem and similar problems for scalar functional differential equations are considered in [6]–[13], [30].

Theorem 17. *The generalized Cauchy problem*

$$\begin{cases} \dot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [a,b], \\ x(c) = \alpha, \end{cases} \tag{13}$$

where $c \in [a,b]$, *is uniquely solvable for all operators* $T^+, T^- \in \mathcal{L}^+$ *such that*

$$\int_a^c (T^+1)(s)\,ds \leqslant \mathcal{T}_a^+, \quad \int_c^b (T^+1)(s)\,ds \leqslant \mathcal{T}_b^+,$$

$$\int_a^c (T^-1)(s)\,ds \leqslant \mathcal{T}_a^-, \quad \int_c^b (T^-1)(s)\,ds \leqslant \mathcal{T}_b^-$$

if and only if

$$\Psi(\mathcal{T}_b^-) < 1 - \mathcal{T}_b^+, \quad \Psi(\mathcal{T}_a^+) < 1 - \mathcal{T}_b^-, \quad \mathcal{T}_a^+\,\mathcal{T}_b^- < (1-\mathcal{T}_b^+)\,(1-\mathcal{T}_a^-),$$

where

$$\Psi(z) = \begin{cases} 0 & \text{if } z \leqslant 1, \\ \dfrac{1}{4}\,(z-1)^2 & \text{if } z > 1. \end{cases}$$

In the last four theorems we present results on the solvability of boundary value problems for systems of functional differential equations. Such problems are considered in [5], [14], [15], [27]–[29].

Theorem 18. *The antiperiodic boundary value problem*

$$\begin{cases} \dot{x} = \sigma_1 T_{12} y + f_1, \\ \dot{y} = \sigma_2 T_{21} x + f_2, \\ x(a) + x(b) = \alpha_1, \quad y(a) + y(b) = \alpha_2, \end{cases}$$

where σ_1, $\sigma_2 \in \{-1, 1\}$, *is uniquely solvable for all operators* $T_{12}, T_{21} \in \mathcal{L}^+$ *such that* $\| T_{12}\| \leqslant C$, $\| T_{21}\| \leqslant D$ *if and only if*

$$C\,D < 4.$$

Theorem 19. *The Cauchy problems*

$$\begin{cases} \dot{x} = -T_{11}x + T_{12}y + f_1, \\ \dot{y} = \quad T_{21}x - T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2, \end{cases}$$

and

$$\begin{cases} \dot{x} = -T_{11}x - T_{12}y + f_1, \\ \dot{y} = -T_{21}x - T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2, \end{cases}$$

are uniquely solvable for all operators $T_{ij} \in \mathcal{L}^+$, $i, j = 1, 2$, *such that*

$$\| T_{11}\| \leqslant A, \quad \| T_{12}\| \leqslant C, \quad \| T_{21}\| \leqslant D, \quad \| T_{22}\| \leqslant B$$

if and only if

$$A < 3, \quad B < 3, \quad C\,D < \varphi(A)\,\varphi(B),$$

where

$$\varphi(t) = \begin{cases} 1 & \text{if } t \in [0, 1], \\ 1 - \dfrac{(t-1)^2}{4} & \text{if } t \in (1, 3]. \end{cases}$$

Theorem 20. *Let* $\max(A, B) \geqslant \frac{1}{5}$. *The Cauchy problems*

$$\begin{cases} \dot{x} = -T_{11}x - T_{12}y + f_1, \\ \dot{y} = \quad T_{21}x - T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2 \end{cases} \tag{14}$$

and

$$\begin{cases} \dot{x} = -T_{11}x + T_{12}y + f_1, \\ \dot{y} = -T_{21}x - T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2 \end{cases} \tag{15}$$

are uniquely solvable for all operators $T_{ij} \in \mathcal{L}^+$, $i, j = 1, 2$, *such that*

$$\| T_{11}\| \leqslant A, \quad \| T_{12}\| \leqslant C, \quad \| T_{21}\| \leqslant D, \quad \| T_{22}\| \leqslant B,$$

if and only if

$$A < 3, \quad B < 3, \quad C\,D < \varphi\big(\min(A, B))\psi(\max(A, B)\big),$$

where

$$\varphi(t) = \begin{cases} 1 & \text{if } t \in [0, 1], \\ 1 - \frac{(t-1)^2}{4} & \text{if } t \in (1, 3], \end{cases} \qquad \psi(t) = \begin{cases} 1 + 1/t & \text{if } t \in (0, 1], \\ 3 - t & \text{if } t \in [1, 3]. \end{cases}$$

If $0 < \max(A, B) < \frac{1}{4}$, *then the inequality*

$$CD < 8 - 16\,\max(A, B)$$

is sufficient for the unique solvability of the problems (14), (15).

Theorem 21. *The Cauchy problem*

$$\begin{cases} \dot{x} = T_{11}x + T_{12}y + f_1, \\ \dot{y} = T_{21}x + T_{22}y + f_2, \\ x(a) = \alpha_1, \quad y(a) = \alpha_2, \end{cases}$$

is uniquely solvable for all operators $T_{ij} = T_{ij}^{+} - T_{ij}^{-}$, $T_{ij}^{+}, T_{ij}^{-} \in \mathcal{L}^{+}$, $i, j = 1, 2$, *such that*

$$\begin{aligned} &\| T_{11}^{+} \| \leqslant A^{+}, \quad \| T_{11}^{-} \| \leqslant A^{-}, \quad \| T_{12}^{+} \| \leqslant C^{+}, \quad \| T_{12}^{-} \| \leqslant C^{-}, \\ &\| T_{21}^{+} \| \leqslant D^{+}, \quad \| T_{21}^{-} \| \leqslant D^{-}, \quad \| T_{22}^{+} \| \leqslant B^{+}, \quad \| T_{22}^{-} \| \leqslant B^{-} \end{aligned}$$

if and only if

$$\begin{aligned} &A^{+} < 1, \quad A^{-} < 1 + 2\sqrt{1 - A^{+}}, \quad B^{+} < 1, \quad B^{-} < 1 + 2\sqrt{1 - B^{+}}, \\ &C^{+}D^{+} < M_1(A^{+}, A^{-}, B^{+}, B^{-}), \quad C^{-}D^{-} < M_1(A^{+}, A^{-}, B^{+}, B^{-}), \\ &C^{-}D^{+} < M_2(A^{+}, A^{-}, B^{+}, B^{-}), \quad C^{+}D^{-} < M_2(A^{+}, A^{-}, B^{+}, B^{-}), \end{aligned}$$

where the functions M_1, M_2 *are defined by the equalities*

$$\begin{aligned} M_1(A^{+}, A^{-}, B^{+}, B^{-}) &= (\varphi(A^{-}) - A^{+})(\varphi(B^{-}) - B^{+}), \\ M_2(A^{+}, A^{-}, B^{+}, B^{-}) &= \inf \Big(k + \frac{1}{k} + m + \frac{1}{m}\Big)\big(R + \sqrt{R^2 + 4S}\big), \end{aligned} \tag{16}$$

where

$$\varphi(t) = \begin{cases} 1 & \text{if } t \in [0, 1], \\ 1 - \frac{(t-1)^2}{4} & \text{if } t \in (1, 3], \end{cases}$$

$$R = \big((1 - a_1^{+})(1 - a_2^{+}) - a_1^{-}(a_2^{-} - 1)\big)\big((1 - b_1^{+})(1 - b_2^{+}) - b_1^{-}(b_2^{-} - 1)\big),$$

$$S = \frac{(1 - a_1^{+})(1 - b_2^{-})}{k} - k\, b_1^{-}(1 - a_1^{+}) + \frac{(1 - a_2^{-})(1 - b_1^{+})}{m} - m\, a_1^{-}(1 - b_2^{+}),$$

the lower bound in (16) *is taken over all* $k, m > 0$ *and over all nonnegative* a_i^{+}, a_i^{-}, b_i^{+}, b_i^{-}, $i = 1, 2$, *such that* $a_1^{+} + a_2^{+} \leqslant A^{+}$, $a_1^{-} + a_2^{-} \leqslant A^{-}$, $b_1^{+} + b_2^{+} \leqslant B^{+}$, $b_1^{-} + b_2^{-} \leqslant B^{-}$.

Remark 22. The function M_2 defined in (16) is computed in Theorems 3, 20 and in [14], [27–29] for some particular cases.

Acknowledgement

The work is supported by the Russian Foundation for Basic Research, (grants # 06-01-00744 and # 07-01-96060).

References

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, Introduction to the theory of functional-differential equations. (Russian) *Nauka, Moscow*, 1991.
2. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, The elements of the contemporary theory of functional differential equations. (Russian) *Methods and Applications. Institute of Computer Science*, 2002.
3. N. V. Azbelev, M. Zh. Alvesh, E. I. Bravyĭ, On singular boundary value problems for a second-order linear functional-differential equation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1999, No. 2, 3–11; English transl.: *Russian Math.* (*Iz. VUZ*) **43**(1999), No. 2, 1–9.
4. E. I. Bravyi, A note on the limit properties of Green functions of boundary value problems for singular functional differential equations of the second order. *Funct. Differ. Equ.* **13**(2006), No. 3-4, 371–380.
5. N. Dilnaya and A. Ronto, Multistage iterations and solvability of linear Cauchy problems. *Miskolc Math. Notes* **4**(2003), No. 2, 89–102.
6. R. Hakl and A. Lomtatidze, A note on the Cauchy problem for first order linear differential equations with a deviating argument. *Arch. Math.* (*Brno*) **38**(2002), no. 1, 61–71.
7. R. Hakl, A. Lomtatidze, and B. Půža, On nonnegative solutions of first order scalar functional differential equations. *Mem. Differential Equations Math. Phys.* **23**(2001), 51–84.
8. R. Hakl, A. Lomtatidze, and B. Půža, On periodical solutions of first order nonlinear fuctional differential equations of non-volterra's type. *Mem. Differential Equations Math. Phys.*, **24** (2001), 83-105.
9. R. Hakl, A. Lomtatidze, B. Půža, New optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations. *Math. Bohem.* **127**(2002), No. 4, 509–524.
10. R. Hakl, A. Lomtatidze, B. Půža, On periodic solutions of first order linear functional differential equations. *Nonlinear Anal.* **49**(2002), no. 7, *Ser. A: Theory Methods*, 929–945.
11. R. Hakl, A. Lomtatidze, and J. Šremr, Some boundary value problems for first order scalar functional differential equations. *Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica 10. Brno: Masaryk University*, 2002.
12. R. Hakl, A. Lomtatidze, and J. Šremr, On an antiperiodic type boundary value problem for first order linear functional differential equations. *Arch. Math.* (*Brno*) **38**(2002), No. 2, 149–160.
13. R. Hakl, A. Lomtatidze, and J. Šremr, On nonnegative solutions of a periodic type boundary value problem for first order scalar functional differential equations. *Funct. Differ. Equ.* **11**(2004), No. 3-4, 363–394.
14. R. Hakl and J. Šremr, On the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators. *Nonlinear Oscil.* (to appear).
15. I. Kiguradze and B. Půža, Boundary value problems for systems of linear functional differential equations. *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica*, 12. *Masaryk University, Brno*, 2003.
16. S. M. Labovskiĭ, Positive solutions of a two-point boundary value problem for a linear singular functional-differential equation. (Russian) *Differentsial' nye Uravneniya* **24**(1988), No. 10, 1695–1704; English transl.: *Differential Equations* **24**(1988), No. 10, 1116–1123.
17. A. Lomtatidze and S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Mem. Differential Equations Math. Phys.* **5**(1995), 125–126.

18. A. Lomtatidze and S. Mukhigulashvili, Some two-point boundary value problems for second order functional differential equation. *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica*, 8. *Masaryk University, Brno*, 2000.
19. A. Lomtatidze and H. Štěpankova, On sign constant and monotone solutions of second order linear functional differential equations. *Mem. Differential Equations Math. Phys.* **35**(2005), 65–90.
20. A. Lomtatidze and P. Vodstrčil, On nonnegative solutions of second order linear functional differential equations. *Mem. Differential Equations Math. Phys.* **32**(2004), 59–88.
21. S. Mukhigulashvili, Two-point boundary value problems for second order functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 1–112.
22. S. V. Mukhigulashvili, On the unique solvability of the Dirichlet problem for a second-order linear functional-differential equation. (Russian) *Differ. Uravn.* **40**(2004), No. 4, 477–484; English transl.: *Differ. Equ.* **40**(2004), No. 4, 515–523.
23. S. Mukhigulashvili, On a periodic boundary value problem for second-order linear functional differential equations. *Bound. Value Probl.* **2005**, No. 3, 247–261.
24. S. Mukhigulashvili, On a periodic boundary value problem for third order linear functional differential equations. *Nonlinear Anal.* **66**(2007), No. 2, 527–535.
25. S. Mukhigulashvili and J. Šremr, On a two-point boundary value problem for the second order linear functional differential equations with monotone operators. *Funct. Differ. Equ.* **13**(2006), No. 3-4, 519–537.
26. T. A. Osechkina, A criterion for the unique solvability of a periodic boundary value problem for a functional-differential equation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **1994**, No. 10, 48–52; English transl.: *Russian Math.* (*Iz. VUZ*) **38**(1994), No. 10, 44–48.
27. J. Šremr, On the initial value problem for two-dimensional systems of linear functional differential equations with monotone operators. *Mathematical Institute, Academy of Sciences of the Czech Republic*, *Preprint* 162, 2005.
28. J. Šremr, Solvability conditions of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators. *Math. Bohem.* (to appear).
29. J. Šremr, On the initial value problem for two-dimensional systems of linear functional differential equations with monotone operators. *Hiroshima Math. J.* (to appear).
30. J. Šremr and P. Šremr, On a two point boundary problem for first order linear differential equations with a deviating argument. *Mem. Differential Equations Math. Phys.* **29**(2003), 75–124.

(Received 25.04.2007)

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