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Short Communications

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ON A PRIORI ESTIMATES OF BOUNDED SOLUTIONS OF SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL INEQUALITIES

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Let $\sigma_i \in \{-1;1\}$ $(i=1,\dots,n)$ and let $a_{il},\ f_i : \mathbb{R} \to \mathbb{R}$ $(i \neq l;\ i,l = 1,\dots,n)$ and $a_{ii} : \mathbb{R} \to \mathbb{R}$ $(i=1,\dots,n)$ be, respectively, nondecreasing and nonincreasing functions.

In this paper we consider the question on a priori estimates of nonnegative solutions $(u_i)_{i=1}^n$ of the system of linear generalized ordinary differential inequalities

$$\sigma_i du_i(t) \leq \sum_{l=1}^n u_l(t) da_{il}(t) + df_i(t) \ \text{ for } \ t \in \mathbb{R} \ \ (i=1,\dots,n) \tag{1}$$

satisfying the condition

$$\limsup_{t\to\infty} u_i(\sigma_i t) < \infty \ \ (i=1,\dots,n). \tag{2}$$

A particular case of the condition (2) is the periodic problem

$$u_i(t+\omega) = u_i(t) \ \text{ for } \ t \in \mathbb{R} \ \ (i=1,\dots,n).$$

We will use these results for the proof of theorems concerning the existence of bounded (on infinite intervals of the real axis) solutions for systems of generalized ordinary differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.g., [1]–[7] and references therein).

These results for systems of linear ordinary differential inequalities belong to I. Kiguradze ([8], [9]).

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Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a,b]$ and $]a,b]$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the set of all real matrices $X = (x_{ij})_{i,j=1}^{n,m}$.

$r(X)$ is the spectral radius of the quadratic $n\times n$-matrix $X = (x_{ij})_{i,j=1}^{n}$.

A matrix-function $X = (x_{ij})_{i,j=1}^{n,m} \to \mathbb{R}^{n\times m}$ is said to be nonnegative, continuous, nondecreasing, etc., if each of its components x_{ij} $(i = 1,\dots,n;$ $j = 1,\dots,m)$ is such.

$\bigvee\limits_a^b(x)$ is the total variation on $[a,b]$ of the function $x : [a,b] \to \mathbb{R}$.

$\mathrm{BV}([a,b],\mathbb{R})$ is the set of all functions $x : [a,b] \to \mathbb{R}$ with bounded total variation.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ is the set of all functions $x : \mathbb{R} \to \mathbb{R}$, such that $\bigvee\limits_a^b(x) < \infty$ for $a < b$ $(a,\, b \in \mathbb{R})$; $x(t-)$ and $x(t+)$ are, respectively, the left and right limits of the function $x : \mathbb{R} \to \mathbb{R}$ at the point $t \in \mathbb{R}$; $d_1x(t) = x(t) - x(t-)$, $d_2x(t) = x(t+) - x(t)$.

$v(x) : \mathbb{R} \to \mathbb{R}$ is the function defined by $v(x)(0) = 0$, $v(x)(t) = \bigvee\limits_0^t(x)$ for $t > 0$ and $v(x)(t) = \bigvee\limits_t^a(x)$ for $t < 0$.

$s_0(x) : \mathbb{R} \to \mathbb{R}$ is the continuous part of the function $x \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$, i.e.,

$$\begin{aligned}
&s_0(x)(0) = 0,\\
&s_0(x)(t) = x(t) - x(0) - \sum_{0<\tau\le t} d_1x(\tau) - \sum_{0\le\tau<t} d_2x(\tau) \ \text{ for } \ t > 0
\end{aligned}$$

and

$$s_0(x)(t) = x(t) - x(0) - \sum_{t<\tau\le 0} d_1x(\tau) - \sum_{t\le\tau<0} d_2x(\tau) \ \text{ for } \ t < 0.$$

$\mathcal{A} : \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}) \times \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}) \to \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ is the operator defined by

$$\begin{aligned}
&\mathcal{A}(x,y)(0) = 0,\\
&\mathcal{A}(x,y)(t) = y(t) + \sum_{0<\tau\le t} d_1x(\tau)\cdot\big(1 - d_1x(\tau)\big)^{-1} d_1y(\tau) -\\
&\qquad\qquad - \sum_{0\le\tau<t} d_2x(\tau)\cdot\big(1 + d_2x(\tau)\big)^{-1} d_2y(\tau) \ \text{ for } \ t > 0,\\
&\mathcal{A}(x,y)(t) = y(t) - \sum_{t<\tau\le 0} d_1x(\tau)\cdot\big(1 - d_1x(\tau)\big)^{-1} d_1y(\tau) +\\
&\qquad\qquad + \sum_{t\le\tau<0} d_2x(\tau)\cdot\big(1 + d_2x(\tau)\big)^{-1} d_2y(\tau) \ \text{ for } \ t < 0
\end{aligned}$$

for every $x \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ such that

$$1+(-1)^j d_j x(t) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j=1,2).$$

If $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, $x : \mathbb{R} \to \mathbb{R}$ and $s < t$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, ds_0(g)(\tau) + \sum_{s<\tau\leq t} x(\tau) d_1 g(\tau) + \sum_{s\leq\tau<t} x(\tau) d_2 g(\tau),$$

where $\int_{]s,t[} x(\tau)\, ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$.

If $s=t$, then we assume

$$\int_s^t x(\tau)\, dy(\tau) = 0.$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)\, dg_2(\tau) \ \text{ for } \ s \leq t.$$

By a solution of the system (1) we understand a vector function $(u_i)_{i=1}^n$, $u_i \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ $(i=1,\dots,n)$ such that

$$\sigma_i\big(u_i(t)-u_i(s)\big) \leq \sum_{l=1}^n \int_s^t u_l(\tau)\, da_{il}(\tau) + f_i(t) - f_i(s) \ \text{ for } \ t \leq s \ \ (i=1,\dots,n).$$

If $s \in \mathbb{R}$, and $\beta \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ is such that

$$1+(-1)^j d_j \beta(t) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j=1,2),$$

then by $\gamma_\beta(\cdot\,,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t) d\beta(t), \quad \gamma(s) = 1.$$

By a solution of this problem we understand a function $\gamma \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$, such that

$$\gamma(t) = 1 + \int_s^t \gamma(\tau)\, d\beta(\tau) \ \text{ for } \ t \in \mathbb{R}.$$

It is known (see [6],[7]) that

$$\gamma_\beta(t,s) = \exp\big(\xi_\beta(t) - \xi_\beta(s)\big) \prod_{s<\tau\leq t} \operatorname{sgn}\big(1 - d_1\beta(\tau)\big) \times$$
$$\times \prod_{s\leq\tau<t} \operatorname{sgn}\big(1 + d_2\beta(\tau)\big) \ \text{ for } \ t > s,$$
$$\gamma_\beta(t,s) = \gamma_\beta^{-1}(s,t) \ \text{ for } \ t < s,$$

where

$$\xi_\beta(t) = s_0(\beta)(t) - s_0(\beta)(0) -$$
$$- \sum_{0<\tau\leq t} \ln\big|1 - d_1\beta(\tau)\big| + \sum_{0\leq\tau<t} \ln\big|1 + d_2\beta(\tau)\big| \ \text{ for } \ t > 0,$$
$$\xi_\beta(t) = s_0(\beta)(t) - s_0(\beta)(0) +$$
$$+ \sum_{t<\tau\leq 0} \ln\big|1 - d_1\beta(\tau)\big| - \sum_{t\leq\tau<0} \operatorname{sgn}\big|1 + d_2\beta(\tau)\big| \ \text{ for } \ t < 0.$$

Remark 1. Let $\beta \in \mathrm{BV}([a,b],\mathbb{R})$ be such that

$$1 + (-1)^j d_j\beta(t) > 0 \ \text{ for } \ t \in [a,b] \ \ (j = 1,2).$$

Let, moreover, one of the functions β, ξ_β and $\mathcal{A}(\beta,\beta)$ be nondecreasing (nonincreasing). The other two functions will be nondecreasing (nonincreasing), as well.

Let $\delta > 0$. We introduce the operators

$$\nu_{1\delta}(\xi)(t) = \sup\Big\{\tau \geq t : \ \xi(\tau) \leq \xi(t+) + \delta\Big\}$$

and

$$\nu_{-1\delta}(\eta)(t) = \inf\Big\{\tau \leq t : \ \eta(\tau) \leq \eta(t-) + \delta\Big\},$$

respectively, on the set of all nondecreasing functions $\xi : \mathbb{R} \to \mathbb{R}$ and on the set of all nonincreasing functions $\eta : \mathbb{R} \to \mathbb{R}$.

If $\sigma = (\sigma_i)_{i=1}^n$, where $\sigma_i \in \{-1;1\}$ $(i = 1,\dots,n)$, then by $\mathcal{N}_+(\sigma)$ $(\mathcal{N}_-(\sigma))$ we denote the set of all $i \in \{1,\dots,n\}$ such that $\sigma_i = 1$ $(\sigma_i = -1)$.

Lemma 1. *Let $c_{ik} : \mathbb{R} \to \mathbb{R}$ $(i \neq k;\ i,k = 1,\dots,n)$ and $c_{ii} : \mathbb{R} \to \mathbb{R}$ $(i = 1,\dots,n)$ be, respectively, nondecreasing and nonincreasing functions, and $\sigma_i \in \{-1;1\}$ $(i = 1,\dots,n)$ be such that*

$$1 + (-1)^j \sigma_i d_j c_{ii}(t) > 0 \ \textit{ for } \ t \in [a,b] \ \ (j = 1,2)$$

and

$$r(S) < 1, \tag{3}$$

where $S = (s_{il})_{i,l=1}^n$,

$$s_{11} = \dots = s_{nn} = 0,$$
$$s_{il} = \sup\bigg\{\bigg|\int_{t_i}^{t} \gamma_{\sigma_i c_{ii}}(t,s)\, dV\big(\mathcal{A}(\sigma_i c_{ii}, c_{il})\big)(s)\bigg| : \ t \in [a,b]\bigg\}$$
$$(i \neq l; \ \ i,l = 1,\dots,n),$$
$$t_i = a \ \textit{ if } \ \sigma_i = 1 \quad \textit{and} \quad t_i = b \ \textit{ if } \ \sigma_i = -1 \ \ (i = 1,\dots,n).$$

Let, moreover, the numbers $\delta_i > 0$ $(i = 1,\dots,n)$ be such that

$$\sigma_i\big(\xi_{\sigma_i c_{ii}}(a) - \xi_{\sigma_i c_{ii}}(b)\big) > \delta_i \ \ (i = 1,\dots,n).$$

Then there exists a positive number $\rho > 0$ such that for every nondecreasing functions $q_i : [a,b] \to \mathbb{R}$ $(i = 1,\dots,n)$ an arbitrary nonnegative solution $u = (u_i)_{i=1}^n$ of the system of linear generalized differential inequalities

$$\sigma_i du_i(t) \le \sum_{l=1}^n u_l(t) dc_{il}(t) + dq_i(t) \ \ \text{for} \ \ t \in [a,b] \ \ (i = 1,\dots,n)$$

admits the estimate

$$\sum_{i=1}^n u_i(t) \le \rho\Big(\rho_0 + \sum_{i=1}^n \rho_i\Big) \ \ \text{for} \ \ t \in [a,b],$$

where

$$\rho_0 = \sum_{i \in \mathcal{N}_+(\sigma)} u_i(a) + \sum_{i \in \mathcal{N}_-(\sigma)} u_i(b),$$

$$\rho_i = \sup\bigg\{\Big|\bigvee_t^{\nu_i(t)} \mathcal{A}(\sigma_i c_{ii}, f_i)\Big| : \ t \in [a,b] \ (i = 1,\dots,n)\bigg\},$$

$$\nu_i(t) \equiv \nu_{\sigma_i \delta_i}(-\xi_{\sigma_i c_{ii}})(t) \ \ (i = 1,\dots,n).$$

Theorem 1. *Let $\sigma_i \in \{-1; 1\}$ $(i = 1,\dots,n)$ be such that*

$$1 + (-1)^j \sigma_i d_j a_{ii}(t) > 0 \ \ \text{for} \ \ t \in \mathbb{R} \ \ (j = 1,2; \ \ i = 1,\dots,n)$$

and the condition (3) *hold, where $S = (s_{il})_{i,l=1}^n$, $s_{11} = \dots = s_{nn} = 0$ and*

$$s_{il} = \sup\bigg\{\Big|\int_{t_i}^t \gamma_{\sigma_i a_{ii}}(t,s)\, dV\big(\mathcal{A}(\sigma_i a_{ii}, a_{il})\big)(s)\Big| : \ t \in \mathbb{R}\bigg\} < \infty$$

$$(i \ne l; \ \ i,l = 1,\dots,n),$$

$$t_i = a \ \ \text{if} \ \ \sigma_i = 1 \quad \text{and} \quad t_i = b \ \ \text{if} \ \ \sigma_i = -1 \ \ (i = 1,\dots,n).$$

Let, moreover,

$$\sigma_i \liminf_{t \to \infty} \big(\xi_{\sigma_i a_{ii}}(t) - \xi_{\sigma_i a_{ii}}(-t)\big) > \delta > 0 \ \ (i = 1,\dots,n)$$

for some $\delta > 0$. Then there exists a positive number $\rho > 0$ such that for any nondecreasing functions $f_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,\dots,n)$ such that

$$\rho_i = \sup\bigg\{\Big|\bigvee_t^{\nu_i(t)} \big(\mathcal{A}(\sigma_i a_{ii}, f_i)\big)\Big| : \ t \in \mathbb{R}\bigg\} < \infty \ \ (i = 1,\dots,n),$$

where $\nu_i(t) \equiv \nu_{\sigma_i \delta}(-\xi_{\sigma_i c_{ii}})(t)$ $(i = 1,\dots,n)$, an arbitrary nonnegative solution of the problem (1), (2) *admits the estimate*

$$\sum_{i=1}^n u_i(t) \le \rho \sum_{i=1}^n (r_i + \rho_i) \ \ \text{for} \ \ t \in \mathbb{R},$$

where

$$r_i = \limsup_{t \to \infty} u_i(\sigma_i t) \ \ (i = 1,\dots,n).$$

Remark 2. If $n = 2$, then the condition (3) in Theorem 2 has the following form:

$$|s_{12}s_{21}| < 1,$$

where s_{12} and s_{21} are defined as in the theorem.

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