Mem. Differential Equations Math. Phys. 38 (2006), 135–137

T. Buchukuri, O. Chkadua, and R. Gachechiladze

SOME DYNAMIC PROBLEMS OF PIEZOELECTRICITY IN DOMAINS WITH CUTS

(Reported on October 17, 2005)

We consider a piezoelectric body which is grounded and mechanically clamped over the one part of the boundary whereas the remaining part of the boundary is mechanically free and electrically isolated. Inside the body there is located a thin metallic electrode connected to the electric source. We investigate the corresponding dynamic boundary-value problem of piezoelectricity in a domain with a cut, prove the existence and uniqueness of solutions and obtain an asymptotic expansion of solutions near the edge of the cut.

Let Ω and Ω_1 ($\overline{\Omega_1} \subset \Omega$)be bounded domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with infinitely smooth boundaries $\partial\Omega$ and $\partial\Omega_1$, respectively. We assume that the boundary $\partial\Omega_1$ of the domain Ω_1 is the union of two surfaces: $\partial\Omega_1 = \overline{S} \cup S_0$ and that the boundary $\partial S = \partial S_0$ is an infinitely smooth curve. Denote $\Omega_2 = \Omega \backslash \overline{\Omega}_1$.

Suppose that the domain $\Omega\backslash\overline{S}$ is filled with anisotropic homogeneous piezoelectric material having a cut at $\overline{S}$.

In the domain $\Omega\backslash\,\overline{S}$, let us consider the system of dynamic equations of piezoelectricity for a homogeneous anisotropic medium [1]:

$$A(\partial_x)u(x,t) - \partial_t^2 u(x,t) = F(x,t) \ \text{ in } \ (\Omega\backslash\,\overline{S}) \times [0,+\infty[\,, \tag{1}$$

where $u = (u_1, u_2, u_3, u_4)$; u_1, u_2, u_3 are the components of the displacement vector, u_4 is the electric potential, F_i, $i = 1, 2, 3$, are the components of the mass force, F_4 is the electric charge density, $A(D_x)$ is the differential operator of the form

$$\begin{aligned} &A(D_x) = \|A_{jk}(D_x)\|_{5\times 5}, \\ &A_{jk}(D_x) = c_{ijlk}\partial_i\partial_l, \quad A_{j4}(D_x) = e_{kjl}\partial_k\partial_l, \\ &A_{4k}(D_x) = -e_{ikl}\partial_i\partial_l, \quad A_{44}(D_x) = \varepsilon_{il}\partial_i\partial_l, \quad j,k = 1,2,3, \end{aligned} \tag{2}$$

where c_{ijlk}, e_{ikl}, ε_{ik} are respectively the elastic, piezoelectric and dielectric constants. In the equalities (1) it is assumed that summation is carried out over the repeated indices. We will follow this rule in the sequel.

The constants in (1)–(2) satisfy the symmetry conditions

$$\begin{aligned} &c_{ijlk} = c_{jilk} = c_{lkij}, \quad e_{kjl} = e_{klj}, \quad \varepsilon_{ik} = \varepsilon_{ki}, \\ &i,j,k,l = 1,2,3, \end{aligned} \tag{3}$$

and the condition of positiveness of the internal energy:

$$\forall(\xi_{ij}), \ \ (\eta_i), \ \ \xi_{ij} = \xi_{ji}, \ \ \exists c_0 > 0 \ \ c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij}, \quad \varepsilon_{ij}\eta_i\eta_j \geq c_0\eta_i\eta_i. \tag{4}$$

Due to (3), (4), the operator $A(D_x)$ is a strongly elliptic operator.

2000 *Mathematics Subject Classification.* 74F15, 74H20, 74H35.

Key words and phrases. Electroelasticity, crack-type dynamic problem, asymptotic expansion of solutions.

Denote by $T(D_y, n)$ the following electromechanical stress operator, the first three components of which coincide with the corresponding components of the mechanical stress and the fourth component is the electric force:

$$
\begin{aligned}
&T(D_y, n) = \|T_{jk}(D_y, n)\|_{4\times 4},\\
&T_{jk}(D_y, n) = c_{ijlk} n_l \partial_i, \quad T_{j4}(D_y, n) = e_{kjl} n_l \partial_k,\\
&T_{4k}(D_y, n) = -e_{ikl} n_i \partial_l, \quad T_{44}(D_y, n) = \varepsilon_{ij} n_j \partial_i, \quad j,k = 1,2,3.
\end{aligned}
$$

Here $n(y) = (n_1(y), n_2(y), n_3(y))$ is the unit normal at the point $y \in \partial\Omega_2$, directed outward from Ω_2.

For a Banach space B, $a > 0$, $m \in \mathbb{N} \cup \{0\}$, we denote by $\overset{\circ}{C}{}^m_a([0, +\infty[\,, B)$ the set of all m-times continuously differentiable B-valued functions on $[0, +\infty[$ satisfying the conditions:

$$
\frac{\partial^l u(0)}{\partial t^l} = 0, \;\; l = 0, \dots, m, \quad \left\| \frac{\partial^l u(t)}{\partial t^l} \right\|_B = O(e^{\alpha t}) \;\; \forall \alpha > a, \;\; l = 0, \dots, m.
$$

Define $C^m_{0,a}([0, +\infty), \mathbb{B})$ as the set of all m-times continuously differentiable $\mathbb{B}$-valued functions on $[0, +\infty)$ satisfying the conditions

$$
\frac{\partial^l u(0)}{\partial t^l} = 0, \quad l = 0, \dots, m-2, \quad \left\| \frac{\partial^l u(t)}{\partial t^l} \right\|_{\mathbb{B}} = O(e^{at}), \;\; l = 0, \dots, m.
$$

(For definitions of these spaces see [2].)

Let us suppose that the considered piezoelectric body is grounded and mechanically clamped over the part S_1 of the boundary $\partial\Omega$, whereas the remaining part S_2 of the boundary $\partial\Omega$ is mechanically free and electrically isolated. A thin metal electrode is located at the cut S and is connected to the electric source with a known potential. Then the corresponding boundary value problem in the space $\overset{\circ}{C}{}^m_a([0, +\infty[\,, [W^1_p(\Omega \backslash S)]^4)$ can be formulated as follows:

$$
\begin{cases}
A(\partial_x) u^(x,t) - \partial_t{}^2 u(x,t) = F(x,t), & (x,t) \in (\Omega \backslash \overline{S}) \times [0, +\infty[\,,\\
\{[T(D_y, n) u(y,t)]_j\}^{\pm} = 0, \quad j = 1,2,3; & (y,t) \in S \times [0, +\infty[\,,\\
\{u_4(y,t)\}^{\pm} = \varphi(y,t), \quad i = 1,2, & (y,t) \in S \times [0, +\infty[\,,\\
\{u(y,t)\}^{+} = 0 & (y,t) \in S_1 \times [0, +\infty[\,,\\
\{[T(D_y, n) u(y,t)]\}^{+} = 0 & (y,t) \in S_2 \times [0, +\infty[\,,\\
u_j(x,0) = \partial_t u_j(x,0) = 0, & x \in \Omega \backslash S, \quad j = 1,2,3,
\end{cases}
\tag{5}
$$

where $F \in C^{m+5}_{0,a}([0, +\infty[\,, [L_{max\{p,2\}}(\Omega \backslash S)]^4)$, $\varphi \in C^{m+7}_{0,a}([0, +\infty[\,, B^{-1/p}_{p,p}(S))$, $p' = p/(p-1)$, $1 < p < \infty$. $\{f\}^+$ denotes the trace of f on $\partial\Omega_2$ from Ω_2 and $\{f\}^-$ denotes the trace of the function f on $\partial\Omega_1$ from Ω_1. $B^{-1/p}_{p,p}(S)$ denotes the Besov space.

We have obtained theorems on the existence and uniqueness of solutions of this dynamic problem by using the Laplace transform, the potential theory and the general theory of pseudodifferential equations on a manifold with boundary.

Employing an asymptotic expansion of solutions of strongly elliptic pseudodifferential equations and an asymptotic expansion of potential-type functions obtained in [3, 4], for sufficiently smooth data of the problem we have obtained a complete asymptotic expansion of the solution near cut's edge ∂S as well as near the curve $L = \partial S_1$, where the type of the boundary conditions changes.

The exponent of the first term of the asymptotic expansion of solutions near the crack edge ∂S is $\frac{1}{2}$, i.e., the singularity of solutions near the crack edge is $\frac{1}{2}$.

In the asymptotic expansion of solutions the time parameter t participates only in the asymptotic coefficients, in particular, in the first coefficient.

We have found an important class of anisotropic bodies, when the oscillation of solutions vanishes near the curve L, the first three terms of the asymptotics do not contain logarithms and the singularity of solutions is calculated by a simple formula. This class is not empty. In particular, we have considered a special class of transversally-isotropic piezoelectric bodies with the elastic constants $c_{11}, c_{33}, c_{13}, c_{44}, c_{66}$, the piezoelectric constant e_{14} and the dielectric constants $\varepsilon_{11}, \varepsilon_{33}$. We assume that the neighbourhood of L is parallel to the isotropic plane. It should be noted that the crystal TeO_2 is a transversally-isotropic body.

In this case we have obtained the following asymptotic properties of solutions near the curve L:

$$c_1\rho^{\gamma_1} + c_2\rho^{\frac{1}{2}\pm i\delta} + c_3\rho^{\gamma_2} + \cdots ,$$

where

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg} A, \quad \gamma_2 = \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg} A,$$

$$A = \sqrt[4]{\frac{\varepsilon_{33}c_{66}}{\varepsilon_{11}c_{44}}}\ \frac{e_{14}}{\sqrt{e_{14}^2 + (\sqrt{\varepsilon_{11}c_{44}} + \sqrt{\varepsilon_{33}c_{66}})^2}}\,.$$

The singularity of solutions depends on the elastic constants, piezoelectric constant, also the dielectric constants and may take any values from the interval $(0;1/2)$. For the general anisotropic case we can say that the singularity of solutions depends also on the geometry of L.

2) Since $\gamma_1 < \frac{1}{2}$, the oscillation vanishes in some neighbourhood of L.

In the case of elasticity theory $\gamma_1 = \frac{1}{2}$ and the oscillation of solutions does not vanish.

Thus in these classes of electro-elastic bodies we have observed the effects which do not take place in classical elasticity.

Naturally arises a question: does there exist an elastic medium for which singularity of solutions near L is equal to $\frac{1}{2}$? Such a medium, for example, is the elastic medium with cubic anisotropy.

Finally note that the boundary value problems of piezoelectricity of a different type in domains with cuts were considered in [5].

References

1. W. Voigt, Lehrbuch der Kristall-Physik. *Teubner, Leipzig,* 1910.
2. D. G. Natroshvili, O. O. Chkadua, and E. M. Shargorodskiĭ, Mixed problems for homogeneous anisotropic elastic media. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **39** (1990), 133–181.
3. O. Chkadua and R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic. *Math. Nachr.* **222** (2001), 79–139.
4. O. Chkadua and R. Duduchava, Asymptotics of functions represented by potentials. *Russ. J. Math. Phys.* **7** (2000), No. 1, 15–47.
5. T. Buchukuri, O. Chkadua, and R. Duduchava, Crack-type boundary value problems of electro-elasticity. *Operator theoretical methods and applications to mathematical physics,* 189–212, *Oper. Theory Adv. Appl.,* 147, *Birkhäuser, Basel,* 2004.

Authors' address:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mails: t_buchukuri@yahoo.com
chkadua@rmi.acnet.ge
rgach@rmi.acnet.ge