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T. Kiguradze

EXISTENCE AND UNIQUENESS THEOREMS ON PERIODIC SOLUTIONS TO MULTIDIMENSIONAL LINEAR HYPERBOLIC EQUATIONS

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In $\mathbb{R}^n$ consider the linear hyperbolic equations

$$u^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha}\in\mathcal{E}^{\boldsymbol{m}}} p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\in\mathcal{O}^{\boldsymbol{m}}} p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})u^{(\boldsymbol{\alpha})} + q(\boldsymbol{x}), \tag{1}$$

and

$$u^{(\boldsymbol{m})} = p_{\boldsymbol{0}}(\boldsymbol{x})u + q(\boldsymbol{x}), \tag{2}$$

where $n \ge 2$, $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ are multi–indeces , and

$$u^{(\boldsymbol{\alpha})} = \frac{\partial^{\alpha_1+\dots+\alpha_n} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

We make use of following notations and definitions.

$\mathbb{Z}_+$ is the set of all nonnegative integers; $\mathbb{Z}_+^n$ is the set of all multiindices $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$; $\|\boldsymbol{\alpha}\| = \alpha_1 + \dots + \alpha_n$; $\boldsymbol{0} = (0, \dots, 0) \in \mathbb{Z}_+^n$.

The inequalities between the multiindices $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ are understood componentwise.

It will be assumed that $\boldsymbol{m} > \boldsymbol{0}$.

If for some multiindex $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ we have $\alpha_{i_1} = \dots = \alpha_{ik} = 0$ $(i_1 < \dots < i_k)$, and $\alpha_{j_1}, \dots, \alpha_{j_{n-k}} > 0$ $(j_1 < \dots < j_{n-k})$, $\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$, then by $\boldsymbol{x}_{\boldsymbol{\alpha}}$ (by $\boldsymbol{x}^{\boldsymbol{\alpha}}$) denote the vector $(x_{i_1}, \dots, x_{i_k}) \in \mathbb{R}^k$ (the vector $(x_{j_1}, \dots, x_{j_{n-k}}) \in \mathbb{R}^{n-k}$). If $\boldsymbol{\alpha} > \boldsymbol{0}$, then in equation (1) by $p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})$ we understand a constant function.

A multiindex $\boldsymbol{\alpha} \in \mathbb{Z}_+^n$ will be called *even*, if all its components are even.

A multiindex $\boldsymbol{\alpha} \in \mathbb{Z}_+^n$ will be called *odd*, if $\|\boldsymbol{\alpha}\|$ is odd.

By $\mathcal{E}^{\boldsymbol{m}}$ and $\mathcal{O}^{\boldsymbol{m}}$, respectively, denote the sets of all even and odd multiindices not exceeding $\boldsymbol{m}$ and different from $\boldsymbol{m}$, i.e.,

$$\mathcal{E}^{\boldsymbol{m}} = \big\{\boldsymbol{\alpha} \in \mathbb{Z}_+^n \setminus \{\boldsymbol{m}\} : \boldsymbol{\alpha} \le \boldsymbol{m},\ \ \alpha_1, \dots, \alpha_n \ \text{ are even}\big\},$$

$$\mathcal{O}^{\boldsymbol{m}} = \big\{\boldsymbol{\alpha} \in \mathbb{Z}_+^n \setminus \{\boldsymbol{m}\} : \boldsymbol{\alpha} \le \boldsymbol{m},\ \ \alpha_1 + \dots + \alpha_n \ \text{ is odd}\big\}.$$

By $\mathcal{S}^{\boldsymbol{m}}$ denote the set of nonzero multiindices $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ whose components either equal to the corresponding components of $\boldsymbol{m}$, or equal to 0, i.e.,

$$\mathcal{S}^{\boldsymbol{m}} = \big\{\boldsymbol{\alpha} \ne \boldsymbol{0} \, : \, \alpha_i \in \{0, m_i\} \ \ (i = 0, \dots, n)\big\}.$$

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ b a vector with positive components. Then by Ω denote the rectangular box $[0, \omega_1] \times \dots \times [0, \omega_n]$ in $\mathbb{R}^n$. Moreover, for an arbitrary multiindex $\boldsymbol{\alpha}$, similarly as we did above, introduce the vectors $\boldsymbol{\omega}_{\boldsymbol{\alpha}} = (\omega_{i_1}, \dots, \omega_{i_k}) \in \mathbb{R}^k$ and $\boldsymbol{\omega}^{\boldsymbol{\alpha}} = (\omega_{j_1}, \dots, \omega_{j_{n-k}}) \in \mathbb{R}^{n-k}$, and the rectangular boxes $\Omega_{\boldsymbol{\alpha}} = [0, \omega_{i_1}] \times \dots \times [0, \omega_{i_k}]$ in $\mathbb{R}^k$ and $\Omega^{\boldsymbol{\alpha}} = [0, \omega_{j_1}] \times \dots \times [0, \omega_{j_{n-k}}]$ in $\mathbb{R}^{n-k}$.

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We say that a function $z : \mathbb{R}^n \to \mathbb{R}$ is $\boldsymbol{\omega}$*–periodic*, if

$$z(x_1, \dots, x_j + \omega_j, \dots, x_n) \equiv z(x_1, \dots, x_n) \quad (j = 1, \dots, n).$$

It will be assumed that the functions $p_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathcal{E}^{\boldsymbol{m}} \cup \mathcal{O}^{\boldsymbol{m}})$ and q, respectively, are $\boldsymbol{\omega}_{\boldsymbol{\alpha}}$–periodic and $\boldsymbol{\omega}$–periodic continuous functions.

Let $\boldsymbol{l} = (l_1, \dots, l_n) \in \mathbb{Z}^n_+$. By $C^{\boldsymbol{l}}$ denote the space of continuous functions $u : \mathbb{R}^n \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \le \boldsymbol{l})$.

By a solution of equation (1) (equation (2)) we will understand a *classical solution*, i.e., a function $u \in C^{\boldsymbol{m}}$ satisfying equation (1) (equation (2)) everywhere in $\mathbb{R}^n$.

In the case, where $n = 2$, $m_1 = m_2 = 1$ ($n = 2$, $m_1 = m_2 = 2$) sufficient conditions for existence and uniqueness of (ω_1, ω_2)–periodic solutions of equation (1) are given in [1–3, 6–8] (in [9, 10]). In the general case the problem on $\boldsymbol{\omega}$–periodic solutions to equations (1) and (2) are little investigated. In the present paper optimal sufficient conditions of existence and uniqueness of $\boldsymbol{\omega}$–periodic solutions to equation (1) (equation (2)) are given. Similar results for higher order nonlinear ordinary differential equations were obtained by I. Kiguradze and T. Kusano [5].

We consider equations (1) and (2) in two cases, where $\boldsymbol{m}$ is either even, or odd. Also note that equations (1) and (2) do contain partial derivatives with even or odd (according to the above definitions) multiindices only (e.g., neither of $\boldsymbol{m}$ and $\boldsymbol{\alpha}$ can equal to $(1, 1, 1, 1)$).

Theorem 1. *Let $\boldsymbol{m}$ be even, and let*

$$(-1)^{\frac{\|\boldsymbol{m}\| + \|\boldsymbol{\alpha}\|}{2}} p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \le 0 \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{\alpha} \in \mathcal{E}^{\boldsymbol{m}}, \tag{3}$$

$$\overline{\mathbb{R}^n \setminus I_{p_{\boldsymbol{0}}}} = \mathbb{R}^n, \tag{4}$$

where $I_{p_{\boldsymbol{0}}} = \{\boldsymbol{x} \in \mathbb{R}^n : p_{\boldsymbol{0}}(\boldsymbol{x}) = 0\}$. Then equation (1) *has at most one $\boldsymbol{\omega}$-periodic solution.*

Theorem 2. *Let $\boldsymbol{m}$ be odd, and let there exist $j \in \{1, 2\}$ such that along with* (4) *the inequality*

$$(-1)^{j + \frac{\|\boldsymbol{\alpha}\|}{2}} p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \le 0 \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{\alpha} \in \mathcal{E}^{\boldsymbol{m}} \tag{5}$$

holds. Then equation (1) *has at most one $\boldsymbol{\omega}$-periodic solution.*

Theorems 1 and 2 almost immediately follow from the following lemma.

Lemma 1. *Let $u \in C^{\boldsymbol{m}}$ be an $\boldsymbol{\omega}$–periodic function. Then*

$$\int_{\Omega^{\boldsymbol{\alpha}}} u^{(\boldsymbol{\alpha})}(\boldsymbol{x})\, u(\boldsymbol{x})\, d\boldsymbol{x}^{\boldsymbol{\alpha}} = (-1)^{\frac{\|\boldsymbol{\alpha}\|}{2}} \int_{\Omega^{\boldsymbol{\alpha}}} \left| u^{\left(\frac{\boldsymbol{\alpha}}{2}\right)}(\boldsymbol{x}) \right|^2 d\boldsymbol{x}^{\boldsymbol{\alpha}} \quad \text{for} \quad \boldsymbol{\alpha} \in \mathcal{E}^{\boldsymbol{m}},$$

$$\int_{\Omega^{\boldsymbol{\alpha}}} u^{(\boldsymbol{\alpha})}(\boldsymbol{x})\, u(\boldsymbol{x})\, d\boldsymbol{x}^{\boldsymbol{\alpha}} = 0 \quad \text{for} \quad \boldsymbol{\alpha} \in \mathcal{O}^{\boldsymbol{m}}.$$

One can easily prove the lemma using integration by parts and taking into consideration $\boldsymbol{\omega}$–periodicity of u.

Proof of Theorem 1. All we need to prove is that if $q(\boldsymbol{x}) \equiv 0$, then equation (1) has only a trivial $\boldsymbol{\omega}$–periodic solution. Indeed, let $q(\boldsymbol{x}) \equiv 0$, and let u be an arbitrary $\boldsymbol{\omega}$–periodic solution of equation (1). After multiplying equation (1) by u and integrating over the rectangular box Ω, by Lemma 1 and condition (3), we get

$$\int_{\Omega} \Big(\left| u^{\left(\frac{\boldsymbol{m}}{2}\right)}(\boldsymbol{x}) \right|^2 + \sum_{\boldsymbol{\alpha} \in \mathcal{E}^{\boldsymbol{m}}} |p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})| \left| u^{\left(\frac{\boldsymbol{\alpha}}{2}\right)}(\boldsymbol{x}) \right|^2 \Big)\, d\boldsymbol{x} = 0. \tag{6}$$

(4) and (6) immediately imply that $u(\boldsymbol{x}) \equiv 0$. ☐

We omit the proof of Theorem 2, since it is similar to the proof of Theorem 1.

Theorem 3. *Let $\boldsymbol{m}$ be even, and let*

$$0 \le (-1)^{\frac{\|\boldsymbol{m}\|}{2}} p_{\boldsymbol{0}}(\boldsymbol{x}) < \frac{(2\pi)^{\|m\|}}{\omega_1^{m_1} \cdots \omega_n^{m_n}}, \qquad \overline{\mathbb{R}^n \setminus I_{p_{\boldsymbol{0}}}} = \mathbb{R}^n. \tag{7}$$

Then equation (2) *has at most one $\boldsymbol{\omega}$-periodic solution.*

To prove the theorem along with Lemma 1 we need the following

Lemma 2. *Let $\boldsymbol{m}$ be even, and let $u \in C^{\boldsymbol{m}}$ be an $\boldsymbol{\omega}$–periodic function. Then*

$$\int_\Omega \left| u^{\left(\frac{\boldsymbol{m}}{2}\right)}(\boldsymbol{x}) \right|^2 d\boldsymbol{x} \le \frac{\omega_1^{m_1} \cdots \omega_n^{m_n}}{(2\pi)^{\|\boldsymbol{m}\|}} \int_\Omega \left| u^{(\boldsymbol{m})}(\boldsymbol{x}) \right|^2 d\boldsymbol{x}. \tag{8}$$

This lemma immediately follows from Wirtinger's inequality ([4], Theorem 258).

Proof of Theorem 3. Assume the contrary: let $q(\boldsymbol{x}) \equiv 0$ and equation (2) have a nontrivial $\boldsymbol{\omega}$–periodic solution u. Then we have

$$u^{(\boldsymbol{m})}(\boldsymbol{x}) = p_{\boldsymbol{0}}(\boldsymbol{x}) u(\boldsymbol{x}) \tag{9}$$

and

$$|u^{(\boldsymbol{m})}(\boldsymbol{x})|^2 = |p_{\boldsymbol{0}}(\boldsymbol{x}) u(\boldsymbol{x})|^2. \tag{10}$$

Multiplying (9) by u, integrating over Ω, by Lemma 1, we get

$$\int_\Omega |p_{\boldsymbol{0}}(\boldsymbol{x})| \left| u(\boldsymbol{x}) \right|^2 d\boldsymbol{x} = \int_\Omega \left| u^{\left(\frac{\boldsymbol{m}}{2}\right)}(\boldsymbol{x}) \right|^2 d\boldsymbol{x}. \tag{11}$$

Integrating (10) over Ω and assuming that $u(\boldsymbol{x}) \not\equiv 0$, by condition (8), we get

$$\int_\Omega \left| u^{(\boldsymbol{m})}(\boldsymbol{x}) \right|^2 d\boldsymbol{x} = \int_\Omega \left| p_{\boldsymbol{0}}(\boldsymbol{x}) u(\boldsymbol{x}) \right|^2 d\boldsymbol{x} < \frac{(2\pi)^{\|m\|}}{\omega_1^{m_1} \cdots \omega_n^{m_n}} \int_\Omega |p_{\boldsymbol{0}}(\boldsymbol{x})| \left| u(\boldsymbol{x}) \right|^2 d\boldsymbol{x}. \tag{12}$$

On the other hand, from (8) and (11) we get the inequality

$$\int_\Omega |p_{\boldsymbol{0}}(\boldsymbol{x})| \left| u(\boldsymbol{x}) \right|^2 d\boldsymbol{x} \le \frac{\omega_1^{m_1} \cdots \omega_n^{m_n}}{(2\pi)^{\|\boldsymbol{m}\|}} \int_\Omega \left| u^{(\boldsymbol{m})}(\boldsymbol{x}) \right|^2 d\boldsymbol{x},$$

which contradicts to (12). The obtained contradiction completes the proof of the theorem. □

Remark 1. In Theorem 3 condition (7) is optimal and it cannot we weakened: strict inequality cannot be replaced by an unstrict one. Indeed, consider the equation

$$u^{(\boldsymbol{m})} = l\, u, \tag{13}$$

where l is a constant. If

$$0 < l < (-1)^m \frac{(2\pi)^{\|\boldsymbol{m}\|}}{\omega_1^{m_1} \cdots \omega_n^{m_n}},$$

then by Theorem 3 equation (13) has only a trivial solution. However, if

$$l = (-1)^m \frac{(2\pi)^{\|\boldsymbol{m}\|}}{\omega_1^{m_1} \cdots \omega_n^{m_n}} \qquad \big(l = 0\big),$$

then it is obvious that the function

$$u(\boldsymbol{x}) = \sin\Big(\frac{2\pi}{\omega_1} x_1\Big) \cdots \sin\Big(\frac{2\pi}{\omega_n} x_n\Big) \quad \big(u(\boldsymbol{x}) = 1\big)$$

is a nontrivial $\boldsymbol{\omega}$–solution of equation (13).

Below we formulate existence theorems.

Theorem 4. *Let $\boldsymbol{m}$ be even, and let along with* (3) *the inequalities*

$$(-1)^{\frac{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|}{2}} \int_{\Omega_{\boldsymbol{\alpha}}} p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})\, d\boldsymbol{x}_{\boldsymbol{\alpha}} < 0 \quad \textit{for} \quad \boldsymbol{\alpha} \in \mathcal{S}^{\boldsymbol{m}}, \tag{14}$$

$$\int_{\Omega} p_{\mathbf{0}}(\boldsymbol{x})\, d\boldsymbol{x} \neq 0$$

hold. Then equation (1) *has one and only one $\boldsymbol{\omega}$-periodic solution.*

Theorem 5. *Let m_1 be the only odd component of the the multiindex $\boldsymbol{m}$, and let there exist $j \in \{1, 2\}$ such that along with* (5) *the inequalities*

$$(-1)^{j+\frac{\|\boldsymbol{\alpha}\|}{2}} \int_{\Omega_{\boldsymbol{\alpha}}} p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})\, d\boldsymbol{x}_{\boldsymbol{\alpha}} < 0 \quad \textit{for} \quad \boldsymbol{\alpha} \in \mathcal{S}^{\boldsymbol{m}}, \tag{15}$$

$$(-1)^j \int_0^{\omega_2} \dots \int_0^{\omega_n} p_{\mathbf{0}}(x_1, x_2, \dots, x_n)\, dx_2 \dots dx_n < 0 \quad \textit{for} \quad x_1 \in \mathbb{R}$$

hold. Then equation (1) *has one and only one $\boldsymbol{\omega}$-periodic solution.*

Remark 2. In Theorems 4 (Theorem 5) condition (14) (condition (15)) is essential and it cannot be weakened. If for at least one $\boldsymbol{\alpha} \in \mathcal{S}^{\boldsymbol{m}}$ $p_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \equiv 0$, then equation (1) may not have an $\boldsymbol{\omega}$–periodic solution. To verify this, consider the equation

$$u^{(2,2,2)} = u^{(2,2,0)} + u^{(2,0,2)} + u^{(0,2,2)} - u^{(0,2,0)} - u^{(0,0,2)} + \sin^2(x_1)\, u - 1. \tag{16}$$

In the case, where $n = 3$, $m_1 = m_2 = m_3 = 2$ and $\omega_1 = \omega_2 = \omega_3 = \pi$, this equation satisfies all of the conditions of Theorem 4, except condition (14). For $\boldsymbol{\alpha} = (2, 0, 0)$ we have $p_{\boldsymbol{\alpha}}(x_2, x_3) \equiv 0$. As a result equation (16) has no (π, π, π)–periodic solution. Assume the contrary: let equation (16) have a (π, π, π)–periodic solution u. By Theorem 1, it is unique, and therefore is independent of x_2 and x_3. Hence u satisfies the equation

$$\sin^2(x_1)\, u - 1 = 0.$$

But the latter equation has only a discontinuous solution. The obtained contradiction proves that equation (16) has no (π, π, π)–periodic solution.

Theorem 6. *Let $\boldsymbol{m}$ be even, and let*

$$0 < (-1)^{\frac{\|\boldsymbol{m}\|}{2}} p_{\mathbf{0}}(\boldsymbol{x}) < \frac{(2\pi)^{\|m\|}}{\omega_1^{m_1} \cdots \omega_n^{m_n}}. \tag{17}$$

Moreover, let $p_{\mathbf{0}}$ and $q \in C^{\boldsymbol{m}}$. Then equation (2) *has one and only one $\boldsymbol{\omega}$-periodic solution.*

Theorem 7. *Let $\boldsymbol{m}$ be even, and let*

$$(-1)^{\frac{\|\boldsymbol{m}\|}{2}} p_{\mathbf{0}}(\boldsymbol{x}) < 0 \quad \textit{for} \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{18}$$

Moreover, let $p_{\mathbf{0}}$ and $q \in C^{\boldsymbol{m}}$. Then equation (2) *has one and only one $\boldsymbol{\omega}$-periodic solution.*

Theorem 8. *Let m be odd, and let there exist a number $j \in \{1, 2\}$ such that*

$$(-1)^j p_{\mathbf{0}}(\boldsymbol{x}) < 0 \quad \textit{for} \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{19}$$

Moreover, let $p_{\mathbf{0}}$ and $q \in C^{\boldsymbol{m}}$. Then equation (2) *has one and only one $\boldsymbol{\omega}$-periodic solution.*

Remark 3. In Theorems 6, 7 and 8 the requirement of additional regularity of functions $p_{\mathbf{0}}$ and q is sharp. If this condition is violated, then equation (2) may not have a $\boldsymbol{\omega}$–periodic classical solution. Indeed, consider the equation

$$u^{(\boldsymbol{m})} = p_{\mathbf{0}}(x_2, \dots, x_n)\, u - p_{\mathbf{0}}^2(x_2, \dots, x_n),$$

where $\boldsymbol{m}$ is even, and $p_{\mathbf{0}}(x_2, \ldots, x_n)$ is an arbitrary continuous $(\omega_2, \ldots, \omega_n)$–periodic function satisfying (18). By Theorem 3, this equation has at most one solution. Hence

$$u(\boldsymbol{x}) = p_{\mathbf{0}}(x_2, \ldots, x_n).$$

But u is a classical solution if and only if $p_{\mathbf{0}} \in C^{\boldsymbol{m}}$.

Remark 4. In Theorems 6, 7 and 8, respectively, the strict inequalities (17), (18) and (19) cannot be replaced by unstrict ones. To verify this, consider the equation

$$u^{(\boldsymbol{m})} = p_{\mathbf{0}}(x_2, \ldots, x_n)\, u - 1,$$

where $\boldsymbol{m}$ is odd and $p_{\mathbf{0}}(x_2, \ldots, x_n)$ is an smooth $(\omega_2, \ldots, \omega_n)$–periodic function such that $p_{\mathbf{0}}(x_2, \ldots, x_n) \geq 0$, $p_{\mathbf{0}}(x_2, \ldots, x_n) \not\equiv 0$. By Theorem 2, this equation has at most one solution. Therefore u is a solution of the equation

$$p_{\mathbf{0}}(x_2, \ldots, x_n)\, u - 1 = 0.$$

But the latter equation has a continuous solution if and only if

$$p_{\mathbf{0}}(x_2, \ldots, x_n) > 0 \quad \text{for} \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

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Author's address:

T. Kiguradze
Florida Institute of Technology
Department of Mathematical Sciences
150 W. University Blvd.
Melbourne, Fl 32901
USA
E-mail: tkigurad@fit.edu