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ON SOME BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Optimal in a sense sufficient conditions are established for the solvability and unique solvability of the boundary value problems of the type

$$u^{(iv)}(t) = g(u)(t),$$

$$u(a) = 0, \quad u(b) = 0, \quad \sum_{k=1}^{2} \big(\alpha_{ik} u^{(k)}(a) + \beta_{ik} u^{(k)}(b)\big) = 0 \quad (i = 1, 2),$$

where $g : C^1([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a continuous operator, α_{ik} and β_{ik} $(i, k = 1, 2)$ are real constants such that

$$\sum_{i=1}^{2} \Big| \sum_{k=1}^{2} (\alpha_{ik} x_k + \beta_{ik} y_k) \Big| > 0 \ \text{ for } \ x_1 x_2 < y_1 y_2.$$

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რეზიუმე. დადგენილია

$$u^{(iv)}(t) = g(u)(t),$$

$$u(a) = 0, \quad u(b) = 0, \quad \sum_{k=1}^{2} \big(\alpha_{ik} u^{(k)}(a) + \beta_{ik} u^{(k)}(b)\big) = 0 \quad (i = 1, 2)$$

სახის სასაზღვრო ამოცანების ამოხსნადობისა და ცალსახად ამოხსნადობის ოპტიმალური საკმარისი პირობები, სადაც $g : C^1([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ უწყვეტი ოპერატორია, ხოლო α_{ik} და β_{ik} $(i, k = 1, 2)$ ისეთი ნამდვილი მუდმივებია, რომ

$$\sum_{i=1}^{2} \Big| \sum_{k=1}^{2} (\alpha_{ik} x_k + \beta_{ik} y_k) \Big| > 0, \ \text{ როცა } \ x_1 x_2 < y_1 y_2.$$

Let $-\infty < a < b < +\infty$, C^1 be the space of continuously differentiable functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_{C^1} = \max\{|u(t)| + |u'(t)| : \ a \le t \le b\}$, L be the space of Lebesgue integrable functions $v : [a,b] \to \mathbb{R}$ with the norm $\|v\|_L = \int_a^b |v(t)|\,dt$, and let $g : C^1 \to L$ be a continuous operator satisfying the condition $g^*(\rho)(\cdot) \in L$ for $0 < \rho < +\infty$, where $g^*(\rho)(t) = \sup\{|g(u)(t)| : \ u \in C^1, \ \|u\|_{C^1} \le \rho\}$.

Consider the functional differential equation

$$u^{(iv)}(t) = g(u)(t) \tag{1}$$

with the boundary conditions

$$u(a) = 0, \ \ u(b) = 0, \ \ \sum_{k=1}^{2} \big(\alpha_{ik} u^{(k)}(a) + \beta_{ik} u^{(k)}(b)\big) = 0 \ \ (i = 1,2), \tag{2}$$

where the constants α_{ik} and β_{ik} $(i,k = 1,2)$ are such that

$$\sum_{i=1}^{2} \Big| \sum_{k=1}^{2} (\alpha_{ik} x_k + \beta_{ik} y_k) \Big| > 0 \ \text{ for } \ x_1 x_2 < y_1 y_2. \tag{3}$$

The particular cases of (1) are the differential equations

$$u^{(iv)}(t) = f\big(t, u(\tau_1(t)), u'(\tau_2(t))\big), \tag{1_1}$$

$$u^{(iv)}(t) = f\big(t, u(t), u'(t)\big), \tag{1_2}$$

and the particular cases of (2) are the boundary conditions

$$u(a) = 0, \ \ u(b) = 0, \ \ \alpha_1 u'(a) + \alpha_2 u''(a) = 0, \ \ \beta_1 u'(b) + \beta_2 u''(b) = 0, \tag{2_1}$$

$$u(a) = 0, \ \ u(b) = 0, \ \ u'(a) = \alpha u'(b), \ \ u''(b) = \alpha u''(a), \tag{2_2}$$

$$u(a) = 0, \ \ u(b) = 0, \ \ u'(a) = u'(b), \ \ u'(a) = u''(b). \tag{2_3}$$

Here $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, $\tau_i : [a,b] \to [a,b]$ $(i = 1,2)$ are measurable functions, $\alpha \ne 0$ and α_i, β_i $(i = 1,2)$ are constants satisfying the inequalities

$$\alpha_1 \alpha_2 \le 0, \ \ \beta_1 \beta_2 \ge 0, \ \ |\alpha_1| + |\alpha_2| > 0, \ \ |\beta_1| + |\beta_2| > 0. \tag{3_1}$$

By $\widetilde{C}^3$ we denote the space of functions $u : [a,b] \to \mathbb{R}$ absolutely continuous along with their first three derivatives, and by $\widetilde{C}^3_0$ we denote the set of all $u \in \widetilde{C}^3$ satisfying the boundary conditions (2). The function $u \in \widetilde{C}^3_0$ is said to be **a solution of the problem (1), (2)** if it almost everywhere on $[a,b]$ satisfies the equation (1).

Theorem 1. *Let there exist $\ell \in [0,1[$ and $\ell_0 \ge 0$, such that for an arbitrary $u \in \widetilde{C}^3_0$ the inequality*

$$\int_a^b g(u)(t) u(t)\,dt \le \ell \int_a^b u''^2(t)\,dt + \ell_0 \tag{4}$$

is fulfilled. Then the problem (1), (2) *has at least one solution.*

To prove this theorem, we will need the following

Lemma 1. *If $\ell \in [0,1[$ and $\ell_0 \geq 0$, then an arbitrary function $u \in \widetilde{C}_0^3$ satisfying the integral inequality*

$$\int_a^b u^{(iv)}(t)u(t)\,dt \leq \ell \int_a^b u''^2(t)\,dt + \ell_0 \tag{5}$$

admits the estimate

$$\|u\|_{C'} \leq r_0, \quad \int_a^b u''^2(t)\,dt \leq r_0^2, \tag{6}$$

where

$$r_0 = (1+b-a)\Big(\frac{\ell_0}{1-\ell}\Big)^{1/2}(b-a)^{1/2}. \tag{7}$$

Proof. According to the formula of integration by parts, by virtue of the conditions (2) and (3) we have

$$\int_a^b u^{(iv)}(t)u(t)\,dt = u'''(b)u(b) - u'''(a)u(a) + u'(a)u''(a) - u'(b)u''(b) +$$
$$+\int_a^b u''^2(t)\,dt = u'(a)u''(a) - u'(b)u''(b) + \int_a^b u''^2(t)\,dt,$$
$$u'(a)u''(a) \geq u'(b)u''(b),$$

and hence

$$\int_a^b u^{(iv)}(t)u(t)\,dt \geq \int_a^b u''^2(t)\,dt.$$

Therefore from the inequality (5) we find that

$$\int_a^b u''^2(t)\,dt \leq \ell \int_a^b u''^2(t)\,dt + \ell_0 \quad \text{and} \quad \int_a^b u''^2(t)\,dt \leq \frac{\ell_0}{1-\ell}\,.$$

On the other hand, by the condition $u(a) = u(b) = 0$ there exists $t_0 \in]a,b[$ such that $u'(t_0) = 0$. Therefore

$$|u'(t)| = \Big|\int_{t_0}^t u''(s)\,ds\Big| \leq (b-a)^{1/2}\Big(\int_a^b u''^2(s)\,ds\Big)^{1/2} \leq$$
$$\leq \Big(\frac{\ell_0}{1-\ell}\Big)^{1/2}(b-a)^{1/2} \quad \text{for } a \leq t \leq b,$$
$$|u(t)| = \Big|\int_a^t u'(s)\,ds\Big| \leq \Big(\frac{\ell_0}{1-\ell}\Big)^{1/2}(b-a)^{3/2} \quad \text{for } a \leq t \leq b.$$

Consequently, the estimate (6) is valid. □

By Lemma 1, the differential equation $u^{(iv)}(t) = 0$ under the boundary conditions (2) has only a trivial solution. Taking this fact into consideration, Corollary 2 of [2] leads to

Lemma 2. *Let there exist a positive constant r such that for every $\lambda \in]0,1[$ an arbitrary solution u of the differential equation*

$$u^{(iv)}(t) = \lambda g(u)(t) \tag{8}$$

satisfying the boundary conditions (2) *admits the estimate*

$$\sum_{i=1}^{4} |u^{(i-1)}(t)| \le r \ \ for \ \ a \le t \le b. \tag{9}$$

Then the problem (1), (2) *has at least one solution.*

Proof of Theorem 1. Let r_0 be the number given by the equality (7), and

$$r = r_0 + 4r_0(b-a)^{-3/2} + 6r_0(b-a)^{-1/2} + (1+b-a)\int_a^b g^*(r_0)(s)\,ds.$$

According to Lemma 2, to prove Theorem 1 it suffices to establish that for every $\lambda \in]0,1[$ an arbitrary solution u of the problem (8), (2) admits the estimate (9).

By virtue of the condition (4), every solution of the problem (8), (2) satisfies the integral inequality (5). This fact by Lemma 1 ensures the validity of the estimates (6). Therefore from (8) we have $|u^{(iv)}(t)| \le g^*(r_0)(t)$ for almost all $t \in [a,b]$. On the other hand, the existence of the points $t_1 \in [a, \frac{3a+b}{4}]$, $t_0 \in [\frac{3b+a}{4}, b]$, $t_0 \in]t_1, t_2[$ such that

$$|u''(t_i)| \le 2r_0(b-a)^{-1/2} \ \ (i=1,2),$$
$$|u'''(t_0)| = (t_2-t_1)^{-1}\big|u''(t_2) - u''(t_1)\big| \le 4r_0(b-a)^{-3/2},$$

is obvious. Therefore

$$|u'''(t)| \le 4r_0(b-a)^{-3/2} + \int_a^b g^*(r_0)(s)\,ds \ \text{ for } \ a \le t \le b,$$
$$|u''(t)| \le 6r_0(b-a)^{-1/2} + (b-a)\int_a^b g^*(r_0)(s)\,ds \ \text{ for } \ a \le t \le b.$$

If along with the above-said we take into account (6), the validity of the estimate (9) becomes clear. □

Theorem 2. *Let there exist* $\ell \in [0,1[$ *such that for arbitrary* u *and* $v \in \widetilde{C}_0^3$ *the inequality*

$$\int_a^b \big(g(u)(t) - g(v)(t)\big)\big(u(t) - v(t)\big)\,dt \le \ell \int_a^b \big(u''(t) - v''(t)\big)^2\,dt \tag{10}$$

is fulfilled. Then the problem (1), (2) *has one and only one solution.*

Proof. For $v(t) \equiv 0$, from (10) we obtain the inequality

$$\int_a^b g(u)(t)u(t)\,dt \le \ell \int_a^b u''^2(t)\,dt + \int_a^b g(0)(t)u(t)\,dt.$$

On the other hand, by virtue of $u(b) = u(a) = 0$ we have

$$|u(t)| \le \frac{b-a}{4}\int_a^b |u''(s)|\,ds \le \int_a^b \Big(\frac{1-\ell}{2\rho}\,u''^2(s) + \frac{(b-a)^2\rho}{32(1-\ell)}\Big)\,ds,$$

where $\rho = 1 + \int_a^b |g(0)(t)|\,dt$. Therefore

$$\int_a^b g(u)(t)u(t)\,dt \le \ell_1 \int_a^b u''^2(t)\,dt + \ell_0,$$

where $\ell_1 = \frac{1+\ell}{2}$, $\ell_0 = \frac{(b-a)^3\rho^2}{32(1-\ell)}$. However, by Theorem 1, the last inequality guarantees the solvability of the problem (1), (2).

It remains for us to prove that the problem (1), (2) has at most one solution. Let u and v be arbitrary solutions of that problem, and $w(t) = u(t) - v(t)$. Then $w \in \widetilde{C}_0^3$. On the other hand, by the condition (10) we have

$$\int_a^b w^{(iv)}(t)w(t)\,dt \le \ell \int_a^b w''^2(t)\,dt$$

whence by Lemma 1 it follows that $w(t) \equiv 0$, and consequently $u(t) \equiv v(t)$. □

Before we proceed to the problem (1), (2), we will cite one lemma which is a simple corollary of Wirtinger's theorem.

Lemma 3. *Let $u : [a,b] \to \mathbb{R}$ be a twice continuously differentiable function such that*

$$u(a) = 0, \quad u(b) = 0. \tag{11}$$

Then

$$\int_a^b u^2(t)\,dt \le \Big(\frac{b-a}{\pi}\Big)^4 \int_a^b u''^2(t)\,dt, \quad \int_a^b u'^2(t)\,dt \le \Big(\frac{b-a}{\pi}\Big)^2 \int_a^b u''^2(t)\,dt. \tag{12}$$

If, however, along with (11) *the condition*

$$u'(a) = u'(b) \tag{13}$$

is fulfilled, then

$$\int_a^b u^2(t)\,dt \le \frac{1}{4}\Big(\frac{b-a}{\pi}\Big)^4 \int_a^b u''^2(t)\,dt, \quad \int_a^b u'^2(t)\,dt \le \Big(\frac{b-a}{2\pi}\Big)^2 \int_a^b u''^2(t)\,dt. \tag{14}$$

Proof. Applying along with (11) the formula of integration by parts and the Schwartz inequality, we obtain

$$\int_a^b u'^2(t)\,dt = \int_a^b u(t)u''(t)\,dt \le \Big(\int_a^b u^2(t)\,dt\Big)^{1/2}\Big(\int_a^b u''^2(t)\,dt\Big)^{1/2}.$$

On the other hand, by Theorem 256 of [1] we have

$$\int_a^b u^2(t)\,dt \le \Big(\frac{b-a}{\pi}\Big)^2 \int_a^b u'^2(t)\,dt. \tag{15}$$

The last two inequalities result in the inequalities (12).

Assume now that along with (11) the condition (13) is fulfilled. Then by Theorem 258 of [1], along with (15) we have

$$\int_a^b u'^2(t)\,dt \le \Big(\frac{b-a}{2\pi}\Big)^2 \int_a^b u''^2(t)\,dt.$$

Consequently, the inequalities (14) are valid. □

We introduce the sets

$$I_0 = \{t \in [a,b] : \ \tau_1(t) = t\}, \quad I_1 = [a,b] \setminus I_0$$

and the numbers

$$\delta_i = \Big(\int_a^b |\tau_i(t) - t| \, dt \Big)^{1/2} \ \ (i = 1, 2).$$

The following theorem holds.

Theorem 3. *Let there exist nonnegative constants* ℓ_i $(i = 1, 2)$ *and a function* $h \in L$ *such that*

$$\Big(\frac{b-a}{\pi} + \delta_1\Big)\Big(\frac{b-a}{\pi}\Big)^3 \ell_1 + \Big(\frac{b-a}{\pi} + \delta_2\Big)\Big(\frac{b-a}{\pi}\Big)^2 \ell_2 < 1 \tag{16}$$

and the conditions

$$f(t,x,y) \operatorname{sgn} x \le \ell_1|x| + \ell_2|y| + h(t) \ \ \text{for} \ \ t \in I_0, \ \ (x,y) \in \mathbb{R}^2, \tag{17}$$

$$|f(t,x,y)| \le \ell_1|x| + \ell_2|y| + h(t) \ \ \text{for} \ \ t \in I_1, \ \ (x,y) \in \mathbb{R}^2 \tag{18}$$

are fulfilled. Then the problem (1), (2) *has at least one solution.*

Proof. We choose $\ell_3 > 0$ in such a way that

$$\ell = \Big(\frac{b-a}{\pi} + \delta_1\Big)\Big(\frac{b-a}{\pi}\Big)^3 \ell_1 + \Big(\frac{b-a}{\pi} + \delta_2\Big)\Big(\frac{b-a}{\pi}\Big)^2 \ell_2 + \ell_3 < 1. \tag{19}$$

If we put

$$g(u)(t) = f\big(t, u(\tau_1(t)), u'(\tau_2(t))\big), \tag{20}$$

then the equation (1_1) takes the form (1). On the other hand, by the conditions (17) and (18), almost everywhere on $[a,b]$ the inequality

$$g(u)(t)u(t) \le \ell_1\big|u(t)u(\tau_1(t))\big| + \ell_2\big|u(t)u'(\tau_2(t))\big| + h(t)|u(t)|$$

is fulfilled. Therefore

$$\int_a^b g(u)(t)u(t) \le$$

$$\le \ell_1 \int_a^b \big|u(t)u(\tau_1(t))\big| \, dt + \ell_2 \int_a^b \big|u(t)u'(\tau_2(t))\big| \, dt + \int_a^b h(t)|u(t)| \, dt. \tag{21}$$

By Lemma 3, the function u satisfies the inequalities (12) from which we find that

$$\int_a^b \big|u(t)u(\tau_1(t))\big| \, dt \le \int_a^b u^2(t) \, dt + \int_a^b |u(t)| \Big| \int_t^{\tau_1(t)} u'(s) \, ds \Big| \, dt \le$$

$$\le \int_a^b u^2(t) \, dt + \Big(\int_a^b u^2(t) \, dt \Big)^{1/2} \Big(\int_a^b \Big(\int_t^{\tau_1(t)} u'(s) \, ds \Big)^2 dt \Big)^{1/2} \le$$

$$\le \int_a^b u^2(t) \, dt + \delta_1 \Big(\int_a^b u^2(t) \, dt \Big)^{1/2} \Big(\int_a^b u'^2(s) \, ds \Big)^{1/2} \le$$

$$\leq \Big(\frac{b-a}{\pi}+\delta_1\Big)\Big(\frac{b-a}{\pi}\Big)^3\int_a^b u''^2(t)\,dt, \tag{22}$$

$$\int_a^b\big|u(t)u'(\tau_2(t))\big|\,dt \leq \int_a^b\big|u(t)u'(t)\big|\,dt+\int_a^b|u(t)|\Big|\int_t^{\tau_2(t)}u'(s)\,ds\Big|\,dt \leq$$
$$\leq \Big(\int_a^b u^2(t)\,dt\Big)^{1/2}\Big[\Big(\int_a^b u'^2(t)\,dt\Big)^{1/2}+\delta_2\Big(\int_a^b u''^2(t)\,dt\Big)^{1/2}\Big] \leq$$
$$\leq \Big(\frac{b-a}{\pi}+\delta_2\Big)\Big(\frac{b-a}{\pi}\Big)^2\ell_2\int_a^b u''^2(t)\,dt \tag{23}$$

and

$$\int_a^b h(t)|u(t)|\,dt=\int_a^b h(t)\Big|\int_a^t u'(s)\,ds\Big|\,dt\leq(b-a)^{1/2}\|h\|_L\Big(\int_a^b u'^2(t)\,dt\Big)^{1/2} \leq$$
$$\leq \frac{(b-a)^{3/2}}{\pi}\,\|h\|_L\Big(\int_a^b u''^2(t)\,dt\Big)^{1/2} \leq \ell_3\int_a^b u''^2(t)\,dt+\ell_0, \tag{24}$$

where $\ell_0=\frac{(b-a)^3}{4\pi^2\ell_3}\,\|h\|_L^2$.

With regard for the inequalities (19) and (22)–(24), from (21) we obtain inequality (4), where $\ell<1$. Consequently, all the conditions of Theorem 1 are fulfilled, which guarantees the solvability of the problem (1), (2). □

Theorem 4. *Let there exist nonnegative, satisfying inequality* (16) *constants* ℓ_1 *and* ℓ_2 *such that the conditions*

$$\big[f(t,x,y)-f(t,\overline{x},\overline{y})\big]\operatorname{sgn}(x-\overline{x}) \leq \ell_1|x-\overline{x}|+\ell_2|y-\overline{y}| \tag{25}$$
$$\text{for } t\in I_0,\ (x,y)\in\mathbb{R}^2,\ (\overline{x},\overline{y})\in\mathbb{R}^2,$$
$$\big|f(t,x,y)-f(t,\overline{x},\overline{y})\big| \leq \ell_1|x-\overline{x}|+\ell_2|y-\overline{y}| \tag{26}$$
$$\text{for } t\in I_1,\ (x,y)\in\mathbb{R}^2,\ (\overline{x},\overline{y})\in\mathbb{R}^2$$

are fulfilled. Then the problem (1), (2) *has one and only one solution.*

Proof. Let $\ell=(\frac{b-a}{\pi}+\delta_1)(\frac{b-a}{\pi})^3\ell_1+(\frac{b-a}{\pi}+\delta_2)(\frac{b-a}{\pi})^2\ell_2$. Then by Theorem 2 and the condition (16), in order to prove Theorem 4 it suffices to establish that the operator g given by the equality (20) for arbitrary u and $v\in\widetilde{C}_0^3$ satisfies the condition

$$\int_a^b\big(g(u+w)(t)-g(u)(t)\big)w(t)\,dt \leq \ell\int_a^b w''^2(t)\,dt. \tag{27}$$

By virtue of (20), (25) and (26), we have

$$\int_a^b\big(g(u+w)(t)-g(u)(t)\big)w(t)\,dt\leq\ell_1\int_a^b\big|w(t)w(\tau_1(t))\big|\,dt+\ell_2\int_a^b\big|w(t)w'(\tau_2(t))\big|\,dt.$$

However, when proving Theorem 3 we have established that an arbitrary function $w\in\widetilde{C}_0^3$ satisfies the condition

$$\ell_1\int_a^b\big|w(t)w(\tau_1(t))\big|\,dt+\ell_2\int_a^b\big|w(t)w'(\tau_1(t))\big|\,dt \leq \ell\int_a^b w''^2(t)\,dt.$$

Consequently, the inequality (27) is valid. □

If $\alpha_{11}=\alpha_1$, $\alpha_{12}=\alpha_2$, $\beta_{11}=\beta_{12}=0$, $\alpha_{21}=\alpha_{22}=0$, $\beta_{21}=\beta_1$, $\beta_{22}=\beta_2$, then by virtue of (3_1) the condition (3) is fulfilled. The same condition is obviously fulfilled for $\alpha_{11}=\beta_{11}=1$, $\beta_{22}=\alpha_{22}=\alpha$, $\alpha_{12}=\alpha_{21}=0$, $\beta_{12}=\beta_{21}=0$. Therefore from Theorems 3 and 4 we have

Corollary 1. *Let there exist nonnegative, satisfying inequality* (16) *constants* ℓ_1 *and* ℓ_2*, such that the conditions* (17) *and* (18) (*the conditions* (25) *and* (26)) *are fulfilled. Then the problem* $(1_1),(2_1)$*, as well as the problem* $(1_1),(2_2)$ *has at least one solution* (*one and only one solution*).

For $\tau_i(t)\equiv t$ $(i=1,2)$, from Theorems 3, 4 and Corollary 1 we obtain

Theorem 5. *Let there exist nonnegative constants* ℓ_1 *and* ℓ_2*, such that*

$$\Big(\frac{b-a}{\pi}\Big)^4\ell_1+\Big(\frac{b-a}{\pi}\Big)^3\ell_2<1 \tag{28}$$

and the condition (17) (*the condition* (25)) *is fulfilled, where* $I_0=[a,b]$. *Then each of the problems* $(1_2),(2)$; $(1_2),(2_1)$ *and* $(1_2),(2_2)$ *has at least one solution* (*one and only one solution*).

As an example, we consider the linear differential equation

$$u^{(iv)}(t)=p_1(t)u(t)+p_2(t)u'(t)+q(t) \tag{29}$$

with Lebesgue integrable coefficients p_1, p_2, $q:[a,b]\to\mathbb{R}$. From Theorem 5 we get

Corollary 2. *Let almost everywhere on* $[a,b]$ *the inequalities*

$$p_1(t)\le\ell_1,\quad |p_2(t)|\le\ell_2, \tag{30}$$

be fulfilled, where ℓ_1 *and* ℓ_2 *are nonnegative constants satisfying the condition* (28). *Then the problem* (29), (2) *and, consequently each of the problems* $(29),(2_1)$ *and* $(29),(2_2)$ *has one and only one solution.*

If $p_1(t)\equiv\ell_1=(\frac{\pi}{b-a})^4$, $p_2(t)\equiv\ell_2=0$ and $\alpha=-1$, then it is obvious that (30) is fulfilled, but instead of (28) we have $(\frac{b-a}{\pi})^4\ell_1+(\frac{b-a}{\pi})^3\ell_2\le 1$. Nevertheless, the homogeneous equation $u^{(iv)}(t)=p_1(t)u(t)+p_2(t)u'(t)$ has the nontrivial solution $u(t)=\sin\frac{\pi(t-a)}{b-a}$ satisfying the boundary conditions (2_2). Therefore there exists $q\in L$ such that the problem $(29),(2_2)$ has no solution.

The above-constructed example shows that in Theorems 1 and 2 the condition $\ell<1$ is optimal, and it cannot be replaced by the condition $\ell\le 1$.

Analogously, in Theorems 3 and 4 and in Corollary 1 (in Theorem 5 and Corollary 2) the strict inequality (16) (the strict inequality (28)) cannot be replaced by the nonstrict inequality.

Theorem 6. *Let there exist nonnegative constants* ℓ_1 *and* ℓ_2 *such that*

$$\Big(\frac{b-a}{\pi}+\delta_1\Big)\Big(\frac{b-a}{\pi}\Big)^3\ell_1+\Big(\frac{b-a}{\pi}+2\delta_2\Big)\Big(\frac{b-a}{\pi}\Big)^2\ell_2<4$$

and the conditions (17) *and* (18) (*the conditions* (25) *and* (26)) *are fulfilled. Then the problem* $(1_1), (2_3)$ *has at least one solution* (*one and only one solution*).

This theorem can be proved just in the same way as Theorems 3 and 4. The only difference in the proof is that instead of the inequalities (12) we use the inequalities (14).

For $\tau_i(t) \equiv t$ $(i = 1, 2)$, from Theorem 6 we have

Theorem 7. *Let there exist nonnegative constants* ℓ_1 *and* ℓ_2*, such that*

$$\Big(\frac{b-a}{\pi}\Big)^4 \ell_1 + \Big(\frac{b-a}{\pi}\Big)^3 \ell_2 < 4 \tag{31}$$

and the condition (17) (*the condition* (25)) *is fulfilled, where* $I_0 = [a, b]$. *Then the problem* $(1_2), (2_3)$ *has at least one solution* (*one and only one solution*).

Corollary 3. *Let almost everywhere on* $[a, b]$ *the inequalities* (30) *be fulfilled, where* ℓ_1 *and* ℓ_2 *are nonnegative constants satisfying the condition* (31)*. Then the problem* $(29), (2_3)$ *has one and only one solution.*

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