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THE DIRICHLET METAHARMONIC GREEN'S FUNCTION FOR UNBOUNDED REGIONS

Abstract. Uniform bounds are obtained for the Dirichlet Green's function for the Helmholtz equation and its derivatives in the case of a two–dimensional unbounded domain with an infinite rough boundary.

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1. Introduction

Since their introduction in 1828, Green's functions have become a fundamental mathematical technique for solving boundary–value problems. Properties of Green's functions and matrices for various elliptic boundary value problems for bounded and unbounded domains with compact boundaries are investigated in detail by many authors (see, e.g., [11], [12], [15], [26], [3], [16], [22], [27], [28], [29], [19], [13], [20], [24] and the references therein). To the authors' best knowledge Green's functions for unbounded regions with boundary extending to infinity have not been treated systematically in the literature (except the particular cases: a cone, a half-space (half–plane) and a strip say; see, e.g., [5], [21]).

Here we consider the Dirichlet Green's function for the Helmholtz equation (reduced wave equation) in a non–locally perturbed half–plane with a one–dimensional, infinite, smooth boundary line. Our goal is to investigate properties of the Green's function and its derivatives in a neighbourhood of the pole and at infinity which are very important in many applications.

Our study is based on the existence and uniqueness results for the corresponding Dirichlet problem in various functional spaces.

In [7], with the help of the appropriate integral equation formulation, it is shown that the Dirichlet problem for all wave numbers and for a non–locally perturbed half–plane with no limit on the boundary curve amplitudes or slopes has exactly one solution satisfying the *upward propagating radiation condition* (UPRC) provided that a Dirichlet datum is bounded and continuous function. The UPRC, suggested by Chandler–Wilde & Zhang [8], generalizes both the *Sommerfeld radiation condition* and the *Rayleigh expansion condition* for diffraction gratings (see also [25], [4], [17], [9]).

An important corollary of these results in the scattering theory is that for a variety of incident fields including the incident plane wave, the Dirichlet boundary–value problem (BVP) for the scattered field has a unique solution (for detail information concerning the history of the problem see, e.g., [7] and the references therein).

In this paper we apply the above results along with the theory of second kind integral equations on unbounded domains, developed in [1] in the case of weighted spaces. This enables us to isolate the so–called principal singular part of the Dirichlet Green's function and its derivatives, and establish uniform bounds for them on a closed, non–locally perturbed half–plane.

2. Preliminary Material

2.1. Here we introduce some notation used throughout.

For $h \in \mathbb{R}$, define $\Gamma_h = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = h\}$ and $U_h^+ = \{x \in \mathbb{R}^2 \mid x_2 > h\}$.

For $V \subset \mathbb{R}^n$ $(n = 1, 2)$ we denote by $BC(V)$ the set of functions bounded and continuous on V, a Banach space under the norm $\|\cdot\|_{\infty,V}$, defined by $\|\psi\|_{\infty,V} := \sup\limits_{x\in V} |\psi(x)|$. We abbreviate $\|\cdot\|_{\infty,\mathbb{R}}$ by $\|\cdot\|_{\infty}$.

For $0 < \alpha \le 1$, we denote by $BC^{0,\alpha}(V)$ the Banach space of functions $\varphi \in BC(V)$, which are uniformly Hölder continuous with exponent α, with the norm $\|\cdot\|_{0,\alpha,V}$ defined by

$$\|\varphi\|_{0,\alpha,V} := \|\varphi\|_{\infty,V} + \sup_{x,y\in V,\, x\neq y}\left[\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}\right].$$

Given $v \in L_\infty(V)$ denote by $\partial_j v$, $j = 1, 2$, the (distributional) derivative $\dfrac{\partial v(x)}{\partial x_j}$ and $\nabla v := (\partial_1 v, \partial_2 v)$.

We denote by $BC^1(V)$ the Banach space

$$BC^1(V) := \{\varphi \in BC(V) \mid \partial_j\varphi \in BC(V),\ j = 1, 2\}$$

under the norm

$$\|\varphi\|_{1,V} := \|\varphi\|_{\infty,V} + \|\partial_1\varphi\|_{\infty,V} + \|\partial_2\varphi\|_{\infty,V}.$$

Further, let

$$BC^{1,\alpha}(V) := \{\varphi \in BC^1(V) \mid \partial_j\varphi \in BC^{0,\alpha}(V),\ j = 1, 2\}$$

denote a Banach space under the norm

$$\|\varphi\|_{1,\alpha,V} := \|\varphi\|_{\infty,V} + \|\partial_1\varphi\|_{0,\alpha,V} + \|\partial_2\varphi\|_{0,\alpha,V}.$$

2.2. Given $f \in BC^{1,\alpha}(\mathbb{R})$, $0 < \alpha \le 1$, with $f_- := \inf\limits_{x_1\in\mathbb{R}} f(x_1) > 0$ and $f_+ := \sup\limits_{x_1\in\mathbb{R}} f(x_1) < +\infty$, define

$$\begin{aligned}
&\Omega_f^+ := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > f(x_1)\},\\
&\Gamma_f := \{(x_1, f(x_1)) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\},\\
&\overline{\Omega_f^+} := \Omega_f^+ \cup \Gamma_f, \quad \Omega_f^- := \mathbb{R}^2 \setminus \overline{\Omega_f^+}.
\end{aligned}$$

Let $\nu(x) = (\nu_1(x), \nu_2(x))$ stand for the unit normal vector to Γ_f at the point $x \in \Gamma_f$ directed out of Ω_f^+, and $\partial_{\nu(x)} = \partial/\partial\nu(x)$ and $\partial_{\tau(x)} = \partial/\partial\tau(x)$ denote the usual normal and tangent derivatives on Γ_f, respectively.

2.3. Denote by

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \ \ (x, y) \in \mathbb{R}^2,\ x \neq y, \tag{2.1}$$

the free–space Green's function (fundamental solution) for the Helmholtz operator $\Delta + k^2$, where $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator and k is a positive constant (a wave number), and $H_m^{(1)}$ is the Hankel function of the first kind of order m.

Definition 2.1. Given a domain $V \subset \mathbb{R}^2$, call $v \in C^2(V) \cap L_\infty(V)$ a radiating solution of the Helmholtz equation in V if $\Delta v + k^2 v = 0$ in V and

$$v(x) = O(r^{-1/2}), \quad \frac{\partial v(x)}{\partial r} - ikv(x) = o(r^{-1/2}), \ \ r = |x|, \tag{2.2}$$

as $r = |x| \to +\infty$, uniformly in $x/|x|$.

The relations (2.2) are the classical Sommerfeld radiation conditions. The set of radiating functions corresponding to the domain V and the parameter k we denote by $\mathrm{Som}(V;k)$.

Definition 2.2. Given a domain $V \subset \mathbb{R}^2$, say that $v : V \to \mathbb{C}$, a solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in V, satisfies the upward propagating radiation condition in V if, for some $h \in \mathbb{R}$ and $\varphi \in L_\infty(\Gamma_h)$, it holds that $U_h^+ \subset V$ and

$$v(x) = 2 \int\limits_{\Gamma_h} \frac{\partial \Phi(x,y)}{\partial y_2} \varphi(y)\, dS_y, \quad x \in U_h^+.$$

The set of functions satisfying UPRC in V with the parameter k we denote by $\mathrm{UPRC}(V;k)$. Note that $\mathrm{Som}(V;k) \subset \mathrm{UPRC}(V;k)$ (for details see [8], [7]).

2.4. Let us introduce the Dirichlet Green's function $G^{(\mathcal{D})}(x,y)$ and the impedance Green's function $G^{(\mathcal{I})}(x,y)$ for the Helmholtz operator $\Delta + k^2$ in the half–plane U_0^+ (for details see [6], [8], [7]):

$$\begin{aligned}
&G^{(\mathcal{D})}(x,y) = \Phi(x,y) - \Phi(x,y'),\\
&G^{(\mathcal{I})}(x,y) = \Phi(x,y) + \Phi(x,y') + P(x-y'), \qquad (2.3)\\
&P(z) := -\frac{ik}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\exp\{i[z_1 t + z_2\sqrt{k^2-t^2}]\}}{\sqrt{k^2-t^2}[\sqrt{k^2-t^2}+k]}\, dt = \\
&\qquad = \frac{|z|e^{ik|z|}}{\pi} \int\limits_0^\infty \frac{t^{-1/2} e^{-k|z|t}\,[|z| + z_2(1+it)]}{\sqrt{t-2i}\,[|z|t - i(|z|+z_2)]^2}\, dt, \quad z \in U_0^+,\\
&x = (x_1,x_2) \in U_0^+, \quad y = (y_1,y_2) \in U_0^+,\\
&y' := (y_1,-y_2) \in U_0^- := \mathbb{R}^2 \setminus \overline{U_0^+}.
\end{aligned}$$

Here and throughout all square roots are taken with non–negative real and imaginary parts.

Note that these Green's functions are radiating in U_0^+: $G^{(\mathcal{D})}(\cdot,y), G^{(\mathcal{I})}(\cdot, y) \in \mathrm{Som}(U_0^+,k)$ and satisfy the Helmholtz equation in U_0^+ (with respect to x and y) and the Dirichlet and the impedance boundary conditions, respectively, on $\Gamma_0 = \partial U_0^+$:

$$\begin{aligned}
&G^{(\mathcal{D})}(x,y) = 0 \quad \text{for} \quad x \in \Gamma_0,\ y \in \overline{U_0^+},\ x \neq y,\\
&\frac{\partial}{\partial x_2} G^{(\mathcal{I})}(x,y) + ikG^{(\mathcal{I})}(x,y) = 0 \quad \text{for} \quad x \in \Gamma_0,\ y \in \overline{U_0^+},\ x \neq y.
\end{aligned}$$

Moreover, $G^{(\mathcal{D})}(x,y) = G^{(\mathcal{D})}(y,x)$, $G^{(\mathcal{I})}(x,y) = G^{(\mathcal{I})}(y,x)$, and there hold the following inequalities for $x, y \in U_0^+$ and $G \in \{G^{(\mathcal{D})}, G^{(\mathcal{I})}\}$:

$$\begin{aligned}
&|G(x,y)|,\ |\nabla_x G(x,y)|,\ |\nabla_y G(x,y)| \leq \\
&\qquad\qquad \leq a_0 \, \frac{(1+x_2)(1+y_2)}{|x-y|^{3/2}} \ \text{ for } \ |x-y| \geq \delta, \\
&|G(x,y)| \leq a_0 \, (1 + |\ln|x-y||) \ \text{ for } \ 0 < |x-y| \leq \delta, \\
&|\nabla_x G(x,y)|,\ |\nabla_y G(x,y)| \leq a_0 \, |x-y|^{-1} \ \text{ for } \ 0 < |x-y| \leq \delta, \\
&|G(x,y)|,\ |\nabla_x G(x,y)|,\ |\nabla_y G(x,y)|,\ |\nabla_x \, \partial_{\nu(y)} G(x,y)| \leq \\
&\qquad\qquad \leq a_1 \, [1 + |x-y|]^{-3/2} \text{for} \quad y \in \Gamma_f, \quad x_2 = h \geq f_+ + \delta,
\end{aligned} \tag{2.4}$$

with $a_0 > 0$ depending only on δ and k, and a_1 depending only on k, δ, Γ_f, and h.

Here and throughout the paper $\delta > 0$ is some fixed number.

2.5. In [7] the following Dirichlet problem is studied completely.

Problem (P). Given $g \in BC(\Gamma_f)$, determine $u \in C^2(\Omega_f^+) \cap C(\overline{\Omega_f^+})$ such that:

(i) u is a solution of the Helmholtz equation, i.e.,

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_f^+;$$

(ii) $[u]^+ = g$ on Γ_f where $[\,\cdot\,]^+$ denotes the limit on Γ_f from Ω_f^+;

(iii) for some $\beta \in \mathbb{R}$,

$$\sup_{x \in \Omega_f^+} x_2^\beta \, |u(x)| < \infty;$$

(iv) $u \in \mathrm{UPRC}(\Omega_f^+; k)$.

In [7] it is shown that the above Dirichlet problem is uniquely solvable for arbitrary $g \in BC(\Gamma_f)$ and $f \in BC^{1,1}(\mathbb{R})$. Moreover, the solution is represented by the generalized double–layer potential

$$u(x) = \int_{\Gamma_f} \partial_{\nu(y)} G^{(\mathcal{I})}(x,y) \, \psi(y) \, dS_y, \quad x \in \Omega_f^+, \tag{2.5}$$

where $G^{(\mathcal{I})}$ is the impedance Green's function for the Helmholtz operator in the half–plane U_0^+, while ψ is an unknown density which solves the integral equation

$$-\frac{1}{2} \, \psi(z) + \int_{\Gamma_f} \partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi) \, \psi(\xi) \, dS_\xi = g(z), \quad z \in \Gamma_f. \tag{2.6}$$

Note that the kernel function $\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)$ of the integral equation (2.6) is bounded on $\Gamma_f \times \Gamma_f$ since $f \in BC^{1,1}(\mathbb{R})$ and the improper integrals in (2.5) and (2.6) are well–defined due to the bounds (2.4).

Further, let

$$\begin{aligned} &\mathcal{B}_{c,M} := \{f \in BC^{1,1}(\mathbb{R}) | \inf_{x_1 \in \mathbb{R}} f(x_1) \geq c > 0, \|f\|_{1,1,\mathbb{R}} \leq M\}, \\ &X_a(\Gamma_f) := \{\varphi \in BC(\Gamma_f) | \varphi(z) = \varphi(z_1, f(z_1)) = O(|z_1|^{-a}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{as } |z_1| \to +\infty\}. \end{aligned} \tag{2.7}$$

In [1] the following assertion as Corollary 5.5 is proved.

Lemma 2.3. *For $1 \leq p \leq \infty$ the integral equation* (2.6) *has exactly one solution $\psi \in L_p(\Gamma_f)$ for every $g \in L_p(\Gamma_f)$ and $f \in \mathcal{B}_{c,M}$. There exists a constant $c^* > 0$, depending only on c, M, and the wave number k, such that $\|\psi\|_{L_p(\Gamma_f)} \leq c^* \|g\|_{L_p(\Gamma_f)}$ for $1 \leq p \leq \infty$, $g \in L_p(\Gamma_f)$, $f \in \mathcal{B}_{c,M}$.*

If $g \in X_a(\Gamma_f)$, for some $a \in [0, 3/2]$, it holds that $\psi \in X_a(\Gamma_f)$. Moreover, for $a \in [0, 3/2]$, there exists a constant $C^*(a)$, depending on a, c, M, and k, such that for all $f \in \mathcal{B}_{c,M}$ and $g \in X_a(\Gamma_f)$, the solution ψ of the integral equation (2.6) *satisfies*

$$\begin{aligned} |\psi(z)| \leq C^*(a)(1 + |z_1|)^{-a} \sup_{\xi_1 \in \mathbb{R}} |(1 + |\xi_1|)^a g(\xi_1, f(\xi_1))|, \\ z = (z_1, z_2) \in \Gamma_f. \end{aligned} \tag{2.8}$$

2.6. Denote by $G_f^{(\mathcal{D})}(y, x)$ the Dirichlet Green's function for the Helmholtz equation in the domain Ω_f^+:

$$G_f^{(\mathcal{D})}(y, x) = G^{(\mathcal{D})}(y, x) - V(y, x), \quad y \in \overline{\Omega_f^+},\ x \in \Omega_f^+,\ y \neq x, \tag{2.9}$$

where $f \in \mathcal{B}_{c,M}$, $G^{(\mathcal{D})}$ is the Dirichlet Green's function for the half–plane U_0^+, and $V(\cdot, x) \in C^2(\Omega_f^+) \cap C(\overline{\Omega_f^+}) \cap$ UPRC$(\Omega_f^+; k)$ solves the following Dirichlet problem

$$(\Delta_y + k^2) V(y, x) = 0 \quad \text{for} \quad y \in \Omega_f^+, \quad x \in \Omega_f^+, \tag{2.10}$$

$$[V(y, x)]^+ = G^{(\mathcal{D})}(y, x) \quad \text{for} \quad y \in \Gamma_f, \quad x \in \Omega_f^+. \tag{2.11}$$

It is evident that, due to the results described in Subsection 2.5, for an arbitrarily fixed $x \in \Omega_f^+$, there exists a unique solution to the BVP (2.10)–(2.11), which is represented in the form of the generalized double–layer potential

$$V(y, x) = \int_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(y, \xi)\Big]\, \psi(\xi, x)\, dS_\xi, \quad y \in \Omega_f^+,\ x \in \Omega_f^+, \tag{2.12}$$

where the unknown density $\psi(\cdot, x)$ is a unique solution of the integral equation on Γ_f

$$\begin{aligned} -\frac{1}{2}\psi(z, x) + \int_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z, \xi)\Big]\, \psi(\xi, x) dS_\xi = \\ = G^{(\mathcal{D})}(z, x), \quad z \in \Gamma_f, \quad x \in \Omega_f^+. \end{aligned} \tag{2.13}$$

With the help of Lemma 2.3 and the bounds (2.4) we will establish uniform bounds for $\psi(z,x)$, when $(z,x)\in\Gamma_f\times\Omega_f^+$, and for $V(y,x)$ and $G_f^{(\mathcal{D})}(y,x)$ and its derivatives, when $(y,x)\in\Omega_f^+\times\Omega_f^+$.

Note that throughout the paper the restrictions on Γ_f of the functions $G_f^{(\mathcal{D})}(y,x)$ and $V(y,x)$ and its derivatives are understood as the corresponding limits from Ω_f^+.

2.7. In what follows we need some technical lemmas which we formulate here for the readers convenience.

We remark here that throughout the paper a_j, b_j, c_j, C_j, and δ_j denote constants depending only on c and M involved in (2.7), on δ involved in (2.4), and on a wave number k (if not otherwise stated). Moreover, we assume that the fixed constant δ involved in the relations (2.4) satisfies the inequality $0<\delta\le\delta^*$ with $\delta^*=\min\{1/2,\ d\}$, where d is a Liapunov radius for all curves Γ_f of the family $\mathcal{B}_{c,M}$ (d depends only on M).

Lemma 2.4. *Let $y\in\overline{\Omega_f^+}$, $x\in\overline{\Omega_f^+}$, and $f\in\mathcal{B}_{c,M}$. Then*

$$I(x,y):=\int\limits_{\Gamma_f}\frac{1}{(1+|x-\xi|)^{3/2}}\,\frac{1}{(1+|y-\xi|)^{3/2}}\,dS_\xi\le\frac{\delta_1}{(1+|x-y|)^{3/2}},$$

where $\delta_1=32\sqrt{1+L^2}$ and $L:=\sup\limits_{\xi_1\in\mathbb{R}}|f'(\xi_1)|$.

Proof. Let $r:=|x-y|$ and denote

$$\Gamma_1:=\{\xi\in\Gamma_f\mid|\xi-x|\le r/2\},\quad\Gamma_2:=\Gamma_f\setminus\Gamma_1.$$

Clearly

$$\xi\in\Gamma_1\implies|y-\xi|\ge|y-x|-|\xi-x|\ge r/2,$$
$$\xi\in\Gamma_2\implies|x-\xi|\ge r/2.$$

Therefore we have

$$I(x,y):=\Bigg(\int\limits_{\Gamma_1}+\int\limits_{\Gamma_2}\Bigg)\frac{dS_\xi}{(1+|x-\xi|)^{3/2}(1+|y-\xi|)^{3/2}}\le$$

$$\le\frac{1}{\big(1+\frac{r}{2}\big)^{3/2}}\int\limits_{\Gamma_1}\frac{dS_\xi}{(1+|x-\xi|)^{3/2}}+\frac{1}{\big(1+\frac{r}{2}\big)^{3/2}}\int\limits_{\Gamma_2}\frac{dS_\xi}{(1+|y-\xi|)^{3/2}}\le$$

$$\le2\left(\frac{2}{1+r}\right)^{3/2}\sup_{x\in\Omega_f^+}\int\limits_{-\infty}^{+\infty}\frac{\sqrt{1+[f'(\xi_1)]^2}\,dS_\xi}{\Big[1+\sqrt{|x_1-\xi_1|^2+|x_2-f(\xi_1)|^2}\Big]^{3/2}}\le$$

$$\le\frac{2^{7/2}\sqrt{1+L^2}}{(1+r)^{3/2}}\int\limits_0^{+\infty}\frac{dt}{(1+t)^{3/2}}\le\frac{32\sqrt{1+L^2}}{(1+r)^{3/2}},$$

whence the proof follows. □

Lemma 2.5. *There exists a positive constant* δ_2 *such that*

$$|\Psi(x,\xi,\eta)|\leq\begin{cases}\dfrac{\delta_2|\xi-\eta|}{[\varrho_x(\xi,\eta)]^2} & \text{for}\quad \varrho_x(\xi,\eta)\leq\delta,\quad |\xi-\eta|\leq\delta,\\[2mm] \dfrac{\delta_2|\xi-\eta|(1+x_2)}{[1+\varrho_x(\xi,\eta)]^{3/2}} & \text{for}\quad \varrho_x(\xi,\eta)\geq\delta,\quad |\xi-\eta|\leq\delta,\end{cases} \tag{2.14}$$

where $\xi,\eta\in\Gamma_f$, $x\in\overline{\Omega_f^+}$, $x\neq\xi$, $x\neq\eta$, $f\in\mathcal{B}_{c,M}$,

$$\varrho_x(\xi,\eta):=\min\{|x-\xi|,\ |x-\eta|\},$$

and Ψ *is one of the following functions*

$$\Psi(x,\xi,\eta)\in\Big\{\nabla_x G^{(\mathcal{I})}(x,\xi)-\nabla_x G^{(\mathcal{I})}(x,\eta),\nabla_x G^{(\mathcal{D})}(x,\xi)-\nabla_x G^{(\mathcal{D})}(x,\eta),$$

$$\nabla_\xi G^{(\mathcal{I})}(x,\xi)-\nabla_\eta G^{(\mathcal{I})}(x,\eta),\ \nabla_\xi G^{(\mathcal{D})}(x,\xi)-\nabla_\eta G^{(\mathcal{D})}(x,\eta),$$

$$\partial_{\nu(\xi)}G^{(\mathcal{I})}(x,\xi)-\partial_{\nu(\eta)}G^{(\mathcal{I})}(x,\eta),\ \partial_{\nu(\xi)}G^{(\mathcal{D})}(x,\xi)-\partial_{\nu(\eta)}G^{(\mathcal{D})}(x,\eta)\Big\}.$$

Proof. We will demonstrate the proof only for one particular case, and let for definiteness

$$\Psi(x,\xi,\eta)=\partial_{x_1}G^{(\mathcal{D})}(x,\xi)-\partial_{x_1}G^{(\mathcal{D})}(x,\eta),\quad \xi,\eta\in\Gamma_f,\ x\in\overline{\Omega_f^+},$$

where $|x-\xi|\geq 8\delta$ and $|\xi-\eta|\leq\delta$. Clearly, we then have $|x-\eta|\geq 4\delta$.

Represent Ψ as follows

$$\Psi(x,\xi,\eta)=\int\limits_\gamma \partial_{\ell(t)}\partial_{x_1}\,G^{(\mathcal{D})}(x,t)dS_t, \tag{2.15}$$

where $\gamma:=\overline{\xi\eta}$ is a straight line segment connecting the points ξ and η, $\ell=(\ell_1,\ell_2)=(\xi-\eta)|\xi-\eta|^{-1}$ is a unit vector and $\partial_{\ell(t)}$ denotes the directional derivative. It is evident that

$$|\Psi(x,\xi,\eta)|=|\xi-\eta|\ \sup_{t\in\gamma}\{\,|\partial_{t_1}\partial_{x_1}\,G^{(\mathcal{D})}(x,t)|+|\partial_{t_2}\partial_{x_1}\,G^{(\mathcal{D})}(x,t)|\,\}. \tag{2.16}$$

The direct calculations along with the recurrence relations for the Hankel functions $[H_0^{(1)}(\mu)]'=-H_1^{(1)}(\mu)$ and $[H_1^{(1)}(\mu)]'=H_0^{(1)}(\mu)-\mu^{-1}H_1^{(1)}(\mu)$ give

$$\begin{aligned}&\partial_{t_1}\partial_{x_1}G^{(\mathcal{D})}(x,t)=\\ &\quad=-\frac{ik}{4}\partial_{t_1}\left\{\frac{x_1-t_1}{|x-t|}H_1^{(1)}(k|x-t|)-\frac{x_1-t_1}{|x-t'|}H_1^{(1)}(k|x-t'|)\right\}=\\ &\quad=-\frac{ik}{4}\left\{-k\,\frac{(x_1-t_1)^2}{|x-t|^2}\,[H_1^{(1)}(k|x-t|)]'+k\,\frac{(x_1-t_1)^2}{|x-t'|^2}\,[H_1^{(1)}(k|x-t'|)]'-\right.\\ &\qquad-\frac{1}{|x-t|}H_1^{(1)}(k|x-t|)+\frac{1}{|x-t'|}H_1^{(1)}(k|x-t'|)+\end{aligned}$$

$$+\frac{(x_1-t_1)^2}{|x-t|^3}H_1^{(1)}(k|x-t|)-\frac{(x_1-t_1)^2}{|x-t'|^3}H_1^{(1)}(k|x-t'|)\Big\}=$$

$$=-\frac{ik}{4}\Big\{\Big[(k+1)\frac{(x_1-t_1)^2}{|x-t|^3}-\frac{1}{|x-t|}\Big]H_1^{(1)}(k|x-t|)-$$

$$-\Big[(k+1)\frac{(x_1-t_1)^2}{|x-t'|^3}-\frac{1}{|x-t'|}\Big]H_1^{(1)}(k|x-t'|)-$$

$$-k(x_1-t_1)^2\Big[\frac{1}{|x-t|^2}-\frac{1}{|x-t'|^2}\Big]H_0^{(1)}(k|x-t|)+$$

$$+\frac{k(x_1-t_1)^2}{|x-t'|^2}\Big[H_0^{(1)}(k|x-t'|)-H_0^{(1)}(k|x-t|)\Big]\Big\}, \tag{2.17}$$

$$\partial_{t_2}\partial_{x_1}G^{(\mathcal{D})}(x,t)=-\frac{ik}{4}(x_1-t_1)\Big\{-k\,\frac{(x_2-t_2)}{|x-t|^2}[H_1^{(1)}(k|x-t|)]'+$$

$$+\frac{(x_2-t_2)}{|x-t|^3}H_1^{(1)}(k|x-t|)-k\,\frac{(x_2+t_2)}{|x-t'|^2}[H_1^{(1)}(k|x-t'|)]'+$$

$$+\frac{(x_2+t_2)}{|x-t'|^3}H_1^{(1)}(k|x-t'|)\Big\}=$$

$$=-\frac{ik}{4}(x_1-t_1)\Big\{(k+1)\frac{(x_2-t_2)}{|x-t|^3}H_1^{(1)}(k|x-t|)-$$

$$-k\,\frac{(x_2-t_2)}{|x-t|^2}H_0^{(1)}(k|x-t|)+(k+1)\frac{(x_2+t_2)}{|x-t'|^3}H_1^{(1)}(k|x-t'|)-$$

$$-k\,\frac{(x_2+t_2)}{|x-t'|^2}H_0^{(1)}(k|x-t'|)\Big\}, \tag{2.18}$$

where $x=(x_1,x_2), t=(t_1,t_2)\in U_0^+,\ t'=(t_1,-t_2)\in U_0^-$.

Due to the asymptotic formulas for the Hankel functions (for a large argument) we easily conclude that

$$H_n^{(1)}(k|x-t'|)-H_n^{(1)}(k|x-t|)=$$

$$=\sqrt{\frac{2}{k\pi}}e^{-i\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}\left[\frac{e^{ik|x-t'|}}{|x-t'|^{1/2}}-\frac{e^{ik|x-t|}}{|x-t|^{1/2}}\right]+O([\varrho_x(t,t')]^{-3/2}).$$

Applying this formula along with the relations

$$q(x,t',t):=|x-t'|-|x-t|=\frac{4x_2t_2}{|x-t'|+|x-t|},$$

$$\frac{1}{|x-t|^{1/2}}-\frac{1}{|x-t'|^{1/2}}=\frac{4x_2t_2}{|x-t|^{1/2}|x-t'|^{1/2}(|x-t'|^{1/2}+|x-t|^{1/2})},$$

$$e^{ik|x-t|}-e^{ik|x-t'|}=e^{ik|x-t|}\Big[1-e^{ik(|x-t'|-|x-t|)}\Big]=$$

$$=-e^{ik|x-t|}\,q(x,t',t)\,\sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\,[q(x,t',t)]^{n-1},$$

and noting that $\varrho_x(t,t')$ can be estimated by $\varrho_x(\xi,\eta)$ from above and from below, we derive from (2.17) and (2.18) for $\xi,\eta\in\Gamma_f$, $x\in\overline{\Omega_f^+}$, $|x-\xi|\geq 8\delta$, and $|\xi-\eta|\leq\delta$

$$\sup_{t\in\gamma}\{\,|\partial_{t_1}\partial_{x_1}G^{(\mathcal{D})}(x,t)|+|\partial_{t_2}\partial_{x_1}G^{(\mathcal{D})}(x,t)|\,\}\leq\frac{c_1\,(1+x_2)}{[1+\varrho_x(\xi,\eta)]^{3/2}}. \tag{2.19}$$

Now let us assume that $|x-\xi|\leq 8\delta$ and $|\xi-\eta|\leq\delta$ $(x\neq\xi,\ x\neq\eta)$. Then $|x-\eta|\leq 12\delta$ and we can always choose in (2.15) a smooth line of integration γ_0 (in the place of γ), such that $|x-t|\geq\varrho_x(\xi,\eta)$ for all $t\in\gamma_0$ and the length of which is estimated by $c_2|\xi-\eta|$. Taking into consideration the representation of the Hankel functions $H_n^{(1)}(\mu)$ $(n=0,1,2,\cdots)$ for a small argument μ

$$H_n^{(1)}(\mu)=d^*+\frac{d_n(n)}{\mu^n}+\frac{d_{n-2}(n)}{\mu^{n-2}}+\cdots+d_n^*(n)\mu^n\ln\mu+d_n^{**}(n)\mu^n+$$
$$+\,d_{n+2}^*(n)\mu^{n+2}\ln\mu+d_{n+2}^{**}(n)\mu^{n+2}+\cdots, \tag{2.20}$$

where d^*,d_j,d_j^*,d_j^{**} are constants, we easily conclude from (2.17) and (2.18)

$$\sup_{t\in\gamma_0}\{\,|\partial_{t_1}\partial_{x_1}G^{(\mathcal{D})}(x,t)|+|\partial_{t_2}\partial_{x_1}G^{(\mathcal{D})}(x,t)|\,\}\leq\frac{c_3}{[\varrho_x(\xi,\eta)]^2}$$
$$\text{for}\quad|x-\xi|\leq 8\delta,\ \ |\xi-\eta|\leq\delta,\ \ x\in\overline{\Omega_f^+},\ \ \xi,\eta\in\Gamma_f. \tag{2.21}$$

Note that for $\delta\leq\varrho_x(\xi,\eta)\leq 8\delta$ there holds the inequality

$$\frac{1}{[\varrho_x(\xi,\eta)]^2}\leq\frac{1}{\delta^2}\leq\frac{1}{\delta^2}\frac{(1+8\delta)^{3/2}}{(1+[\varrho_x(\xi,\eta)]^{3/2}}.$$

Therefore, from (2.15), (2.16), (2.19), and (2.21) the inequality (2.14) follows. □

Lemma 2.6. *Let $f\in\mathcal{B}_{c,M}$ and $\Gamma_f(x,\lambda):=\{\xi\in\Gamma_f\mid|x_1-\xi_1|\leq\lambda\}$ with some finite constant $\lambda>0$. Then*

$$\Big|\int_{\Gamma_f(x,\lambda)}\partial_{x_j}G(\xi,x)\,dS_\xi\Big|\leq\delta_3,\ \ j=1,2,\ \ \forall x\in\overline{\Omega_f^+},\quad G\in\{G^{(\mathcal{D})},G^{(\mathcal{I})}\},$$

where δ_3 is a constant depending only on λ, k, and M.

Proof. A principal part of the functions $\partial_{x_j}G^{(\mathcal{D})}(\xi,x)$ and $\partial_{x_j}G^{(\mathcal{I})}(x,\xi)$ is a singular kerel $-(2\pi)^{-1}\partial_{x_j}\ln|\xi-x|$ and the remainder can be estimated as $O(|\xi-x|\ln|\xi-x|)$. Therefore the proof immediately follows from the well–known properties of Cauchy type integrals and harmonic logarithmic potentials (see, e.g., [23], [9], [10]). □

Lemma 2.7. *For $\forall f \in \mathcal{B}_{c,M}$ and a positive constant A there exists a constant δ_4, depending only on A, M, δ, and k, such that*

$$\int\limits_{\Gamma_f} |G(\xi,x)|\, dS_\xi \le \delta_4, \quad \int\limits_{\Gamma_f} \big|\partial_{\nu(\xi)} G(\xi,x)\big|\, dS_\xi \le \delta_4, \ \forall x \in \overline{\Omega_f^+},\ x_2 \le A,$$

where $G \in \{G^{(\mathcal{D})}, G^{(\mathcal{I})}\}$ and δ_3 is a constant depending only on λ, k, and M.

Proof. We recall that the kernels $\partial_{\nu(\xi)} G^{(\mathcal{D})}(\xi,x)$ and $\partial_{\nu(\xi)} G^{(\mathcal{I})}(x,\xi)$ have the function $-(2\pi)^{-1}\partial_{\nu(\xi)} \ln|\xi - x|$ as their principal singular part for $|\xi - x| < 1$, while they behave as $O(|\xi - x|^{-3/2})$ for $|\xi - x| \ge 1$. Now the proof easily follows from the well–known properties of harmonic logarithmic potentials (see, e.g., [14], [17], [9], [10]). □

Lemma 2.8. (i) *Let $\xi, \eta, z \in \Gamma_f$ be different points and $f \in \mathcal{B}_{c,M}$. Moreover, let $|\xi-\eta| + |\xi - z| + |\eta - z| \le A$ with some finite positive constant A. Then*

$$\big|\partial_{\nu(\xi)} G(z,\xi) - \partial_{\nu(\eta)} G(z,\eta)\big| \le \frac{\delta_5\,|\xi-\eta|}{\varrho_z(\xi,\eta)},$$

where the constant δ_5 depends only on A, M, δ, and k.

(ii) *Let $\xi, z \in \Gamma_f$ and $f \in \mathcal{B}_{c,M}$. Then*

$$\big|\partial_{\nu(\xi)} G(z,\xi)\big| \le \frac{\delta_6}{(1 + |z-\xi|)^{3/2}}. \tag{2.22}$$

(iii) *There exists a positive constant δ_7 such that*

$$\big|\partial_{z_j}\partial_{x_p} G(z,x)\big| \le \begin{cases} \dfrac{\delta_7(1+z_2)(1+x_2)}{(1+|z-x|)^{3/2}} & \text{for} \quad |z-x| \ge \delta, \\[2mm] \dfrac{\delta_7}{|z-x|^2} & \text{for} \quad |z-x| \le \delta, \end{cases} \tag{2.23}$$

$$\big|\partial_{z_j}\partial_{x_p} G(z,x) - \partial_{y_j}\partial_{x_p} G(y,x)\big| \le$$

$$\le \begin{cases} \dfrac{\delta_7|z-y|(1+x_2)}{[1+\varrho_x(z,y)]^{3/2}} & \text{for} \quad \varrho_x(z,y) \ge \delta, \quad |z-y| \le \delta, \\[2mm] \dfrac{\delta_7|z-y|}{[\varrho_x(z,y)]^3} & \text{for} \quad \varrho_x(z,y) \le \delta, \quad |z-y| \le \delta, \end{cases} \tag{2.24}$$

$$G \in \{G^{(\mathcal{D})},\ G^{(\mathcal{I})}\}, \quad x,y,z \in U_0^+.$$

Proof. (i) It suffices to prove the corresponding inequality for a principal singular part of the normal derivative of the impedance Green's function $\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)$, i.e., to show that

$$\big|\partial_{\nu(\xi)} \ln|z-\xi| - \partial_{\nu(\eta)} \ln|z-\eta|\big| \le \frac{\delta_5^*\,|\xi-\eta|}{\varrho_z(\xi,\eta)},$$

which is a well–known property of the logarithmic kernel (see, e.g., [14], [18], [2], [10]).

(ii) The inequality (2.22) is a ready consequence of the bounds (2.4) and the relation

$$\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi) = -(2\pi)^{-1}\partial_{\nu(\xi)} \ln|z-\xi| + O(|z-\xi|\ln|z-\xi|) \quad \text{for} \quad |z-\xi| \le \delta.$$

(iii) The inequality (2.23) follows from the explicit expressions of the second order derivatives $\partial_{z_j}\partial_{x_p} G(z,x)$ (cf. (2.17) and (2.18)), while (2.24) can be established by the reasoning applied in the proof of Lemma 2.5. □

3. Uniform Bounds for the Density Function $\psi(z,x)$ and Its Derivatives

3.1. Let us introduce the integral operator

$$(\mathcal{K}\varphi)(z) := \int\limits_{\Gamma_f} \left[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right] \varphi(\xi)\, dS_\xi, \quad z \in \Gamma_f, \tag{3.1}$$

and rewrite (2.13) as follows

$$-\frac{1}{2}\,\psi(z,x) + (\mathcal{K}\psi(\cdot,x))(z) = G^{(\mathcal{D})}(z,x), \quad z \in \Gamma_f, \quad x \in \Omega_f^+. \tag{3.2}$$

Apply the operator $\mathcal{K}$ to the equality (3.2) to obtain

$$-\frac{1}{2}\,\mathcal{K}\psi(\cdot,x) + \mathcal{K}\left(\mathcal{K}\psi(\cdot,x)\right) = \mathcal{K}G^{(\mathcal{D})}(\cdot,x),$$

whence it follows that $\mathcal{K}\psi$ solves the integral equation (2.6) with the right–hand side function $g = \mathcal{K}G^{(\mathcal{D})}(\cdot,x)$. Therefore, due to Lemma 2.3, we have (see (2.8))

$$|(\mathcal{K}\psi(\cdot,x))(z)| = \left| \int\limits_{\Gamma_f} \left[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right] \psi(\xi,x)\, dS_\xi \right| \le$$

$$\le C^*(a)(1+|z_1-x_1|)^{-a} \sup_{y\in\Gamma_f} \{(1+|y_1-x_1|)^a |Q(y,x)|\} \;\; \forall a \in [0,3/2], \tag{3.3}$$

where $x \in \Omega_f^+$ and

$$Q(z,x) := \left(\mathcal{K}G^{(\mathcal{D})}(\cdot,x)\right)(z) = \int\limits_{\Gamma_f} \left[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right] G^{(\mathcal{D})}(\xi,x)\, dS_\xi. \tag{3.4}$$

Lemma 3.1. *There exists a positive constant a_2, such that*

$$|Q(z,x)| \le \; a_2\, \frac{(1+x_2)}{(1+|z-x|)^{3/2}}, \quad z \in \Gamma_f, \;\; x \in \Omega_f^+, \quad f \in \mathcal{B}_{c,M}. \tag{3.5}$$

Proof. We establish the required estimate in several steps.

Step 1. Assume that $z \in \Gamma_f$ and $x \in \Omega_f^+$ with $r := |z_1 - x_1| \geq 3\delta$, and denote

$$\Gamma_1 := \{\xi \in \Gamma_f \mid |\xi_1 - z_1| \leq r/3\},$$
$$\Gamma_2 := \{\xi \in \Gamma_f \mid |\xi_1 - x_1| \leq r/3\},$$
$$\Gamma_3 := \Gamma_f \setminus \{\Gamma_1 \cup \Gamma_2\}.$$

Clearly,

$$\xi \in \Gamma_1 \quad \Longrightarrow \quad |\xi_1 - x_1| \geq |x_1 - z_1| - |z_1 - \xi_1| \geq \frac{2r}{3} \geq 2\delta,$$
$$\xi \in \Gamma_2 \quad \Longrightarrow \quad |\xi_1 - z_1| \geq \frac{2r}{3} \geq 2\delta,$$
$$\xi \in \Gamma_3 \quad \Longrightarrow \quad |\xi_1 - x_1| \geq \frac{r}{3} \geq \delta, \quad |\xi_1 - z_1| \geq \frac{r}{3} \geq \delta.$$

Decompose $Q(z,x)$ as follows

$$Q(z,x) = Q^{(1)}(z,x) + Q^{(2)}(z,x) + Q^{(3)}(z,x), \tag{3.6}$$

where

$$Q^{(p)}(z,x) = \int\limits_{\Gamma_p} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\; G^{(\mathcal{D})}(\xi,x)\, dS_\xi, \quad p = 1,2,3.$$

Due to the estimates (2.4) and boundedness of the kernel function $\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)$ on $\Gamma_f \times \Gamma_f$ we get

$$\begin{aligned}|Q^{(1)}(z,x)| &\leq \int\limits_{\Gamma_1} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big|\; \Big|G^{(\mathcal{D})}(\xi,x)\Big|\; dS_\xi \leq \\ &\leq \int\limits_{\Gamma_1} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big|\; \frac{a_0(1+x_2)(1+\xi_2)}{|x-\xi|^{3/2}}\, dS_\xi \leq \\ &\leq \frac{a_0(1+x_2)(1+M)}{\Big[\big(\frac{2r}{3}\big)^2 + \inf\limits_{\xi\in\Gamma_f} |x_2-\xi_2|^2\Big]^{3/4}} \int\limits_{\Gamma_f} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big|\; dS_\xi.\end{aligned}$$

By Lemma 2.7 it is easy to see that

$$\sup_{z\in\Gamma_f} \int\limits_{\Gamma_f} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big|\; dS_\xi =: b_0 < +\infty, \tag{3.7}$$

where the constant b_0 depends only on M and k. Therefore

$$|Q^{(1)}(z,x)| \leq \frac{a_0\, b_0\, (1+M)(1+x_2)}{\Big[\frac{4r^2}{9} + \inf\limits_{\xi\in\Gamma_f} |x_2-\xi_2|^2\Big]^{3/4}} \leq \frac{5\, a_0\, b_0\, (1+M)\, (1+x_2)}{\Big[r + \inf\limits_{\xi\in\Gamma_f} |x_2-\xi_2|\Big]^{3/2}}. \tag{3.8}$$

Note that

(i) $x_2 > f_+ + 2 \;\Longrightarrow\; \inf\limits_{\xi\in\Gamma_f} |x_2 - \xi_2| = x_2 - \sup\limits_{\xi\in\Gamma_f} \xi_2 = x_2 - f_+ \geq$

$$\geq \frac{2}{f_+ + 2}\, x_2 \geq \frac{2}{f_+ + 2}\,(x_2 - z_2) \text{ with } z_2 = f(z_1) \;\; \forall z_1 \in \mathbb{R}; \tag{3.9}$$

$$\text{(ii)} \quad \tau \geq \varepsilon > 0 \implies \tau \geq \frac{\varepsilon}{1+\varepsilon}\,(1+\tau); \tag{3.10}$$

$$\text{(iii)} \quad \{\tau \geq \varepsilon \geq 0,\ 0 \leq t \leq A\} \implies \frac{1}{1+\tau} \leq \frac{1+\varepsilon+A}{1+\varepsilon}\,\frac{1}{1+t+\tau} \leq \leq \frac{1+\varepsilon+A}{1+\varepsilon}\,\frac{1}{1+\sqrt{t^2+\tau^2}}. \tag{3.11}$$

With the help of (3.9) and (3.10) we get from (3.8)

$$|Q^{(1)}(z,x)| \leq \begin{cases} c_1\, \dfrac{1+x_2}{(1+|x_1-z_1|+|x_2-z_2|)^{3/2}} & \text{for } x_2 \geq f_+ + 2, \\[2ex] c_2\, \dfrac{1}{(1+|x_1-z_1|)^{3/2}} & \text{for } x_2 \leq f_+ + 2, \end{cases} \tag{3.12}$$

where $c_1 = \kappa_1^{-3/2} c_2$ and $c_2 = 5a_0 b_0 (1+M)$ with $\kappa_1 = \min\left\{\dfrac{2}{f_+ + 2}, \dfrac{3\delta}{1+3\delta}\right\} < 1$.

In view of (3.11) it follows from (3.12) that

$$|Q^{(1)}(z,x)| \leq \begin{cases} c_1\, \dfrac{1+x_2}{(1+|x-z|)^{3/2}} & \text{for } x_2 \geq f_+ + 2, \\[2ex] c_2 \left(\dfrac{1+3\delta+2(1+f_+)}{1+3\delta}\right)^{3/2} \dfrac{1+x_2}{(1+|x-z|)^{3/2}} & \\ & \text{for } x_2 \leq f_+ + 2, \end{cases}$$

i.e.,

$$|Q^{(1)}(z,x)| \leq c_3\, \frac{1+x_2}{(1+|x-z|)^{3/2}} \quad \text{for } z \in \Gamma_f,\ x \in \Omega_f^+,\ |x_1 - z_1| \geq 3\delta, \tag{3.13}$$

where $c_3 = \max\left\{c_1, c_2 \left(\dfrac{1+3\delta+2(1+f_+)}{1+3\delta}\right)^{3/2}\right\}$.

Quite analogously we derive

$$|Q^{(3)}(z,x)| = \int\limits_{\Gamma_3} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)|\, |G^{(\mathcal{D})}(\xi,x)|\, dS_\xi \leq$$

$$\leq \int\limits_{\Gamma_3} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)|\, \frac{a_0(1+M)(1+x_2)}{|x-\xi|^{3/2}}\, dS_\xi \leq$$

$$\leq \frac{a_0\, b_0\, (1+M)(1+x_2)}{\left[\frac{1}{\sqrt{2}}\left(\frac{r}{3} + \inf\limits_{\xi \in \Gamma_f} |x_2 - \xi_2|\right)\right]^{3/2}} \leq$$

$$\le c_4 \frac{(1+x_2)}{1+|x-z|]^{3/2}}, \quad z\in\Gamma_f, \ \ x\in\Omega_f^+, \ \ |x_1-z_1|\ge 3\delta. \quad (3.14)$$

Further, we estimate $Q^{(2)}(z,x)$ for $x_2 < f_+ + 2$ (see (2.4) and (3.10))

$$|Q^{(2)}(z,x)| = \int\limits_{\Gamma_2} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)|\,|G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le$$

$$\le \int\limits_{\Gamma_2} \frac{a_0(1+M)^2}{|z-\xi|^{3/2}}\,|G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le$$

$$\le \int\limits_{\Gamma_2} \frac{a_0(1+M)^2}{\Big[\big(\frac{2r}{3}\big)^2 + (z_2-\xi_2)^2\Big]^{3/4}}\,|G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le$$

$$\le \frac{a_0(1+M)^2}{\big(\frac{2r}{3}\big)^{3/2}} \int\limits_{\Gamma_2} |G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le$$

$$\le \frac{a_0(1+M)^2}{\big(\frac{2}{3}\big)^{3/2}\Big(\frac{3\delta}{1+3\delta}\Big)^{3/2}(1+|x_1-z_1|)^{3/2}} \int\limits_{\Gamma_f} |G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le$$

$$\le \frac{c_5}{[1+|x_1-z_1|]^{3/2}}, \quad (3.15)$$

where $z\in\Gamma_f$, $x\in\Omega_f^+$, $|x_1-z_1|\ge 3\delta$, $x_2 < f_+ + 2$, and $c_5 = \dfrac{a_0(1+M)^2 b_1}{\Big(\frac{2\delta}{1+3\delta}\Big)^{3/2}}$ with

$$b_1 := \sup_{x\in\Omega_f^+,\, x_2\le f_+ + 2} \int\limits_{\Gamma_f} |G^{(\mathcal{D})}(\xi,x)|\,dS_\xi < +\infty. \quad (3.16)$$

Now let $x_2 \ge f_+ + 2$ and denote $q := x_2 - z_2 \ge 2$ for $z=(z_1,z_2)\in\Gamma_f$. By (2.4), (3.9), and (3.10) we get

$$|Q^{(2)}(z,x)| \le \int\limits_{\Gamma_2} \frac{a_0(1+M)^2}{|z-\xi|^{3/2}}\,\frac{a_0(1+x_2)(1+M)}{|\xi-x|^{3/2}}\,dS_\xi \le$$

$$\le a_0^2(1+M)^3(1+x_2)\int\limits_{\Gamma_2} \frac{1}{|z_1-\xi_1|^{3/2}}\,\frac{1}{(|\xi_1-x_1|^2+|\xi_2-x_2|^2)^{3/4}}\,dS_\xi \le$$

$$\le c_6\,\frac{1+x_2}{(1+r)^{3/2}} \int\limits_{\Gamma_2} \frac{dS_\xi}{(|\xi_1-x_1|+q)^{3/2}} \le c_7\,\frac{1+x_2}{(1+r)^{3/2}} \int\limits_0^{r/3} \frac{dt}{(t+q)^{3/2}} \le$$

$$\leq c_7 \frac{1+x_2}{(1+r)^{3/2}} \int\limits_0^r \frac{dt}{(t+q)^{3/2}} = c_7 \frac{1+x_2}{(1+r)^{3/2}} \left[\frac{-2}{(t+q)^{1/2}}\right]_0^r =$$
$$= 2c_7 \frac{1+x_2}{(1+r)^{3/2}} \left[\frac{1}{q^{1/2}} - \frac{1}{(r+q)^{1/2}}\right] =$$
$$= 2c_7 \frac{1+x_2}{(1+r)^{3/2}} \frac{r}{q^{1/2}(r+q)^{1/2}[q^{1/2}+(r+q)^{1/2}]} \leq$$
$$\leq 2c_7 \frac{1+x_2}{(1+r)^{1/2}q^{1/2}(r+q)} = 2c_7 \frac{1+x_2}{\sqrt{q+rq}(r+q)} \leq$$
$$\leq 2c_7 \frac{1+x_2}{(r+q)^{3/2}} \leq c_8 \frac{1+x_2}{(1+r+q)^{3/2}} =$$
$$= c_8 \frac{1+x_2}{(1+|x_1-z_1|+|x_2-z_2|)^{3/2}}, \tag{3.17}$$
$$z\in\Gamma_f,\ \ x\in\Omega_f^+,\ \ |x_1-z_1|\geq 3\delta,\ \ x_2\geq f_+ + 2.$$

From (3.15) and (3.17) as above with the help of (3.11) we obtain

$$|Q^{(2)}(z,x)| \leq c_9 \frac{1+x_2}{(1+|x-z|)^{3/2}},\ \ z\in\Gamma_f,\ \ x\in\Omega_f^+,\ \ |x_1-z_1|\geq 3\delta. \tag{3.18}$$

The equality (3.6) along with (3.13), (3.14), and (3.18) leads to the estimate

$$|Q(z,x)| \leq a_2^* \frac{1+x_2}{(1+|x-z|)^{3/2}} \ \text{ for } z\in\Gamma_f,\ \ x\in\Omega_f^+,\ \ |x_1-z_1|\geq 3\delta, \tag{3.19}$$

where $a_3^* = c_3 + c_4 + c_9$.

Step 2. Now, let $z\in\Gamma_f$, $x\in\Omega_f^+$, and $r=|z_1-x_1|\leq 3\delta$. First we consider the case when $x_2\geq f_+ + 2$. Applying (2.4), (3.7), and (3.11) we have from (3.4)

$$|Q(z,x)| \leq \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)|\,|G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \leq$$
$$\leq \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)|\, \frac{a_0(1+x_2)(1+M)}{|x-\xi|^{3/2}}\,dS_\xi \leq$$
$$\leq \frac{a_0 b_0(1+M)(1+x_2)}{(x_2-f_+)^{3/2}} \leq \frac{c_{10}\,(1+x_2)}{(1+|x-z|)^{3/2}}, \tag{3.20}$$
$$z\in\Gamma_f,\quad x\in\Omega_f^+,\quad |x_1-z_1|\leq 3\delta,\quad x_2\geq f_+ + 2.$$

In the case $x_2\leq f_+ + 2$ we get (see (3.16))

$$|Q(z,x)| \leq \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)|\,|G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \leq$$

$$\le b_1 \sup_{z,\xi\in\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)| \le$$

$$\le \left[b_1 \sup_{z,\xi\in\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)| \right] \frac{(1+3\delta+2f_+ +2)^{3/2}(1+x_2)}{(1+|x-z|)^{3/2}} \le$$

$$\le \frac{c_{11}\,(1+x_2)}{(1+|x-z|)^{3/2}} \tag{3.21}$$

$$\text{for} \quad z\in\Gamma_f, \quad x\in\Omega_f^+, \quad |x_1-z_1|\le 3\delta, \quad x_2\le f_+ +2.$$

Combining (3.20) and (3.21) we obtain

$$|Q(z,x)| \le a_2^{**} \frac{1+x_2}{(1+|x-z|)^{3/2}} \quad \text{for } z\in\Gamma_f,\ x\in\Omega_f^+,\ |x_1-z_1|\le 3\delta$$

with $a_3^{**}=\max\{c_{10},\ c_{11}\}$, which along with (3.19) completes the proof. □

Now let us return back to the relation (3.3) to estimate the function $(\mathcal{K}\psi(\cdot,x))\,(z)$.

Lemma 3.2. *There exists a positive constant* a_3*, such that*

$$|(\mathcal{K}\psi(\cdot,x))\,(z)| \le a_3 \frac{(1+x_2)}{(1+|z-x|)^{3/2}} \quad \text{for } z\in\Gamma_f,\ x\in\Omega_f^+,\ f\in\mathcal{B}_{c,M}. \tag{3.22}$$

Proof. The inequalities (3.3) and (3.5) imply

$$|(\mathcal{K}\psi(\cdot,x))\,(z)| \le C^{**}(a) \frac{(1+x_2)}{(1+|z_1-x_1|)^a} \sup_{y\in\Gamma_f} \left\{ \frac{(1+|y_1-x_1|)^a}{(1+|y-x|)^{3/2}} \right\}, \tag{3.23}$$
$$z\in\Gamma_f,\ x\in\Omega_f^+,\ \forall a\in[0,3/2],$$

where $C^{**}(a)=a_2\,C^*(a)$ depends only on M, δ, k and a.

For $z\in\Gamma_f$, $x\in\Omega_f^+$, and $x_2\le f_+ +2$ we have from (3.23)

$$|(\mathcal{K}\psi(\cdot,x))\,(z)| \le C^{**}(3/2) \frac{1+x_2}{(1+|z_1-x_1|)^{3/2}} \sup_{y\in\Gamma_f} \left[\frac{1+|y_1-x_1|}{1+|y-x|} \right]^{3/2} \le$$
$$\le C^{**}(3/2) \frac{1+x_2}{(1+|z_1-x_1|)^{3/2}} \le a_3^* \frac{1+x_2}{(1+|z-x|)^{3/2}} \tag{3.24}$$

with $a_3^*=C^{**}(3/2)(3+2f_+)^{3/2}$, since

$$\frac{1}{1+|z_1-x_1|} \le \frac{1+2(f_+ +1)}{1+|z-x|}$$

due to (3.11) (with $\varepsilon=0$ and $A=2(f_+ +1)$).

If $z\in\Gamma_f$, $x\in\Omega_f^+$, and $x_2\ge f_+ +2$, then by (3.9)

$$\sup_{y\in\Gamma_f} \frac{(1+|y_1-x_1|)^a}{(1+|y-x|)^{3/2}} \le \sup_{y\in\Gamma_f} \frac{1}{(1+|y-x|)^{(3-2a)/2}} \le$$

$$\le \sup_{y\in\Gamma_f} \frac{1}{(1+|x_2-f(y_1)|^2)^{(3-2a)/4}} \le$$

$$\le \frac{1}{(1+\inf\limits_{y\in\Gamma_f}|x_2-f(y_1)|^2)^{(3-2a)/4}} \le$$

$$\le \frac{1}{(1+\mu_1^{-2}|x_2-z_2|^2)^{(3-2a)/4}} \le$$

$$\le \mu_1^{(3-2a)/2} \frac{1}{(1+|x_2-z_2|^2)^{(3-2a)/4}} \le$$

$$\le \mu_1^{(3-2a)/2} \frac{2^{(3-2a)/4}}{(1+|x_2-z_2|)^{(3-2a)/2}} \le$$

$$\le \frac{c_1}{(1+|x_2-z_2)|)^{(3-2a)/2}}$$

$$\text{with} \quad c_1 = 2\,\mu_1^{(3-2a)/2} \quad \text{and} \quad \mu_1 = (f_+ + 2)/2.$$

Therefore, from (3.23)

$$|(\mathcal{K}\psi(\cdot,x))\,(z)| \le \ C^{**}(a)\,c_1\ \frac{(1+x_2)}{(1+|z_1-x_1|)^a(1+|x_2-z_2|)^{(3-2a)/2}}$$

$$\text{for} \quad z\in\Gamma_f, \ \ x\in\Omega_f^+, \ \ x_2 \ge f_+ + 2.$$

In turn this relation yields (with $a = 3/2$ and $a = 0$, respectively)

$$(1+|z_1-x_1|)^{3/2}\,|(\mathcal{K}\psi(\cdot,x))\,(z)| \le \ C^{**}(3/2)\,c_1\ (1+x_2),$$

$$(1+|z_2-x_2|)^{3/2}\,|(\mathcal{K}\psi(\cdot,x))\,(z)| \le \ C^{**}(0)\,c_1\ (1+x_2),$$

whence, due to the inequality

$$(1+|z_1-x_1|)^{3/2} + (1+|z_2-x_2|)^{3/2} > (1+|z-x|)^{3/2},$$

we conclude

$$|(\mathcal{K}\psi(\cdot,x))\,(z)| \le a_3^{**}\ \frac{(1+x_2)}{(1+|z-x|)^{3/2}};\ \text{for } z\in\Gamma_f,\ x\in\Omega_f^+,\ x_2\ge f_+ +2, \tag{3.25}$$

with $a_3^{**} = 2\ \max\{C^{**}(3/2)\,c_1,\ C^{**}(0)\,c_1\}$.

From (3.24) and (3.25) the inequality (3.22) follows with $a_3 = \max\{a^*, a^{**}\}$. □

Lemma 3.3. *There exists a positive constant a_4, such that for $f \in \mathcal{B}_{c,M}$*

$$|\psi(z,x)| \le \begin{cases} \dfrac{a_4(1+x_2)}{(1+|z-x|)^{3/2}} & \text{for} \quad |z-x| \ge \delta, \quad z\in\Gamma_f, \quad x\in\Omega_f^+, \\ a_4\,(1+|\ln|z-x|\,|) & \text{for} \quad |z-x| \le \delta, \quad z\in\Gamma_f, \quad x\in\Omega_f^+. \end{cases}$$

Proof. From (3.2) we get

$$\psi(z,x) = 2\left(\mathcal{K}\psi(\cdot,x)\right)(z) - 2G^{(\mathcal{D})}(z,x), \quad z \in \Gamma_f, \ \ x \in \Omega_f^+ .$$

Therefore the proof follows from Lemma 3.2 and the estimates (2.4). □

3.2. In this subsection we will establish bounds for the first order derivatives of the function $\psi(z,x)$ with respect to the variable x.

Put

$$\psi_j(z,x) := \partial_{x_j}\psi(z,x), \quad j = 1,2. \tag{3.26}$$

Due to the invertibility and continuity of the operator $-\frac{1}{2}\,I + \mathcal{K}$ (see Lemma 2.3 and (3.1)), where I stands for the identical operator, we easily see that $\psi_j(z,x)$ solves the integral equation

$$\left(-\frac{1}{2}\,I + \mathcal{K}\right)\psi_j(z,x) := -\frac{1}{2}\,\psi_j(z,x) + \left(\mathcal{K}\psi_j(\cdot,x)\right)(z) = \partial_{x_j}G^{(\mathcal{D})}(z,x) \tag{3.27}$$

for $z \in \Gamma_f$, $x \in \Omega_f^+$, $j = 1,2$. This equation is obtained by the formal differentiation with respect to x_j of the equation (2.13) (i.e., (3.2)).

By Lemma 2.3 we conclude that the integral equation (3.27) is uniquely solvable for any $x \in \Omega_f^+$ in the space $X_a(\Gamma_f)$ with arbitrary $a \in [0, 3/2]$. To obtain uniform bounds for $\psi_j(z,x)$ on $\Gamma_f \times \Omega_f^+$ we apply the same approach as above. To this end first we investigate the behaviour of the function

$$\begin{aligned} Q_j(z,x) :&= \partial_{x_j} Q(z,x) = \Big(\mathcal{K}\partial_{x_j}G^{(\mathcal{D})}(\cdot,x)\Big)(z) = \\ &= \int\limits_{\Gamma_f} \Big[\partial_{\nu(\xi)}G^{(\mathcal{I})}(z,\xi)\Big]\, \partial_{x_j}G^{(\mathcal{D})}(\xi,x)\, dS_\xi, \qquad (3.28) \\ & z \in \Gamma_f, \ x \in \Omega_f^+, \ j = 1,2. \end{aligned}$$

Lemma 3.4. *There exists a positive constant a_5, such that for $f \in \mathcal{B}_{c,M}$*

$$|Q_j(z,x)| \le \begin{cases} \dfrac{a_5(1+x_2)}{(1+|z-x|)^{3/2}} & \text{for} \quad |z-x| \ge \delta, \quad z \in \Gamma_f, \quad x \in \Omega_f^+, \\ a_5\left(1 + |\ln|z-x||\right) & \text{for} \quad |z-x| \le \delta, \quad z \in \Gamma_f, \quad x \in \Omega_f^+. \end{cases}$$

Proof. The scheme of the proof is the same as in the proof of Lemma 3.1, but it needs some modifications since the integral over Γ_f of the function $\left|\partial_{x_j}G^{(\mathcal{D})}(\xi,x)\right|$ is not any more uniformly bounded on Ω_f^+ and, moreover, the restriction of the same function on $\Gamma_f \times \Gamma_f$ is a strictly singular kernel (we recall that the function $\partial_{\nu(\xi)}G^{(\mathcal{D})}(\xi,x)$ is bounded on $\Gamma_f \times \Gamma_f$).

Step 1. Let $z \in \Gamma_f$ and $x \in \Omega_f^+$ with $r := |x_1 - z_1| \ge 4\delta$. Further let $\Gamma_f = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 &:= \{\xi \in \Gamma_f \mid |\xi_1 - z_1| \le r/4\}, \\ \Gamma_2 &:= \{\xi \in \Gamma_f \mid |\xi_1 - x_1| \le r/4\}, \\ \Gamma_3 &:= \Gamma_f \setminus \{\Gamma_1 \cup \Gamma_2\}, \end{aligned}$$

and denote

$$Q_j^{(p)}(z,x) := \int\limits_{\Gamma_p} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\, \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi, \quad p=1,2,3.$$

Clearly, $Q_j(z,x) = Q_j^{(1)}(z,x) + Q_j^{(2)}(z,x) + Q_j^{(3)}(z,x)$. Applying the same arguments as above and considering separately the two cases, $x_2 \geq f_+ + 2$ and $x_2 \leq f_+ + 2$, with the help of (2.4), (3.7), and (3.11) we easily get

$$\Big|Q_j^{(1)}(z,x)\Big| \leq \int\limits_{\Gamma_1} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big| \, \frac{a_0(1+x_2)(1+M)}{|x-\xi|^{3/2}}\, dS_\xi \leq$$

$$\leq \frac{a_0 b_0 (1+x_2)}{\Big[\big(\frac{3r}{4}\big)^2 + \inf\limits_{\xi\in\Gamma_f} |x_2-\xi_2|^2\Big]^{3/4}} \leq \frac{c_1(1+x_2)}{[1+|z-x|]^{3/2}} \tag{3.29}$$

$$\text{for} \quad z\in\Gamma_f, \ \ x \in \Omega_f^+, \ \ |z_1 - x_1| \geq 4\delta.$$

Further we estimate $Q_j^{(2)}(z,x)$. For $\lambda > 0$ let us introduce

$$\Omega_f^+(\lambda) := \{x \in \Omega_f^+ \mid \operatorname{dist}(x,\Gamma_f) \geq \lambda\} = \Omega_f^+ \setminus \bigcup_{z\in\Gamma_f} B(z,\lambda), \tag{3.30}$$

where $B(z,\lambda)$ is a circle centered at z and radius λ.

First we assume that $x \in \Omega_f^+(\delta)$. Note that for $\xi \in \Gamma_2$ we have $|x-\xi| \geq \delta$ and $|z_1 - \xi_1| \geq 3\delta$, and therefore due to (2.4) and (3.10), and Lemma 2.4

$$\Big|Q_j^{(2)}(z,x)\Big| \leq \int\limits_{\Gamma_2} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big| \, \Big|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\Big|\, dS_\xi \leq$$

$$\leq \int\limits_{\Gamma_2} \frac{a_0(1+M)^2}{|z-\xi|^{3/2}}\, \frac{a_0(1+M)(1+x_2)}{|\xi-x|^{3/2}}\, dS_\xi \leq$$

$$\leq a_0^2 (1+M)^3 \Big(\frac{1+\delta}{\delta}\Big)^3 (1+x_2) \int\limits_{\Gamma_2} \frac{dS_\xi}{[1+|z-\xi|]^{3/2}[1+s|x-\xi|]^{3/2}} \leq$$

$$\leq \frac{c_2(1+x_2)}{[\,1+|z-x|\,]^{3/2}} \tag{3.31}$$

with $c_2 = a_0^2(1+M)^3\left(\frac{1+\delta}{\delta}\right)^3 \delta_1$ (see Lemma 2.4).

Now, consider the case: $x \in \Omega_f^+ \setminus \Omega_f^+(\delta)$ and denote by $\overline{x} = (\overline{x}_1, \overline{x}_2) = (\overline{x}_1, f(\overline{x}_1)) \in \Gamma_f$ the point of Γ_f nearest to x, i.e., $|x-\overline{x}| = \operatorname{dist}(x,\Gamma_f)$.

Decompose Γ_2 into two parts: $\Gamma_2 := \Gamma_2' \cup \Gamma_2''$ where

$$\Gamma_2' := \{\xi \in \Gamma_2 \mid |x_1 - \xi_1| \leq \delta\},$$
$$\Gamma_2'' := \Gamma_2 \setminus \Gamma_2' = \{\xi\in\Gamma_2 \mid \delta \leq |x_1-\xi_1| \leq r/4\}.$$

Note that $\overline{x} \in \Gamma_2'$ since $|x_1 - \overline{x}_1| \le |x - \overline{x}| \le \delta$. Moreover,

$$\begin{aligned} \xi \in \Gamma_2' &\Longrightarrow |z_1 - \xi_1| \ge |x_1 - z_1| - |x_1 - \xi_1| \ge \frac{3r}{4} \ge 3\delta, \\ &\quad\;\; |z - \xi| \ge \frac{3r}{4} \ge 3\delta, \\ \xi \in \Gamma_2'' &\Longrightarrow |x - \xi| \ge \delta, \\ \xi \in \Gamma_f &\Longrightarrow |\xi - \overline{x}| \le |\xi - x| + |x - \overline{x}| \le 2\,|\xi - x|. \end{aligned} \tag{3.32}$$

We proceed as follows

$$\begin{aligned} \Big|Q_j^{(2)}(z,x)\Big| \le \bigg| &\int\limits_{\Gamma_2'} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi) - \partial_{\nu(\overline{x})} G^{(\mathcal{I})}(z,\overline{x})\Big]\, \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi + \\ &+ \partial_{\nu(\overline{x})} G^{(\mathcal{I})}(z,\overline{x}) \int\limits_{\Gamma_2'} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \bigg| + \\ &+ \bigg| \int\limits_{\Gamma_2''} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\, \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \bigg|. \end{aligned}$$

Due to Lemmas 2.5, 2.6, and 2.4, and estimates (2.4), (3.10), and (3.32) we obtain

$$\begin{aligned} \big|Q_j^{(2)}(z,x)\big| \le &\int\limits_{\Gamma_2'} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi) - \partial_{\nu(\overline{x})} G^{(\mathcal{I})}(z,\overline{x})\Big|\, \Big|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\Big|\, dS_\xi + \\ &+ \Big|\partial_{\nu(\overline{x})} G^{(\mathcal{I})}(z,\overline{x})\Big| \bigg| \int\limits_{\Gamma_2'} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \bigg| + \\ &+ \int\limits_{\Gamma_2''} \Big|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big|\, \Big|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\Big|\, dS_\xi \le \\ \le &\int\limits_{\Gamma_2'} \frac{\delta_2 |\xi - \overline{x}|}{[1 + \varrho_z(\xi,\overline{x})]^{3/2}}\, \frac{a_0}{|\xi - \overline{x}|}\, dS_\xi + \\ &+ \frac{a_0 (1+M)^2}{|z - \overline{x}|^{3/2}} \bigg| \int\limits_{\Gamma_2'} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \bigg| + \\ &+ \bigg| \int\limits_{\Gamma_2''} \frac{a_0 (1+M)^2}{|z - \xi|^{3/2}}\, \frac{a_0 (1+M)(1+x_2)}{|\xi - x|^{3/2}}\, dS_\xi \bigg| \le \\ \le &\frac{2\delta_2 a_0}{\big(1 + \frac{r}{2}\big)^{3/2}}\, |\Gamma_2'| + \frac{2^{3/2} a_0 (1+M)^2}{r^{3/2}} \delta_3 + \end{aligned}$$

$$+\, a_0(1+M+\delta)^3 \left(\frac{1+2\delta}{2\delta}\right)^3 \int\limits_{\Gamma_2''} \frac{dS_\xi}{(1+|z-\xi|)^{3/2}(1+|\xi-x|)^{3/2}} \leq$$

$$\leq \frac{c_3}{[1+|z-x|]^{3/2}} \leq \frac{c_3\,(1+x_2)}{[1+|z-x|]^{3/2}}, \tag{3.33}$$

where $|\Gamma_2'|$ denotes the length of the arc Γ_2'. Combining (3.31) and (3.33) we get

$$\left|Q_j^{(2)}(z,x)\right| \leq c_4\, \frac{1+x_2}{[1+|z-x|]^{3/2}} \quad \text{for} \quad z\in\Gamma_f,\ \ x\in\Omega_f^+,\ \ |z_1-x_1|\geq 4\delta. \tag{3.34}$$

Note that

$$\xi\in\Gamma_3 \quad\Longrightarrow\quad |x-\xi|\geq \frac{r}{4}\geq\delta, \quad |z-\xi|\geq\frac{r}{4}\geq\delta.$$

Therefore with the help of Lemma 2.4 and relations (2.4) we derive

$$\begin{aligned}\left|Q_j^{(3)}(z,x)\right| &\leq \int\limits_{\Gamma_3} \left|\partial_{\nu(\xi)}G^{(\mathcal{I})}(z,\xi)\right| \left|\partial_{x_j}G^{(\mathcal{D})}(\xi,x)\right| dS_\xi \leq \\ &\leq \int\limits_{\Gamma_3} \frac{a_0(1+M)^2}{(1+|z-\xi|)^{3/2}}\, \frac{a_0(1+M)(1+x_2)}{(1+|\xi-x|)^{3/2}}\, dS_\xi \leq \\ &\leq a_0^2(1+M)^3\,\delta_1\, \frac{1+x_2}{(1+|z-x|)^{3/2}} \\ &\qquad \text{for} \quad z\in\Gamma_f,\ \ x\in\Omega_f^+,\ \ |z_1-x_1|\geq 4\delta.\end{aligned} \tag{3.35}$$

From (3.29), (3.34), and (3.35) it follows that

$$\begin{aligned}|Q_j(z,x)| &\leq \left|Q_j^{(1)}(z,x)\right| + \left|Q_j^{(2)}(z,x)\right| + \left|Q_j^{(3)}(z,x)\right| \leq \\ &\leq a_5^*\frac{1+x_2}{[1+|z-x|]^{3/2}} \quad \text{for} \quad z\in\Gamma_f,\ \ x\in\Omega_f^+,\ \ |z_1-x_1|\geq 4\delta.\end{aligned} \tag{3.36}$$

Step 2. Let us now consider the case $r=|z_1-x_1|\leq 4\delta$.

Sub–step 2.1. First we provide that $x\in\Omega_f^+(\delta)$ (see (3.30)), so that $|x-\xi|\geq\delta$ for $\xi\in\Gamma_f$. Therefore,

$$|Q_j(z,x)| \leq \int\limits_{\Gamma_f} \left|\partial_{\nu(\xi)}G^{(\mathcal{I})}(z,\xi)\right| \left|\partial_{x_j}G^{(\mathcal{D})}(\xi,x)\right| dS_\xi. \tag{3.37}$$

If, in addition, $x_2\geq f_+ + 2$, then applying the same arguments as in the proof of the inequality (3.21), we get

$$|Q_j(z,x)| \leq \frac{c_5(1+x_2)}{[1+|z-x|]^{3/2}} \quad \text{for} \quad z\in\Gamma_f, \quad x\in\Omega_f^+(\delta), \quad x_2>f_+ + 2, \quad |z_1-x_1|\leq 4\delta. \tag{3.38}$$

For $x \in \Omega_f^+(\delta)$ with $x_2 < f_+ + 2$, we have from (3.37)

$$|Q_j(z,x)| \le \int\limits_{\Gamma_f} \left|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\right| \, dS_\xi \quad \sup_{z,\xi\in\Gamma_f} \left|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right|.$$

It is evident that

$$\begin{gathered} \sup_{z,\xi\in\Gamma_f} \left|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right| \; =: b_2 < +\infty, \\ \sup_{x\in\Omega_f^+(\delta),\, x_2\le f_+ + 2} \int\limits_{\Gamma_f} \left|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\right| \, dS_\xi \; =: b_3 < +\infty. \end{gathered} \tag{3.39}$$

Thus, for $x \in \Omega_f^+(\delta)$ with $x_2 \le f_+ + 2$ and $|z_1 - x_1| \le 4\delta$, we have

$$|Q_j(z,x)| \le b_2 b_3 \le b_2 b_3 (1 + 4\delta + 2f_+ + 2)^{3/2} \, \frac{1 + x_2}{[1 + |z - x|]^{3/2}}. \tag{3.40}$$

From (3.38) and (3.40) it follows

$$|Q_j(z,x)| \le c_6 \, \frac{1 + x_2}{(1 + |z - x|)^{3/2}} \quad \text{for} \quad x \in \Omega_f^+(\delta), \;\; |z_1 - x_1| \le 4\delta. \tag{3.41}$$

Sub-step 2.2. Further, we assume that $x \in \Omega_f^+ \setminus \Omega_f^+(\delta)$. Recall that $|z_1 - x_1| \le 4\delta$ and put

$$\Gamma_1 := \{\xi \in \Gamma_f \mid |x_1 - \xi_1| \le 8\delta\}, \quad \Gamma_2 := \Gamma_f \setminus \Gamma_1. \tag{3.42}$$

It is evident that

$$\xi \in \Gamma_2 \quad \Longrightarrow \quad |x - \xi| \ge 8\delta, \quad |z - \xi| \ge 4\delta. \tag{3.43}$$

We represent $Q_j(z,x)$ as follows

$$Q_j(z,x) = I_1(z,x) + I_2(z,x) \tag{3.44}$$

where

$$I_j(z,x) := \int\limits_{\Gamma_j} \left[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right] \, \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \, dS_\xi, \quad j = 1,2. \tag{3.45}$$

Using (3.43), (2.4), (3.10)

$$\begin{gathered} I_2(z,x) \le \int\limits_{\Gamma_2} \frac{a_0(1+M)^2}{|z-\xi|^{3/2}} \, \frac{a_0(1+M)^2}{|\xi - x|^{3/2}} \, dS_\xi \le \\ \le a_0^2 (1+M)^4 \left(\frac{1+4\delta}{4\delta}\right)^3 \int\limits_{\Gamma_2} \frac{dS_\xi}{(1 + |z-\xi|)^{3/2} (1 + |\xi - x|)^{3/2}} \le \\ \le a_0^2 (1+M)^4 \left(\frac{1+4\delta}{4\delta}\right)^3 \int\limits_{\Gamma_f} \frac{dS_\xi}{(1 + |z-\xi|)^{3/2} (1 + |\xi - x|)^{3/2}} \le c_7 \\ \text{for} \quad x \in \Omega_f^+ \setminus \Omega_f^+(\delta), \;\; |z_1 - x_1| \le 4\delta. \end{gathered} \tag{3.46}$$

Estimation of $I_1(z,x)$ is a little bit complicated and needs subtle reasoning related to the composition of singular kernels (cf. [24] and the references therein). To obtain the required estimates for $I_1(z,x)$ in what follows we consider the two possible cases separately

$$\text{(i)} \quad |x-\overline{x}| \le 6^{-1}|z-x|, \tag{3.47}$$

$$\text{(ii)} \quad |x-\overline{x}| \ge 6^{-1}|z-x|, \tag{3.48}$$

where $\overline{x}$ is a point on Γ_1 nearest to x.

Clearly, $|x-\overline{x}| \le \delta$ and $|z_1-\overline{x}_1| \le 5\delta$.

Sub–step 2.2.(i). First let us assume that (3.47) holds and devide Γ_1 into two disjoint parts (see (3.42))

$$\Gamma_1 = \Gamma_{11} \cup \Gamma_{12} \ \text{ with } \ \Gamma_{11} := \{\xi \in \Gamma_1 \| \xi - \overline{x}| \le 2\varrho\}, \ \ \Gamma_{12} := \Gamma_1 \setminus \Gamma_{11}, \tag{3.49}$$
$$\varrho := \min\{\delta,\, 6^{-1}|x-z|\}.$$

It is easy to see that Γ_{11} is connected, the distance from the end points of Γ_1 to Γ_{11} is greater than δ, and, moreover,

$$\xi \in \Gamma_{11} \Longrightarrow |z-\xi| \ge 2^{-1}|x-z| \ge 3\varrho, \quad |\xi - x| \ge 2^{-1}|\xi - \overline{x}|, \tag{3.50}$$

$$\xi \in \Gamma_{12} \Longrightarrow |x-\xi| \ge 2^{-1}|\xi-\overline{x}| \ge \varrho, \quad |\xi_1 - \overline{x}_1| \ge 2(1+L^2)^{-1/2}\varrho. \tag{3.51}$$

We represent I_1 as follows (see (3.45))

$$I_1(z,x) = I_{11}(z,x) + I_{12}(z,x) \tag{3.52}$$

where

$$I_{1q}(z,x) := \int\limits_{\Gamma_{1q}} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\, \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi, \quad q=1,2. \tag{3.53}$$

With the help of Lemma 2.8, (3.39), and (3.50) we have

$$\begin{aligned}
|I_{11}(z,x)| \le{} & \left| \int\limits_{\Gamma_{11}} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi) - \partial_{\nu(\overline{x})} G^{(\mathcal{I})}(z,\overline{x})\Big]\, \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \right| + \\
& + \Big|\partial_{\nu(\overline{x})} G^{(\mathcal{I})}(z,\overline{x})\Big| \left| \int\limits_{\Gamma_{11}} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \right| \le \\
& \le \int\limits_{\Gamma_{11}} \frac{\delta_5 a_0}{|\xi - x|} \frac{|\xi - \overline{x}|}{\varrho_z(\xi,\overline{x})}\, dS_\xi + b_2 \left| \int\limits_{\Gamma_{11}} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \right| \le \\
& \le \frac{4\delta_5 a_0}{3\varrho} \int\limits_{\Gamma_{11}} dS_\xi + b_2 \left| \int\limits_{\Gamma_{11}} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \right|.
\end{aligned} \tag{3.54}$$

Note that

$$\int\limits_{\Gamma_{11}} dS_\xi \le \int\limits_{\alpha}^{\beta} \sqrt{1+[f'(\xi_1)]^2}\, d\xi_1 \le \sqrt{1+L^2}\,(\beta-\alpha),$$

where $\beta = \overline{x}_1 + 2\varrho$, $\alpha = \overline{x}_1 - 2\varrho$.

Therefore

$$\int\limits_{\Gamma_{11}} dS_\xi \le 4\sqrt{1+L^2}\,\varrho. \tag{3.55}$$

Further, in accordance with the relation

$$\partial_{x_j} G^{(\mathcal{D})}(\xi,x) = -\frac{1}{2\pi}\,\frac{x_j-\xi_j}{|x-\xi|^2} + O(|x-\xi|\,\ln|x-\xi|),$$

we easily conclude that

$$\int\limits_{\Gamma_{11}} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi = -\frac{1}{2\pi}\int\limits_{\Gamma_{11}} \frac{x_j-\xi_j}{|x-\xi|^2}\, dS_\xi + O(1).$$

On the other hand the integral $-\frac{1}{2\pi}\int_{\Gamma_{11}}\frac{x_j-\xi_j}{|x-\xi|^2}\,dS_\xi$ $(j=1,2)$ can be represented as either the real or imaginary part of the Cauchy type integral

$$\Psi(\tau) := -\frac{1}{2\pi}\int\limits_{\Gamma_{11}} \frac{e^{-i\theta(\zeta)}d\zeta}{\tau-\zeta}, \quad \zeta=\xi_1+i\xi_2, \quad \tau = x_1+ix_2, \quad d\zeta = e^{i\theta(\zeta)}\,dS,$$

where $\theta(\zeta)$ is the angle between the tangent line at $\zeta\in\Gamma_{11}$ and x_1 axis. Rewrite this integral as follows (see, e.g., [23], §11)

$$\begin{aligned}\Psi(\tau) &= -\frac{1}{2\pi}\int\limits_{\Gamma_{11}} \frac{e^{-i\theta(\zeta)}-e^{-i\theta(\overline{\tau})}}{\tau-\zeta}\,d\zeta - \frac{1}{2\pi}\,e^{-i\theta(\overline{\tau})}\int\limits_{\Gamma_{11}}\frac{d\zeta}{\tau-\zeta} = \\ &= -\frac{1}{2\pi}\,e^{-i\theta(\overline{\tau})}\ln\frac{\tau-\tau_1}{\tau-\tau_0} + O(1),\end{aligned}$$

where $\overline{\tau} = \overline{x}_1 + i\overline{x}_2$, and τ_0 and τ_1 are the end points of the arc Γ_{11}. By relations (3.47) and (3.49) and since $x\in\Omega_f^+\setminus\Omega_f^+(\delta)$, i.e., $|x-\overline{x}|\le\delta$, we deduce that $\varrho\le|\tau-\tau_j|\le 4\varrho$, and consequently

$$\frac{1}{4} \le \frac{|\tau-\tau_1|}{|\tau-\tau_0|} \le 4.$$

Therefore

$$|\Psi(\tau)| \le c_8$$

with a positive constant c_8 depending only on M and δ.

Therefore we conclude that

$$\Bigg|\int\limits_{\Gamma_{11}} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\,dS_\xi\Bigg| \le c_9. \tag{3.56}$$

Now, from (3.54), (3.55), and (3.56) it follows that

$$|I_{11}(z,x)| \le c_{10}. \tag{3.57}$$

Further, we estimate $I_{12}(z,x)$ using (3.53), (3.39), and (3.51)

$$\begin{aligned}|I_{12}(z,x)| &\le \int\limits_{\Gamma_{12}} \left|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right| \left|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\right| dS_\xi \le \\ &\le a_0\, b_2 \int\limits_{\Gamma_{12}} \frac{dS_\xi}{|\xi - x|} \le 2\, a_0\, b_2 \int\limits_{\Gamma_{12}} \frac{dS_\xi}{|\xi_1 - \overline{x}_1|} \le \\ &\le 2\, a_0\, b_2 \sqrt{1+L^2} \left\{ \int\limits_{-16\delta}^{-2\varrho/\sqrt{1+L^2}} \frac{dt}{|t|} + \int\limits_{2\varrho/\sqrt{1+L^2}}^{16\delta} \frac{dt}{|t|} \right\} \le \\ &\le c_{11}\, [1 + |\ln \varrho|]. \end{aligned} \tag{3.58}$$

Finally, from (3.52), (3.57), and (3.58) we obtain

$$|I_1(z,x)| \le c_{12}\, [1 + |\ln \varrho|]$$

with $c_{12} = \max\{c_{10},\, c_{11}\}$, whence

$$|I_1(z,x)| \le c_{12}\, [1 + |\ln \delta|]\, [1 + |\ln |z - x|\, |]. \tag{3.59}$$

Sub–step 2.2.(ii). Now we assume that there holds (3.48) and estimate $I_1(z,x)$. To this end, divide the arc Γ_1 determined by (3.42) into two disjoint subsets: $\Gamma_1 = \sigma_{11} \cup \sigma_{12}$, where

$$\sigma_{11} := \{\xi \in \Gamma_1 \mid |\xi - z| \le \varrho/4\}, \quad \sigma_{12} := \Gamma_1 \setminus \sigma_{11}$$

with $\varrho := 6^{-1}|z - x| \le |x - \overline{x}| \le \delta$.

It can be easily shown that the distance from the end points of Γ_1 to σ_{12} is greater than 2δ.

In view of (3.45), (2.4), and (3.39) we have

$$\begin{aligned}|I_1(z,x) &\le \int\limits_{\Gamma_1} \left|\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\right| \left|\partial_{x_j} G^{(\mathcal{D})}(\xi,x)\right| dS_\xi \le \\ &\le a_0\, b_2 \int\limits_{\Gamma_1} \frac{dS_\xi}{|\xi - x|} = a_0\, b_2 \left[\int\limits_{\sigma_{12}} + \int\limits_{\sigma_{12}} \right] \frac{dS_\xi}{|\xi - x|}.\end{aligned}$$

It is easy to show that

$$\begin{aligned} \xi \in \sigma_{11} \quad &\Longrightarrow \quad |\xi - x| \ge |\overline{x} - x| \ge 6^{-1}|x - z| = \varrho, \\ \xi \in \sigma_{12} \quad &\Longrightarrow \quad |\xi - x| \ge 2^{-1}|\xi - \overline{x}|, \end{aligned}$$

$$\int\limits_{\sigma_{11}} dS_\xi \leq 2\sqrt{1+L^2}\;\varrho,$$

$$\int\limits_{\sigma_{12}} \frac{dS_\xi}{|\overline{x}-\xi|} \leq \int\limits_{\sigma_{12}} \frac{dS_\xi}{|\overline{x}_1-\xi_1|} \leq \sqrt{1+L^2}\left[\;\int\limits_{-16\delta}^{-\varrho/\sqrt{1+L^2}} + \int\limits_{\varrho/\sqrt{1+L^2}}^{16\delta}\;\right]\frac{dt}{|t|}$$
$$\leq c_{13}\,[1+|\ln\varrho|]$$

with positive constant c_{13} depending only on δ and L. Taking into consideration these relations we easily deduce

$$|I_1(z,x)| \leq a_0\, b_2\,[\,2\sqrt{1+L^2}+c_{13}\,(1+|\ln\varrho|)\,],$$

i.e.,

$$|I_1(z,x)| \leq c_{14}\,[\,1+|\ln\varrho|\,]. \tag{3.60}$$

Now, if we combine the results (3.59) and (3.60), we get

$$|I_1(z,x)| \leq c_{15}\,[\,1+|\,\ln|z-x|\,|\,] \tag{3.61}$$

for $z\in\Gamma_f$ and $x\in\Omega_f^+\setminus\Omega_f^+(\delta)$ with $|z_1-x_1|\leq 4\delta$.

Finally, from (3.44), (3.46), and (3.61) it follows that

$$|Q_j(z,x)| \leq c_{16}\,[\,1+|\,\ln|z-x|\,|\,] \tag{3.62}$$

$$\text{for}\quad z\in\Gamma_f \quad\text{and}\quad x\in\Omega_f^+\setminus\Omega_f^+(\delta)\quad\text{with}\quad |z_1-x_1|\leq 4\delta. \tag{3.63}$$

Note that for x and z satisfying the relations (3.63) there holds the inequality

$$|z-x| \leq \sqrt{(4\delta)^2+(f_+ +2)^2} =: A_1,$$

and, therefore, if we assume that $|x-z|\geq\delta$, then (3.62) and (3.63) yield

$$|Q_j(z,x)| \leq c_{16}\,[\,1+|\,\ln\delta\,|+|\ln A_1|\,](1+A_1)^{3/2}\,\frac{1+x_2}{[1+|x-z|]^{3/2}},$$

which along with (3.36), (3.41), and (3.62) completes the proof. □

Corollary 3.5. *There exists a positive constant a_6, such that*

$$|(\mathcal{K}Q_j(\cdot,x))\,(z)| \leq a_6\,\frac{1+x_2}{[1+|x-z|]^{3/2}},\quad z\in\Gamma_f,\quad x\in\Omega_f^+.$$

Proof. It is a ready consequence of Lemmas 3.1 and 3.4. □

Now, let us return back to the equation (3.27) and apply to the both sides the operator $\mathcal{K}$ (see (3.26) and (3.28))

$$-\frac{1}{2}\,(\mathcal{K}\psi_j(\cdot,x))\,(z)+\mathcal{K}\,(\mathcal{K}\psi_j(\cdot,x))\,(z) = Q_j(z,x).$$

Due to Lemmas 3.2, 3.3, 3.4, and Corollary 3.5 we have

$$\left|\left(\mathcal{K}^2\psi_j(\cdot,x)\right)(z)\right| = \left|2^{-1}\,(\mathcal{K}\psi_j(\cdot,x))\,(z)+Q_j(z,x)\right| \leq$$

$$\leq a_7 \, \frac{1+x_2}{[1+|z-x|]^{3/2}} \quad \text{for} \quad z \in \Gamma_f, \quad x \in \Omega_f^+,$$

and

$$|(\mathcal{K}\psi_j(\cdot,x))\,(z)| = \left| 2^{-1}\, \psi_j(z,x) + \partial_{x_j} G^{(\mathcal{D})}(z,x) \right| \leq$$

$$\leq \begin{cases} a_8 \, \dfrac{1+x_2}{[1+|z-x|]^{3/2}} & \text{for} \quad z \in \Gamma_f, \quad x \in \Omega_f^+, \quad |z-x| \geq \delta, \\ a_8 \, [1+|\,\ln|z-x|\,|] & \text{for} \quad z \in \Gamma_f, \quad x \in \Omega_f^+, \quad |z-x| \leq \delta. \end{cases} \tag{3.64}$$

The relations (3.27) and (3.64) show that the principal singular part of $\psi_j(z,x)$ near the pole x and at infinity (as $|z-x| \to +\infty$) is the function $-2\,\partial_{x_j} G^{(\mathcal{D})}(z,x)$. Therefore we can prove the following proposition.

Lemma 3.6. *There exists a positive constant a_9, such that*

$$|\psi_j(z,x)| \leq \begin{cases} \dfrac{a_9\,(1+x_2)}{[1+|z-x|]^{3/2}} & \text{for} \quad z \in \Gamma_f, \quad x \in \Omega_f^+, \quad |z-x| \geq \delta, \\ \dfrac{a_9}{|z-x|} & \text{for} \quad z \in \Gamma_f, \quad x \in \Omega_f^+, \quad |z-x| \leq \delta. \end{cases}$$

Proof. It is a ready consequence of estimates (3.64) and (2.4). □

3.3. In this subsection we will study the behaviuor of the tangent derivatives of the functions $\psi(z,x)$ and $\nabla_x \psi(z,x)$ with respect to the variable $z \in \Gamma_f$ for $(z,x) \in \Gamma_f \times \Gamma_h$. Throughout this subsection we assume that h is a positive constant satisfying the inequality $h > f_+ + \delta$ and $f \in \mathcal{B}_{c,M}$.

We start with the following

Lemma 3.7. *Let*

$$K^{(2)}(y,\eta) := \int_{\Gamma_f} \left[\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) \right] \, \partial_{\nu(\eta)} G^{(\mathcal{I})}(\xi,\eta) \, dS_\xi, \quad y,\eta \in \Gamma_f. \tag{3.65}$$

There exists a positive constant δ_8, such that

$$|K^{(2)}(y',\eta) - K^{(2)}(y'',\eta)| \leq \delta_8 \, \frac{|y'-y''|\,(1+|\,\ln|y'-y''|\,|\,)}{[1+\varrho_\eta(y',y'')]^{3/2}}, \tag{3.66}$$

$$|K^{(2)}(y,\eta') - K^{(2)}(y,\eta'')| \leq \delta_8 \, \frac{|\eta'-\eta''|\,(1+|\,\ln|\eta'-\eta''|\,|\,)}{[1+\varrho_y(\eta',\eta'')]^{3/2}}, \tag{3.67}$$

for $y,\eta,y',y'',\eta',\eta'' \in \Gamma_f$, $|y'-y''| \leq \delta$, and $|\eta'-\eta''| \leq \delta$.

Proof. We will prove the inequality (3.66). The proof of (3.67) is verbatim.

It is evident that (see Lemma 2.8)

$$|K^{(2)}(y',\eta) - K^{(2)}(y'',\eta)| \leq$$

$$\leq \int_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y',\xi) - \partial_{\nu(\xi)} G^{(\mathcal{I})}(y'',\xi)| \, |\partial_{\nu(\eta)} G^{(\mathcal{I})}(\xi,\eta)| \, dS_\xi \leq$$

$$\leq \delta_6 \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y',\xi) - \partial_{\nu(\xi)} G^{(\mathcal{I})}(y'',\xi)| \, \frac{1}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi. \tag{3.68}$$

We divide Γ_f into four subsets. Without loss of generality let us assume that $y_1' < y_1''$, and put $r := 2^{-1}|y_1' - y_1''|$ and

$$\Gamma_1 := \{\xi \in \Gamma_f \mid \xi_1 \leq -8\delta + y_1'\} \cup \{\xi \in \Gamma_f \mid \xi_1 \geq 8\delta + y_1''\},$$

$$\Gamma_2 := \{\xi \in \Gamma_f \mid -8\delta + y_1' \leq \xi_1 \leq y_1' - r\} \cup$$

$$\cup \{\xi \in \Gamma_f \mid y_1'' \leq \xi_1 \leq y_1'' + 8\delta\},$$

$$\Gamma_3 := \{\xi \in \Gamma_f \mid y_1' - r \leq \xi_1 \leq y_1' + r\},$$

$$\Gamma_4 := \{\xi \in \Gamma_f \mid y_1' + r \leq \xi_1 \leq y_1'' + r\}.$$

Further, let

$$I_j(z,x) := \int\limits_{\Gamma_j} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y',\xi) - \partial_{\nu(\xi)} G^{(\mathcal{I})}(y'',\xi)| \, \frac{1}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi,$$

$$j = \overline{1,4}.$$

Applying Lemmas 2.4, 2.5, 2.8, and (3.39) we obtain

$$|I_1(z,x)| \leq \int\limits_{\Gamma_1} \frac{\delta_2(1+M)|y'-y''|}{[1+\varrho_\xi(y',y'')]^{3/2}} \, \frac{1}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi \leq$$

$$\leq \int\limits_{\Gamma_f} \left[\frac{1}{(1+|\xi-y'|)^{3/2}} + \frac{1}{(1+|\xi-y''|)^{3/2}} \right] \frac{\delta_2(1+M)|y'-y''|}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi \leq$$

$$\leq \delta_1 \delta_2 (1+M) \, |y'-y''| \, \frac{1}{[1+\varrho_\eta(y',y'')]^{3/2}},$$

$$|I_2(z,x)| \leq \int\limits_{\Gamma_2} \frac{\delta_5 \, |y'-y''|}{\varrho_\xi(y',y'')} \, \frac{1}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi \leq$$

$$\leq \delta_5 |y'-y''| \int\limits_{\Gamma_2} \left[\frac{1}{|\xi-y'|} + \frac{1}{|\xi-y''|} \right] \frac{1}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi \leq$$

$$\leq \frac{\delta_5 \, |y'-y''|}{\left(1 + \inf\limits_{\xi\in\Gamma_2} |\xi-\eta|\right)^{3/2}} \int\limits_{\Gamma_2} \left[\frac{1}{|\xi-y'|} + \frac{1}{|\xi-y''|} \right] dS_\xi \leq$$

$$\leq \frac{c_1 |y'-y''| \, (1 + |\ln|y'-y''||)}{[1+\varrho_\eta(y',y'')]^{3/2}},$$

$$|I_3(z,x)| + |I_4(z,x)| \leq \int\limits_{\Gamma_3 \cup \Gamma_4} \frac{2\, b_2}{(1+|\xi-\eta|)^{3/2}} \, dS_\xi \leq$$

$$\leq \frac{2\,b_2}{\Big(1+\inf\limits_{\xi\in\Gamma_3\cup\Gamma_4}|\xi-\eta|\Big)^{3/2}}\int\limits_{\Gamma_3\cup\Gamma_4} dS_\xi \leq$$

$$\leq \frac{4\,b_2\sqrt{1+L^2}\,|y'-y''|}{\Big(1+\inf\limits_{\xi\in\Gamma_3\cup\Gamma_4}|\xi-\eta|\Big)^{3/2}} \leq \frac{c_2\,|y'-y''|}{[1+\varrho_\eta(y',y'')]^{3/2}},$$

whence the proof follows. □

Now we are in a position to show the Hölder continuity property for $\psi(z,x)$ and $\nabla_x\psi(z,x)=(\psi_1(z,x),\psi_2(z,x))$ (see (3.26)).

Lemma 3.8. *There exist positive constants* a_{10} *and* a_{11}*, such that*

$$|\chi(z,x)| \leq \frac{a_{10}}{[1+|z-x|]^{3/2}}, \tag{3.69}$$

$$|\chi(z',x)-\chi(z'',x)| \leq \frac{a_{11}\,|z'-z''|\,(1+|\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}, \tag{3.70}$$

$$\text{for}\quad z,z',z''\in\Gamma_f,\quad x\in\Gamma_h,\quad |z'-z''|\leq\delta,$$

where $\chi\in\{\psi,\,\psi_1,\,\psi_2\}$.

Proof. Step 1. The inequalities (3.69) follow immediately from Lemmas 3.3 and 3.6.

Step 2. Let us show the inequality (3.70) for the function ψ. From (3.2) we obtain

$$\psi(z,x) = -2G^{(\mathcal{D})}(z,x) - 4\left(\mathcal{K}G^{(\mathcal{D})}(\cdot,x)\right)(z) + 4\left(\mathcal{K}^2\psi(\cdot,x)\right)(z). \tag{3.71}$$

Note that the kernel function of the integral operator $\mathcal{K}^2$ is the function $K^{(2)}$ given by (3.65). Taking into consideration that $z',z''\in\Gamma_f$ and $x\in\Gamma_h$ with the help of (2.4) and the inequality (3.69) we get from (3.71)

$$\begin{aligned}|\psi(z',x)-\psi(z'',x)| &\leq 2\,|G^{(\mathcal{D})}(z',x)-G^{(\mathcal{D})}(z'',x)|+\\ &+4\int\limits_{\Gamma_f}|\partial_{\nu(\xi)}G^{(\mathcal{I})}(z',\xi)-\partial_{\nu(\xi)}G^{(\mathcal{I})}(z'',\xi)|\,\frac{a_0(1+M)(1+h)}{[1+|\xi-x|]^{3/2}}\,dS_\xi+\\ &+4\int\limits_{\Gamma_f}|K^{(2)}(z',\xi)-K^{(2)}(z'',\xi)|\,\frac{a_{10}}{[1+|\xi-x|]^{3/2}}\,dS_\xi.\end{aligned} \tag{3.72}$$

By the same approach as in the proof of Lemma 2.5 we can easily deduce that

$$|G^{(\mathcal{D})}(z',x)-G^{(\mathcal{D})}(z'',x)| \leq \frac{c_1\,|z'-z''|}{[1+\varrho_x(z',z'')]^{3/2}} \tag{3.73}$$

for $z',z''\in\Gamma_f$ and $x\in\Gamma_h$.

The second summand in the right–hand side of (3.72) can be estimated as the integral (3.68) in the proof of Lemma 3.7

$$\int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(z',\xi) - \partial_{\nu(\xi)} G^{(\mathcal{I})}(z'',\xi)| \, \frac{1}{[1+|\xi - x|]^{3/2}} \, dS_\xi \leq$$

$$\leq \frac{c_2\,|z'-z''|\,(1+|\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}. \tag{3.74}$$

The third summand in the right–hand side of (3.72) we can estimate with the help of Lemmas 3.7 and 2.4

$$\int\limits_{\Gamma_f} |K^{(2)}(z',\xi) - K^{(2)}(z'',\xi)| \, \frac{1}{[1+|\xi - x|]^{3/2}} \, dS_\xi \leq$$

$$\leq \int\limits_{\Gamma_f} \frac{\delta_6\,|z'-z''|\,|(1+|\ln|z'-z''|\,|)}{[1+\varrho_\xi(z',z'')]^{3/2}} \, \frac{1}{[1+|\xi - x|]^{3/2}} \, dS_\xi \leq$$

$$\leq \delta_6\,|z'-z''|(1+|\ln|z'-z''||) \left\{ \int\limits_{\Gamma_f} \frac{dS_\xi}{[1+|\xi - z'|]^{3/2}\,[1+|\xi - x|]^{3/2}} + \right.$$

$$\left. + \int\limits_{\Gamma_f} \frac{dS_\xi}{[1+|\xi - z''|]^{3/2}[1+|\xi - x|]^{3/2}} \right\} \leq$$

$$\leq \frac{2\,\delta_1\,\delta_6\,|z'-z''|\,(1+|\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}. \tag{3.75}$$

It is easy to see that (3.73)–(3.75) imply (3.70) for the function ψ.

The proof for the function ψ_j is verbatim due to the equation (3.27), the inequality (3.69) and Lemma 2.5. □

In what follows we show that the solutions ψ and ψ_j of the integral equations (3.2) and (3.27) possess the tangent derivatives (on Γ_f) which are Hölder continuous.

Lemma 3.9. *Let $f \in \mathcal{B}_{c,M}$ and $x \in \Gamma_h$. Then the solutions ψ and ψ_j of the integral equations (3.2) and (3.27) possess the tangent derivatives with respect to z on Γ_f and there exists a positive constant a_{12}, such that*

$$|\partial_{\tau(z)}\,\chi(z,x)| \leq \frac{a_{12}}{[1+|z-x|\,]^{3/2}},$$

$$|\partial_{\tau(\xi)}\,\chi(\xi,x) - \partial_{\tau(\eta)}\,\chi(\eta,x)| \leq \frac{a_{12}\,|\xi-\eta|^\alpha}{[1+\varrho_x(\xi,\eta)]^{3/2}},$$

$$\text{for} \quad z,\xi,\eta \in \Gamma_f, \quad x \in \Gamma_h, \quad |\xi-\eta| \leq \delta, \quad j=1,2, \quad \forall \alpha \in [0,1),$$

where $\chi \in \{\psi,\,\psi_1,\,\psi_2\}$.

Proof. We will prove the lemma for the function ψ_j since the proof for the ψ is verbatim.

We first show that the function

$$P(z,x) := (\mathcal{K}\psi_j(\cdot,x))(z) = \\ = \int\limits_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\, \psi_j(\xi,x)\, dS_\xi, \quad z\in\Gamma_f, \quad x\in\Gamma_h, \tag{3.76}$$

possesses the tangent derivative with respect to z on Γ_f.

To this end we introduce a cut off function

$$\varphi\in C^\infty(\mathbb{R}), \qquad \varphi(t_1) := \begin{cases} 1 & \text{for} \quad -2\le t_1\le 2,\\ 0 & \text{for} \quad |t_1|\ge 4, \end{cases} \tag{3.77}$$

$$0\le \varphi(t_1)\le 1, \quad \sup_{t_1\in\mathbb{R}} \{|\varphi(t_1)| + |\varphi'(t_1)| + |\varphi''(t_1)|\} =: N < +\infty.$$

Represent $P(z,x)$ as

$$P(z,x) = P_1(z,x) + P_2(z,x), \tag{3.78}$$

where

$$P_1(z,x) := \int\limits_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\; \psi_j(\xi,x)\, [1-\varphi(z_1-\xi_1)]\, dS_\xi,$$
$$P_2(z,x) := \int\limits_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)\Big]\; \psi_j(\xi,x)\, \varphi(z_1-\xi_1)\, dS_\xi. \tag{3.79}$$

Note that

$$\partial_{\tau(z)} P_1(z,x) = \int\limits_{\Gamma_1(z)} \{[\partial_{\tau(z)}\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)]\,[1-\varphi(z_1-\xi_1)] - \\ - [\partial_{\nu(\xi)} G^{(\mathcal{I})}(z,\xi)]\, \partial_{\tau(z)}\varphi(z_1-\xi_1)\}\; \psi_j(\xi,x)\, dS_\xi,$$

where $\Gamma_1(z) := \{\xi\in\Gamma_f \mid |z_1-\xi_1|\ge 2\}$.

Therefore due to (2.4), (3.77), and Lemmas 3.6, 2.8(iii), and 2.4 we have

$$|\partial_{\tau(z)} P_1(z,x)| = \int\limits_{\Gamma_1(z)} \frac{a_0+\delta_7}{(1+|z-\xi|)^{3/2}}\, \frac{a_9(1+N)(1+h)}{(1+|\xi-x|)^{3/2}}\, dS_\xi \le \\ \le \frac{c_1}{(1+|z-x|)^{3/2}}, \tag{3.80}$$

$$|\partial_{\tau(z')} P_1(z',x) - \partial_{\tau(z'')} P_1(z'',x)| \le \\ \le \int\limits_{\Gamma_f} \frac{c_2\,|z'-z''|}{[1+\varrho_\xi(z',z'')]^{3/2}\,(1+|\xi-x|)^{3/2}}\, dS_\xi \le$$

$$\leq \frac{c_3\,|z'-z''|}{[1+\varrho_x(z',z'')]^{3/2}} \tag{3.81}$$

$$\text{for}\quad z,z',z''\in\Gamma_f,\quad x\in\Gamma_h,\quad |z'-z''|\leq\delta.$$

Further we study the differentiability of the function $P_2(z,x)$ given by (3.79). To this end let us first note that the line of integration in (3.79) is actually the finite arc $\Gamma_2(z) := \{\xi\in\Gamma_f \mid |\xi_1 - z_1| \leq 4\}$. Therefore, the length of the arc is bounded by the constant $4\sqrt{1+L^2}$. We represent $P_2(z,x)$ as follows

$$P_2(z,x) = P_{21}(z,x) + P_{22}(z,x) \tag{3.82}$$

with

$$P_{21}(z,x) := \int\limits_{\Gamma_2(z)} R(z,\xi)\,\varphi(z_1-\xi_1)\psi_j(\xi,x)\,dS_\xi,$$

$$P_{22}(z,x) := -\frac{1}{2\pi}\int\limits_{\Gamma_2(z)} \left(\partial_{\nu(\xi)}\ln|z-\xi|\right)\,\varphi(z_1-\xi_1)\;\psi_j(\xi,x)\;dS_\xi,$$

where $R(z,\xi)$ is the so–called regular part of the normal derivative of the impedance Green's function

$$R(z,\xi) := \partial_{\nu(\xi)}G^{(\mathcal{I})}(z,\xi) + \frac{1}{2\pi}\partial_{\nu(\xi)}\ln|z-\xi| =$$

$$= \partial_{\nu(\xi)}\left(\kappa_1|z-\xi|^2\ln|z-\xi| + \kappa_2|z-\xi|^2 + \right.$$

$$\left. + \,\kappa_3|z-\xi|^4\ln|z-\xi| + \kappa_4|z-\xi|^4 + \cdots\right); \tag{3.83}$$

here κ_j are quite definite constants (depending on k, δ and M).

From (3.83) it follows that

$$|\partial_{\tau(z)}R(z,\xi)| \leq c_4|z-\xi|\,|\ln|z-\xi||, \tag{3.84}$$

$$|\partial_{\tau(z')}R(z',\xi) - \partial_{\tau(z'')}R(z'',\xi)| \leq$$

$$\leq c_4|z'-z''|(1+|\ln|z'-\xi|| + |\ln|z''-\xi||), \tag{3.85}$$

$$\text{for}\;\; z,z',z'',\xi\in\Gamma_f,\;\; |z'-z''|\leq\delta,$$

$$|z-\xi|\leq A,\quad |z'-\xi|\leq A,\quad |z''-\xi|\leq A,$$

where $A>0$ is some constant and c_4 depends only on M, δ, k, and A.

With the help of (3.83), (3.84), and (3.85) along with Lemma 3.6 we deduce

$$|\partial_{\tau(z)}P_{21}(z,x)| \leq \int\limits_{\Gamma_2(z)} |\partial_{\tau(z)}[R(z,\xi)\,\varphi(z_1-\xi_1)]|\;|\psi_j(\xi,x)|\,dS_\xi \leq$$

$$\leq c_5\int\limits_{\Gamma_2(z)}\frac{dS_\xi}{(1+|\xi-x|)^{3/2}} \leq \frac{4c_5\sqrt{1+L^2}}{\left(1+\inf\limits_{\xi\in\Gamma_2(z)}|\xi-x|\right)^{3/2}} \leq$$

$$\leq \frac{c_6}{[1+|z-x|\,]^{3/2}}\,, \quad z \in \Gamma_f, \quad x \in \Gamma_h, \tag{3.86}$$

$$\begin{aligned}
&|\partial_{\tau(z')} P_{21}(z',x) - \partial_{\tau(z'')} P_{21}(z'',x)| \leq \\
&\leq \int\limits_{\Gamma_f} |\partial_{\tau(z')}[R(z',\xi)\varphi(z'_1-\xi_1)] - \partial_{\tau(z'')}[R(z'',\xi)\varphi(z''_1-\xi_1)]\psi_j(\xi,x)|dS_\xi \leq \\
&\leq c_7\,|z'-z''| \int\limits_{\Gamma_2(z')\cup\Gamma_2(z'')} \frac{(1+|\,\ln|z'-\xi|\,|+|\,\ln|z''-\xi|\,|)}{(1+|\xi-x|)^{3/2}}\,dS_\xi \leq \\
&\leq \frac{c_8\,|z'-z''|}{[1+|z-x|\,]^{3/2}}.
\end{aligned} \tag{3.87}$$

Before we start to study the function $P_{22}(z,x)$ let us recall some basic properties of the harmonic logarithmic kernel:

(i) $\partial_{\tau(z)}\partial_{\nu(\xi)} \ln|z-\xi| = \partial_{\nu(z)}\partial_{\tau(\xi)} \ln|z-\xi| =$

$$= -\frac{\nu(\xi)\cdot\tau(z)}{|z-\xi|^2} + 2\,\frac{[\nu(\xi)\cdot(\xi-z)]\,[\tau(z)\cdot(\xi-z)]}{|z-\xi|^4},$$

(ii) $K^*(z,\xi) := \partial_{\tau(z)}\,[(\partial_{\nu(\xi)} \ln|z-\xi|)\,\varphi(z_1-\xi_1)], \quad |K^*(z,\xi)| \leq \dfrac{c_1^*}{|z-\xi|},$

$$\begin{aligned}
&|K^*(z',\xi) - K^*(z'',\xi)| \leq \frac{c_1^*\,|z'-z''|}{[\varrho_\xi(z',z'')]^2},\\
&|K^*(z,\xi') - K^*(z,\xi'')| \leq \frac{c_1^*\,|\xi'-\xi''|}{[\varrho_z(\xi',\xi'')]^2},
\end{aligned} \tag{3.88}$$

where $\xi,\xi',\xi'',z,z',z'' \in \Gamma_f$, $|z'-z''| \leq \delta$, and $|\xi'-\xi''| \leq \delta$; the constant c_1^* depends only on M. Moreover, the function

$$g^*(z) := \partial_{\tau(z)} \int\limits_{\Gamma_2(z)} \partial_{\nu(\xi)} \ln|z-\xi|\,dS_\xi \tag{3.89}$$

exists for all $z \in \Gamma_f$ and is a Lipschitz continuous function, i.e.,

$$|g^*(z') - g^*(z'')| \leq c_2^*\,|z'-z''| \quad \text{for} \quad |z'-z''| \leq \delta \tag{3.90}$$

with the constant c_2^* depending only on M.

Taking into consideration that $K^*(z,\xi)$ has a compact support $\Gamma_2(z)$ with respect to ξ for a fixed z and applying the differentiation formula for potential type integrals with weakly singular kernels we get (see, e.g., [10], Theorem A32, formula (A19), Remarks A33 and A34):

$$\partial_{\tau(z)} P_{22}(z,x) = -\frac{1}{2\pi} \int\limits_{\Gamma_f} K^*(z,\xi)\,[\psi_j(\xi,x) - \psi_j(z,x)]\,dS_\xi -$$

$$- \frac{1}{2\pi}\, \psi_j(z,x)\, g^*(z),$$

whence it follows that (see Lemma 3.8, (3.88), (3.89), and (3.90))

$$|\partial_{\tau(z)} P_{22}(z,x)| \le \\ \le \frac{1}{2\pi} \int\limits_{\Gamma_2(z)} \frac{c_1^*}{|z-\xi|}\, \frac{a_{10}(1+M)|z-\xi|\,(1+|\,\ln|z-\xi|\,|)}{[1+\varrho_x(\xi,z)]^{3/2}}\, dS_\xi + \\ + \frac{1}{2\pi}\, \frac{a_{10}}{[1+|z-x|]^{3/2}} \sup_{z\in\Gamma_f} |g^*(z)| \le \frac{c_9}{[1+|z-x|]^{3/2}}. \tag{3.91}$$

Let us now consider the difference

$$\partial_{\tau(z')} P_{22}(z',x) - \partial_{\tau(z'')} P_{22}(z'',x) = I_1 + I_2 + I_3 + \\ + \frac{1}{2\pi}\, [\psi_j(z'',x)\, g^*(z'') - \psi_j(z',x)\, g^*(z')], \tag{3.92}$$

where

$$I_1 := \int\limits_{\Gamma_2'} K^*(z',\xi)\, [\psi_j(\xi,x) - \psi_j(z',x)]\, dS_\xi,$$

$$I_2 := \int\limits_{\Gamma_2'} K^*(z'',\xi)[\psi_j(\xi,x) - \psi_j(z'',x)] dS_\xi,$$

$$I_3 := \int\limits_{\Gamma_f\setminus\Gamma_2'} \{K^*(z',\xi)[\psi_j(\xi,x)-\psi_j(z',x)] - K^*(z'',\xi)[\psi_j(\xi,x)-\psi_j(z'',x)]\} dS_\xi,$$

$$\Gamma_2' := \{\xi\in\Gamma_f \mid |\xi_1 - z_1'| \le 4\sqrt{1+L^2}|z_1' - z_1''|\}.$$

Clearly, $|z_1' - z_1''| \le |z' - z''| \le \sqrt{1+L^2}\, |z_1' - z_1''|$.

Using the relations (3.88) and Lemma 3.8 we derive

$$|I_1| \le \int\limits_{\Gamma_2'} \frac{c_1^*}{|z'-\xi|}\, \frac{a_{10}\, |\xi - z'|(1+|\,\ln|\xi - z'|\,|)}{[1+\varrho_x(\xi,z')]^{3/2}}\, dS_\xi \le \\ \le \frac{c_1^* a_{10}}{[1+\varrho_x(z',z'')]^{3/2}} \int\limits_{\Gamma_2'} (1+|\,\ln|\xi - z'|\,|)\, dS_\xi \le \\ \le \frac{c_{10}\, |z'-z''|\,(1+|\,\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}, \tag{3.93}$$

and, analogously,

$$|I_2| \le \frac{c_{11}\, |z'-z''|\,(1+|\,\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}. \tag{3.94}$$

Further we have (see (3.88) and Lemma 3.8)

$$|I_3| \leq \int\limits_{\Gamma_f\setminus\Gamma_2'} \{|\,[K^*(z',\xi) - K^*(z'',\xi)]\,[\psi_j(\xi,x) - \psi_j(z',x)]\,|+$$

$$+\,|\,K^*(z'',\xi)\,[\psi_j(z'',x) - \psi_j(z',x)]\,|\}\,dS_\xi \leq$$

$$\leq \int\limits_{\Gamma_2^*} \frac{c_1^*\,|z'-z''|}{[\varrho_\xi(z',z'')]^2}\;\frac{a_{10}|\xi-z'|(1+|\,\ln|\xi-z'|\,|)}{[1+\varrho_x(\xi,z')]^{3/2}}\,dS_\xi +$$

$$+\,\frac{a_{10}|z'-z''|(1+|\,\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}\int\limits_{\Gamma_2^*}\frac{c_1^*}{|z''-\xi|}\,dS_\xi,$$

where $\Gamma_2^* := \{\xi\in\Gamma_f\setminus\Gamma_2' \mid \min\{z_1',z_1''\}-4<\xi_1<\max\{z_1',z_1''\}+4\}$.

Note that,

$$\begin{aligned}\xi\in\Gamma_2^* \implies & |\xi-z'|\leq\sqrt{1+L^2}\,|\xi_1-z_1'|\leq\\ &\qquad\leq\sqrt{1+L^2}\,(4+|z_1'-z_1''|\leq\sqrt{1+L^2}\,(4+\delta),\\ & |\xi-z''|\leq\sqrt{1+L^2}\,(4+\delta),\\ & |\xi_1-z_1'|\geq 4\sqrt{1+L^2}\,|z_1'-z_1''|\geq 4\,|z'-z''|,\\ & |\xi-z'|\geq 4\,|z'-z''|,\quad |\xi-z''|\geq 3\,|z'-z''|.\end{aligned}$$

Therefore,

$$|I_3|\leq\frac{c_{12}\,|z'-z''|}{[1+\varrho_x(z',z'')]^{3/2}}\int\limits_{\Gamma_2^*}\frac{1+|\,\ln|\xi-z'|\,|}{|\xi-z'|}\,dS_\xi+$$

$$+\,\frac{c_{13}|z'-z''|(1+|\ln|z'-z''\|)^2}{[1+\varrho_x(z',z'')]^{3/2}}\leq$$

$$\leq\frac{c_{14}|z'-z''|(1+|\ln|z'-z''\|)^2}{[1+\varrho_x(z',z'')]^{3/2}}. \tag{3.95}$$

With the help of Lemma 3.8 and (3.90) we get

$$\begin{aligned}&|\psi_j(z'',x)\;g^*(z'')-\psi_j(z',x)\;g^*(z')|\leq\\ &\quad\leq|\psi_j(z'',x)-\psi_j(z',x)|\,|g^*(z'')|+|\psi_j(z',x)|\,|g^*(z'')-g^*(z')|\leq\\ &\quad\leq\frac{c_{15}\,|z'-z''|\,(1+|\,\ln|z'-z''|\,|)}{[1+\varrho_x(z',z'')]^{3/2}}.\end{aligned}\tag{3.96}$$

In view of (3.92)–(3.96) we conclude

$$|\partial_{\tau(z')}P_{22}(z',x)-\partial_{\tau(z'')}P_{22}(z'',x)|\leq\frac{c_{16}|z'-z''|(1+|\ln|z'-z''\|)^2}{[1+\varrho_x(z',z'')]^{3/2}}. \tag{3.97}$$

By estimates (3.86), (3.87), (3.91), and (3.97) from (3.82) we have

$$
\begin{aligned}
|\partial_{\tau(z)}P_2(z,x)| &\le \frac{c_{17}}{[1+|z-x|^{3/2}}, \\
|\partial_{\tau(z')}P_2(z',x)-\partial_{\tau(z'')}P_2(z'',x)| &\le \frac{c_{17}|z'-z''|(1+|\ln|z'-z''||)^2}{[1+\varrho_x(z',z'')]^{3/2}}.
\end{aligned} \tag{3.98}
$$

Finally, the estimates (3.80), (3.81), and (3.98) along with (3.78) lead to the relations

$$
\begin{aligned}
|\partial_{\tau(z)}P(z,x)| &\le \frac{c_{18}}{(1+|z-x|)^{3/2}}, \\
|\partial_{\tau(z')}P(z',x)-\partial_{\tau(z'')}P(z'',x)| &\le \frac{c_{18}|z'-z''|(1+|\ln|z'-z''||)^2}{[1+\varrho_x(z',z'')]^{3/2}}
\end{aligned} \tag{3.99}
$$

for $z, z', z'' \in \Gamma_f$, $x \in \Gamma_h$, $|z'-z''| \le \delta$.

Now the proof of the lemma follows from (2.4), (3.76), (3.99), Lemmas 2.5 and 2.8, and the equality (cf. (3.2), (3.27))

$$
\psi_j(z,x) = -2\,\partial_{x_j}G^{(\mathcal{D})}(z,x) + 2\,(\mathcal{K}\psi_j(\cdot,x))\,(z), \quad z\in\Gamma_f, \quad x\in\Gamma_h. \qquad \square
$$

4. Uniform Bounds for the Dirichlet Green's Function and Its Derivatives

In this section we will estimate the so–called *regular part* $V(y,x)$ of the Dirichlet Green's function defined by (2.12) and its derivatives. This leads to the corresponding uniform bounds for the function $G_f^{(\mathcal{D})}(y,x)$ on $\Omega_f^+ \times \Omega_f^+$.

4.1. We start with the following proposition.

Theorem 4.1. *Let $f \in \mathcal{B}_{c,M}$ and $(y,x) \in \Omega_f^+ \times \Omega_f^+$. Then there exists a positive constant C_1, such that*

$$
|V(y,x)| \le \begin{cases} \dfrac{C_1(1+y_2)(1+x_2)}{[1+|x-y|]^{3/2}} & \text{for} \quad |x-y|+|x-\overline{x}|+|y-\overline{y}|\ge\delta, \\ C_1[1+|\ln|x-y||] & \text{for} \quad |x-y|+|x-\overline{x}|+|y-\overline{y}|\le\delta, \end{cases} \tag{4.1}
$$

where $V(y,x)$ is a unique solution of the Dirichlet boundary value problem (2.10)–(2.11)*, and $\overline{x}$ and $\overline{y}$ are points on Γ_f nearest to x and y, respectively.*

Proof. As we have mentioned above (see Subsection 2.6), the solution $V(\cdot,x)$ of the BVP (2.10)–(2.11) can be represented (in Ω_f^+) by formula (2.12) with the density function $\psi(\cdot,x)$ which solves the integral equation (2.13). We rewrite (2.12) as follows

$$
\begin{aligned}
V(y,x) &= \int\limits_{\Gamma_f} \Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\;\psi(\xi,x)\,dS_\xi = \\
&= -2\int\limits_{\Gamma_f} \Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\;G^{(\mathcal{D})}(\xi,x)\,dS_\xi +
\end{aligned}
$$

$$+\,2\int\limits_{\Gamma_f}\Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\ (\mathcal{K}\psi(\cdot,x))(\xi)\,dS_\xi,\quad y,x\in\Omega_f^+\times\Omega_f^+. \quad (4.2)$$

Due to the estimates (2.4) and Lemma 3.2 it follows from (4.2) that a principal singular part of $V(y,x)$ is the function $-2W(y,x)$, where

$$W(y,x)=\int\limits_{\Gamma_f}\Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\ G^{(\mathcal{D})}(\xi,x)\,dS_\xi \quad (4.3)$$

(cf. the estimates (2.4) and (3.22) for the function $G^{(\mathcal{D})}(\xi,x)$ and $\mathcal{K}\psi(\cdot,x))(\xi)$, respectively). Therefore it suffices to obtain the bounds (4.1) for the function $W(y,x)$.

To this end we consider separately the four possible cases:

$$\begin{array}{ll}\text{(i)}\ \ x_2\ge f_++2,\ \ y_2\ge f_++2, & \text{(ii)}\ \ x_2\ge f_++2,\ \ y_2\le f_++2,\\ \text{(iii)}\ \ x_2\le f_++2,\ \ y_2\ge f_++2, & \text{(iv)}\ \ x_2\le f_++2,\ \ y_2\le f_++2.\end{array} \quad (4.4)$$

Case (i). Since in the case under consideration $|x-\xi|\ge\delta$ and $|y-\xi|\ge\delta$ for any $\xi\in\Gamma_f$, with the help of bounds (2.4) and Lemma 2.4 along with relations (3.9)–(3.11) we have

$$|W(y,x)|\le\int\limits_{\Gamma_f}\frac{c_1(1+y_2)(1+x_2)}{[\,1+|\xi-y|\,]^{3/2}\,[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi\le$$

$$\le\frac{c_2(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}}.$$

Case (ii). Let $r:=|x-y|$ and consider the two sub–cases:

$$\text{(ii)}_1\quad r\ge 2(f_++2),\qquad \text{(ii)}_2\quad r\le 2(f_++2).$$

Sub–case (ii)_1. We denote

$$\Gamma_1:=\{\xi\in\Gamma_f\mid |y-\xi|\le r/2\},\quad \Gamma_2=\Gamma_f\setminus\Gamma_1.$$

It is evident that

$$\xi\in\Gamma_1\ \Longrightarrow\ |x-\xi|\ge\frac{r}{2}\ge\delta,$$

$$\xi\in\Gamma_2\ \Longrightarrow\ |y-\xi|\ge\frac{r}{2}\ge\delta,\ \ |x-\xi|\ge\delta,$$

since $f_++2\ge\delta$.

Therefore by (2.4), (4.3), and Lemmas 2.7 and 2.4 we derive

$$|W(y,x)|\le\int\limits_{\Gamma_1}|\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)|\,\frac{a_0(1+M)(1+x_2)}{[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi\le$$

$$\le\int\limits_{\Gamma_2}\frac{a_0(1+M)^2}{[\,1+|\xi-y|\,]^{3/2}}\,\frac{a_0(1+M)(1+x_2)}{[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi\le$$

$$\leq \frac{c_3(1+x_2)}{[\,1+r\,]^{3/2}} \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\, dS_\xi +$$

$$+\, c_4(1+x_2) \int\limits_{\Gamma_f} \frac{dS_\xi}{[\,1+|\xi-y|\,]^{3/2}[\,1+|\xi-x|\,]^{3/2}} \leq$$

$$\leq \frac{c_5(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}}.$$

Sub–case (ii)$_2$. Note that $x_2 \leq |x_2-y_2| + |y_2| \leq 3(f_+ + 2)$ and apply again the bounds (2.4) and Lemma 2.7 to obtain

$$|W(y,x)| \leq \int\limits_{\Gamma_1} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\, \frac{a_0(3+M)(1+x_2)}{[\,1+|\xi-x|\,]^{3/2}}\, dS_\xi \leq c_6 \leq$$

$$\leq c_6[1+2(f_+ + 2)]^{3/2}\, \frac{(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}}.$$

Case (iii). It is quite similar to Case (ii) and we easily arrive at the inequalities obtained in Sub–cases (ii)$_1$ and (ii)$_2$.

Case (iv). Put $r_1 := |y_1 - x_1|$ and consider the two cases:

$$\text{(iv)}_1 \quad r_1 \geq 2\delta, \qquad\qquad \text{(iv)}_2 \quad r_1 \leq 2\delta.$$

Sub–case (iv)$_1$. Denote

$$\Gamma_1 := \{\xi \in \Gamma_f \mid |\xi_1 - x_1| \leq r_1/2\},$$
$$\Gamma_2 := \{\xi \in \Gamma_f \mid |\xi_1 - y_1| \leq r_1/2\},$$
$$\Gamma_3 := \Gamma_f \setminus \{\Gamma_1 \cup \Gamma_2\}.$$

It is evident that

$$\xi \in \Gamma_1 \implies |y-\xi| \geq \frac{r_1}{2} \geq \delta,$$

$$\xi \in \Gamma_2 \implies |x-\xi| \geq \frac{r_1}{2} \geq \delta,$$

$$\xi \in \Gamma_3 \implies |x-\xi| \geq \frac{r_1}{2} \geq \delta,\;\; |x-\xi| \geq \frac{r_1}{2} \geq \delta.$$

Therefore in view of (2.4), (3.9)–(3.11), Lemmas 2.4 and 2.7 we deduce

$$|W(y,x)| \leq \int\limits_{\Gamma_1} \frac{a_0(3+M)^2}{[\,1+|\xi-y|\,]^{3/2}}\, |G^{(\mathcal{D})}(\xi,x)|\, dS_\xi +$$

$$+ \int\limits_{\Gamma_2} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\, \frac{a_0(3+M)^2}{[\,1+|\xi-x|\,]^{3/2}}\, dS_\xi +$$

$$+\int\limits_{\Gamma_3} \frac{a_0^2\,(3+M)^4}{[\,1+|\xi-y|\,]^{3/2}[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi \leq$$

$$\leq \frac{c_7(1+x_2)(1+y_2)}{[\,1+|x-y|\,]^{3/2}}.$$

Sub–case (iv)$_2$. Since $r_1=|x_1-y_1|\leq 2\delta$ we have $|x-y|^2\leq 4[\delta^2+(f_+ + 2)^2] =: A_1^2$.

We consider the two possible cases

$$|x-\overline{x}|\leq \kappa_0|x-y| \tag{4.5}$$

and

$$|x-\overline{x}|\geq \kappa_0|x-y|, \tag{4.6}$$

where, as above, $\overline{x}$ is a point on Γ_f nearest to x and $\kappa_0=(12A_1)^{-1}$.

Let there hold the inequality (4.5) and represent $W(y,x)$ as follows

$$W(y,x)=W_1(y,x)+W_2(y,x)+W_3(y,x), \tag{4.7}$$

where

$$W_j(y,x):=\int\limits_{\Gamma_j}\Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\;G^{(\mathcal{D})}(\xi,x)\,dS_\xi,\quad j=1,2,3,$$

with

$$\begin{aligned}
&\Gamma_1:=\{\xi\in\Gamma_f \mid |\xi_1-y_1|\geq 4\delta\},\\
&\Gamma_2:=\{\xi\in\Gamma_f \mid |\xi_1-y_1|\leq 4\delta,\ |\xi-\overline{x}|\leq 2\kappa_0|x-y|\},\\
&\Gamma_3:=\{\xi\in\Gamma_f \mid |\xi_1-y_1|\geq 4\delta,\ |\xi-\overline{x}|\geq 2\kappa_0|x-y|\}.
\end{aligned} \tag{4.8}$$

Clearly

$$\xi\in\Gamma_1 \quad\Longrightarrow\quad |\xi_1-x_1|\geq 2\delta,\ \ |\xi_1-y_1|\geq 4\delta,$$

and therefore, by (2.4)

$$|W_1(y,x)|\leq \int\limits_{\Gamma_1}\frac{a_0^2(3+M)^4}{[\,1+|\xi-y|\,]^{3/2}[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi\leq c_8. \tag{4.9}$$

Before we start to estimate the functions W_2 and W_3 let us note that

1) $\xi\in\Gamma_2\cup\Gamma_3 \quad\Longrightarrow |\xi-x|\leq[(8\delta)^2+(f_+ +2)^2]^{1/2}=:A_2,\ |\xi-y|\leq A_2;$

2) $\xi\in\Gamma_2 \qquad\quad\Longrightarrow |\xi-y|\geq(1-3\kappa_0)|x-y|\geq 2^{-1}|x-y|,$

$$|\xi-x|\geq 2^{-1}|\overline{x}-\xi|\geq 2^{-1}|\overline{x}_1-\xi_1|,$$

$$\left|\ln\frac{|\xi-x|}{2A_2}\right|\leq\left|\ln\frac{|\xi-\overline{x}|}{4A_2}\right|\leq\left|\ln\frac{|\xi_1-x_1|}{4A_2}\right|;$$

3) $\xi\in\Gamma_3$ $\Longrightarrow$ $|\xi-x|\geq|\xi-\overline{x}|-|x-\overline{x}|\geq\kappa_0|x-y|$,

$$\left|\ln\frac{|\xi-x|}{2A_2}\right|\leq\left|\ln\frac{\kappa_0|y-x|}{2A_2}\right|.$$

We employed here that $|\ln t_1|$ is a decreasing function for $t_1\in(0,1)$.

Using these relations along with (2.4) and Lemma 2.7 we deduce

$$|W_2(y,x)|+|W_3(y,x)|\leq\int\limits_{\Gamma_2}\frac{a_0^2[1+|\ln|\xi-x||]}{|\xi-y|}\,dS_\xi+$$
$$+\int\limits_{\Gamma_3}|\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)|\,a_0\,[1+|\ln|\xi-x||]\,dS_\xi\leq$$
$$\leq c_9\left[\frac{1}{|x-y|}\int\limits_{-2\kappa_0|x-y|}^{2\kappa_0|x-y|}[1+|\ln|t_1||]\,dt_1+\right.$$
$$\left.+[1+|\ln|x-y||]\int\limits_{\Gamma_f}|\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)|\,dS_\xi\right]\leq$$
$$\leq c_{10}\,[1+|\ln|x-y||].\tag{4.10}$$

It is evident that (4.7), (4.9), and (4.10) yield

$$|W(y,x)|\leq c_{11}\,[1+|\ln|x-y||]\tag{4.11}$$

for x and y satisfying conditions $(\mathrm{iv})_2$ and (4.5).

Now we assume that there holds (4.6) (along with $(\mathrm{iv})_2$) and decompose $W(y,x)$ as follows

$$W(y,x)=W_1^*(y,x)+W_2^*(y,x),\tag{4.12}$$

where

$$W_j^*(y,x):=\int\limits_{\Gamma_j^*}\Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\;G^{(\mathcal{D})}(\xi,x)\,dS_\xi,\quad j=1,2;$$

here $\Gamma_1^*:=\Gamma_1$ and $\Gamma_2^*:=\Gamma_f\setminus\Gamma_1^*$, where Γ_1 is defined by (4.8).

By the same arguments as above (see (4.9)) we easily deduce

$$|W_1^*(y,x)|\leq c_{12}.\tag{4.13}$$

Note that (see (4.6) and the items 1) and 3) above)

$$\xi\in\Gamma_2^*\Longrightarrow|\xi-x|\geq|x-\overline{x}|\geq\kappa_0|x-y|,$$
$$|\xi-x|\geq[(8\delta)^2+(f_++2)^2]^{1/2}=A_2,$$

$$\left|\ln\frac{|\xi-x|}{2A_2}\right|\leq\left|\ln\frac{\kappa_0|x-y|}{2A_2}\right|.$$

Therefore with the help of (2.4) and Lemma 2.7 we get

$$|W_2^*(y,x)| \le a_0 \int\limits_{\Gamma_2^*} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\,[1+|\,\ln|\xi-x|\,|]\,dS_\xi \le$$

$$\le c_{13}\,[1+|\,\ln|x-y|\,|] \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\,dS_\xi \le$$

$$\le c_{14}\,[1+|\,\ln|x-y|\,|]. \tag{4.14}$$

Due to (4.12), (4.13), and (4.14) we arrive at the inequality (4.11) for x and y satisfying the conditions (4.6) and $(\mathrm{iv})_2$.

Further we show that, if either

$$|x-\overline{x}| \ge \delta, \quad x_2 \le f_+ + 2, \quad y_2 \le f_+ + 2, \quad |x_1 - y_1| \le 2\delta, \tag{4.15}$$

or

$$|y-\overline{y}| \ge \delta, \quad x_2 \le f_+ + 2, \quad y_2 \le f_+ + 2, \quad |x_1 - y_1| \le 2\delta, \tag{4.16}$$

then $W(y,x)$ is uniformly bounded by a constant depending only on M, δ, and k. Indeed, by (2.4) and Lemma 2.7 we have

$$|W(y,x)| \le \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\,\frac{a_0(1+M)(1+x_2)}{[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi \le$$

$$\le a_0(3+M)^2 \int\limits_{\Gamma_f} |\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)|\,dS_\xi \le a_0(3+M)^2\delta_4$$

if x and y satisfy (4.15), and

$$|W(y,x)| \le \int\limits_{\Gamma_f} \frac{a_0(1+M)(1+y_2)}{[\,1+|\xi-y|\,]^{3/2}}|G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le$$

$$\le a_0(3+M)^2 \int\limits_{\Gamma_f} |G^{(\mathcal{D})}(\xi,x)|\,dS_\xi \le a_0(3+M)^2\delta_4$$

if x and y satisfy (4.16), i.e.,

$$|W(y,x)| \le \frac{c_{15}(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}}$$

if x and y satisfy either (4.15) or (4.16).

Thus, $W(x,y)$ has a logarithmic singularity $O(\ln|x-y|)$ if and only if the both points x and y are situated on Γ_f, and the uniform bounds given in (4.1) hold. □

Corollary 4.2. *Let $f \in \mathcal{B}_{c,M}$ and $(y,x) \in \Omega_f^+ \times \Omega_f^+$. Then there exists a positive constant C_2, such that*

$$|G_f^{(\mathcal{D})}(y,x)| \leq \begin{cases} \dfrac{C_2(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}} & \text{for} \quad |x-y| \geq \delta, \\ C_2\,[1+|\,\ln|x-y|\,|\,] & \text{for} \quad |x-y| \leq \delta, \end{cases}$$

where $G_f^{(\mathcal{D})}(y,x)$ is the Dirichlet Green's function for the Helmholtz operator given by (2.9).

Proof. It is a ready consequence of the bounds (2.4) and Theorem 4.1. □

4.2. Here we investigate the first order derivatives of the function $V(y,x)$ and $G_f^{(\mathcal{D})}(y,x)$. Due to the symmetry property of the Green's function $G_f^{(\mathcal{D})}(y,x) = G_f^{(\mathcal{D})}(x,y)$, it suffices to study the properties of derivatives with respect to the variable x.

From (4.2) it follows that for $(x,y) \in \Omega_f^+ \times \Omega_f^+$

$$\partial_{x_j} V(y,x) = \int\limits_{\Gamma_f} \Big[\,\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)\Big]\ \partial_{x_j}\psi(\xi,x)\,dS_\xi =$$

$$= -2W_j(y,x) + 2\int\limits_{\Gamma_f} \Big[\,\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)\Big]\ (\mathcal{K}\partial_{x_j}\psi(\cdot,x))(\xi)\,dS_\xi, \quad j=1,2,$$

where $W_j(y,x) := \partial_{x_j} W(y,x)$ (see (4.3)), i.e.,

$$W_j(y,x) = \int\limits_{\Gamma_f} \Big[\,\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)\Big]\ \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\,dS_\xi, \quad j=1,2. \tag{4.17}$$

Theorem 4.3. *Let $f \in \mathcal{B}_{c,M}$ and $(y,x) \in \Omega_f^+ \times \Omega_f^+$. Then there exists a positive constant C_3, such that*

$$|\partial_{x_j} V(y,x)| \leq \begin{cases} \dfrac{C_3(1+y_2)(1+x_2)}{[1+|x-y|]^{3/2}} & \text{for} \ |x-y|+|x-\overline{x}|+|y-\overline{y}| \geq \delta, \\ \dfrac{C_3}{|x-y|} & \text{for} \ |x-y|+|x-\overline{x}|+|y-\overline{y}| \leq \delta, \end{cases} \tag{4.18}$$

where $\overline{x}$ and $\overline{y}$ are points on Γ_f nearest to x and y, respectively.

Proof. Due to the estimates (3.64) and (2.4) we see that a principal singular part of $\partial_{x_j} V(y,x)$ is $-2W_j(y,x)$. Therefore in what follows we will study the behaviour of the function $W_j(y,x)$ defined by (4.17).

As in Theorem 4.1 we will consider separately the four possible cases (i)–(iv) described by the inequalities (4.4).

Cases (i)–(ii) can be considered by the same arguments as in the proof of Theorem 4.1 to obtain

$$|W_j(y,x)| \leq \frac{c_1(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}} \tag{4.19}$$

for $\{x_2 \geq f_+ + 2,\ y_2 \geq f_+ + 2\}$ or $\{x_2 \geq f_+ + 2,\ y_2 \leq f_+ + 2\}$.

Case (iii), i.e., $\{x_2 \leq f_+ + 2,\ y_2 \geq f_+ + 2\}$.

If $|x - \overline{x}| \geq \delta$, with the help of (4.17), (2.4), and Lemma 2.4 we easily derive an inequality of type (4.19).

If $|x - \overline{x}| \leq \delta$, then we represent $W_j(y,x)$ as follows

$$W_j(y,x) = W_j^{(1)}(y,x) + W_j^{(2)}(y,x),$$

where

$$W_j^{(p)}(y,x) = \int\limits_{\Gamma_p} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) \Big]\ \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi, \quad p = 1,2$$

with

$$\Gamma_1 := \{\xi \in \Gamma_f \mid |\xi - \overline{x}| \geq 2\delta\}, \quad \Gamma_2 := \Gamma_f \setminus \Gamma_1.$$

Since

$$\xi \in \Gamma_1 \implies |\xi - x| \geq \delta,$$

by the bounds (2.4) and Lemma 2.4 we get

$$|W_j^{(1)}(y,x)| \leq \frac{c_2(1+y_2)(1+x_2)}{[\,1 + |x-y|\,]^{3/2}}.$$

Since

$$\xi \in \Gamma_2 \implies \varrho_y(\xi,\overline{x}) \geq 4^{-1}|x-y|,$$

for $W_j^{(2)}(y,x)$, in view of (2.4), (3.9)–(3.11), and Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
|W_j^{(2)}(y,x)| &\leq \int\limits_{\Gamma_2} \Big| \partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) - \partial_{\nu(\overline{x})} G^{(\mathcal{I})}(y,\overline{x}) \Big|\ \Big| \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \Big|\ dS_\xi + \\
&\quad + |\partial_{\nu(\overline{x})} G^{(\mathcal{I})}(y,\overline{x})| \left| \int\limits_{\Gamma_2} \partial_{x_j} G^{(\mathcal{D})}(\xi,x)\, dS_\xi \right| \leq \\
&\leq \int\limits_{\Gamma_2} \frac{\delta_2 |\xi - \overline{x}|(1+y_2)}{[1 + \varrho_y(\xi,\overline{x})]^{3/2}}\ \frac{a_0}{|\xi - x|}\, dS_\xi + \\
&\quad + \frac{\delta_3\, a_0\, (1+y_2)(1+M)}{[\,1+|x-y|\,]^{3/2}} \leq \frac{c_3(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}}.
\end{aligned}$$

Combining the results obtained we see that in the case under consideration the estimate of type (4.19) holds true.

Case (iv), i.e., $\{x_2 \leq f_+ + 2,\ y_2 \leq f_+ + 2\}$.

In addition, if one of the following conditions

$$1)\ |x - \overline{x}| \geq \delta, \qquad 2)\ |y - \overline{y}| \geq \delta, \qquad 3)\ |y - x| \geq \delta$$

holds, then with the help of bounds (2.4), and Lemmas 2.4, 2.5, 2.6, 2.7, along with (3.9)–(3.11) and arguments applyed in the previous case, we arrive at the inequality of type (4.18) (cf. the proof of Lemma 3.4). Therefore it remains to consider the situation when

$$|x-\overline{x}|\le\delta,\qquad |y-\overline{y}|\le\delta,\qquad |y-x|\le\delta. \tag{4.20}$$

In what follows we assume that the inequalities (4.20) hold and let

$$\gamma_1:=\{\xi\in\Gamma_f \mid |\xi_1-x_1|\ge 4\delta\},$$

$$\gamma_2:=\{\xi\in\Gamma_f \mid |\xi_1-x_1|\le 4\delta\}=\Gamma_f\setminus\gamma_1.$$

Clearly,

$$W_j(y,x)=W_{j1}(y,x)+W_{j2}(y,x), \tag{4.21}$$

with

$$W_{jp}(y,x):=\int\limits_{\gamma_p}\Big[\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\,\partial_{x_j}G^{(\mathcal{D})}(\xi,x)\,dS_\xi,\quad p=1,2. \tag{4.22}$$

Since

$$\xi\in\gamma_1 \implies \varrho_\xi(x,y)\ge\delta,$$

we easily show that

$$|W_{j1}(y,x)|\le\int\limits_{\gamma_1}\frac{c_4\;dS_\xi}{[\,1+|\xi-y|\,]^{3/2}[\,1+|\xi-x|\,]^{3/2}}\le c_5. \tag{4.23}$$

Further we will estimate $W_{j2}(y,x)$. To this end we consider the two possible cases:

$$(a)\;\; |x-\overline{x}|\le 8^{-1}|x-y|,\qquad\qquad (b)\;\; |x-\overline{x}|> 8^{-1}|x-y|. \tag{4.24}$$

Sub–case: (4.20) and (4.24) (a).

Denote $r:=|x-y|$ and

$$\sigma_1:=\{\xi\in\gamma_2 \mid |\xi-\overline{x}|\le 4^{-1}r\},\qquad \sigma_2:=\gamma_2\setminus\sigma_1.$$

Clearly,

$$\xi\in\gamma_2 \implies |x-\xi|\ge 2^{-1}|\overline{x}-\xi|,$$

$$\xi\in\sigma_1 \implies |\xi-y|\ge\frac{5r}{8},\quad |\xi-\overline{y}|\ge\frac{r}{2},$$

$$\xi\in\sigma_2 \implies |\xi-x|\ge\frac{r}{8}.$$

We represent $W_{j2}(y,x)$ in the form

$$W_{j2}(y,x)=I_1(y,x)+I_2(y,x),$$

where

$$I_q(y,x):=\int\limits_{\sigma_q}\Big[\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\,\partial_{x_j}G^{(\mathcal{D})}(\xi,x)\,dS_\xi,\quad q=1,2.$$

Applying the inequalities (2.4), Lemmas 2.5, 2.7, and 2.8(ii) along with the arguments employed in the proof of inequalities (3.55) and (3.56), we derive

$$
\begin{aligned}
|I_1(y,x)| \leq & \int\limits_{\sigma_1} \Big| \partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) - \partial_{\nu(\overline{x})} G^{(\mathcal{I})}(y,\overline{x}) \Big| \; \Big| \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \Big| \, dS_\xi + \\
& + |\partial_{\nu(\overline{x})} G^{(\mathcal{I})}(y,\overline{x})| \left| \int\limits_{\sigma_1} \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \, dS_\xi \right| \leq \\
\leq & \int\limits_{\sigma_2} \frac{c_5 |\xi - \overline{x}|}{[\varrho_y(\xi,\overline{x})]^2} \, \frac{1}{|\xi - x|} \, dS_\xi + \frac{c_6}{|y-\overline{x}|} \left| \int\limits_{\sigma_1} \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \, dS_\xi \right| + \\
& + c_7 \left[\frac{1}{|x-y|^2} \int\limits_{\sigma_1} dS_\xi + \frac{1}{|y-x|} \left| \int\limits_{\sigma_1} \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \, dS_\xi \right| \right] \leq \\
\leq & \frac{c_8}{|x-y|}, \\
|I_2(y,x)| \leq & \int\limits_{\sigma_2} \Big| \partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) \Big| \; \Big| \partial_{x_j} G^{(\mathcal{D})}(\xi,x) \Big| \, dS_\xi \leq \\
\leq & \frac{c_9}{|x-y|} \int\limits_{\sigma_2} \Big| \partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) \Big| \, dS_\xi \leq \\
\leq & \frac{c_{10}}{|x-y|}.
\end{aligned}
$$

These relations lead to the inequality

$$
|W_{j2}(y,x)| \leq \frac{c_{11}}{|x-y|}. \tag{4.25}
$$

Sub–case: (4.20) and (4.24) (b).

In the case in question we have

$$
\xi \in \gamma_2 \implies |x-\xi| \geq |\overline{x}-x| \geq 8^{-1}|x-y|,
$$

and therefore from (4.22) by (2.4) and Lemma 2.7 it follows

$$
|W_{j2}(y,x)| \leq \int\limits_{\gamma_2} \Big| \partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi) \Big| \, \frac{a_0}{|\xi - x|} \, dS_\xi \leq \frac{c_{12}}{|x-y|}. \tag{4.26}
$$

Now, from (4.21), (4.23), (4.25), and (4.26) we see that in the case under consideration

$$
|W_j(y,x)| \leq \frac{c_{13}}{|x-y|}.
$$

Thus we have established the inequality (4.18) for the principal singular part of the function $\partial_{x_j} V(y,x)$. This completes the proof. □

Corollary 4.4. *Let f and $G_f^{(\mathcal{D})}(y,x)$ be the same as in Corollary* 4.2. *Then there exists a positive constant C_4, such that*

$$|\nabla_x G_f^{(\mathcal{D})}(y,x)|,\ |\nabla_y G_f^{(\mathcal{D})}(y,x)| \leq \begin{cases} \dfrac{C_4(1+y_2)(1+x_2)}{[\,1+|x-y|\,]^{3/2}} & \text{for} \quad |x-y| \geq \delta, \\ \dfrac{C_4}{|x-y|} & \text{for} \quad |x-y| \leq \delta, \end{cases}$$

where $(y,x) \in \Omega_f^+ \times \Omega_f^+$.

Proof. It readily follows from the symmetry property of the Green's function $G_f^{(\mathcal{D})}(y,x) = G_f^{(\mathcal{D})}(x,y)$, Theorem 4.3, formula (2.9) and bounds (2.4). □

4.3. In this subsection we will find uniform bounds for the second order derivatives of the Green's function $G_f^{(\mathcal{D})}(y,x)$ for $y \in \Gamma_f$ and $x_2 > f_+$.

First we prove the following proposition.

Theorem 4.5. *Let $f \in \mathcal{B}_{c,M}$, $y \in \Gamma_f^+$, and $x \in \Gamma_h$ with constant $h \geq f_+ + \delta$. Then there exists a positive constant C_5, such that*

$$|\partial_{x_j}\partial_{\nu(y)} V(y,x)| \leq \frac{C_5}{[\,1+|x-y|\,]^{3/2}},$$

where $V(y,x)$ is the regular part of the Dirichlet Green's function $G_f^{(\mathcal{D})}(y,x)$ given by (2.12)–(2.13).

Proof. Let φ be the function given by the relations (3.77). In view of (2.12) we have

$$V(y,x) = V_1(y,x) + V_2(y,x), \tag{4.27}$$

where

$$V_1(y,x) := \int_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)\Big]\, [1-\varphi(\xi_1-y_1)]\, \psi(\xi,x)\, dS_\xi, \tag{4.28}$$

$$V_2(y,x) := \int_{\Gamma_f} \Big[\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)\Big]\, \varphi(\xi_1-y_1)\, \psi(\xi,x)\, dS_\xi. \tag{4.29}$$

Note that the integrands in (4.28) and (4.29) vanish when $|\xi_1-y_1| \leq 2$ and $|\xi_1-y_1| \geq 4$, respectively, due to properties of the function φ.

Denote

$$\Gamma_1 := \{\xi \in \Gamma_f \mid |\xi_1-y_1| \geq 2\},$$

$$\Gamma_2 := \{\xi \in \Gamma_f \mid |\xi_1-y_1| \leq 4\}.$$

From (4.28) it follows that

$$\partial_{x_j}\partial_{\nu(y)} V_1(y,x) =$$
$$= \int_{\Gamma_1} \Big[\partial_{\nu(y)}\partial_{\nu(\xi)} G^{(\mathcal{I})}(y,\xi)\Big]\, [1-\varphi(\xi_1-y_1)]\, \psi_j(\xi,x)\, dS_\xi +$$

$$+\int_{\Gamma_1}\Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\,\Big]\;[\nu_1(y)\,\varphi'(\xi_1-y_1)]\,\psi_j(\xi,x)\,dS_\xi,$$

where $\psi_j(\xi,x)=\partial_{x_j}\psi(\xi,x)$, $j=1,2$, and $(y,x)\in\Omega_f^+\times\Omega_f^+$.

With the help of Lemmas 2.4, 2.8(iii), 3.6 and (3.77), and since the integrand is a continuous function of $y\in\Omega_f^+$, we obtain

$$|\partial_{x_j}\partial_{\nu(y)}V_1(y,x)|=c_1\int_{\Gamma_1}\frac{1}{[\,1+|y-\xi|\,]^{3/2}}\,\frac{1}{[\,1+|\xi-x|\,]^{3/2}}\,dS_\xi\le$$

$$\le\frac{c_2}{[\,1+|x-y|\,]^{3/2}},\quad y\in\Gamma_f,\quad x\in\Gamma_h; \tag{4.30}$$

here and in what follows c with subscript denotes positive constants depending only on c, M, k, δ, and h.

Further, we estimate $\partial_{x_j}\partial_{\nu(y)}V_2(y,x)$, $j=1,2$. To this end recall that (see (2.1), (2.3), (2.20))

$$G^{(\mathcal{I})}(y,\xi)=-\frac{1}{2\pi}\ln|y-\xi|+\Psi(y,\xi)$$

$$\text{for}\ \ y\in\Omega_f^+,\ \ \xi\in\Gamma_f,\ \ 0<|y-\xi|<A=\text{const}, \tag{4.31}$$

where $\Psi(\cdot,\cdot)$ and its first order derivatives are continuous and the second order derivatives possess a logarithmic singularity.

The equalities (4.29) and (4.31) imply

$$\partial_{x_j}\partial_{\nu(y)}V_2(y,x)=\int_{\Gamma_f}\Big\{\Big[\,\partial_{\nu(y)}\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\,\Big]\;\varphi(\xi_1-y_1)\;-$$

$$-\Big[\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\,\nu_1(y)\varphi'(\xi_1-y_1)\Big\}\,\psi_j(\xi,x)dS_\xi=$$

$$=X_1(y,x)+X_2(y,x), \tag{4.32}$$

where $x\in\Gamma_h$, $y\in\Omega_f^+\setminus\Omega_f^+(\delta)$ (see (3.30)), and

$$X_1(y,x)=\int_{\Gamma_f}\big\{\big[\,\partial_{\nu(y)}\partial_{\nu(\xi)}\Psi(y,\xi)\big]\;\varphi(\xi_1-y_1)\;-$$

$$-\;\Big[\,\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)\Big]\;\nu_1(y)\varphi'(\xi_1-y_1)\Big\}\,\psi_j(\xi,x)\,dS_\xi, \tag{4.33}$$

$$X_2(y,x)=-\frac{1}{2\pi}\int_{\Gamma_f}\big[\,\partial_{\nu(y)}\partial_{\nu(\xi)}\ln|y-\xi|\big]\;\varphi(\xi_1-y_1)\,\psi_j(\xi,x)\,dS_\xi.$$

With the help of (3.77), Lemmas 2.7 and 3.6, and properties of the function $\Psi(y,x)$, we deduce

$$\begin{aligned}|X_1(y,x)| \le & \frac{c_3}{\Big[1+\inf\limits_{\xi\in\Gamma_2}|\xi-x|\Big]^{3/2}}\times \\ & \times\int\limits_{\Gamma_2}\Big\{[1+|\ln|y-\xi||]+|\partial_{\nu(\xi)}G^{(\mathcal{I})}(y,\xi)|\Big\}\,dS_\xi \le \\ & \le \frac{c_4}{[1+|x-y|]^{3/2}} \quad \text{for} \quad y\in\Gamma_f, \quad x\in\Gamma_h. \end{aligned} \tag{4.34}$$

Next we estimate $X_2(y,x)$. Note that $\partial_{\nu(y)}\partial_{\nu(\xi)}\ln|y-\xi|$ is a hypersingular kernel on Γ_f and we have to transform the expression (4.33) before we pass to the limit as y approaches Γ_f.

It can be easily checked that for an arbitrary two times differentiable function g there holds the equality

$$\partial_{\nu(y)}\partial_{\nu(\xi)}\,g(y-\xi)=(\nu(\xi)\cdot\nu(y))\;\Delta_y g(y-\xi)-\partial_{\tau(y)}\partial_{\tau(\xi)}\,g(y-\xi).$$

Since $\ln|\xi-y|$ is a harmonic function for $y\ne\xi$, we have

$$\partial_{\nu(y)}\partial_{\nu(\xi)}\ln|y-\xi|=\partial_{\tau(y)}\partial_{\tau(\xi)}\ln|y-\xi|.$$

Therefore for $x\in\Gamma_h$ and $y\in\Omega_f^+\setminus\Omega_f^+(\delta)$ we can rewrite X_2 in the form (with the help of the integration by parts formula)

$$\begin{aligned} X_2(y,x) &= \frac{1}{2\pi}\int\limits_{\Gamma_f}[\partial_{\tau(\xi)}\partial_{\tau(y)}\ln|y-\xi|]\varphi(\xi_1-y_1)\psi_j(\xi,x)dS_\xi = \\ &= -\frac{1}{2\pi}\int\limits_{\Gamma_f}[\partial_{\tau(y)}\ln|y-\xi|]\partial_{\tau(\xi)}[\varphi(\xi_1-y_1)\psi_j(\xi,x)]dS_\xi = \\ &= X_2^{(1)}(y,x)+X_2^{(2)}(y,x), \end{aligned} \tag{4.35}$$

$$X_2^{(1)}(y,x):=-\frac{1}{2\pi}\int\limits_{\Gamma_f}[\partial_{\tau(y)}\ln|y-\xi|][\partial_{\tau(\xi)}\varphi(\xi_1-y_1)]\psi_j(\xi,x)dS_\xi, \tag{4.36}$$

$$X_2^{(2)}(y,x):=-\frac{1}{2\pi}\int\limits_{\Gamma_f}[\partial_{\tau(y)}\ln|y-\xi|]\varphi(\xi_1-y_1)[\partial_{\tau(\xi)}\psi_j(\xi,x)]dS_\xi. \tag{4.37}$$

Due to properties of the harmonic logarithmic potential (see [23], §12, §14) we can assume $y\in\overline{\Omega_f^+\setminus\Omega_f^+(\delta)}$ in (4.35)–(4.37) (remark that the potentials (4.36) and (4.37) have no jumps when y crosses the line Γ_f and that the function $\partial_{\tau(y)}\ln|y-\xi|$ is a singular kernel on Γ_f). Moreover, the line of integration in (4.36) is the arc

$$\Gamma_2^{**}:=\{\xi\in\Gamma_2 \mid 2\le|\xi_1-y_1|\le 4\}.$$

By Lemma 3.6 then we have for $y \in \Gamma_f$ and $x \in \Gamma_h$

$$|X_2^{(1)}(y,x)| \le \int\limits_{\Gamma_f} \frac{c_4\, dS_\xi}{|y-\xi|\,[1+|\xi-x|]^{3/2}} \le \frac{c_4}{[1+|y-x|]^{3/2}}. \tag{4.38}$$

It remains to estimate $X_2^{(2)}(y,x)$, which can be rewritten as follows

$$X_2^{(2)}(y,x) = -\frac{1}{2\pi}\int\limits_{\Gamma_f} [\partial_{\tau(y)} \ln|y-\xi|]\, \varphi(\xi_1-y_1)\, [\partial_{\tau(\xi)}\psi_j(\xi,x) -$$

$$- \partial_{\tau(y)}\psi_j(y,x)]\, dS_\xi -$$

$$- \frac{1}{2\pi}\, [\partial_{\tau(y)}\psi_j(y,x)] \int\limits_{\Gamma_f} [\partial_{\tau(y)} \ln|y-\xi|]\, \varphi(\xi_1-y_1)\, dS_\xi.$$

Applying Lemma 3.9 (with $\alpha = 1/2$) and the inequality (see, e.g., [23], §14)

$$\Bigg|\int\limits_{\Gamma_f} [\partial_{\tau(y)} \ln|y-\xi|]\, \varphi(\xi_1-y_1)\, dS_\xi\Bigg| \le \Bigg|\int\limits_{\Gamma_2\setminus\Gamma_2^{**}} \partial_{\tau(y)} \ln|y-\xi|\; dS_\xi\Bigg| +$$

$$+ \int\limits_{\Gamma_2^{**}} |\partial_{\tau(y)} \ln|y-\xi||\; dS_\xi \le c_6,$$

we derive

$$|X_2^{(2)}(y,x)| \le c_7 \Bigg[\int\limits_{\Gamma_2} \frac{1}{|\xi-y|}\, \frac{|\xi-y|^{1/2}}{[1+\varrho_x(\xi,y)]^{3/2}}\, dS_\xi + \frac{1}{[1+|y-x|]^{3/2}}\Bigg] \le$$

$$\le \frac{c_8}{[1+|y-x|]^{3/2}} \quad \text{for} \quad y \in \Gamma_f, \quad x \in \Gamma_h. \tag{4.39}$$

Consequently, in view of (4.35), (4.38), and (4.39) we get

$$|X_2(y,x)| \le \frac{c_9}{[1+|y-x|]^{3/2}} \quad \text{for} \quad y \in \Gamma_f, \quad x \in \Gamma_h. \tag{4.40}$$

From (4.32), (4.34), and (4.40) it follows that

$$|\partial_{x_j}\partial_{\nu(y)} V_2(y,x)| \le \frac{c_{10}}{[1+|y-x|]^{3/2}} \quad \text{for} \quad y \in \Gamma_f, \quad x \in \Gamma_h,$$

which along with (4.30) and (4.27) completes the proof. □

Corollary 4.6. *Let f and h be the same as in Theorem* 4.6. *Then there exists a positive constant C_6, such that*

$$|\partial_{x_j}\partial_{\nu(y)} G_f^{(\mathcal{D})}(y,x)| \le \frac{C_6}{[\,1+|x-y|\,]^{3/2}}, \quad \text{for} \quad y \in \Gamma_f, \quad x \in \Gamma_h,$$

where $G_f^{(\mathcal{D})}(y,x)$ is the Dirichlet Green's function given by (2.9).

Proof. It follows from Theorem 4.5 and the bounds (2.4). □

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