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GEOMETRIC ASPECTS OF RIEMANN–HILBERT PROBLEMS

Abstract. We discuss some recent generalizations of the Riemann-Hilbert transmission problem and their connections with certain geometric objects appearing in the theory of loop groups and infinite-dimensional Grassmanians. In particular, we describe in some detail the geometric model for the totality of elliptic Riemann-Hilbert problems in terms of Fredholm pairs of subspaces suggested by B.Bojarski. A number of fundamental geometric and topological properties of related infinite-dimensional Grassmanians are established in this context, with a special emphasis on relations to the theory of Fredholm structures. Several generalizations of the classical Riemann-Hilbert problem are also discussed. The main attention is given to linear conjugation problems for compact Lie groups, Riemann-Hilbert problems for generalized Cauchy-Riemann systems, and nonlinear Riemann-Hilbert problems for solutions of generalized Cauchy-Riemann systems. Some geometric aspects of the Riemann-Hilbert monodromy problem for ordinary differential equations with regular singular points are discussed in brief.

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Key words and phrases: Holomorphic function, Riemann–Hilbert transmission problem, Töplitz operator, Birkhoff factorization, partial indices, Fredholm operator, Fredholm index, compact operator, Shatten ideal, Hilbert–Schmidt operator, compact Lie group, linear representation, loop group, holomorphic vector bundle, Fredholm structure, C*-algebra, Hilbert module, Clifford algebra, Dirac operator, generalized Cauchy–Riemann operator, Shapiro–Lopatinski condition, analytic disc, totally real submanifold, complex point, hyper-holomorphic cell, Riemann–Hilbert monodromy problem, regular singular point.

რეზიუმე. განხილულია რიმან-ჰილბერტის სასაზღვრო ამოცანის განზოგადებები და მათი კავშირი ზოგიერთ გეომეტრიულ ობიექტთან მარყუჟთა ჯგუფთა და უსასრულო განზომილებიან გრასმანიანის თეორიებში. კერძოდ, აღწერილია ბოიარსკის მიერ შემოთავაზებული გეომეტრიული მოდელი ელიფსური რიმან-ჰილბერტის ამოცანების ერთობლიობისათვის. ამ კონტექსტში დადგენილია შესაბამისი გრასმანიანების ზოგიერთი მნიშვნელოვანი გეომეტრიული და ტოპოლოგიური თვისება.

მოყვანილია რიმან-ჰილბერტის კლასიკური ამოცანის ზოგიერთი განზოგადება. ძირითადი ყურადღება დათმობილი აქვს წრფივი შეუღლების ამოცანას ლის ჯგუფებისთვის, რიმან-ჰილბერტის ამოცანებს განზოგადებული კოში-რიმანის სისტემებისთვის და არაწრფივ რიმან-ჰილბერტის ამოცანებს განზოგადებული კოში-რიმანის ამონახსნებისთვის.

განხილულია აგრეთვე რიმან-ჰილბერტის მონოდრომიული პრობლემის გეომეტრიული ასპექტები რეგულარული განსაკუთრებულობების მქონე ჩვეულებრივი დიფერენციალური განტოლებებისთვის

Introduction

As is well known, the classical *Riemann–Hilbert (transmission) problem* (or *linear conjugation problem for holomorphic functions*) has deep and far reaching connections with many important problems in analysis and geometry (see, e.g., [122], [61], [19], [20], [21]). Thus in addition to a comprehensive analytic theory [122], it also has some natural global geometric aspects. In particular as was suggested in [20] (see also [21]), the totality of elliptic Riemann–Hilbert problems permits a visual geometric description in terms of *Fredholm pairs of subspaces* of an appropriate functional space.

This geometric interpretation enabled one to study various global aspects of the Riemann–Hilbert problem in an abstract setting, which has eventually led to some conceptual developments [22], [23], [85], [86] and non-trivial geometric results about certain infinite-dimensional Grassmanians [20], [22], [88], [90], [163]. Closely related concepts and constructions appeared useful in the geometric theory of loop groups of compact Lie groups [136], [86]. The goal of this paper is to present a coherent exposition of those geometric results and discuss some new developments in the same direction.

We begin with a brief recollection of basic facts about the classical Riemann–Hilbert problems. The main emphasis is on the factorization of matrix-functions on the unit circle (*Birkhoff factorization theorem* [136]) and the group of invertible abstract singular integral operators [20].

We proceed by studying the geometric model of the set of elliptic Riemann–Hilbert problems suggested by B.Bojarski [20]. The related Grassmanians and operator groups are introduced and their topology is studied. In particular, we describe the homotopy type of the *Fredholm Grassmanians*, show that they can be endowed with smooth manifold structures, and explain how one can put them in the context of *Fredholm structures*. Similar results are obtained for the group of invertible abstract singular operators. Our exposition of these topics essentially relies on results of [20], [24], [136], [58], [86], but we present some new results as well.

In the next five sections we describe some recent developments related to geometric aspects of linear conjugation problems. They arise from several natural generalizations of the Riemann–Hilbert problem in various directions. Those generalizations can be roughly divided in two groups: linear (Sections 2,3,4) and nonlinear ones (Sections 5,6). Our exposition of these topics is based on the papers [86], [88], [151], [160], [95].

In the second section we develop the Fredholm theory of the so-called *Riemann–Hilbert problem with coefficients in a compact Lie group* which was studied in the author's papers [86], [88]. Our derivation of the main results is based on the *generalized Birkhoff factorization theorem* for regular loops on a compact Lie group obtained in [136].

It should be noted that most of the results of this section can be derived from the classical Fredholm theory by realizing the group of coefficients as a matrix group. However the invariant approach suggests a more flexible setting and reveals some new aspects of the topic which did not arise in the classical setting. For example, in this way one obtains a natural way of constructing Fredholm structures on loop groups and Birkhoff strata, which suggests a number of non-trivial questions and perspectives. In should be noted that similar problems are treated in recent papers of D.Freed [58], [59] and G.Misiolek [120], [121] by essentially different methods.

Another type of "linear" generalization of the Riemann–Hilbert problem is considered in the third section. This generalization was suggested by the present author in the framework of *Hilbert modules over C*-algebras* [88], [90].

It turns out that one can determine the homotopy type of the related Grassmanian in terms of the K-groups of the basic algebra. This result enables one to obtain natural invariants of families of elliptic Riemann–Hilbert problems and extends some aspects of the Birkhoff factorization in the context of C*-algebras. Its proof requires a considerable portion of the theory of Hilbert modules over C*-algebras which was founded in papers of A.Fomenko and A.Mishchenko [119], and G.Kasparov [79].

Most of the technical results needed for our purposes can be found in the papers of A.Mishchenko and his school [118], [156], [114]. Closely related results on the structure of Grassmanians over C*-algebras can be found in recent papers [170], [43] but those authors did not consider relations to Riemann–Hilbert problems.

The general theory of boundary value problems for elliptic systems suggests a natural formulation of a local boundary value problem for a wide class of first order systems with constant coefficients [17], [162] which specializes to the Riemann–Hilbert problem in the case of the usual Cauchy-Riemann system. Thus it is natural to refer to those multidimensional boundary value problems as to Riemann–Hilbert problems for first order systems. As was revealed by recent developments in Clifford analysis [31], this analogy is quite far-reaching for the so-called *generalized Cauchy-Riemann systems* introduced by E.Stein and G.Weiss [150], in particular for the systems associated with Euclidean Dirac operators.

A version of Fredholm theory can be developed for those multidimensional Riemann–Hilbert problems which satisfy the Shapiro-Lopatinski condition [151]. Some effective methods of verifying this condition were developed by I.Stern [151], [152]. This topic suggested some interesting problems and it has gained considerable attention in last forty years [162], [9], [151]. Notice that the problem of describing those generalized Cauchy-Riemann systems which possess elliptic local boundary value problems appeared sufficiently difficult and remained unsolved for a long time [151]. Its solution became eventually possible [90] due to the recent advances in *operator K-theory* [10], [71].

In the fourth section we describe some results of I.Stern [151], [152], and of the present author [90], [91], which contribute to the Fredholm theory of multi-dimensional Riemann–Hilbert problems. In particular, following [151] we derive explicit criteria of fredholmness and present a comprehensive list of generalized Cauchy-Riemann systems possessing elliptic Riemann–Hilbert problems. This list extends the one presented in [152]. It was obtained by the present author using some recent results from operator K-theory [10], [71]. The results of this section seem interesting by their own and provide a background for our approach to non-linear Riemann–Hilbert problems for Cauchy-Riemann systems described in the last section.

The theory of non-linear Riemann–Hilbert problems for holomorphic functions is nowadays a well-developed topic of complex analysis [143], [159]. It is naturally connected with the so-called *analytic discs attached to totally real submanifolds* [16], [56] and non-linear singular integral equations [159]. Attached analytic and pseudo-analytic discs play important role in M.Gromov's approach to some problems of symplectic geometry [72].

A good understanding of the structure of solutions to non-linear Riemann–Hilbert problems is important in many aspects of this topic. Especially interesting are the cases when a *target manifold* is globally foliated by the boundaries of attached analytic discs. In the Section 5 we describe a class of non-linear Riemann–Hilbert problems which possess this property. This class was investigated in the papers [160], [95], and it seems to provide a reasonable starting point for investigating multi-dimensional non-linear Riemann–Hilbert problems.

Up to our knowledge there only exist a few papers devoted to multi-dimensional non-linear Riemann–Hilbert problems, basically in the case when the non-linearity enters through a small perturbation of a linear problem [11], [165]. It seems natural to ask if it is possible to apply the paradigm of *analytic discs attached to totally real submanifolds* in some multi-dimensional settings. This suggests considering *hyper-holomorphic cells* with boundaries in a given submanifold.

Apparently, for such an attempt to be reasonable it is necessary to choose a submanifold (*target manifold*) in an appropriate way. A natural approach to this problem, based on our results for linear Riemann–Hilbert problems, is described in the sixth section. We become able to indicate a class of targets which give rise to certain non-linear Fredholm operators describing the local structure of attached hyper-holomorphic cells. A number of seemingly interesting open problems may be formulated in this setting and we discuss some of them at the end of the section.

A classical topic closely related to the *Riemann–Hilbert transmission problem* (RHTP) is the so-called *Riemann–Hilbert monodromy problem for differential equations with regular singularities* [36], [7]. It is formulated as the problem of constructing a system of ordinary differential equations with *regular* or *Fuchs type* singularities and prescribed monodromy [7]. This problem is often referred to as *Hilbert's 21st problem* because it appeared in the famous list of problems formulated by Hilbert in the beginning of previous century.

The theory of *Riemann–Hilbert monodromy problem* (RHMP) exhibits a number of deep and important geometric aspects some of which are intrinsically connected with the main topics of this paper. As is well known [139], [7], any concrete (i.e., with a given *monodromy data* [7]) RHMP can be reduced to a RHTP with a piecewise constant matrix function as a coefficient, in other words, the solution can be constructed in terms of Birkhoff factorization of piecewise constant matrix-functions.

Thus the relation between these two problems is so natural and direct that they can be considered as two different disguises of the same problem. For this reason, any discussion of the RHTP would be essentially incomplete without mentioning at least some aspects of the RHMP so it seemed reasonable to the author to include some of those in the present paper.

More precisely, we touch upon a few topics concerned with regular systems of differential equations on Riemann surfaces and holomorphic vector bundles over Riemann surfaces which are closely related with the geometric stuff considered in previous sections. In the last section we present a brief discussion of those topics, including some background from the theory of differential equations with regular singular points. The exposition in this section is closely related to the topics studied by G.Giorgadze in [63], [64], [65].

In a single paper it is of course impossible to present a comprehensive exposition of all those topics we touched upon. Additional information on these and other geometric aspects of Riemann–Hilbert problems can be found, e.g., in [58], [59], [23], [39], [34], [35], [98], [100], [101], [102], [113], [159], [93], [7], [25], [26], [65].

The author benefited from discussions of various aspects of Riemann–Hilbert with a number of experts and colleagues. Especially extensive and useful were discussions with B.Bojarski who initiated many of ideas and concepts discussed in this paper. Some of the results on the global geometric structure of Riemann–Hilbert problems and Fredholm Grassmanians were obtained jointly with B.Bojarski [24]. Results of section 5 were obtained jointly with E.Wegert and I.Spitkovsky [160] with whom the author had numerous discussions about various aspects of Birkhoff factorization and nonlinear analysis. Our discussion of regular systems in the last section is based on some results and papers of G.Giorgadze, who courteously suggested to use them in the present paper.

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1. Riemann–Hilbert Problems and Fredholm Grassmanians

The classical formulation of the Riemann—Hilbert problem is related to the decomposition of the extended complex plane $\overline{\mathbb{C}}$ (*Riemann sphere*) into two complementary domains with a smooth common boundary Γ. In the simplest and the most classical case one just takes the decomposition

$$\overline{\mathbb{C}} = D_+ \cup \mathbb{T} \cup D_-,$$

where D_+ is the unit disc, $\mathbb{T}$ stands for the unit circle, and D_- is the complementary domain containing the infinite point ∞ (the North Pole of Riemann sphere). Let $A(D_\pm) = C(\overline{D}_\pm) \cap H(D_\pm)$ denote the set of all complex valued functions which are continuous in the closure of the corresponding domain and holomorphic inside. The set of vector functions of length $n \geq 1$ with all their components in $A(D_\pm)$ is denoted by $A^n(D_\pm)$.

A fundamental problem of complex analysis, known as Riemann—Hilbert transmission problem or linear conjugation problem, is to describe the totality of piecewise holomorphic (vector) functions $(X_+, X_-) \in A^n(D_+) \times A^n(D_-)$, with the normalizing condition $X_-(\infty) = 0$, such that their boundary values on $\mathbb{T}$ satisfy the transmission (or linear conjugation) condition

$$X_+(t) = G(t)X_-(t) + h(t), \quad t \in \mathbb{T}, \tag{1.1}$$

where $h(t)$ is a given (vector) function and $G(t)$ is a given continuous matrix function of the size $(n \times n)$ on $\mathbb{T}$.

The same problem can be of course formulated on any Riemann surface but we stick here to the zero genus case (Riemann sphere) as above. Solutions may be considered in various functional spaces. Actually, in order to obtain a reasonable theory it is necessary to impose some additional regularity conditions on the functions in question. In the classical setting one works with the functions which are Hölder continuous [122], [61].

Discussion of those regularity conditions and functional spaces is beyond the scope of this paper. In fact for us it is basically sufficient that the problem can be described by some Fredholm operator between certain Hilbert or Banach spaces so in the sequel we tacitly assume that all ingredients satisfy some regularity conditions which guarantee fredholmness of the corresponding operators in question. For example, the problem can be placed in a Hilbert space context by working with square-integrable functions and this is well-suited for studying global geometric aspects of the problem.

Solvability and other properties of this problem are very well understood in various classes of functional spaces (see, e.g., [122]). For example, the problem is Fredholm in appropriate L^2-spaces if the coefficient matrix $G(t)$ is non-degenerate at every point of the unit circle and belongs to some Hölder class. The index of such problem appears to be equal to the *winding number* (topological degree) of the determinant det $G(t)$, i.e., it is equal to the divided by 2π increment of the argument of $\det G(t)$ along the unit circle [122].

One can also express the kernel and cokernel dimension in terms of the so-called ***partial indices*** of the matrix function $G(t)$. Those are defined in terms of ***Birkhoff factorization*** of non-degenerate matrix functions on the circle [122], [19].

The famous Birkhoff factorization theorem states that a sufficiently regular (e.g., Hölder continuous) non-degenerate matrix function on $\mathbb{T}$ can be represented in the form

$$G(t) = G_+(t)\text{diag}(z^{\mathbf{k}})G_-(t), \tag{1.2}$$

where matrix functions $G_\pm(t)$ are of the same regularity class, non-degenerate, and holomorphic in domains $D_\pm$ respectively, $G_-(\infty)$ is the identity matrix, and

$$\text{diag}(z^{\mathbf{k}}) = \text{diag}(z^{k_1}, \dots, z^{k_n}), \quad k_1, \dots, k_n \in \mathbb{Z},$$

is a diagonal matrix function on $\mathbb{T}$ [122], [158].

Integer numbers k_i are called (left) partial indices [122], [158] (or exponents [38], [136]) of matrix function $G(t)$. For a given matrix function $G(t)$, there can exist different factorizations of the form 1.2 but (left) partial indices are uniquely defined up to the order [158]. Analogously one can define a right Birkhoff factorization of $G(t)$ and right partial indices. We will only deal with the left factorizations because they are well-suited for investigation of Riemann–Hilbert problems of the form (1.1).

Partial indices exhibit quite non-trivial behaviour. Right partial indices need not be equal to the left ones. However for sufficiently regular (rational, Hölder) matrix functions, the sum of all left partial indices (***left total index***) is equal to the analogously defined ***right total index***. Actually, both the left and right total index are equal to the Fredholm index of the corresponding Riemann–Hilbert problem (1.1).

In fact, even for very regular (smooth, rational) matrix functions their collections of left and right partial indices are practically independent of each other (except the restriction that both total indices should be equal). For example, it was proved in [53] that, for each two integer vectors $k, l \in \mathbb{Z}$ with $\sum k_i = \sum l_i$ there exists a non-degenerate rational matrix-function on the unit circle whose vectors of left and right partial indices are k and l respectively.

At the same time if the algebraic degrees of the numerator and denominator of each element of a rational matrix function are bounded by a fixed integer N then one can get upper estimates for the modulus of the left and right indices [63], thus in this case differences between right and left indices can not be arbitrary. If one drops regularity requirements and considers almost everywhere non-degenerate matrix functions with bounded measurable coefficients then each pair of integer vectors can serve as collections of left and right partial indices of such a matrix function [30].

Partial indices are closely related to the properties of holomorphic vector bundles over the Riemann sphere [19], [21], [63]. The problem of computing (left or right) partial indices of a concrete matrix function is far from trivial because in most cases they are not topological invariants and one has to take into account the analytic properties of a given matrix function. After several decades of gradual progress, this problem was eventually solved for several important classes of matrix functions [32], [113].

Recently these results were simplified and generalized in [5]. We reproduce here some results of [5] because they enable one to effectively compute partial indices in most of situations appearing in practice. Thus the problem of computing partial indices nowadays can be considered as an algorithmically solvable one. Moreover, these results, besides being important and instructive by themselves, suggest further interesting problems some of which are described below.

More precisely, following [5] we present the formulae with the aid of which one can calculate the left and right partial indices of continuous matrix functions (or, which is the same, *matrix loops*) and relate them to the splitting type of the corresponding holomorphic vector bundle.

Let Γ be a smooth closed positively oriented loop in $\mathbb{C}P^1$ which separates $\mathbb{C}P^1$ into two connected domains U_+ and U_-. Suppose $0 \in U_+$ and $\infty \in U_-$. In this situation one can investigate the Riemann–Hilbert transmission problem with coefficient $f : \Gamma \to GL_n(\mathbb{C})$ of Hölder class.

To investigate solvability of such a problem one needs some effective methods of finding partial indices of the coefficient matrix. We now describe an algorithm based on results of [32] and [5].

Let $a : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be a continuous and invertible matrix function on the contour Γ. Its partial indices may be found using the so-called *power moments* of a [32].

Definition 1.1 ([32], [5]). A power moment of matrix function $a(t)$ with respect to contour Γ is defined as the matrix

$$c_j = \frac{1}{2\pi i} \int_\Gamma t^{-j-1} a^{-1}(t) dt, \qquad j \in \mathbb{Z}.$$

Let $k = \mathrm{ind}_\Gamma a(t)$ and consider the family of block Töplitz matrices T_l:

$$\begin{pmatrix} c_l & c_{l-1} & \cdots & c_{-2k} \\ c_{l+1} & c_l & \cdots & c_{-2k+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_0 & c_{-1} & \cdots & c_{-2k-l} \end{pmatrix}, \quad l \in \mathbb{Z}.$$

It turns out that the partial indices of matrix function $a(t)$ can be derived from the ranks of matrices T_l. Suppose that a matrix function $a(t)$ admits analytic continuation to U_+ and has in that domain p poles $z_1, \dots, z_p$ of multiplicities $\kappa_1, \dots, \kappa_p$. Then the matrix function $\tilde{a}(t) = \prod_{j=1}^p (z - z_j)^{\kappa_j} a(t)$ is analytic and

$$\mathrm{ind}_\Gamma \det \tilde{a}(t) = \mathrm{ind}_\Gamma \det a(t) + Nn,$$

where $N = \kappa_1 + \kappa_2 + \cdots + \kappa_p$ is the total multiplicity of poles of $a(t)$. The index of $\tilde{a}(t)$ is denoted by κ, i.e. $\kappa = k + Nn$.

For such a matrix function, left and right partial indices k_j^{l}, k_j^{r}, $j = 1, \ldots, n$, can be expressed by explicit formulae.

Theorem 1.1 ([32], [5]). *The left and right partial indices of matrix function $a(t)$ are given by the formulae*

$$k_j^{\mathrm{r}} = \operatorname{card}\{l \mid n + r_{-l-1} - r_{-l} \leq j - 1, \quad l = 2\kappa, 2\kappa - 1, \ldots 0\} - 1,$$

$$k_j^{\mathrm{l}} = 2\kappa + 1 - \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \leq j - 1, \quad l = 2\kappa, 2\kappa - 1, \ldots, 0\}.$$

$j = 1, \ldots, n$, r_l is the rank of the Töplitz matrix T_{-l} and it is assumed that $r_{-2\kappa-1} = 0$.

One can now derive various conclusions about the structure of partial indices of a given matrix function.

Proposition 1.1 ([5]). *Let $N > 0$.*

i) *Suppose $N > 2\kappa$, then all right partial indices of the meromorphic function $a(t)$ are negative.*
ii) *Suppose $N \leq 2\kappa$, then in order for the inequalities $k_j^{\mathrm{r}} \leq 0$ to hold for all j, it is necessary and sufficient that*

$$r_{N-2\kappa} \leq r_{N-2\kappa-1} + 1.$$

If among the right partial indices occur both negative and positive numbers, then let us introduce the following numbers: $\alpha = \sum_{k_j^{\mathrm{r}}<0} |k_j^{\mathrm{r}}|$ and $\beta = \sum_{k_j^{\mathrm{r}}>0} |k_j^{\mathrm{r}}|$. As is well known, the numbers α and β are dimensions of the kernel and cokernel of the Fredholm operator of a given RHTP [122], [61]. The possibility of calculating these numbers in terms of matrices T_{-k} is guaranteed by the following statement.

Proposition 1.2 ([5]). *Suppose that among the partial indices of meromorphic matrix function $a(t)$ there are positive as well as negative numbers. Then*

$$\alpha = (N+1)n - r_{N-2\kappa}, \beta = \kappa + n - r_{N-2\kappa}.$$

In order to establish solvability of a concrete Riemann–Hilbert problem it is important to have some criteria for existence of non-negative partial indices.

Theorem 1.2 ([5]). *In order that among the partial indices of meromorphic matrix function appear non-negative ones, it is necessary and sufficient that the following conditions hold:*

i) $\operatorname{ind}_\Gamma \det a(t) \geq 0$,
ii) $r_{N-2\kappa} = (N+1)n$.

As is well known the situation is especially simple if the coefficient of a RHTP admits the so-called *canonical factorization* [113]. The above results enable one to get a simple effective criterion for existence of canonical factorization.

Theorem 1.3 ([5]). *In order that the meromorphic matrix function $a(t)$ has a canonical factorization, it is necessary and sufficient that*

i) $\operatorname{ind}_\Gamma \det a(t) = 0$,

ii) $r_{-(2n-1)N} = (N+1)n$.

The concept of stability of a matrix function naturally arises from the relation between RHTP and holomorphic vector bundles [139], [20]. In concrete cases one can effectively stability by the same approach.

Theorem 1.4 ([5]). *Suppose $\kappa = \operatorname{ind}_\Gamma \det a(t) + Nn = 0$. Then the right partial indices of the meromorphic matrix function $a(t)$ are equal to $-N$ and hence $a(t)$ is stable.*

One can obtain a similar result when $\kappa \neq 0$. Define the integers q and r, $0 \leq r < |q|$ by the relation

$$\operatorname{ind}_\Gamma \det a(t) = nq + r.$$

Then for the stability of partial indices it is necessary and sufficient that

$$\begin{aligned} r_{\kappa+q-2k} &= (\kappa+q+1)n, \\ r_{\kappa+q-2k+1} &= (\kappa+q+1)n + r. \end{aligned}$$

Proofs of the above results follow more or less directly from the general formulae for partial indices presented in Theorem 1.1. One can extend the list of corollaries of the latter theorem. In particular, as was shown in [65] one can obtain some general estimates for partial indices in terms of the above data.

Theorem 1.5 ([65]). *For the left partial indices $\lambda_j = k_j^l$, the following estimate is valid*

$$-N \leq \lambda_j \leq 2\operatorname{ind}_\Gamma \det a(t) + N(2n-1) + 1,$$

where N is the number of poles counted with multiplicities and n is the dimension of $a(t)$.

Proof. We present a proof of this estimate following [65]. Observe first that from the above formulae for k_j^l one has

$$\lambda_j = 2\operatorname{ind}_\Gamma \det a(t) + N(2n-1) + 1 - d_j,$$

where $d_j = \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \leq j-1, l = 2\kappa, 2\kappa-1, \dots, 0\}$.

This implies that

$$\begin{aligned} \max_{1\leq j\leq n} \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \leq j-1, \quad l = 2\kappa, \dots, 0\} = \\ = \max \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \leq 0, \quad l = 2\kappa, \dots, 0\} = 2\kappa + 1 \end{aligned}$$

and

$$\begin{aligned} \min_{1\leq j\leq n} \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \leq j-1, \quad k = 2\kappa, \dots, 0\} = \\ = \min \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \leq 0, \quad l = 2\kappa, \dots, 0\} = 0, \end{aligned}$$

which proves the above inequalities. □

An important task is to work out algorithms for explicit construction of the factors $f_\pm$ in the Birkhoff factorization. This issue is far from being solved in general but it should be noted that detailed results for (2×2) matrix functions were obtained in [50], [77], [78]. A comprehensive review of these and other analytic aspects of Riemann–Hilbert transmission problem can be found in [100].

Thus one can conclude that the theory of Birkhoff factorization of a single matrix function is sufficiently well developed. We now turn to some geometric problems of global nature naturally associated with Birkhoff factorization theorem.

For $K = (k_1, k_2, \dots, k_n)$, denote by Ω_K the Birkhoff stratum in the group Ω of based Hölder loops on $GL(n, \mathbb{C})$. In general the factors in Birkhoff factorization are not unique, but if one fixes f^+ (or f^-) then f^- (respectively f^+) is uniquely defined [68].

Let $\Omega_\pm$ denote the subgroup consisting of boundary values of matrix functions holomorphic in U_+, U_- respectively (in the latter case we require that a matrix function is regular at infinity and tends to the identity matrix at infinity). The Banach Lie group $\Omega^+ \times \Omega^-$ acts analytically on Ω via

$$f \overset{\alpha}{\longmapsto} h_1 f h_2^{-1}, \quad f \in \Omega, \quad h_1 \in \Omega^+, \quad h_2 \in \Omega^-.$$

It is clear, that the orbit of the diagonal matrix d_K by the action α is Ω_K.

In [38] it was proved that the stability subgroup H_K of f under the action α consists of those pairs (h_1, h_2) of upper triangular matrix-functions where the (i, j)-th entry in h_1 is a polynomial in z of degree at most $(k_1 - k_2)$ and $f = h_1 f h_2^{-1}$. Hence the subgroup H_K has the finite dimension

$$\dim H_K = \sum_{k_i \geq k_j} (k_i - k_j + 1).$$

Choose now any pair $(h_1, h_2) \in H_K$ and consider the holomorphic vector bundle on $\mathbb{C}P^1$ which is obtained by the covering of the Riemann sphere $\mathbb{C}P^1$ by three open sets $\{U^+, U^-, U_3 = \mathbb{C}P^1 \setminus \{0, \infty\}\}$, with transition functions

$$g_{13} = h_1 : U^+ \cap U_3 \to GL(n, \mathbb{C}),$$

$$g_{23} = h_2 d_K : U^- \cap U_3 \to GL(n, \mathbb{C}).$$

Denote this bundle by $E \to \mathbb{C}P^1$. From the Birkhoff factorization theorem it follows that every holomorphic vector bundle splits into direct sum of line bundles

$$E \cong E(k_1) \oplus \cdots \oplus E(k_n).$$

Remark 1.1. The possibility of decomposing of a holomorphic vector bundle into the direct sum of line bundles was proved by A. Grothendieck without applying the Birkhoff theorem so there exist two independent proofs of this important fact.

The numbers $k_1, \dots, k_n$ are the Chern numbers of the line bundles $E(k_1)$, $\dots, E(k_n)$. we order them in such way that $k_1 \geq \cdots \geq k_n$. The integer-valued vector $K = (k_1, \dots, k_n) \in \mathbb{Z}^n$ is called the *splitting type* of the holomorphic vector bundle E. It completely defines the holomorphic type of the bundle E. The set of all matrix loops with a fixed collection of left partial indices K is called *Birkhoff stratum* of type K and denoted by Ω_K [68], [19]. Birkhoff strata have interesting geometric and topological properties [68], [19], [21], [38] some of which will be described below.

Taking into account the relations between partial indices $\kappa_1, \ldots, \kappa_n$ of the matrix-function $f \in \Omega$ and the splitting type of holomorphic vector bundle E it is easy to see that Birkhoff strata Ω_K numerate the holomorphy types of vector bundles over $\mathbb{C}P^1$.

Theorem 1.6 ([68], [19]). *There is a one-to-one correspondence between the strata Ω_K and isomorphism classes of holomorphic vector bundles on $\mathbb{C}P^1$.*

Denote by $O(E)$ the sheaf of germs of holomorphic sections of the bundle E, then the solutions of the RHTP can be interpreted of the zeroth cohomology group $H^0(\mathbb{C}P^1, O(E))$. Therefore the number l of the linearly independent solutions is $\dim H^0(\mathbb{C}P^1, O(E))$. Moreover, the (total) Chern number $c_1(E)$ of the bundle E is equal to index $\det G(t)$. In particular one obtains a well known criterion of solvability of RHTP.

Theorem 1.7 ([139]). *A RHTP has solutions if and only if $c_1(E) \geq 0$ and the number l of linearly independent solutions is*

$$l = \dim H^0(\mathbb{C}P^1, O(E)).$$

Notice that, besides being interesting by themselves, the explicit formulae for the partial indices given in Theorem 1.1 suggest further perspectives and problems. Consider for example the set R_N of non-degenerate rational matrix functions on $\mathbf{T}$ with the degrees of the numerator and denominator bounded from above by a certain number N. From the aforementioned formulae for the partial indices it follows that the range of the vector of right partial indices of matrices from R_N is finite (cf. [64]). One can wonder what is the maximal possible "distance" between the vectors of right and left partial indices for matrix functions from R_N. Some estimates of such type were obtained in [64].

Furthermore, one may ask what can be the topological types of the intersections $\Omega_K \cap R_N$. For small N, they should admit a complete description. On the other hand, for N large enough, it is natural to conjecture that the topological type should be the same as for Ω_K itself. Up to the author's knowledge, both these issues remain uninvestigated.

From the preceding discussion it is clear that properties of a concrete Riemann–Hilbert problem may be studied by well developed methods so we will not further discuss those classical topics in this paper. As it was realized relatively recently, the set of all Riemann–Hilbert problems possesses interesting geometric properties and permits non-trivial geometric descriptions in the framework of global analysis (see, e.g., [20], [21], [136]). The same is true for certain natural subsets of this set, e.g., for Birkhoff strata [21], [38].

In this way there emerged several interesting geometric settings and approaches which will be our main concern in the sequel. We begin with recalling an abstract geometric model for the set of ***elliptic*** Riemann–Hilbert problems suggested in [20]. Following an established tradition, we say that a Riemann–Hilbert problem is elliptic if it is described by a Fredholm operator (i.e., an operator T with a closed image and finite-dimensional kernel and cokernel [46]). The index of a Riemann–Hilbert problem is defined as the index of the corresponding Fredholm operator [122], [46], i.e., the difference between the dimension of kernel and codimension of the image:

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T.$$

For simplicity and brevity we choose the framework of operators acting in Hilbert spaces. However most of our constructions and results remain valid for a wide class of Banach spaces. As will be shown in the sequel, they can be also generalized in the context of Hilbert modules over C^*-algebras [90].

Let H be a complex Hilbert space and M, N be its closed infinite-dimensional subspaces.

Definition 1.2 ([80], [20]). A pair $\mathbf{P} = (M, N)$ is called a Fredholm pair (FP) if $M + N$ is a closed subspace of finite codimension $b_{\mathbf{P}}$, and $\dim(M \cap N) = a_{\mathbf{P}}$ is also finite. If this is the case, then the difference $a_{\mathbf{P}} - b_{\mathbf{P}} = i(M, N)$ is called the index of Fredholm pair $\mathbf{P}$.

The concept of Fredholm pair was introduced in 60-ties by T.Kato [80] who established in particular that such pairs and their indices are stable with respect to continuous deformations of the subspaces in question. For a precise formulation of this property see [80] or [20].

In order to characterize Fredholm pairs, certain classes of bounded linear operators in H were introduced in [20]. Let $L(H)$ denote the algebra of bounded linear operators in H and $GL(H)$ denote the group of operators possessing a bounded inverse. Let J be a fixed two-sided ideal in $L(H)$. For example, one can take the (unique closed two-sided) ideal K of compact (completely continuous) operators or the subideal K_0 consisting of finite rank operators.

For a given operator $S \in L(H)$, let $C(S, J)$ denote the subalgebra of operators $A \in L(H)$ such that the commutator $[A, S] = AS - SA$ belongs to the ideal J. The intersection $C(S, J) \cap GL(H)$ will be denoted by $GL(S, J)$, clearly it is a subgroup of $GL(H)$ (not necessarily a closed one).

As was explained in [20] the classical singular integral operators and linear conjugation problems can be interpreted as elements of the algebra $C(P, K)$, where P is an orthogonal projector with infinite dimensional image and kernel. Many topological properties of such operators and related Grassmanians remain valid if one changes the ideal K by certain subideal J as above.

Definition 1.3 ([20]). Let P be an orthogonal projection on a closed subspace in H such that $\dim \operatorname{im} P = \dim \ker P = \infty$. The algebra $C(P, J)$ is called the algebra of abstract singular operators associated with ideal J, $K_0 \subset J \subset K$.

In the sequel we will be mainly interested in the group of invertible (abstract) singular operators $GL(P, J)$. If M is a closed linear subspace of H and $A \in GL(H)$ an invertible operator in H, then $A(M)$ denotes the image of M under A and we think of it as a subspace M rotated by A. Let P_M denote any projection onto M, i.e., the range of P_M is M and $(P_M)^2 = P_M$. Of course there exist many projectors with the given range M. In a Hilbert space, the condition that P_M is self-adjoint (or orthogonal) specifies it in a unique way but we do not assume that P_M is orthogonal. We consider the complementary projections $P = P_M$, $Q = Id - P$ and present a useful characterization of Fredholm pairs which was obtained in [20].

Theorem 1.8 ([20]). *A pair (M, N) of closed subspaces of a Hilbert space is a Fredholm pair if and only if it has the form $(M, A(ker P))$ for some projection P as above and some operator $A \in G(P, K)$. The operator $\Phi \in L(H)$ defined by the formula*

$$\Phi(x) = Px + AQx \tag{1.3}$$

is a Fredholm operator with $ind\,\Phi = i(M, N)$. Any operator of this form in $L(H)$ is a Fredholm operator.

It turns out that in many problems it becomes necessary to consider the set of all Fredholm pairs with a fixed first subspace. In other words, one chooses a closed infinite dimensional and infinite codimensional subspace M and considers the so-called Fredholm Grassmanian consisting of all subspaces N such that (M, N) is a Fredholm pair (cf. [136], Ch.7). This is actually a "leaf" in the Grassmanian of all Fredholm pairs and one may represent the whole Grassmanian as a fibration with a fiber isomorphic to this leaf.

This definition permits several useful modifications which we present following [136]. Consider a complex Hilbert space decomposed in an orthogonal direct sum $H = H_+ \oplus H_-$ and choose a real number $s \geq 1$. For further use we need a family of subideals in $K(H)$ which is defined as follows (cf. [58]).

Recall that for any bounded operator $A \in L(H)$ the product A^*A is a non-negative self-adjoint operator, so it has a well-defined square root $|A| = (A^*A)^{1/2}$ (see, e.g., [145]). If A is compact, then A^*A is also compact and $|A|$ has a discrete sequence of eigenvalues

$$\mu_1(A) \geq \mu_2(A) \geq \cdots$$

tending to zero. The $\mu_n(A)$ are called singular values of A. For a finite $s \geq 1$ one can consider the expression (sth norm of A)

$$||A||_s = \left[\sum_{j=1}^{\infty}(\mu_j(A))^s\right]^{1/s} \tag{1.4}$$

and define the sth Schatten ideal K_s as the collection of all compact operators A with a finite sth norm (s-summable operators) [145].

Using elementary inequalities it is easy to check that K_s is really a two-sided ideal in $L(H)$. These ideals are not closed in $L(H)$ with its usual norm topology but if one endows K_s with the sth norm as above then K_s becomes a Banach space [145]. Two special cases are well-known: K_1 is the ideal of trace class operators and K_2 is the ideal of Hilbert-Schmidt operators. For $s = 2$, the above norm is called the Hilbert-Schmidt norm of A and it is well known that $K_2(H)$ endowed with this norm becomes a Hilbert space (see, e.g., [145]). Obviously $K_1 \subset K_s \subset K_r$ for $1 < s < r$ so one obtains a chain of ideals starting with K_1. For convenience we set $K_\infty = K$ and obtain an increasing chain of ideals K_s with $s \in [1, \infty]$.

Of course one can introduce similar definitions for a linear operator A acting between two different Hilbert spaces, e.g., for an operator from one subspace M to another subspace N of a fixed Hilbert space H. In particular we can consider the classes $K_s(H_\pm, H_\mp)$. Let us also denote by $F(M, N)$ the space of all Fredholm operators from M to N.

Definition 1.4 ([136]). The sth Fredholm Grassmanian of a polarized Hilbert space H is defined as

$$\begin{aligned} Gr_F^s(H) = \{W \subset H :\ & \pi_+|W \text{ is an operator from } F(W, H_+), \\ & \pi_-|W \text{ is an operator from } K_s(W, H_-)\}. \end{aligned}$$

These Grassmanians are of the major interest for us. Actually, many of their topological properties (e.g., the homotopy type discussed below) do not depend on the number s appearing in the definition. On the other hand, more subtle properties like manifold structures and characteristic classes of Gr_F^s do depend on s in a quite essential way. As follows from the discussion in [58] this is a delicate issue and we circumvent it by properly choosing the context.

As follows from the results of [136], it is especially convenient to work with the Grassmanian $Gr_F^2(H)$ defined by the condition that the second projection π_- restricted to W is a Hilbert-Schmidt operator. Following [136] we denote it by $Gr_r(H)$ and call the restricted Grassmanian of H.

Fredholm Grassmanians appear to have interesting analytic and topological properties. It turns out that Grassmanian Gr_F^s can be turned into Banach manifolds modelled on Schatten ideal K_s. In particular $Gr_r(H)$ has a natural structure of a Hilbert manifold modelled on the Hilbert space $K_2 = K_2(H)$ [136]. All these Grassmanians have the same homotopy type (see Theorem 1.9 below). Moreover certain natural subsets of Grassmanians Gr^s_F can be endowed with so-called *Fredholm structures* [47], which suggests in particular that one can define various global topological invariants of $Gr^s_F(H)$.

Definition 1.3 also yields a family of subgroups $GL^s = GL(\pi_+, K_s)$ of $GL(\pi_+, K)$ ($s \geq 1$). For our purposes especially important is the subgroup $GL(\pi_+, K_2)$ which naturally acts on $Gr_r(H)$.

Definition 1.5 ([136]). The restricted linear group $GL_r(H)$ is defined as the subgroup of $GL(\pi_+, K)$ consisting of all operators A such that the commutator $[A, \pi_+]$ belongs to the Hilbert-Schmidt class $K_2(H)$.

From the very definition it follows that GL^s acts on Gr^s and by merely an examination of the proof of Theorem 1.8 given in [20] (cf. also [136], Ch.7) one finds out that these actions are transitive. In order to give the most convenient description of the isotropy subgroups of these actions, we follow the presentation of [136], denote by $U(H)$ the group of unitary operators, and introduce a subgroup $U^s(H) = U(H) \cap GL^s(H)$ consisting of all unitary operators from GL^s. For $s = 2$ this subgroup is denoted by U_r. Now the description of isotropy groups is available by the same way of reasoning which was applied in [136] for $s = 2$.

Proposition 1.3. *The subgroup $U^s(H)$ acts transitively on $Gr^s(H)$ and the isotropy subgroup of the subspace H_+ is isomorphic to $U(H_+) \times U(H_-)$.*

From the existence of a polar decomposition for a bounded operator on H it follows that subgroup $U^s(H)$ is a retract of GL^s and it is straightforward to obtain similar conclusions for the actions of GL^s.

Corollary 1.1. *The group GL^s acts transitively on the Grassmanian $Gr^s(H)$ and the isotropy groups of this action are contractible.*

Thus such an action obviously defines a fibration with contractible fibers and it is well known that for such fibrations the total space (GL^s) and the base (Gr^s) are homotopy equivalent [46].

Corollary 1.2. *For any $s \geq 1$, the Grassmanian Gr^s and the group GL^s have the same homotopy type. In particular, GL_r is homotopy equivalent to Gr_r.*

Remark 1.2. As we will see in the next section, all the groups $GL(\pi_+, J)$ have the same homotopy type for any ideal J between K_0 and K. In particular, this is true for every Schatten ideal K_s. Thus all the above groups and Grassmanians have the same homotopy type.

We are now ready to have a closer look at the topology of Gr^s and GL^s which will be our main concern in the rest of this section.

The homotopy type of GL_r and Gr_r is described in the following statement which was obtained in [86], [163], and [136]. This gives an answer to a question posed in [20]. The proof presented below follows the lines of [136].

Theorem 1.9. *For any $s \in [1, \infty]$, the homotopy groups of the group GL^s and Fredholm Grassmanian Gr^s are given by the formulae*

$$\pi_0 \cong \mathbf{Z}; \ \ \pi_{2k+1} \cong \mathbf{Z}, \ \pi_{2k+2} = 0, \ k \geq 0. \tag{1.5}$$

Proof. In virtue of Corollary 1.2, it is sufficient to determine the homotopy type of GL^s which we denote simply by G. To this end let us consider a certain fibration

$$p_1 : GL^s \to F(H_+, H_+) \times K_s(H_+, H_-)$$

defined in the following way.

Write any element (operator) $A \in GL^s$ as a (2×2)-matrix of operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to the given polarization of H (thus a is a bounded operator from to H_+ to H_+ and so on).

Then define $p_1(A)$ as the first column of this matrix, i.e., $p_1(A) = (a, c)$. It is evident that the image $X = Imp_1$ is an open subset of the target space. Introduce now a subgroup $G_1 \subset G$ of elements of G defined by upper-triangular matrices of the form

$$\begin{pmatrix} I_+ & b \\ 0 & d \end{pmatrix},$$

where I_+ denotes the identity operator on H_+.

Lemma 1.1. *The subgroup G_1 is contractible.*

Indeed, notice first that in this representation the operator d is always invertible, in other words the set of possible d-s appearing in the last formula is exactly $GL(H_-)$. As to b it can be an arbitrary operator from $K_s(H_-, H_+)$. Thus the subgroup G_1 as a topological space is homeomorphic to the product $GL(H_-) \times K_s(H_-, H_+)$. By Kuiper's theorem [104], the first factor is contractible and the second factor, being a vector space, is also contractible. Thus we conclude that G_1 is contractible.

Now it is straightforward to verify our next claim.

Lemma 1.2. *$p_1(A) = p_1(A')$ if and only if there exists a $T \in G_1$ such that $A = A'T$.*

Thus we conclude that X is the homogeneous space G/G_1 which is apparently a fibration with the fibers isomorphic to G_1. As was already explained above, this implies that G is homotopy equivalent to X.

Consider now the mapping $\pi_1 : X \to F(H_+, H_+)$ defined as the restriction of the first projection, i.e., $\pi_1(a, c) = a$. We want to show that this is also a surjective mapping with contractible fibers. Then, by the same reasoning as above, we will be able to conclude that G is homotopy equivalent to $F(H_+, H_+)$. Since it is well known that the homotopy groups of the latter space are exactly those as were given in the statement of the theorem, this would complete the proof.

Thus we see that it remains to verify next two lemmas.

Lemma 1.3. *Each $a \in F(H_+, H_+)$ can appear as an left upper corner element of a two-by-two matrix above.*

Lemma 1.4. *For each $a \in F(H_+, H_+)$, the set of all c such that $(a, c)^*$ can appear as the first column of a matrix representing an element of G, coincides with the set of all $c \in K_s(H_+, H_-)$ such that $c|\ker a$ is injective. The set of all such c is a contractible subset in $K_s(H_+, H_-)$.*

The first of these two lemmas follows from a well-known procedure of regularizing of a Fredholm operator. One takes any embedding c of $\ker a$ into H_- and takes b to be a finite rank operator from H_- onto $(im\, a)^\perp$. Then one can obtain an appropriate d by taking any epimorphism of H_- onto $\ker b$ with the kernel im$c$. It is trivial to check that this really defines an operator from $GL(\pi_+, K_0)$ so this construction does the job simultaneously for all ideals K_s with $s \geq 1$ and the first lemma is proved.

Moreover, from this argument it becomes evident that the only restriction on c in order that it could "accompany" a given a in GL^s is that it maps $\ker a$ injectively into H_- (again no matter which ideal K_s is considered). On the other hand if c appears as the lower-left corner element of such a matrix then its kernel should be trivial.

The last statement of the last lemma follows from the fact that the set of all such c is apparently homeomorphic to the set of all n-tuples of linearly independent vectors (i.e., n-frames) in H_-, where $n = \dim \ker a$. As is well known all spaces of frames are contractible [46] so we obtain the desired conclusion. This completes the proof of the theorem. □

As was shown in [136], the *restricted Grassmanian* $Gr^2(H)$ has also a remarkable structure of a cellular complex (CW-complex) which is closely related to the so-called partial indices [19] and gives a visual interpretation of certain phenomena discussed in [19], [20]. Moreover, Fredholm Grassmanians can be turned into differentiable manifolds, which enables one to construct an analogue of the Morse theory and recover in this way the cellular structure obtained from the partial indices [136], [88]. We describe here a simple explicit way of introducing differentiable manifold structures on Fredholm Grassmanians Gr^s following the exposition of this topic in [136].

Theorem 1.10. *For any finite $s \geq 1$, the Grassmanian $Gr^s(H)$ is a differentiable manifold modelled on Banach space $K_s(H)$.*

Proof. We first construct a natural atlas on Gr^s (cf. [136] for $s = 2$). Notice that the graph of every s-summable operator $w : H_+ \to H_-$ belongs to Gr^s. Since the sum of a Fredholm operator and an s-summable operator is a Fredholm operator, one concludes that, for every $W \in Gr^s$, the graph of any s-summable operator from W to $W^\perp$ also belongs to Gr^s. Such graphs constitute an open subset $U_W \in Gr_r$ consisting of all W' such that the orthogonal projection $W' \to W$ is an isomorphism. Obviously this open subset is in a one-to-one correspondence with the space $K_s(W, W^\perp)$ of s-summable operators from W to $W^\perp$, which defines an atlas on Gr^s.

We now describe an explicit form of the transition diffeomorphisms of this atlas and verify that this atlas really defines a structure of a differentiable manifold, i.e., differentials $D(g_i \circ g_j^{-1})(p)$ are bounded operators in $K_s(H)$. This would apparently complete the proof.

Let U_V and U_W be the open sets in Gr^s corresponding to the spaces $H_1 = K_s(V, V^\perp)$ and $H_2 = K_s(W, W^\perp)$. Let us show that the images H_{12} and H_{21} of the intersection $U_V \cap U_W$ in these spaces are open and the corresponding "change of coordinates" $H_{12} \to H_{21}$ is continuously differentiable.

Let us consider the identity transformation of H as an operator

$$V \oplus V^\perp \to W \oplus W^\perp$$

and write it in the form of a two-by-two matrix of operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to these direct sum decompositions. Here a is an operator from V to W, and so on (cf. the proof of Theorem 1.9).

From the fact that both V and W belong to Gr^s it follows easily that the diagonal terms a, d are Fredholm operators while b and c are operators of K_s class. Suppose now that a subspace $L \in U_V \cap U_W$ is simultaneously the graph of operators $T_1 : V \to V^\perp$ and $T_2 : W \to W^\perp$. Then operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ T_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ T_2 \end{pmatrix} q$$

should coincide as operators from V to $W \oplus W^\perp$ for some isomorphism $q : V \to W$. This implies that

$$T_2 = (c + dT_1)(a + bT_1)^{-1}. \tag{1.6}$$

The last relation apparently shows that T_2 is a continuous function of T_1 on the open set $H_{12} = \{T_1 \in H_1 : a + bT_1 \text{ is invertible}\}$.

This means that the atlas U_W really defines on Gr^s a structure of a topological manifold and it remains to verify that the differentials of coordinate changes in this atlas do exist and they are bounded linear operators as operators in $K_s(H)$. To this end let us compute the differential of T_2 as a function of T_1. By a standard application of Leibniz rule for operator-valued functions one obtains:

$$DT_2(T_1) = d(a + bT_1)^{-1} - (c + dT_1)(a + bT_1)b(a + bT_1)^{-1}.$$

Now one can make a straightforward examination the linear operator in $K_s(H)$ defined as the multiplication by the right hand side of this formula, using the Neumann series for the inverse $(a + bT_1)^{-1}$, and verify that it defines a bounded linear operator on $K_s(H)$. Thus this atlas really defines a differentiable manifold structure on Gr^s and the proof is complete. □

Remark 1.3. In the case $s = 2$ the same atlas defines a *holomorphic Hilbert manifold* structure on Gr_r (modelled on the Hilbert space $K_2(H)$ with its Hilbert-Schmidt norm) (cf. [136]).

Remark 1.4. Apparently one can introduce similar operator groups and Grassmanians in any Banach space. The above results remain valid for wide classes of separable Banach spaces with basis and contractible general linear group but here we cannot dwell upon that issue.

The restricted Grassmanian $Gr^2(H)$ possesses further interesting geometric properties and it is especially important in physical applications [115], [116], [136], [148]. For this reason we now describe some important differential geometric properties of $Gr_r = Gr^2(H)$ which can be established by the same methods as above. First of all we collect some of its properties as a particular case of the results already obtained.

Corollary 1.3. *The restricted Grassmanian Gr_r is a homogeneous space under the actions of U_r and GL_r. If W is a closed complex subspace of H and P_W denotes the orthoprojector on W then $W \in Gr_r$ if and only if the difference $P_W - P_+$ is a Hilbert Schmidt operator.*

The first statement was already proved. As to the second one, notice that both conditions given in it obviously imply that W is of infinite dimension and codimension. Thus there exist a $U \in U(H)$ such that $W = U \cdot H_+$. Then of course $P_W = UP_+U^{-1}$ and it follows by a direct calculation that $P_W - P_+$ is Hilbert-Schmidt if and only if $[U, P_+]$ is Hilbert-Schmidt, which exactly means that $U \in U_r$.

We can now list the basic geometric properties of Gr_r.

Theorem 1.11 (cf. [136], [148]).

1. *The restricted Grassmanian $Gr_r(H)$ is a complex-analytic manifold modelled on the separable Hilbert space $L^2(H_+, H_-)$.*

2. *The actions of $GL_r(H)$ and U_r on Gr_r are complex analytic and real analytic respectively.*

3. *The linear isotropy representation at the point H_+ is described by the map*

$$Ad : G_+ \to GL(L^2(H_+, H_-)), AdT \cdot S = T_{11} \circ S \circ T_{22}^{-1}, S \in L^2(H_+, H_-).$$

4. *The connected components of Gr_r consist of subspaces W with a fixed value of $indP_+| W$, in other words, they are given by the sets ($\kappa \in \mathbb{Z}$)*

$$G_r^k = \{W : ind(P_+| W : W \to H_+) = k\}.$$

5. *There exist natural holomorphic embeddings of the total Grassmanians*

$$G(\mathbb{C}^{2N}) = \cup_{n=0}^{2N} G_n(\mathbb{C}^{2N})$$

into Gr_r such that their images $Gr_r(N)$ form an increasing sequence of subsets and the union $\cup_{N\geq 1} Gr_r(N) = Gr_r(\infty)$ is dense in Gr_r. Moreover, for all N and k the intersection $Gr_r(N) \cap G_r^k$ is biholomorphic to $G_{N+k}(\mathbb{C}^{2N})$.

Corollary 1.4 ([136]). *All holomorphic functions on Gr_r are locally constant.*

Proof. Since U_r acts by holomorphic transformations, it is sufficient to consider a function f which is holomorphic on G_r^0, the connected component of Gr_r containing H_+. Given now two points W_1 and W_2 in $G_r^0 \cap Gr_r(\infty)$ there exists an N such that $W_1, W_2 \in G_r^0 \cap Gr_r(N)$. Since the latter is biholomorphic to a connected compact complex manifold $G_N(\mathbb{C}^{2N})$, the function f takes the same values on W_1 and W_2. Thus f is constant on $G_r^0 \cap Gr_r(\infty)$ and the result follows from the density of the latter set in G_r^0. □

As was shown by several authors (see [136], [58], [148], [167]), the restricted Grassmanian Gr_r possesses remarkable differential geometric properties. Most of them follow from the important observation that Gr_r carries a natural U_r-invariant Kählerian structure which we will now describe following [136] and [148].

Recall that the so-called *Schwinger term* [148] is defined as a bilinear form on the Lie algebra u_r of U_r given by

$$s(A,B) = tr(A_{12}B_{21} - B_{12}A_{21})$$

where $A, B \in u_r$ are represented as (2×2)-matrices with respect to a fixed polarization. Define then a real-valued antisymmetric bilinear form on u_r by setting

$$\tilde{\Omega}(A,B) = (-\mathbf{i})s(A,B).$$

Then one can verify that the bilinear form $\tilde{\Omega}$ vanishes on the subalgebra $u(H_+) \times u(H_-)$ and is invariant under the linear isotropy representation of $U(H_+) \times U(H_-)$. Hence it descends to a form Ω_+ on

$$u_r/u(H_+) \times u(H_-) \cong L^2(H_+, H_-) \cong T_{H_+}Gr_r$$

which is invariant under $U(H_+) \times U(H_-)$.

Notice that there also exists a natural complex structure J_+ on $T_{H_+}Gr_r$ which is also $U(H_+) \times U(H_-)$-invariant, namely: $J_+T = \mathbf{i}T$ for all $T \in L^2(H_+, H_-)$. This in the usual way produces a $U(H_+) \times U(H_-)$-invariant Kählerian structure on on Gr_r.

The main properties of the restricted Grassmanian which can be formulated in this context, are collected in the following statement.

Theorem 1.12 ([148]). *The restricted Grassmanian is a Hermitian symmetric space, in particular it is geodesically complete. The geodesic exponential map Exp at the point H_+ is given by*

$$\pi \circ \exp : T_{H_+}Gr_r \to Gr_r,$$

where exp *is the exponential map of U_r and π is the projection $U_r \to Gr_r$. The Riemann curvature tensor of Gr_r is completely fixed by its value in the point H_+ where it can be given by a certain Töplitz-like operator.*

Comparing the above formulae with the ones defining the Kähler metric on the based loop group investigated by D.Freed [58], one can observe that they are completely similar. This similarity is not of course occasional, the link between the both structures being given by the Grassmanian embedding of a based loop group [136]. It is now obvious that various properties of loop groups can derived from analogous properties of the restricted Grassmanian. We confine ourselves to this short remark because exploiting this relation further does not fit the main topic of this paper. Our aim was just to show that Fredholm Grassmanians and loop groups can be studied using several independent approaches each of which is rich enough to deserve a separate detailed discussion.

In this spirit, we proceed by mentioning that another fruitful framework for discussing various geometric properties of Fredholm Grassmanians emerges from the Fredholm structures theory [46]. As was observed in [86], [88], certain dense subsets of these Grassmanians can be endowed with natural Fredholm structures.

This fact seems remarkable since a Fredholm structure on an infinite dimensional manifold enables one to introduce non-trivial global geometric and topological invariants of this manifold. The reason for this circumstance is that Fredholm Grassmanians are closely related to loop groups of compact Lie groups [136] and such loop groups can be endowed with some natural Fredholm structures [85], [86], [58]. Our discussion of this issue is based on the results of [86] and [58] but we present them with a view toward Fredholm Grassmanians.

For simplicity we only consider the classical case corresponding to the loop group of unitary group U_n. Recall that Riemann–Hilbert problems for arbitrary compact Lie groups were studied in [86]. Some results of [86] are presented in the next section. The discussion below is actually applicable for arbitrary compact Lie groups.

Recall that a Fredholm structure on an (infinite-dimensional) Banach manifold M modelled on a Banach space E is defined as by an atlas (U_i, g_i) on M such that for any point $p \in g_j(U_i \cap U_j)$ the differential (Frechet derivative) of the transition diffeomorphism $D(g_i \circ g_j^{-1})(p)$ is an invertible operator of the form "identity + compact" [47].

Appearance of a natural Fredholm structure on an infinite dimensional Banach manifold is a remarkable event as such structures possess various interesting global geometric and topological invariants (curvatures, characteristic classes) [46]. An important result due to J.Elworthy and A.Tromba states that a Fredholm structure on M can be constructed from a Fredholm mapping $M \to H$ with zero index and also from certain smooth families of zero index Fredholm operators parameterized by the points of M [48].

These facts were used in [85], [88] to construct Fredholm structures on loop groups. It was done using the families of Fredholm operators parameterized by regular loops. In virtue of the results of [85], [88], to each regular loop f one can assign a Fredholm operator associated with the Riemann–Hilbert problem R_f defined by loop f (i.e., f is the coefficient of the problem R_f). The following statement follows from the results of [85], [88] combined with the main result of [48].

Theorem 1.13. *With any (complex) linear representation γ of U_n one can associate a Fredholm structure F_γ on the group L^1U_n of H^1-loops on U_n.*

Here the loop group L^1U_n is endowed with the usual H^1-norm [120]. It is easy to verify that with this norm it becomes a Hilbert Lie group. We do not reproduce here details of the argument in [85], [88] because they were performed in the framework of the Fredholm theory of generalized Riemann–Hilbert problems which involves a lot of technicalities irrelevant to the main subject of this paper.

A nice geometric explanation of the existence of Fredholm structures on loop groups can be given using some recent results of G.Misiolek [120], [121]. The main result of [120] yields in particular that the exponential map of the group LG of H^1-loops on a compact Lie group G is a Fredholm map of index zero. This local result enables one to obtain a Fredholm atlas on the loop group by merely taking the inverse of the exponential map at identity and spreading this chart to any point of LG by left shifts.

More precisely, there exists a local chart (U, ϕ) at the unit of LG such that $\phi : U \to V$ is a diffeomorphism on some open subset in the space of loops LA on the Lie algebra A of group G, and for any $x \in U$ the differential $D\phi_(x)$ is an invertible operator of the form "identity + compact". From the same result of G.Misiolek it follows that differentials of left shifts L_g by elements of LG are also of the same form "identity + compact". Let us construct a chart (U_g, ϕ_g) at $g \in LG$ by setting $U_g = L_g(U)$, $\phi_g = \phi \circ (L_g)^{-1}$. This obviously gives an atlas on LG and it is not difficult to check that differentials of the transitions mappings of this atlas also belong to the Fredholm subgroup. In this way one obtains a Fredholm structure corresponding to the adjoint representation of a loop group.

This argument was worked out jointly with G.Misiolek in the August of 2001 during a Banach Center workshop on non-linear differential equations. Details and applications will appear in a forthcoming paper by G.Khimshiashvili and G.Misiolek.

Actually Fredholm structures on loop groups come from various sources. An interesting geometric way of constructing Fredholm structures on loop groups was suggested by D.Freed [58].

The construction used by D.Freed reveals certain differential geometric aspects of loop groups which are apparently interesting from the viewpoint of Riemann–Hilbert problems. For this reason, we briefly discuss some ingredients of his construction and their relation to the concepts used in the theory of Riemann–Hilbert problems.

Let now ΩG denote the group of based (i.e. the number 1 maps to the identity of G) smooth loops on a compact Lie group G with Lie algebra A. Recall that any real number s one can define the (Sobolev) H_s metric on ΩG by

$$(X, Y)_{H_s} = \int_T (\nabla^s X(x), Y(x))_A dx, \quad X, Y \in \Omega A,$$

where ∇ denotes the Laplace operator d^*d on T and $(\cdot,\cdot)_A$ denotes the inner product on A given by minus the Killing form on A [136].

The Hilbert space completion of the smooth loops in this inner product is denoted by $\Omega_s A$. As is well known, H_s loops are continuous for $s > 1/2$ [136]. In this range one also obtains in the standard way [136] the corresponding completions $\Omega_s G$ which are Hilbert manifolds modelled on $\Omega_s A$ [136], [58]. As was proved in [59], loop groups $\Omega_s G$ are *Hilbert Lie groups* for $s > 1/2$. In fact, in many aspects it is also important to consider the $H_h = H_{1/2}$ metric (h for "one half") on ΩG.

As was shown in [58], this metric is a *homogeneous Kähler metric* [105]. The corresponding Kähler structure on ΩG is most easily described by exhibiting its complex and symplectic structures and then observing that the metric defined by those is exactly the H_h metric.

The almost complex structure on ΩG is evident from the decomposition of its complexified tangent space Lie algebra $\Omega A_{\mathbb{C}} = M_+ \oplus M_-$ into the direct sum of two subspaces consisting of loops with only positive (negative) Fourier coefficients, i.e. we consider M_+ as the holomorphic tangent space and M_- as the antiholomorphic tangent space. Alternatively, one can define a J operator on the tangent space ΩA, that is, an operator whose square is equal to -1.

Denote by D the operator $d/d\phi = \mathbf{i}zd/dz$. Notice that its kernel is trivial on the space of based loops ΩA and the operator $|D|$ is the square root of the positive Laplacian $-d^2/d\phi^2$. This implies that $J = D/|D|$ has its square equal to minus the identity. One concludes that ΩG is a complex manifold by applying an infinite dimensional version of the Newlander-Nirenberg theorem [105]. It is possible since the torsion tensor of this almost complex structure vanishes [58]. The same conclusion can be derived from the generalized Birkhoff theorem obtained in [136].

There also exists a left invariant symplectic form ω on ΩG. It is described by defining it on the Lie algebra ΩA by the formula

$$\omega(X,Y) = \frac{1}{2\pi}\int_T (X'', Y)_A.$$

Here X, Y are interpreted as elements of ΩA.

The form ω is nondegenerate since D has no kernel on based loops and one can check by standard computation that ω is smooth (see [136]). Summarizing all this we conclude that ΩG is an infinite dimensional Kähler manifold and ω is the Kähler form for the Kähler metric

$$(X,Y) = \frac{1}{2\pi}\int_T \left(\left|\frac{d}{d\phi}\right| X(\phi), Y(\phi)\right)_A d\phi.$$

Comparing with the above formula for Sobolev metrics we see that this is precisely the H_h metric. Using its specific properties an elegant formula for the corresponding Kähler connection was established in [58].

Denote by $\mathbf{T}$ the circle group, then $\mathbf{T}$ acts on the space of free loops LG by rotations and one can form the semidirect product of $\mathbf{T}$ and LG which we denote EG. Obviously,

$$\Omega G = EG/(\mathbf{T} \times G)$$

so EG naturally acts on ΩG. By the general principles of Lie group actions, to this action corresponds an infinitesimal action which assigns a vector field ξ_Z to each element $Z \in EG$ (as is well known, this vector field is the it right invariant extension of Z to EG). Evaluation at the identity in ΩG gives the map which identifies $M_{\mathbb{C}} = M_+ \oplus M_-$ with the complexified tangent space to ΩG.

Let ∇ denote the Kähler connection and ∇^L the Lie derivative. Then, again by general principles, the difference $\nabla_{\xi_Z} - \nabla^L_{\xi_Z}$ is tensorial hence it defines a linear transformation on M. Since both derivatives preserve the complex structure, after complexification this transformation separately preserves M_+ ad M_-. Denoting by EA the semidirect product of $\mathbb{R}$ and LA, we obtain a map $\phi : EA \to L(M_+)$ as the $\mathbb{C}$-linear extension of the map defined by

$$Z \mapsto \nabla_{\xi_Z} - \nabla^L_{\xi_Z}|M_+.$$

D.Freed expressed the Kähler (H_h) connection by indicating a remarkable explicit formula for ϕ in terms of Töplitz operators [58]. Families of such operators eventually enabled D.Freed to construct a Fredholm structure on ΩG whose characteristic classes coincided with those defined using the Chern-Weil theory for the Kähler H_h metric [59].

This circumstance is especially remarkable in the context of our approach because as was explained in [86], [88] Fredholm structures defined by Riemann–Hilbert problems can be also induced from certain families of Töplitz operators. Due to the considerable technical complicity of these results we do not present here a detailed discussion of them but we wish to emphasize that the approach of D.Freed can be used to establish some subtle geometric and topological properties of Birkhoff strata in H_h metric. In particular, one obtains a general approach to computing their curvatures and characteristic classes and it is interesting to verify if this leads to the same results as were obtained in [38] for the groups of Hölder loops.

Another intriguing open problem is to investigate whether those "Freedholm" structures are equivalent (concordant in the sense of [48]) to some of ones obtained from parameterizing the loop groups by families of Riemann–Hilbert problems as in [86], [88]. Actually, there is some evidence that the structures constructed by D.Freed correspond to the case of Riemann–Hilbert problems with respect to the adjoint representation of the group.

Since each Fredholm Grassmanian Gr^s by its very definition gives rise to a family of Fredholm operators parameterized by it, one can consider the Fredholm structure defined by this family. It is the natural to conjecture that those *Fredholm structures on Fredholm Grassmanians* can be related with the preceding results using the so-called Grassmanian models of loop groups [136]. As will be explained in Section 2, the interpretation of a loop as a coefficient of Riemann–Hilbert problem gives a natural mapping of an appropriate loop group LG into the group $GL(\pi_+, K)$. In virtue of the above discussion (cf. also [136]) it is clear that by posing proper regularity conditions on a loop f one can achieve that the rotation of subspace H_+ by the operator of multiplication by f gives a subspace in one of Grassmanians Gr^s. In this way one obtains a natural mapping of LG into Gr^s which is called the *Grassmanian model* (or *Grassmanian embedding*) of a loop group [136].

Some properties of these models follow from the preceding discussion, others were established in [136], basically for the case of the restricted Grassmanian $Gr_r(H)$. In particular it is well known that the group of continuously differentiable loops can be embedded in Gr_r [136]. However the topology of LG and the one induced on its image as a subset of Fredholm Grassmanian do not coincide in general. In fact, it is an interesting and difficult analytical problem to find exact regularity conditions which guarantee that the corresponding loop group can be realized in Gr^s for a concrete s (see examples presented in [136], Ch.7).

We avoid discussion of this problem by concentrating our attention on the group of smooth (infinitely differentiable) loops $L_\infty U_n$ which is the smallest of interesting groups of that kind. Its image under the above embedding is called the smooth (Fredholm) Grassmanian Gr_∞. It is easy to check that it lies in each Fredholm Grassmanian Gr^s. A more interesting circumstance is that it is homotopy equivalent to each of them [136] so it captures important global properties of these Grassmanians.

Now one can transplant various structures from $L_\infty U_n$ to Gr_∞. In particular it is evident that $L_\infty U_n$ can be endowed with Fredholm structures which are just the restrictions of the Fredholm structures on H^1-loops provided by Proposition 1.13 so we obtain the same conclusion for the smooth Grassmanian.

Proposition 1.4. *With each linear (finite dimensional) representation of U_n one can associate a Fredholm structure on the smooth Grassmanian $Gr_\infty(H)$.*

Of course one may ask whether it is possible to extend these structures to ambient Grassmanians Gr^s but this problem involves some delicate analytic issues which will be discussed elsewhere.

As an example of perspectives suggested by these results let us formulate another natural problem. From the mentioned result of Elworthy and Tromba and Proposition 1.13 it follows that there exists an index zero Fredholm mapping of the loop group $L^1 U_n$ in Hilbert space. It would be interesting and instructive to find an explicit construction of such a mapping. The same problem can be formulated for all compact Lie groups. It would be also interesting to find such a mapping from the smooth Grassmanian $Gr_\infty(H)$ in its model space.

Also, it is well known that for a Fredholm manifold M one can define its characteristic classes $ch_k(M) \in H^{2k}(M, \mathbb{Z})$ [59]. A natural and important problem is to identify these classes in the cohomology of M. In our setting this problem permits a particularly nice formulation.

As was already mentioned, the smooth Grassmanian has the same homotopy type as Fredholm Grassmanians $Gr^s(H)$ so their cohomology rings are isomorphic and the structure of these rings is well-known [47]. It is also well known (see [47], [58]) that any Fredholm Hilbert manifold has well-defined Chern classes ch_j which are classes in the even-dimensional cohomology of this manifold. Combining these two observations we conclude that a Fredholm structure on the smooth Grassmanian defines certain classes in $H^{2j}(Gr^s(H))$. Thus we come to the problem of computing these classes for the structures F_γ described above. Some results in this direction were obtained in [58], [59]. It is remarkable that such Chern classes can be represented by some differential forms using traces of appropriate products of operators from Schatten classes [58], which indicates an intriguing analogy with the non-commutative geometry of A.Connes [33].

We now recall the main result of [59] and explain its relation to the geometric models for Riemann–Hilbert problems discussed in this section. Recall that the group of units of the Calkin algebra $Q(H) = L(H)/K(H)$ can be naturally identified with the factor-group $Q^* = GL(H)/GK(H)$. As is well known it is a Banach Lie group modelled on its Lie algebra $Q(H)$ which is actually a C*-algebra [42].

One can analogously define factor-groups $Q^s = GL/GL^s$ for each $s \geq 1$. Since L^s is an ideal in $L(H)$, GL^s is a normal subgroup in $GL(H)$ so the quotient Q^s is a group. Unfortunately it cannot be made into a Banach Lie group because L^s is not closed in $L(H)$, which implies that the Lie algebra L/L^s is not Hausdorff in the quotient topology. So one has to regard Q^s just as an abstract group. Nevertheless these are god objects because they are closely related to Q^* which has a number of useful topological interpretations [47].

As wa shown by D.Freed [59], the hidden "nice" structure of groups Q^s can be revealed by considering some special homomorphisms $G \to Q^s$ of a Banach Lie group G which factor through the projection $\pi : F_0(H) \to Q^s$. This setting is actually a particular case of the notion of *Fredholm representations* considered by A.S.Mishchenko and his followers (cf. [118], [147]).

In such a context D.Freed was able to prove a general theorem providing useful information on the Chern classes of the GK^s-bundle emerging on G via pull-back from $F_0(H)$ (recall that $F_0(H)$ is the classifying space for all $GK^s(H)$-bundles [47]). Denote the Chern classes of the universal $GK(H)$-bundle by ch_l.

Theorem 1.14 ([59]). *Let G be a Banach Lie group with the Lie algebra $Lie(G)$. Suppose that $T : G \to F_0(H)$ is a smooth map such that the composition $\pi \circ T : G \to Q^s$ is a homomorphism, i.e. $T(g)T(h) - T(gh) \in L^s$ for all $g, h \in G$. Assume further that the map $(g, h) \mapsto T(g)T(h) - T(gh)$ is a smooth map into L^s. Let $T' : Lie(G) \to L(H)$ be the differential of T at* *the identity, and define the left-invariant L^s-valued 2-form on G by putting*

$$\Omega(X,Y) = [T'(X), T'(Y)] - T'[X,Y], X, Y \in Lie(G).$$

Then for all $l \geq s$ the cohomology class T^*ch_l is represented invariantly by the form

$$\gamma_l = -(\mathbf{i}/2\pi)^l (l!)^{-1} tr(\Omega^l).$$

Here the elements of $Lie(G)$ are understood as left invariant vector fields on G which, in the definition of Ω are evaluated at the identity of G. The assumptions in the theorem guarantee that the trace $tr(\Omega^l)$ exists for all $l \geq s$. A detailed proof of this theorem may be found in [59].

Of course it is not quite obvious how to construct such maps T as required in the theorem. Nevertheless, a natural source of examples is provided by Fredholm Hilbert Lie groups G introduced above. Recall that a Fredholm structure on G in a standard way defines a map $G \to F_0(H)$. One just takes an F_0-map of $F : G \to H$ (which exists according to the criterion given in [48]) and takes as T the map defined by the family of differentials $d_x F, x \in G$. In many cases this map satisfies conditions of the theorem for some s and one becomes able to compute some components of the total Chern class of the corresponding Fredholm structure.

An example of such kind emerges from the "Kählerian" Fredholm structures on loop groups constructed in [58]. This enabled D.Freed to compute the total Chern class of such a Fredholm structure in the case of $SU(n)$ [58].

Fredholm structures defined by families of Riemann–Hilbert problems in some cases also generate maps T which satisfy conditions of the theorem [89]. Actually, in most cases such families only satisfy a weaker condition which amounts to saying that "long commutators" of operators $T(g)$ belong to some group GL^s. In [89] the same is expressed by saying that the algebra of operators generated by $T(g)$ satisfies a *polynomial identity modulo compact operators*. This situation is not covered by Freed's theorem but still it is very close to the notion of *Fredholm module* which plays an important role in the non-commutative differential geometry of A.Connes [33]. So one may hope that the methods of computing Chern character developed in [33] can be applied in this situation. Unfortunately this idea has not yet found sufficient development so we delay discussion of the topic for the future.

It seems also worthy of noting that some properties of *partial indices* of Riemann–Hilbert problems can be formulated in the language of Fredholm structures. As is well known (see [21]), the collections of matrix functions with the fixed partial indices, usually called *Birkhoff strata* [21], [136], define an interesting stratification of the loop group. Using Grassmanian models of loop groups and Riemann–Hilbert problems described above, one obtains the corresponding strata in the smooth Grassmanian Gr_∞ and restricted Grassmanian Gr_r (cf. [136]).

Using the well-known properties of partial indices [20], one can show that Birkhoff strata are complex analytic submanifolds of the finite codimension in the group $LGL_n(\mathbb{C})$ of loops on the complex general linear group

$GL_n(\mathbb{C})$ [38]. Similar conclusions can be derived for Birkhoff strata considered in the group $\Omega GL_n(\mathbb{C})$ of based loops [38]. Analogous results in the context of Fredholm Grassmanians were obtained in [136], [88]. One can actually describe the homotopy type of a Birkhoff stratum B_K in terms of the indexing vector K [38], [136].

In order to be more precise, denote for a while by G the group of L^1-loops on $GL_n(\mathbb{C})$ and denote by B_K the subset of all loops with a given collection of (left) partial indices K.

Theorem 1.15 ([38]). *B_K is a locally closed complex-analytic submanifold of G of a finite codimension and its codimension is equal to*

$$\sum_{k_i > k_j} (k_i - k_j - 1).$$

The homotopy type of B_K in terms of K can be described as follows. Let g_K denote the diagonal matrix function from 1.2 corresponding to the integer vector K. Denote by $GL(K)$ the centralizer in $GL_n(\mathbb{C})$ of the image of g_k and by $\triangle : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ the diagonal embedding. Finally, denote by X_K the factor space (as a topological space)

$$(GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))/\triangle(GL(K)),$$

where in the denominator stays the image of $GL(K)$ under the diagonal embedding. Notice that this is a finite dimensional space of the type well suited for applying usual methods of algebraic topology which enable one to compute many topological invariants of this space and eventually determine its homotopy type.

Theorem 1.16 ([38]). *B_K is homotopy equivalent to X_K.*

Further results on the topological type of Birkhoff strata can be found in [38], [58]. Most of the preceding discussion is applicable to Birkhoff strata in the loop group LG of an arbitrary compact Lie group G (cf. the discussion of G-exponents in the next section).

We proceed by explaining how one can approach the study of Birkhoff strata in the context of Fredholm structures. To this end one can use a slight generalization of the notion of a *proper Fredholm submanifold* introduced in [131], [154].

The generalization we have in mind can be naturally formulated in the setting of *Riemannian Hilbert manifolds.* In order to be more precise let us recall some relevant notions from the theory of infinite dimensional manifolds.

Let M be a smooth infinite dimensional manifold modelled on a separable Hilbert space H. As is well known, one can introduce then all basic concepts of differential geometry (vector fields, bracket operation, p-forms, tensors, etc.) in the same way as for finite dimensional manifolds [105].

A Riemannian metric on M is defined as a smooth section g of $S^2(T * M)$ such that $g(x)$ is an inner (scalar) product on T_xM equivalent to the inner product on H for all $x \in M$. If such a g is given then (M, g) is called a *Riemannian Hilbert manifold.* It is well known that there exists a unique torsion free *connection* (sometimes spelled as *connexion*) (in the usual sense of differential geometry [105]) compatible with the metric g. Such a connection is called the *Levi-Civita connection* of g.

Let M, N be Hilbert manifolds, g a Riemannian metric on N and ∇ the Levi-Civita connection of g. A smooth map is called an *immersion* if $d_x f$ is injective and $d_x f(T_x M)$ is a closed linear subspace of $T_{f(x)}N$ for all $x \in M$. Then the restriction of $g(x)$ to $d_x f(T_x M)$ defines a Riemannian metric on M. The *normal bundle* $\nu(M)$ of $f(M)$ in N is defined as the bundle over M with fibers $\nu_x(M)$ isomorphic to the orthogonal complement of $d_x f(T_x M)$ in $T_{f(x)}N$ [105]. An immersion which is globally one-to-one is called an *embedding* and its image is called a submanifold in N. We will only deal with submanifolds of *finite codimension*, i.e. $d_x f(T_x M)$ has finite codimension in $T_{f(x)}N$ (for all $x \in M$).

For an embedded submanifold M one can in a standard way define its *tubular neighbourhood* $t(M)$ and exponential mapping *exp* : $t(M) \to N$ [105]. Finally, using the natural identification of $t(M)$ with a (locally trivial) fibration of open normal disks (balls) $D_r(M) = D_r(\nu_(M))$ of (sufficiently small) radius r in $\nu(M)$ one can define the so-called *end point map* $Y_r : D_r(M) \to N$.

Notice that in the particular case when N is just the model space H, our definition coincides with the definition of the end point map for submanifolds of Hilbert space used in [131], [154]. Since $D_r(M)$ is also a Riemannian Hilbert manifold one can speak of Fredholm mappings of $D_r(M)$ into any other Hilbert manifold. Thus in particular it make sense to speak of Fredholm maps into N.

Definition 1.6. M is called a *proper Fredholm* submanifold (*PF-submanifold*) of N if the end point map $Y_r : D_r(M) \to N$ is proper and Fredholm for all sufficiently small $r > 0$.

Many examples of proper Fredholm submanifolds of Hilbert space can be found in [131], [154]. Their curvature operators have remarkable compactness properties [154] and they can be successfully studied with differential geometric methods [154]. This concept is also well suited for the case when the ambient manifold N is endowed with a Fredholm structure. This situation repeatedly appears in the context of loop groups and Fredholm Grassmanians so we introduce some relevant concepts.

Definition 1.7. An infinite dimensional Riemannian Hilbert manifold M is called a *Fredholm Riemannian Hilbert manifold* (*FRH-manifold* for short) if M is endowed with a compatible Fredholm atlas. If M is simultaneously a Lie group and the metric is invariant then M is called a *Fredholm Hilbert Lie group* (*FHL-group*).

It is now clear that one can also define the notion of a *Fredholm action* of a FHL-group. From the preceding discussion it follows that these concepts are "non-vacuous". In particular, the smooth Fredholm Grassmanians give examples of FRH-manifolds. Moreover as was explained above, the groups of H^1-loops on compact Lie groups appear to have natural Fredholm structures arising from the exponential mapping [120]. It is then easy to check that they actually give examples of FHL-groups. Finally, an important example of Fredholm action of a FHL-group is provided by the action of a loop group on the appropriate space of connections described in [131], [154]. Actually, some actions appearing in the context of integrable systems [121] can also be interpreted as Fredholm actions of FHL-groups.

Remark 1.5. These definitions permit a lot of variations (e.g. in the setting of more general Banach manifolds) and the range of applicability of these concepts may be substantially extended (e.g. in the framework of *Euler equations on infinite dimensional Lie groups* [94] and *Poisson Hilbert Lie groups*) but we cannot dwell upon such developments in the present paper.

Having these concepts at hand we can formulate some related results with a view to applications to Birkhoff strata in loop groups.

Proposition 1.5. *A PF-submanifold of finite codimension of a FRH-manifold inherits a Fredholm structure whose equivalence class is uniquely determined by the embedding.*

This follows from the Proposition 2.8 of [154]. Indeed, the Fredholm structure constructed there on a submanifold M of Hilbert space was defined by the family of curvature operators of M. Since these operators can be computed from an arbitrarily small tubular neighbourhood of M one can easily check that the argument used in [154] can be applied in our situation.

Taking into account the general definition of a *Fredholm submanifold* of a given Fredholm manifold [48], this proposition can be expressed by saying that a PF-submanifold is a Fredholm submanifold. In the framework of Fredholm structures theory one can introduce and investigate various geometric properties of Fredholm submanifolds.

In order to apply all this to Birkhoff strata, notice that the exponential mapping and curvature operators of loop groups and Birkhoff strata were computed in [120], [58]. Translated into our language, results of [120], [58] mean that the end point maps of Birkhoff strata are proper Fredholm so they are PF-submanifolds of loop groups (as was shown above they have finite codimension). Now the preceding proposition enables us to place Birkhoff strata in the Fredholm structures context. As was shown in [47], for each closed Fredholm submanifold its fundamental class is well defined as a class in the cohomology of ambient Fredholm manifold.

Proposition 1.6. *The Birkhoff strata are Fredholm submanifolds of $Gr_\infty(H)$ and each of them has a well-defined fundamental class in the even-codimensional cohomology of $Gr_r(H)$.*

Some computations of the fundamental classes of Birkhoff strata can be found in [38]. In particular, S.Disney managed to compute the fundamental classes for all collections of partial indices with not more than three different components k_i. It would be interesting to complete his results by finding a general formula for the fundamental class of B_K in $H^*(LGL_n(\mathbb{C}))$.

Remark 1.6. Taking into account the fact that the stratification of LG given by Birkhoff strata is of a complex-analytic nature [38] one may wish to investigate its properties along the usual lines of stratification theory [70]. In particular, it is interesting to check if this stratification satisfies Whitney conditions [70]. Some further results about the geometry of Birkhoff strata can be derived from the results about the structure of the so-called *isoparametric submanifolds* obtained by R.Palais and H.Terng [131], [154].

Using the above approach for loop groups and Fredholm Grassmanians associated with compact Lie groups [136] one can generalize Theorem 1.13 in this context. The formulation which we present follows from the results of [88] which in turn are based on the Fredholm theory for Riemann–Hilbert problems developed in [85], [88]. The existence follows from the Fredholmness of the corresponding linear conjugation problem for G [86]. Recall that for any compact Lie group one can naturally define the smooth Grassmanian Gr^G_∞ lying in $Gr^G_r(H)$.

Proposition 1.7. *For each linear representation γ of a compact Lie group G, the smooth Grassmanian $Gr^G_\infty(H)$ has a canonical Fredholm structure F_γ induced by γ.*

As was already mentioned, for any Fredholm structure on a complex Banach manifold one can define its Chern classes [47], so we become able to introduce some global topological invariants of such Grassmanians.

Corollary 1.5. *For each even k, there exists a canonical cohomology class in $H^{2k}(Gr^G_\infty(H))$ which can be defined as the Chern class of the canonical Fredholm structure F_γ.*

It is now evident that one can formulate a number of natural questions related to such Fredholm structures. It is the author's hope that the approach described in this section can lead to new insights about global properties of the classical Riemann–Hilbert problems and geometric objects naturally associated with them. In the next two sections we discuss some generalizations of the classical Riemann–Hilbert problems which naturally arise in the framework of this approach.

2. Riemann–Hilbert Problems for Compact Lie Groups

In this section we describe a generalization of the Riemann–Hilbert problem which was recently introduced by the present author [86] in the framework of the geometric theory of loop groups [136].

Consider again the Riemann sphere $\mathbb{P} = \overline{\mathbb{C}}$ decomposed as the union of the unit disc D_+, unit circle $\mathbb{T}$, and exterior domain D_-, which contains the infinite point ∞ denoted by N ("north pole"). The main idea is to permit more general coefficients in the transmission equation (1.1). It is natural take as coefficient a function on the circle with values in a compact Lie group G, and to search for piecewise holomorphic mappings with values in a given representation space of G. The precise statement of the problem is given below and the rest of the section is devoted to its investigation.

It appears that in the case of a Lie group G one can develop a reasonable theory analogous to the classical one [158] which relies on the recent generalization by A.Pressley and G.Segal [136] of the well-known factorization theorem due to G. Birkhoff [122].

It is easy to indicate several natural regularity conditions for a coefficient which guarantee that the problem is described by a Fredholm operator in appropriate functional spaces. One can also obtain an index formula (Theorem 2.2) in terms of so-called *partial G-indices* (or *G-exponents*) which is a direct generalization of the corresponding classical result [122]. A natural framework for our discussion is provided by a generalized Birkhoff factorization theorem and Birkhoff stratification of a loop group so we have to present first some auxiliary concepts and results.

Let G be a connected compact Lie group of the rank p with the Lie algebra A. As is well known [136], each of such groups has a complexification $G_{\mathbb{C}}$ with the Lie algebra $A_{\mathbb{C}} = A \otimes \mathbb{C}$. This circumstance is very important as it provides complex structures on loop groups and this is the main reason why our discussion is restricted to compact groups.

Let LG denote the group of continuous based (i.e., sending the number 1 to the unit of G) loops on G endowed with the point-wise multiplication and usual topology [136]. We need some regularity conditions on loops and for the sake of simplicity let us first assume that all loops in loop groups under consideration are (at least once) continuously differentiable.

For an open set U in $\mathbb{P}$ let $A(U, \mathbb{C}^n)$ denote the subset of $C(\overline{U}, \mathbb{C}^n)$ formed by those vector-functions which are holomorphic in U, where as usual $\overline{U}$ denotes the closure of U.

Assume also that we are given a fixed linear representation r of the group G in a vector space V. For our purposes it is natural to assume that V is a complex vector space. Notice that for a compact group G, one has a complete description of all its complex linear representations [1].

We are now in a position to formulate our generalization. Namely having fixed a loop $f \in LG$, the (homogeneous) *generalized Riemann–Hilbert problem* (GRHP) R_f with coefficient f is formulated as a question about the existence and description of pairs $(X_+, X_-) \in A(D_+, V) \times A(D_-, V)$ with $X_-(N) = 0$ satisfying the transmission condition on $\mathbb{T}$

$$X_+(z) = r(f(z)) \cdot X_-(z). \tag{2.1}$$

For any loop h on V, we also obtain an inhomogeneous problem $R_{f,h}$ (with the right-hand side h) by replacing the transition equation 2.1 by the condition

$$X_+(z) - r(f(z)) \cdot X_-(z) = h(z). \tag{2.2}$$

In other words, we are interested in the kernel and cokernel of the natural linear operator T_f expressed by the left-hand side of the formula 2.2 and acting from the space of piecewise holomorphic vector-functions on $\mathbb{P}$

with values in V into the loop space LV. To avoid discussing regularity conditions, when dealing with the inhomogeneous GRHP it will be always assumed that the loop h is Hölder-continuous, which is a standard assumption in the classical theory [122], [61].

Remark 2.1. In the particular case when $G = U(n)$ is the unitary group we get that $G_{\mathbb{C}} = GL(n, \mathbb{C})$ is the general linear group. If we take r to be the standard representation on $\mathbb{C}^n$, then the equations (2.1) and (1.1) coincide and we obtain the classical Riemann–Hilbert problem. Note that even in this classical case one obtains a plenty of such problems at the expense of taking various representations of $U(n)$, and the result below can be best illustrated in this situation.

Needless to say, the same picture is observed for all groups but as a matter of fact only irreducible representations of simple groups are essential. Moreover, the exceptional groups of Cartan's list will also be excluded and the remaining groups will be termed as "classical simple groups".

It would not be appropriate to reproduce and discuss here all necessary concepts and constructions from the theory of Lie groups. All necessary results on Lie groups, in a form suitable for our purposes, are contained in a book of J. Adams [1] and we repeatedly refer to this book in the sequel.

Let f be a loop on G. We would like to associate with f some numerical invariants analogous to the classical partial indices. To this end let us choose a maximal torus T^p in G and a system of positive roots. Then following [136] one can define the nilpotent subgroups $N_0^{\pm}$ of $G_{\mathbb{C}}$ whose Lie algebras are spanned by the root vectors of $A_{\mathbb{C}}$ corresponding to the positive (respectively negative) roots. We also introduce subgroups $L^{\pm}$ of $LG_{\mathbb{C}}$ formed by the loops which are the boundary values of holomorphic mappings of the domain B_+ (respectively B_-) into the group $G_{\mathbb{C}}$, and the subgroups $N^{\pm}$ consisting of the loops from L^+ (respectively L_-) such that $f(0)$ belongs to N_0^+ (respectively $f(N)$ belongs to N_0^-).

The following fundamental result was proved in [136].

Decomposition Theorem. *Let G be a classical simple compact Lie group, and $H = L^2(T, A_{\mathbb{C}})$ be the polarized Hilbert space with $H = H_+ \oplus H_-$, where H_+ is the usual Hardy space of boundary values of holomorphic loops on $A_{\mathbb{C}}$. Then we have the following decomposition of the groups of based loops LG:*

(i) *LG is the union of subsets B_K indexed by the lattice of holomorphisms of T into the maximal torus T^p.*

(ii) *B_K is the orbit of $K \cdot H_+$ under N^- where the action is defined by the usual adjoint representation of G. Every B_K is a locally closed contractible complex submanifold of finite codimension d_K in LG, and it is diffeomorphic to the intersection L_K^+ of N^- with $K \cdot L_1^- \cdot K^{-1}$, where L_1^- consists of loops equal to the unit at the infinite point N.*

(iii) *The orbit of* $K \cdot H_+$ *under* N^+ *is a complex cell* C_K *of dimension* d_K. *It is diffeomorphic to the intersection* L_K^+ *of* N^+ *with* $K \cdot L_1^- \cdot K^{-1}$, *and meets* B_K *transversally at the single point* $K \cdot H_+$.

(iv) *The orbit of* $K \cdot H_+$ *under* $K \cdot L_1^- \cdot K^{-1}$ *is an open subset* U_K *of* LG, *and the multiplication of loops gives a diffeomorphism from* $B_K \times C_K$ *into* U_K.

The proof presented in [136] is of a geometric nature and reveals a number of new aspects of Birkhoff factorization some of which were further developed in [59], [88]. Recall that in the classical case this result reduces essentially to the Birkhoff factorization theorem for matrix loops [158].

Let us use this theorem in our setting. For a loop f on G, the (left) Birkhoff factorization is defined as its representation in the form

$$f = f_+ \cdot H \cdot f_- \,, \tag{2.3}$$

where $f_\pm$ belong to the corresponding group $L^\pm G$ and H is some homomorphism of $\mathbb{T}$ into T^p.

Now it is evident that the points (ii) and (iv) of the above theorem imply the following existence result.

Proposition 2.1. *Every differentiable loop in a classical simple compact group has a factorization.*

The same is true for Hölder loops and for some wider classes of loops [136]. Note that we could also introduce the right factorization with the reversed order of f_+ and f_- and the result would also be valid. Our choice of the factorization type is consistent with the problem under consideration.

Taking into account that any homomorphism H from 2.3 is determined by a sequence of p integer numbers $(k_1, \dots, k_p)$, we get that this sequence can be associated with any loop f. These integers are called (left) *G-exponents* (or *partial G-indices*) of f. Their collection will be denoted by $K(f)$.

It is easy to prove that $K(f)$ (up to the order) does not actually depend neither on the terms of the representation (2.3) nor on the choice of the maximal torus. For a given maximal torus, the proof of this fact can be obtained as in the classical case, while the independence on the choice of a maximal torus follows from the well-known fact that any two maximal tori are conjugate [1].

The exponents provide basic analytical invariants of loops and have some topological interpretations.

Proposition 2.2. *Two loops lie in the same connected component of* LG *if and only if they have the same sum of exponents.*

This follows easily from the contractibility of subgroups $L^\pm$ and the point (ii) of the Decomposition Theorem.

Remark 2.2. In the classical case when $G = U(n)$ we obtain the usual partial indices, and Proposition 2.2 reduces to the evident observation that the connected components of LU_n are classified by the sum of partial indices which is known to coincide with the increment of the determinant argument of a matrix function along the unit circle [122].

Having at hand exponents of loops we may identify each subset B_K with the collection of loops having a given collection of G-exponents equal to K (up to the order) and use the corresponding decomposition of LG in the topological study of GRHP. Note that in the classical theory of RHP the geometry of B_K (so-called Birkhoff strata) was the subject of an intensive investigation [19], [68]. Later B. Bojarski proposed an approach in the spirit of global analysis [20] which can also be treated from the viewpoint of the theory of Fredholm structures [47].

Theorem 2.1 ([86], [58]). *For any classical simple compact group G, the group of based H^1-loops ΩG carries a natural Fredholm structure with respect to which all strata B_K are contractible Fredholm submanifolds of ΩG.*

This result provides a manifestation of close connections between the geometric theory of Riemann–Hilbert problems and global analysis discussed in [22], [23]. Its proof makes an essential use of results obtained in [136], [85]. Another proof was given by D.Freed [58]. We mention here some of corollaries which are in the spirit of our exposition.

Corollary 2.1. *The inclusion of each stratum B_K into LG defines a cohomological fundamental class $[B_K]$ in $H^*(LG)$.*

This is an immediate consequence of the cohomology theory for Fredholm manifolds [47].

Remark 2.3. In the particular case when $G = U(n)$ this fact was established by S. Disney [38] without referring to Fredholm structures.

This corollary leads to the purely topological problem of computing such fundamental classes in terms of the classical description of $H^*(LG)$ given by R. Bott [28]. For $G = U(n)$, some of those fundamental classes were computed in the same work of S. Disney [38]. Further progress in this topic was obtained in [58], [93]. The general problem of computing these fundamental classes seems to remain unsolved.

Another type of problem arises in connection with the aforementioned Grassmanian models for LG. We only describe it in the classical case when $G = U_n$ and r is its canonical representation on $\mathbb{C}^n$.

Recall that given a polarized Hilbert space $H = H_+ \oplus H_-$, one may introduce the Fredholm Grassmanian $Gr_+(H)$ consisting of all subspaces in H such that the orthogonal projection on H_+ is Fredholm and the complementary projection on H_- is of Hilbert-Schmidt class [136]. For the canonical representation of $G = U_n$, H_+ can be taken to coincide with the usual Hardy subspace in the space H of square-integrable vector functions on the unit circle [61], [136]. This is of course just a special case of Definition 1.4.

Note that any two elements of $Gr_+(H)$ form then a Fredholm pair in the sense of Definition 1.2 and the index of such a pair is well-defined. As was shown in [136], smooth loop groups are embedded in Gr_+ so that for any two loops the index of the pair of images of H_+ is defined, and one may try to compute it in terms of exponents. It is sufficient to do that in the case when one of the subspaces coincides with H_+, which corresponds to the constant loop. From the computation of the virtual dimension of an element in $Gr_+(H)$ [136] it is easy to derive a formula for the index of a Fredholm pair of such subspaces which, in virtue of [20], coincides with the index of the corresponding Riemann–Hilbert problem.

Corollary 2.2. *The index under consideration is equal to the sum of exponents of the given loop.*

For any natural n, let B_n denote the union of all Birkhoff strata with the sum of exponents equal to n. Collection together all observations, we see finally that the images of sets B_n lie in various connected components of Gr_+ and n is equal to the Fredholm index of pr_+.

There are also some intersecting differential geometric aspects of the Fredholm stratification. In particular, one may compute the curvature of B_K in terms of the corresponding Toeplitz operators [58] and check that they provide examples of the so-called isoparametric submanifolds of LG [154]. These strata are among a few known examples of non-linear Fredholm submanifolds of a geometrically interesting Banach manifold.

In order to obtain a formula for the index of a GRHP, it is apparently necessary to take into account the influence of a given representation r, which is not difficult to do because for classical compact groups all irreducible representations are determined by their highest weights [1]. At the same time it is clear that the index of a GRHP behaves additively with respect to taking direct sums of representations. Thus it is sufficient to assume that r is irreducible with the highest weight $w(r)$.

Recall also that with any weight w of the group G one can associate the so-called elementary symmetric sum $S(w)$ [1] and evaluate it on any integer vector with p components.

Theorem 2.2. *Let r be an irreducible representation of a classical simple compact Lie group G. Then the index of a GRHP with a differentiable loop f as coefficient is given by*

$$\mathrm{ind}P_f = S(w(r))(k_1(f), \dots, k_p(f)), \tag{2.4}$$

where $w(r)$ is the highest weight of representation r.

Proof. Standard facts about decompositions of characters of irreducible representations imply that it is sufficient to prove 2.4 only for the so-called basic representations, i.e., such that their classes in the representation ring $R(G)$ form a set of algebraic generators of $R(G)$ [1].

According to Proposition 2.1 we may write representation (2.3) for the coefficient loop f. Then we insert (2.3) in the equation (2.1) and collect the "+"-marked terms in the left-hand side, which gives us an equivalent but much more convenient form of the transmission condition:

$$(r(f_+))^{-1}(X_+) = r(H)r(f_-)(X_-).$$

We introduce now the new functions

$$Y_+ = (r(f_+))^{-1}(X_+),\ Y_- = r(f_-)(X_-)$$

and notice that they are holomorphic in the same domains as X_+, X_- respectively. This means that we can equivalently solve the new Riemann–Hilbert problem for the pair (Y_+, Y_-) with respect to the action of T^r.

Since the representations of tori always decompose into the direct sum of irreducible representations which are one-dimensional, we conclude that the above factorization enables us to reduce the given problem to the collection of ones having the same form as classical Riemann–Hilbert problems.

It remains only to determine the exponents of the one-dimensional components of the representation $r(H)$. This can be easily done using the effective description of basic representations available for all classical groups [1]. These descriptions are similar for all classical groups and therefore we shall consider only one case, say, for A_k series.

Then basic representations are of the form $s^{(i)}$ with $1 \leq i \leq k$, where the exponent in brackets denotes the exterior degree of the standard representation on $\mathbb{C}^k$ [1]. The character of $s^{(i)}$ is given by the ith elementary symmetric function in k indeterminates and the Weyl group W reduces to the symmetric group S_k. It follows that this character coincides with the elementary symmetric sum of the highest weight $S(x_1 + \cdots + x_i)$ and the exponents of one-dimensional representations of T^p are given by the terms of the corresponding symmetric sum for exponents of the loop f.

Notice now that each of the arising one-dimensional problems is simply a classical Riemann–Hilbert problem for vector-functions vanishing at infinity. By a classical result [122], the index of such a problem is equal to the exponent of the coefficient. Collecting all these observations we obtain the desired index formula. □

Remark 2.4. The Fredholm property of operator T_f is automatically available due to the regularity and invertibility of the coefficient f. This easily follows from the description of our problem in terms of holomorphic principal $G_{\mathbb{C}}$-bundles on P, which is presented below.

In terms of G-exponents one can also obtain a more precise description of the space of solutions to GRHP.

Corollary 2.3. *The dimension of the kernel of a GRHP R_p is equal to the sum of all positive terms in the formal symmetric sum of exponents of its coefficient f.*

This follows from the proof of the theorem, since the corresponding fact for one-dimensional reductions of our problem is well known [158].

We would also like to indicate one aspect of the loop group theory where our result seems useful. Namely, a given loop, generally speaking, can be attributed to various ambient loop groups by considering some natural embeddings of the groups under consideration (e.g., $U(n) \subset O(2n)$). This may change both the highest weight and exponents and there arises an interesting problem of describing possible changes of exponents under such embeddings of coefficient groups.

This problem is not of a merely theoretical interest because it is closely related with the problem of effective computation of the exponents and explicit factorization of a given loop. A natural way is to realize the group in question as a matrix group by considering the matrix realization of the representation r involved in the definition of GRHP, and then compute the partial indices of the corresponding matrix function using the well-known results of [32] and [113].

It should be noted that for wide classes of matrix functions there exist effective algorithms for such computations which are easy to implement on computer [32], [5]. Then it would remain to take into account the changes of exponents caused by embedding G in a matrix group.

It should be noted that in some cases one can explicitly compute the factors entering into Birkhoff factorization even for certain discontinuous loops. Interesting results in this direction may be found in [49], [50], [77], [78].

The same problems can be formulated with respect to arbitrary homomorphisms of the groups under consideration. For example, it would be interesting to investigate the role played by the *Dynkin index* of such a homomorphism. We delay a discussion of this issue to the future and pass to another geometric topic related with GRHP.

It is well known that an adequate language for classical RHP is provided by holomorphic vector bundles over $\mathbb{P}$ [136]. In the case of an arbitrary compact group G there exists a natural connection between GRHP and principal $G_{\mathbb{C}}$-bundles over the Riemann sphere [136]. This connection works in both directions. In particular, the results on the structure of solutions of GRHP enable one to get some information on deformations of $G_{\mathbb{C}}$-bundles.

Theorem 2.3. *The base of the versal deformation of a holomorphic principal $G_{\mathbb{C}}$-bundle corresponding to a loop f has dimension d_K, where $K = K(f)$ is the collection of exponents of f. Moreover, it is given by the formula*

$$d_K = \sum_{k_i > k_j} (k_i - k_j - 1). \tag{2.5}$$

This can be derived from the geometric description of the Birkhoff stratification provided by the above Decomposition Theorem. Indeed, each stratum corresponds to a fixed isomorphism class of the bundles under consideration [136]. In fact, the point (iii) of the Decomposition Theorem shows that such strata possess natural transversals which, due to the smoothness of strata, yield the germs of base of versal deformation [132]. At the same time, the dimension d_K may be computed by the general technique of the deformation theory in terms of the first cohomology group of $\mathbb{P}$ with coefficients in the adjoint representation of G [132]. The formula (2.5) then follows from (2.3) and Serre duality [136].

Corollary 2.4. *A holomorphic principal $G_{\mathbb{C}}$-bundle is (holomorphically) trivial if and only if all exponents of the corresponding loop are equal to zero.*

Corollary 2.5. *A holomorphic principal $G_{\mathbb{C}}$-bundle is stable if and only if all pairwise differences of its exponents do not exceed* 1.

One can also explicitly compute various cohomology groups associated with such $G_{\mathbb{C}}$-bundles in terms of exponents which appears useful, for example, in some applications of GRHP to nonlinear equations [39]. One can also use the exponents for investigating which principal bundles can be realized as subbundles of a given $G_{\mathbb{C}}$-bundle. Notice that for U_n-bundles a complete solution of this problem was obtained in [141].

It is worth noting that it is also possible to formulate some reasonable non-linear versions of GRHP: one has to consider the equation 2.1 with respect to more general actions of compact groups, e.g., on their homogeneous spaces. In this context one can obtain some conditions in terms of the exponents which guarantee solvability of a non-linear problem [83].

In conclusion of this section it seems appropriate to point out that recently there appeared a number of papers which use the generalized Birkhoff factorization for solving non-linear equations. Some results of such kind are presented in [136].

A spectacular application of Birkhoff factorization to proving the existence of solutions of a non-linear equation on Lie group can be found in the paper of H.Doi [39]. The literature devoted to applications of Riemann–Hilbert problems to non-linear equations and integrable systems is very ample and there is no possibility to discuss this issue here. We just indicate a recent book of P.Deift [34] which contains a detailed exposition of several advanced applications of Riemann–Hilbert to analysis and differential equations.

3. Linear Conjugation Problems over C^*-Algebras

In this section we introduce certain geometric objects over C^*-algebras which are relevant to the homotopy classification of abstract elliptic problems of linear conjugation. The abstract problem of linear conjugation was introduced by B. Bojarski [20] as a natural generalization of the classical Riemann–Hilbert problem for holomorphic vector-functions. As was later realized by the author [84], the whole issue fits nicely into Fredholm structures theory [47], more precisely into the homotopy theory of operator groups started by R. Palais [130] and developed by M. Rieffel [137] and K. Thomsen [155].

Similar geometric objects appear in loop groups theory, K-theory, and the geometric aspects of operator algebras, and have recently gained considerable attention [136], [27], [18], [147], [170], [114]. This circumstance enabled the author [84], [91] to develop a geometric approach to abstract linear conjugation problems presented in this section.

Recall that in 1979 B. Bojarski formulated a topological problem which appeared important in his investigation of Riemann–Hilbert transmission problems [20]. That topological problem was later solved independently in [81] and [163] (cf. also [136]). Moreover, these results were used in studying several related topics of global analysis and operator theory [27], [164], [82], [84].

An important advantage of the geometric formulation of elliptic transmission problems in terms of Fredholm pairs of subspaces of a Hilbert space given in [20], was that it permitted various modifications and generalizations. Thus it became meaningful to consider similar problems in more general contexts [84]. Along these lines, the present author was able to develop some aspects of Fredholm structures theory [46] in the context of Hilbert C^*-modules [79], [119], [114], which led to some progress in the theory of generalized transmission problems [84], [91].

Such an approach enables one, in particular, to investigate elliptic transmission problems over an arbitrary C^*-algebra. Clearly, this gives a wide generalization of the original setting used in [20], [163], [27], [81], since the latter corresponds to the case in which the algebra is taken to be the field of complex numbers $\mathbb{C}$. This also generalizes the geometric models for classical Riemann–Hilbert problems considered in the previous section of this paper. It may be added that this approach enabled the author to clarify the homotopy classification of abstract singular and bisingular operators over C^*-algebras [84].

Notice also that the setting of transmission problems over C^*-algebras includes the investigation of families of elliptic transmission problems parameterized by a (locally) compact topological space X. In fact, this corresponds to considering transmission problems over the algebra of continuous functions on the parameter space $C(X)$, and classification of families of elliptic problems of such kind becomes a special case of our general results.

To make the presentation concise, we freely use the terms and constructions from a number papers on related topics, especially from [20], [18], [79], [119], [117], and [156]. An exhaustive description of the background and necessary topological notions is contained in [18], [79], [119], [117], [114]. Actually, all necessary results can be found in the recent book [114] which contains a detailed description of the theory of Hilbert modules over $C*$-algebras.

We pass now to the precise definitions needed to formulate a generalization of a geometric approach to transmission problems suggested by B.Bojarski [20]. We use essentially the same concepts as in [20], but sometimes in a slightly different form adjusted to the case of Hilbert C^*-modules.

Let A be a unital C^*-algebra. Denote by H_A the standard Hilbert module over A, i.e.,

$$H_A = \left\{ \{a_i\}, \quad a_i \in A, \quad i = 1, 2, \dots \ : \ \sum_{i=1}^{\infty} a_i a_i^* \in A \right\}. \qquad (3.1)$$

Since there exists a natural A-valued scalar product on H_A possessing usual properties [114], one can introduce direct sum decompositions and consider various types of bounded linear operators on H_A. Denote by $B(H_A)$ the collection of all A-bounded linear operators having A-bounded adjoints. This algebra is one of the most fundamental objects in Hilbert C^*-modules theory [79], [119], [114].

As is well known, $B(H_A)$ is a Banach algebra and it is useful to consider also its group of units $GB = GB(H_A)$ and the subgroup of unitaries $U = U(H_A)$. For our purpose it is important to have adjoints, which, as is explained, e.g., in [114], is not the case for an arbitrary bounded operator on the Hilbert A-module H_A. In particular, for this algebra we have an analog of the polar decomposition [114], which implies that $GB(H_A)$ is retractable to $U(H_A)$. Thus these two operator groups are homotopy equivalent, which is important for our consideration.

Compact linear operators on H_A are defined to be A-norm limits of finite rank linear operators [114]. Their collection is denoted by $K(H_A)$.

Recall that one of the central object in B. Bojarski's approach [20] is a special group of operators associated with a fixed direct sum decomposition of a given complex Hilbert space. With this in mind, we fix a direct sum decomposition of Hilbert A-modules of the form $H_A = H_+ + H_-$, where H_+ and H_- are both isomorphic to H_A as A-modules. As is well known, any operator on H_A can be written as a (2×2)-matrix of operators with respect to this decomposition. We denote by π_+ and π_- the natural orthogonal projections defined by this decomposition.

Introduce now the subgroup $GB_r = GB_r(H_A)$ of $GB(H_A)$ consisting of operators whose off-diagonal terms belong to $K(H_A)$. Let $U_r = U_r(H_A)$ denote the subgroup of its unitary elements. To relate this to transmission problems, we must have an analog of the restricted (Fredholm) Grassmanian introduced in [136]. In fact, this is practically equivalent to working with Fredholm pairs of subspaces which was used in [20]. To implement all this in our generalized setting, some technical preliminaries are needed.

Recall that there is a well-defined notion of a finite rank A-submodule of a Hilbert A-module [119]. This enabled A. Mishchenko and A. Fomenko to introduce the notion of a Fredholm operator in a Hilbert A-module by requiring that its kernel and image be finite-rank A-submodules [119]. It turns out that many important properties of usual Fredholm operators remain valid in this context, too. Thus, if the collection of all Fredholm operators on H_A is denoted by $F(H_A)$, then there exists a canonical homomorphism $\mathrm{ind}_A : F(H_A) \longrightarrow K_0(A)$, where $K_0(A)$ is the usual topological K-group of the basic algebra A [18] (if there is no possibility of confusion we write ind instead of ind_A).

This means simply that Fredholm operators over C^*-algebras have indices obeying the usual additivity law. In the sequel, we will freely refer to the detailed exposition of these results in [117], [114].

Granted the above technicalities, we can now introduce a special Grassmanian $Gr_+ = Gr_+(H_A)$ associated with the given decomposition. It consists of all A-submodules V of H_A such that the projection π_+ restricted on V is Fredholm while the projection π_- restricted on V is compact. Using the analogs of the local coordinate systems for $Gr_+(H_{\mathbf{C}})$ constructed in [136], we can verify that $Gr_+(H_A)$ is a Banach manifold modelled on the Banach space $K(H_A)$. For our purpose it suffices to consider Gr_+ as a metrizable topological space with the topology induced by the standard one on the infinite Grassmanian $Gr^\infty(A)$.

Now the problem that we are interested in is to investigate the topology of $Gr_+(H_A)$ and $GB_r(H_A)$. Note that for $A = \mathbb{C}$ this is the problem formulated by B. Bojarski in [20].

The main topological results about these objects can be formulated as follows.

Theorem 3.1. *The group $GB_r(H_A)$ acts transitively on $Gr_+(H_A)$ with contractible isotropy subgroups.*

Theorem 3.2. *All even-dimensional homotopy groups of $Gr_+(H_A)$ are isomorphic to the index group $K_0(A)$ while its odd-dimensional homotopy groups are isomorphic to the Milnor group $K_1(A)$.*

Of course, the same statements hold for the homotopy groups of $GB_r(H_A)$, since by Theorem 3.1 these two spaces are homotopy equivalent. We formulate the result for $Gr_+(H_A)$ because it is the space of interest for transmission problems theory.

The homotopy groups of $GB_r(H_A)$ were first computed by the author in [84] without considering Grassmanians. Later, similar results were obtained by S. Zhang [170] in the framework of K-theory. The contractibility of isotropy subgroups involved in Theorem 3.1 in the case $A = \mathbb{C}$ was established in [136].

In the proof of Theorem 3.1, we will obtain more precise information on the structure of isotropy subgroups. It should also be noted that the contractibility of isotropy subgroups follows from a fundamental result on C^*-modules called the generalization of Kuiper's theorem for Hilbert C^*-modules, which was obtained independently by E. Troitsky [156] and J. Mingo [117]. Particular cases of Theorem 3.2 for various commutative C^*-algebras A may be useful to construct classifying spaces for K-theory.

The solution of Bojarski's original problem is now immediate (cf. [81], [163], [136]).

Corollary 3.1. *Even-dimensional homotopy groups of the collection of classical Riemann–Hilbert problems are trivial while odd-dimensional ones are isomorphic to additive group of integers $\mathbb{Z}$.*

Note that the non-triviality of these groups can be interpreted in terms of the so-called *spectral flow* of order zero pseudo-differential operators, which has recently led to some interesting developments by B. Booss and

K.Wojciechowsky [27]. This has shed a new light on the Atiyah-Singer index formulas in the odd-dimensional case. Similar results hold for abstract singular operators over A (for the definition of abstract singular operators see [84]).

Corollary 3.2. *Homotopy groups of invertible singular operators over a unital C^*-algebra A are expressed by the relations*

$$\begin{aligned} \pi_0 \cong K_0(A), \quad \pi_1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus K_1(A); \\ \pi_{2n} \cong K_0(A), \quad \pi_{2n+1} \cong K_1(A), \ n \in \mathbb{Z}_+. \end{aligned} \tag{3.2}$$

Specifying this result for the algebras of continuous functions one can, in particular, compute the homotopy classes of invertible classical singular integral operators on arbitrary regular closed curves in the complex plane $\mathbb{C}$ (see [81], [84] for the precise definitions).

Corollary 3.3. *If $K \subset \mathbb{C}$ is a smooth closed curve with k components, then homotopy groups of invertible classical singular integral operators on K are expressed by the relations (where n is natural and arbitrary):*

$$\pi_0 \cong \mathbb{Z}, \quad \pi_1 \cong \mathbb{Z}^{2k+1}; \quad \pi_{2n} = 0, \quad \pi_{2n+1} \cong \mathbb{Z}. \tag{3.3}$$

As shown in [84], this information also enables one to find homotopy classes and index formulas for the so-called bisingular operators. The latter can be defined by purely algebraic means, starting from the algebra of abstract singular operators. One is thus led to the notion of a bisingular operator over a C^*-algebra and to the description of homotopy classes of elliptic bisingular operators. The notion was introduced in [84] and the description of index ranges follows from the results of this paper.

Corollary 3.4. *Abstract elliptic bisingular operators over a C^*-algebra A are homotopically classified by their indices taking values in $K_0(A)$. The index homomorphism is an epimorphism onto $K_0(A)$.*

As is well known, the usual bisingular operators correspond to certain pseudo-differential operators on the two-torus $\mathbb{T}^2$ [42]. In a similar manner, one may recover some of the known results on homotopy groups of invertible pseudo-differential operators over other two-surfaces [44].

One can also obtain an index formula for abstract bisingular operators in terms of homotopy classes of their operator-valued symbols which can be described by Theorem 3.2. For brevity, the results concerning the index formulas for bisingular operators will not be presented here.

Theorem 3.1 is proved below after developing the necessary geometric constructions over C^*-algebras. We also present the outlines of proofs of Theorem 3.2 and corollaries.

It is standard in C^*-algebras theory to identify subspaces with projections. Thus direct sum decompositions of the type described above correspond to the so-called infinite Grassmanian over A which can be written as

$$Gr^{\infty}(A) = \Big\{p \in B(H_A): \ p = p^2 = p^* \ \text{and} \ p \sim Id \sim Id - p\Big\}, \qquad (3.4)$$

where '$\sim$" denotes the well-known equivalence relation for projections introduced by Murray and von Neumann [18].

Fixing such a decomposition is equivalent to fixing a projection with image and kernel being A-modules of infinite rank. Having fixed such a projection p which will play the role of the projection π_+ introduced above, one can readily verify the useful characterization of GB_r.

Lemma 3.1. $GB_r(H_A) = \{x \in B(H_A) : xp - px \in A \otimes K(H)\}$, *where $K(H)$ stands for the ideal of compact operators in the usual separable complex Hilbert space H.*

The aforementioned (2×2)-matrix representation of $x \in B(H_A)$ can be rewritten as

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad (3.5)$$

where $x_{11} = pxp$, $x_{12} = px(1-p)$, $x_{21} = (1-p)xp$, $x_{22} = (1-p)x(1-p)$.

It is obvious now that $GB_r(H_A)$ is $*$-isomorphic to the group of units of the C^*-algebra consisting of (2×2)-matrices over $B(H_A)$ whose off-diagonal entries are the elements of $A \otimes K(H)$. Further, from the existence, additivity, and stability properties of the Fredholm index (see diagram 3.6 below) it follows that, for $x \in GB_r(H_A)$, both x_{11} and x_{22} should be Fredholm operators with the opposite indices, which is important for the sequel.

Using simple algebraic identities for such (2×2)-matrices (explicitly written in [136] for matrices over $B(H)$), and the fact that $K(H_A)$ is an ideal in $B(H_A)$, it is easy to verify that if such a (2×2)-matrix is applied to an element V of $Gr_+(H_A)$, then the restriction to V of the first projection π_+ is transformed into $x_{11}\pi_+ + x_{12}$ and thus remains Fredholm, while the restriction to V of the second projection gives $x_{22}\pi_- + x_{21}$ and remains compact. This means that xV is again in $Gr_+(H_A)$ and we have proved

Lemma 3.2. *The restricted linear group $GB_r(H_A)$ acts on the special Grassmanian $Gr_+(H_A)$.*

Now, it is evident that to determine the isomorphy class of stability subgroups it is sufficient to identify it for a "coordinate submodule" H_+ in $GB_r(H_A)$. It readily follows from the existence of polar decompositions that the latter subgroup is homotopy equivalent to the isotropy subgroup of H_+ in the restricted unitary group $U_r(H_A)$ (which acts on $Gr_+(H_A)$ as a subgroup of $GB_r(H_A)$).

Analyzing the description of a similar isotropy subgroup in the case of the usual Hilbert space given in [136], one easily finds that in view of the above technical results for Hilbert C^*-modules the same conclusion holds in our case, too.

Lemma 3.3. *The stability subgroup of* H_+ *in* $U_r(H_A)$ *is isomorphic to* $U(H_+) \times U(H_-)$.

Recall that the latter group is contractible according to the result of E. Troitsky and J. Mingo [156], [117].

To prove Theorem 3.1 it remains to check the transitivity, which is the most delicate part of the proof. We will use the method of proof from [136] adapted to our situation. Note that Fredholm operators with vanishing indices can be transformed into invertible ones by a compact perturbation. The corresponding statement for Hilbert C^*-modules is expressed by the so-called *fundamental commutative diagram* of Fredholm structures theory [47], [114]. In our case it has the form

$$\begin{array}{ccccccc} GB(H_A) & \longrightarrow & F_0(H_A) & \longrightarrow & F(H_A) & \longrightarrow & B(H_A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ GB(H_A)/GK(H_A) & \longrightarrow & G_0 & \longrightarrow & G & \longrightarrow & Q_A\,. \end{array} \tag{3.6}$$

Here $F(H_A)$ and $F_0(H_A)$ stand for semigroups of all Fredholm operators and those with the zero index, respectively. G denotes the group of units of the factor-algebra $Q_A = B(H_A)/K(H_A)$ (Calkin algebra over A) and G_0 is its identity component. The right vertical arrow is the Calkin factorization and the left one is the factor-homomorphism on the factor-group below. The upper arrows are inclusions.

The commutativity of this diagram is well known to experts and follows from the facts established in [47], [114] (cf. also [18]). Also, it is a standard verification that the left lower corner horizontal arrow is a homeomorphism. In topological terms, the latter fact means that G_0 is the classifying space for the K-functor [18] and its homotopy groups are isomorphic to the corresponding K-groups of the basic algebra. This conclusion is explained in full detail in [47], [114].

Let us now return to our situation and take an A-submodule V belonging to the special Grassmanian $Gr_+(H_A)$. By definition, there exists a Fredholm operator $T \in B(H_+, H_-)$ such that V is its graph, i.e., is the set of points (x, Tx) with respect to the given decomposition of H_A.

To prove the transitivity of action, it is sufficient to obtain a (2×2)-matrix $M \in GB_r(H_A)$ of the form described above such that $M(H_+) = V$. For this, we consider first the diagonal matrix $\mathrm{diag}(T, T^*)$, where T^* is the adjoint operator of T. From the additivity property of the Fredholm index it follows that this matrix has the zero index when considered as an element of $B(H_A)$. Considering its class in the space of (2×2)-matrix over the Calkin algebra B/K, one sees that it is invertible. Thus our diagram shows that this matrix can be turned into the invertible one by a compact perturbation. In other words, there exist compact off-diagonal terms x_{21}, x_{12} such that our diagonal matrix completed with such entries becomes invertible as an operator on H_A. This already implies the existence of the desired matrix M. One could also finish the proof arguing as in [136], Ch. 6.

The simplest way to verify Theorem 3.2 is as follows. One notices that Proposition 6.2.4 of [136] suggests that $GB_r(H_A)$ should be homotopy equivalent to $F(H_+)$. This would already prove Theorem 3.2 because from diagram 3.6 it follows that homotopy groups of $F(H_A)$ are isomorphic with K-theory of A. In fact, it may be actually proved that $GB_r(H_A)$ is homotopy equivalent to $F(H_+)$, using a suitable modification of the argument from [136], Ch. 6. However, in order to make this argument rigorous one needs to develop a substantial portion of Hilbert modules theory. For the sake of brevity, here we prefer another way, more algebraic in spirit, which closely follows the lines of [170]. In doing so we will borrow freely the concepts and results from [117], [170], and [114].

Throughout this section we will use the identification of direct sum decompositions with projections and fix an element $p \in B(H_A)$ with $p = p^2 = p^*$. Below we will omit some tedious details which are standard for the theory of operator algebras and K-theory.

As was explained, it suffices to compute the homotopy groups of the restricted linear group $GB_r(H_A)$. Denote by $GB_r^0(H_A)$ its identity component. As is well known, in dealing with K-theory invariants, it is useful to consider the conjugations by unitary operators. With this in mind, we introduce the notation $UpU^* = \{vpv^* : v \in U_r(H_A)\} = \{vpv^* : v \in GB_r(H_A),\ vv^* = v^*v = Id\}$. The following simple proposition is verified using the standard techniques of K-theory (cf. [18]).

Lemma 3.4. *$U(upu^*)U^*$ is the path component of UpU^* containing upu^*.*

One also has an equivalent description of the K_0-functor which was already used in [117] and [170].

Lemma 3.5. *For any such $p \in Gr^\infty(A)$, the fundamental group $\pi_1(UpU^*)$ is isomorphic to $K_0(A)$.*

Indeed, later we will produce an explicit isomorphism between these two groups in terms of some partial isometries associated with elements of $GB_r(H_A)$, which plays an important role in the argument.

Following [170], a unitary operator $x \in U_r(H_A)$ will be called p-adapted if both off-diagonal terms of the corresponding (2×2)-matrix (see formula 3.5) are some partial isometries in $A \otimes K(H)$.

It is easy to calculate some associated projections needed in the sequel.

Lemma 3.6 ([170]). *If x is a p-adapted unitary, then $p - x_{11}x_{11}^*$, $p - x_{11}^*x_{11}$, $(Id - p) - x_{22}x_{22}^*$, $(Id - p) - x_{22}^*x_{22}$ are projections in $A \otimes K(H)$.*

The following results from [170] amount to a partial isometry description of the K-functor. Equivalent statements can be found in [117] and [114]. A similar factorization for the case $A = C$ was also used in [136].

Proposition 3.1. *Any $X \in GB_r(H_A)$ can be represented as*

$$x = (Id + k) \cdot \mathrm{diag}(z_1, z_2) \cdot u, \tag{3.7}$$

where $k \in A \otimes K(H)$, the second factor is invertible, and u is a p-adapted unitary.

Recall that according to one of the basic constructions any partial isometry $b \in A \otimes K(H)$ defines a class $[bb^*] \in K_0(A)$ [18]. The following proposition follows from this construction and the equivalence relation in $K_0(A)$.

Proposition 3.2. *The class*

$$[u_{12}u_{12}^*] - [u_{21}u_{21}^*] \in K_0(A) \tag{3.8}$$

is independent of a p-adapted unitary u entering into a representation of a given $x \in GB_r(H_A)$ in form 3.7.

Now we are able to define the mappings giving the desired group isomorphisms. Our strategy is to consider the group GB_r as a fibration over its homogeneous space GB_r/GK and, next, to compute the homotopy groups of GB_r/GK, since the homotopy groups of the fibre $GK(H_A)$, being the standard participants in K-theory, are well known.

Observe first that representation 3.7 implies the equality of cosets $x \cdot GK(H_A) = u \cdot GK(H_A)$ of the elements x and u with respect to the subgroup $GK(H_A)$. By Lemma 3.5, for such u we have the following direct sum of projections:

$$(p - u_{12}u_{12}^*) \oplus (u_{12}u_{12}^*). \tag{3.9}$$

As is well known, direct sums do not have any influence on the stable equivalence relation involved in the definition of $K_0(A)$. In other words, it is meaningful to assign to element 3.9 the class

$$[u_{12}u_{12}^*] - [u_{21}u_{21}^*] \in K_0(A) \tag{3.10}$$

A connection between the considered basic topological spaces is established by

Lemma 3.7. *The element defined by* (3.9) *belongs to UpU^*.*

For the proof it suffices to observe that this statement follows from Proposition 3.1 in [170] by which for any two projections $r_1, r_2 \in A \otimes K(H)$ there exists unitary $w \in GK(H_A)$ such that $wpw^* = (p - r_1) \oplus r_2$.

By virtue of these lemmas we arrive at the basic correspondence giving the desired isomorphism at the level of fundamental groups. Below it is assumed that the base point of GB_r is the identity, and that of UpU^* is p.

Proposition 3.3. *The maps defined by the relations*

$$\begin{aligned} u \cdot GB_r(H_A) &\mapsto [(p - u_{12}u_{12}^*) \oplus u_{21}u_{21}^*]_{UpU^*} \mapsto \\ &\mapsto [u_{21}u_{21}^*] - [u_{12}u_{12}^*] \in K_0(A) \end{aligned} \tag{3.11}$$

are the bijections inducing the isomorphisms

$$\pi_0(GB_r)(= GB_r(H_A)/GK(H_A)) \cong \pi_0(UpU^*) \cong K_0(A). \tag{3.12}$$

Now the results concerning the computation of higher homotopy groups can be formulated as follows (cf. [170]).

Proposition 3.4. *For any natural n one has the isomorphisms*

$$\pi_{2n+1}(GB_r(H_A)) \cong \pi_{2n+1}(UpU^*) \cong K_1(A), \tag{3.13}$$

$$\pi_{2n+2}(GB_r(H_A)) \cong \pi_{2n+2}(UpU^*) \cong K_0(A). \tag{3.14}$$

These isomorphisms can be verified by means of the long exact sequence of homotopy groups associated with a natural operator fibration over UpU^* with the contractible total space $U_\infty(A)$ which is, as above, the group of unitaries in the unitization of $A \otimes K(H)$.

To this end, we consider the map defined by $u \mapsto upu^*$. Clearly, its fibers are all isomorphic with the commutant of p in U, i.e., $(p')_U = \{u \in U_\infty(A) : up = pu\}$. It is also simple to check that this map is a submersion and, according to an infinite-dimensional generalization of Ehresmann's theorem [46], defines a locally trivial fibration with the fiber p'.

The long exact homotopy sequence of this fibration breaks, as usual, into short exact sequences:

$$0 \to \pi_{k+1}(UpU^*) \to \pi_k(p') \to \pi_k(U_\infty(A)) \to 0. \tag{3.15}$$

Since the homotopy groups of the stabilized unitary group $U_\infty(A)$ are isomorphic to the K-groups of A, these exact sequences immediately imply that $\pi_{2n+2}(UpU^*) \cong K_0(A)$ and $\pi_{2n+1} \cong K_1(A)$. Recalling that UpU^* is weakly homotopy equivalent to $GB_r(H_A)$, we obtain the desired conclusion.

Now Theorem 3.2 becomes an immediate consequence of Propositions 3.3 and 3.4.

We will make a few comments on the formulations and proofs of the corollaries.

Corollary 3.1 is simply a special case of Theorem 3.2, where $A = C(S^1)$ is the algebra of continuous functions on the unit circle, which is clear from the interpretation of Riemann–Hilbert problems given in [21]. By a similar reasoning, Corollary 3.3 follows from Corollary 3.2.

Corollary 3.2 can be derived from Theorem 3.2 using the scheme of [84], where the same result for classical singular integral operators on closed contours was derived from the solution of the Bojarski's problem formulated above. To do that, we need first to clarify which one of several possible definitions of abstract singular operators (cf. [20], [135]) is actually appropriate in our setting.

We will use a modification of the approach of [135] (cf. [84]). Fix an invertible operator $U \in GB(H_A)$ with the properties:

1. Both operators U and U^{-1} have spectral radii equal to 1;
2. There exists a projector $p \in GB(H_A), p \sim Id \sim Id - p$, such that

$$Up = pUp, \quad Up \neq pU, \quad pU^{-1} = pU^{-1}p\,; \tag{3.16}$$

3. $\operatorname{coker}(U \mid im\ p)$ is an A-module of finite rank.

There are many such operators. For example, one may take the right shift in a Hilbert A-module and the projector on the "positive half-space" (these are the abstract counterparts of multiplication by the independent variable and the Hardy projector from the theory of classical singular integral operators [135]). Denote by $R(U)$ the C^*-subalgebra generated by U and U^{-1}. It is trivial to verify that for any $T \in R(U)$ the commutator $[T, p] = Tp - pT$ is compact, i.e., $[T, p] \in K(H_A)$.

Moreover, the information about A-Fredholm operators contained in diagram 3.6 enables one to apply the arguments from [135] and obtain a description of invertible elements in $R = R(U)$.

Proposition 3.5. *Invertible operators are dense in* $R(U)$. *They are characterized by the condition that at least one of their restrictions on im p or im* $(Id - p)$ *is a semi-Fredholm operator* [135].

Following [135], any operator of the form

$$T = Lp + Mq + C, \tag{3.17}$$

where $q = 1 - p$, L, $M \in R(U)$, $C \in K(H_A)$, is called an abstract singular operator over A (associated with the pair (U, p)). The totality of all such operators is denoted by $S(U)$.

This is a true generalization of the usual singular operators which are obtained when $A = \mathbb{C}$, U is the unitary operator of multiplication by an independent variable in $H = L_2(S^1)$, and p is the Hardy projector (for details see [135]).

A standard application of Gelfand theory [18] provides symbols of singular operators which are functions on the spectrum of U. Assuming U to be unitary, it follows that with any operator T of form 3.17 one may naturally associate a pair of continuous functions $h(T) = (h(L), h(M))$ on the unit circle. A symbol is called nondegenerate if both its components are nowhere vanishing on S^1. As usual, the index ind $h(T)$ of such a nondegenerate symbol is defined as the difference of argument increments of its components along S^1. Thus we can now formulate the key characterization of elliptic singular operators.

Proposition 3.6. *An operator* $T \in S(U)$ *of form* 3.17 *is Fredholm if and only if its symbol is nondegenerate, i.e., both its coefficients* L, M *are invertible operators.*

After the above preparations, the proof runs in complete analogy with that from [135]. To compute $\pi_*(S(U))$ over A one has only to compute the homotopy groups of pairs of invertible operators in $GR(U)$. The latter group being homotopy equivalent to $GB_r(H_A)$ with $\pi_+ = p$, the answer is provided by Theorem 3.2. Adding the groups from the latter theorem to the homotopy groups of nondegenerate symbols computed in [81], one obtains Corollary 3.2.

Finally, Corollary 3.4 can be obtained from Corollary 3.2 using the scheme of [81], where this was done for the classical counterparts of our results. However, this requires a lot of technical preparation. In particular, one needs to generalize the tensor product construction of conventional bisingular operators from the algebra of pseudodifferential operators on the unit circle (see [42]). These technicalities are rather tedious and require a separate presentation.

Note that in the geometry of Hilbert C^*-modules there are some related topics which admit a nice presentation in terms of special Grassmanians and transmission problems. Part of these results can be found in [170], [114].

Here we will discuss only one topic most closely related to the geometric study of elliptic transmission problems [20], [136]. The point is that our Theorem 3.2 suggests that there should exist a finer geometric structure of the Grassmanian $Gr_+(H_A)$ expressed in terms of a stratification similar to the Birkhoff stratification by partial indices of invertible matrix-functions on the unit circle [68], [19] which plays a prominent role in the classical theory of transmission problems [61], [158].

Such a stratification can be constructed using the geometric language developed in this paper. To this end, let us fix a path component Gr_γ of the Grassmanian $Gr_+(H_A)$ corresponding to a certain element $\gamma \in K_0(A)$. By Proposition 3.3 it is clear that γ is essentially the Fredholm index of the projection π_+ restricted to any element V of this component.

Since $K_0(A)$ is a group, it is reasonable to consider all pairs $(\alpha, \beta) \in K_0 \times K_0$, where $\alpha - \beta = \gamma$. For any pair denote by $B_{\alpha,\beta}$ the subset of all V such that the following relations hold for classes in $K_0(A)$ (recall that any projective A-module generates a class in $K_0(A)$):

$$[\ker \pi_+ \mid V] = \alpha, \quad [\operatorname{coker} \pi_+ \mid V] = \beta. \tag{3.18}$$

Evidently, such a collection is a subset of the given component. Obviously,

$$Gr_\gamma = \bigcup B_{\alpha,\beta}. \tag{3.19}$$

The path component Gr_γ being arbitrary, we obtain a natural decomposition of the special Grassmanian Gr_+ which is similar to the classical Birkhoff stratification [19], [68] (in fact, our decomposition is cruder, which can be seen in the case of classical transmission problems with respect to the unit circle). Of course, it is tempting to verify which properties of the Birkhoff stratification are still valid in our generalized setting and to generalize some of the results on its geometric structure obtained in the classical case [21], [136]. This topic awaits further investigation but certain results are already available of which we present only two.

Proposition 3.7. *All $B_{\alpha,\beta}$ are Banach analytic subspaces of $Gr_+(H_A)$ in the sense of* [41].

Proposition 3.8. *Decomposition* 3.19 *is a complex analytic stratification of $Gr_+(H_A)$* [41].

These results are of the technical nature and require a big portion of the Banach analytic geometry in the spirit of [41] which would be irrelevant to the present exposition. We give them only to indicate more connections with nontrivial geometric problems one of which will be formulated below.

Note that a less precise version of Proposition 3.7 was obtained in the classical case ($A = \mathbb{C}$) by S. Disney [38]. The classical counterpart of Proposition 3.8 was implicitly used by B. Bojarski [19] in his investigation of the stability properties of partial indices.

We conclude with a purely geometric problem suggested by our constructions, which leads to highly nontrivial homological computations even in the classical case [38], [58]. Recall that a complex analytic subset of a complex Banach manifold has a well-defined cohomological fundamental class in the cohomology of the ambient manifold [47]. A discussion of the orientation classes for K-theory in [48] shows that the same is valid for extraordinary cohomological theories like K-theory. Hence fundamental classes of $B_{\alpha,\beta}$ are well defined and there arises a problem of computing them in terms of K-theory. As was mentioned, some results for the classical case were obtained in [38], but our knowledge of these fundamental classes is still very poor.

An intriguing open problem is to construct a finer analytic stratification of the special Grassmanian $Gr_+(H_A)$ similar to that in [136] to obtain more topological invariants for transmission problems. There is some evidence that this should be possible for commutative A.

Our constructions and results can also be interpreted in terms of Fredholm structures over A. Granted diagram 3.6, the basic notions of this theory can be introduced as in [47]. Several important results of Fredholm structures theory have direct analogs for structures over A. In particular, a family of A-Fredholm operators parameterized by points of a manifold M defines an A-Fredholm structure on M.

Applying this result to our restricted Grassmanian $Gr_+(H_A)$, one obtains an A-Fredholm structure on it. Moreover, the Birkhoff strata $B_{\alpha,\beta}$ are Fredholm submanifolds with respect to this structure, and following [38] one can introduce their Chern classes and express them as pull-backs of universal classes carried by the classifying bundle for A-Fredholm structures. Some results in this direction were obtained in [87].

4. Riemann–Hilbert Problems in Higher Dimensions

Since holomorphic functions can be considered as solutions to the Cauchy-Riemann system in the plane, a natural way of introducing multi-dimensional generalizations of the classical Riemann–Hilbert problems is related with considering elliptic first order systems of differential equations with constant coefficients on Euclidean spaces of higher dimension [162]. For brevity, such systems will be called simply *elementary elliptic systems* (EES).

Given such a system S one can take a smooth domain D_+ in the source space of the system, choose a matrix function G (of proper size) on the boundary ∂D_+ and look for couples $X_\pm$ of solutions to the system S in domains D_+ and D_- (the complement of D_+) satisfying the linear conjugation condition of the form (1.1). This gives a natural analogue of the Riemann–Hilbert problem considered as a problem of linear conjugation.

In many cases it is also important to consider a more general type of (local) boundary value problems for EES which, by a slight abuse of terminology, are also called Riemann–Hilbert problems for EES [151], [84]. Linear conjugation problems appear to be a particular case of such problems [151], [84]. In multi-dimensional setting, those generalized Riemann–Hilbert problems exhibit more complicated behaviour than linear conjugation problems. In particular, a longstanding problem was to characterize those EES which posses elliptic Riemann–Hilbert problems [152], while for linear conjugation problems this issue is substantially more simple (see Theorem 4.8 below).

For these reasons in the sequel we pay main attention to generalized Riemann–Hilbert problems. They possess a number of remarkable properties and have gained considerable attention [17], [125], [31]. In particular, such problems for Euclidean Dirac operators play significant role in Clifford analysis [31], [27], [62], [125].

In many situations it is desirable that such problems could be described by Fredholm operators in appropriate functional spaces. The general theory of elliptic boundary value problems indicates a natural approach to this topic [162], [27]. In particular, they can be reduced to a system of integral equations on the boundary [162] and this approach appeared quite effective in the case of linear conjugation problems for EES defining quaternionic regular functions [142].

However this approach does not automatically lead to effective conditions of Fredholmness and there does not seem to exist a version of Fredholm theory for Riemann–Hilbert problems applicable in the case of an arbitrary EES. Actually, it is well known that such systems do not always possess local elliptic boundary value problems [17] and some natural boundary value problems fail to be Fredholm.

In line with the discussion in previous sections, we will only deal with Fredholm boundary value problems for a class of especially well-behaved EES called *generalized Cauchy-Riemann systems* (GCRS) which were introduced by E.Stein and G.Weiss [150]. Even for such systems, the problem of describing those of them which possess elliptic Riemann–Hilbert problems is quite non-trivial and remained unsolved for a long time (for a comprehensive review of the topic see [151]). A good understanding of this issue would open a way of generalizing many results of previous sections to higher dimensions so it is of a major importance for our approach and we will consider it in some detail.

More precisely, we discuss the Riemann–Hilbert problems for generalized Cauchy-Riemann systems. In order to provide a visual description of the class of GCRSs let us first present some basic results about the structure of such systems. The results presented below are scattered in several sources [150], [62], [151], [152] so bringing them together seemed to be reasonable.

It should be also noted that at present there exist two approaches to the Fredholm theory of Riemann–Hilbert problems for GCRS. The first one is direct and uses an explicit form of the Shapiro-Lopatinski condition for

GCRS obtained in [152]. The second approach used in the author's papers [91], [93] is more sophisticated. It relies on some recent results about elliptic operators and K-theory [10], [71].

The direct approach permits a self-contained exposition of many aspects of Fredholm theory for GRHPs so we present the main results available within approach of [151], [152]. However the second approach is indispensable in order to complete the list of GCRS possessing elliptic GRHP (see Theorem 4.6, 4.7 below) but it uses rather complicated topological machinery in the spirit of K-homology approach to boundary value problems [10], [71]. It would be hardly possible to give a reasonable exposition of this topic in a paper of such length so we just present the main results and briefly mention the related concepts.

To begin with, consider a general *homogeneous elliptic first order system with constant coefficients* of the form

$$(EES) \qquad \sum_{j=0}^{n} M_j \frac{\partial w}{\partial x_j} = 0\,,$$

where M_j are constant complex $(m \times m)$-matrices and w is a differentiable mapping from $\mathbb{R}^{n+1}$ to $\mathbb{C}^m$.

Definition 4.1 ([150]). If for every differentiable solution w of the system (EES) all of its components w_j, $j = 1, \ldots, m$, are harmonic functions, then (EES) is called a generalized Cauchy–Riemann system.

The ellipticity of such an (EES) becomes an easy consequence [150].

Theorem 4.1. *Every generalized Cauchy–Riemann system of the form* (EES) *is elliptic, i.e.,*

$$\det\left(\sum_{j=0}^{n} \lambda_j M_j\right) \neq 0$$

for all $\lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

Indeed, if this is not so, then there would exist two vectors $\lambda \neq 0$ as above, and $\nu \neq 0$, $\nu \in \mathbb{C}^m$, such that

$$\left(\sum_{j=0}^{n} \lambda_j M_j\right)\nu = 0.$$

From that one can conclude by straightforward calculations that the vector-function

$$w(x) = \left(\exp \sum_{j=0}^{n} \lambda_j x_j\right)\nu$$

is a non-harmonic solution and therefore (EES) is not a generalized Cauchy–Riemann system.

Remark 4.1. If (EES) is elliptic, then every matrix M_j is necessarily invertible. By a left multiplication by M_0^{-1} the system (EES) is transformed to

$$E\,\frac{\partial w}{\partial x_0}+A_1\,\frac{\partial w}{\partial x_1}+\cdots+A_n\,\frac{\partial w}{\partial x_n}=0 \tag{4.1}$$

with $A_j=M_0^{-1}M_j$, $j=0,\dots,n$, $A_0=E=$ identity.

In the sequel we always consider generalized Cauchy–Riemann systems which are already of the form (4.1, which apparently does not cause the loss of generality. The following two theorems characterize GCRS by the properties of their coefficient matrices.

Theorem 4.2 ([151]). *Let the system* 4.1 *be a generalized Cauchy–Riemann system, then the coefficient matrices satisfy the relations*

$$\begin{gathered}A_j^2=-E,\quad j=1,\dots,n,\\ A_iA_j+A_jA_i=0,\quad i,j=1,\dots,n,\quad i\neq j.\end{gathered} \tag{4.2}$$

Proof. The proof is direct and instructive so we present it following [151]. Putting $A_0=E$ consider the function

$$w(x)=2x_ix_jb-(x_i^2A_i^{-1}A_j+x_j^2A_j^{-1}A_i)b,\quad i\neq j,$$

where b is an arbitrary vector from $\mathbb{C}^m$. Then w is a solution of 4.1 because we have

$$\begin{aligned}&E\,\tfrac{\partial w}{\partial x_0}+A_1\,\tfrac{\partial w}{\partial x_1}+\cdots+A_n\,\tfrac{\partial w}{\partial x_n}=A_i\,\tfrac{\partial w}{\partial x_i}+A_j\,\tfrac{\partial w}{\partial x_j}=\\ &=A_i(2x_j-2x_iA_i^{-1}A_j)b+A_j(2x_i-2x_jA_j^{-1}A_i)b=0\end{aligned}$$

Furthermore, one has

$$\Delta w=-2(A_i^{-1}A_j+A_j^{-1}A_i)b.$$

Assuming (4.1) to be a generalized Cauchy–Riemann system means that $\Delta w=0$ must hold. This and the fact that b can be chosen arbitrarily, yields

$$A_i^{-1}A_j+A_j^{-1}A_i=0,\quad i,j=0,\dots,n,\quad i\neq j. \tag{4.3}$$

Putting $i=0$ we obtain

$$A_j^{-1}=-A_j\quad\text{and}\quad A_j^2=-E\quad\text{for}\quad j=1,\dots,n.$$

This combined with 4.3 gives

$$A_iA_j+A_jA_i=0,\quad i,j=1,\dots,n,\quad i\neq j.$$

So the desired relations are verified. □

Theorem 4.3 ([151]). *Let w be a solution of the system* (4.1) *from the Sobolev space $W_2^1(G)$ and let the coefficient matrices of* (4.1) *satisfy the relations*

$$A_iA_j+A_jA_i=-2\delta_{ij}E,\quad i,j=1,\dots,n. \tag{4.4}$$

Then w is a harmonic vector in the domain G.

Proof. For a twice continuously differentiable solution w we have

$$\left(E\,\frac{\partial}{\partial x_0}-\sum_{j=1}^{n}A_j\,\frac{\partial}{\partial x_j}\right)\left(E\,\frac{\partial}{\partial x_0}-\sum_{j=1}^{n}A_j\,\frac{\partial}{\partial x_j}\right)w(x)=0.$$

A formal calculation using 4.4 shows that the second-order operator is just the Laplacian and so $w\in C^2(G)$ has to be harmonic. Now, assuming that $w\in W_2^1(G)$, the Weyl lemma [162] can be used in a standard way to complete the proof. □

Remark 4.2. The above algebraic relations are of course the the famous relations for the generators of a Clifford algebra Cl_n [62]. The generalized Cauchy–Riemann systems are often defined by postulating the algebraic relations 4.4 so they can be interpreted in the context of Clifford analysis [31]. Thus A_j are often consider not as matrices but as so-called ***hyper-complex units*** (generators of a Clifford algebra). Here we work exclusively with matrices because this permits to apply the same considerations to more general systems. Hyper-complex systems can of course be transformed into matrix notation by writing every hyper-complex component as a single equation. The dimension of the obtained coefficient matrices is always of power of two.

Thus in the sequel we deal with systems of the form

$$\begin{gathered}E\,\frac{\partial w}{\partial x_0}+A_1\,\frac{\partial w}{\partial x_1}+\cdots+A_n\,\frac{\partial w}{\partial x_n}=0,\\ A_iA_j+A_jA_j=-2\delta_{ij}E,\end{gathered}\tag{4.5}$$

where A_j are constant complex $(m\times m)$-matrices and w is a vector-function with values in $\mathbb{C}^m$.

In fact, by taking a closer look at the coefficient matrices one can specify all generalized Cauchy–Riemann systems explicitly. A natural way to do that is by referring to the representation theory of Clifford algebras [62]. Indeed, it is easy to see that such matrices A_i define a representation of a Clifford algebra Cl_n of an n-dimensional vector space V [62]. This interpretation of matrices A_i leads to an explicit description of their structure since representations of Clifford algebras are completely known.

We do not reproduce here the general definition of a Clifford algebra [62]. For our purposes it is sufficient to recall that by choosing an orthonormal basis e_i in V one can describe Cl_n as the associative algebra generated by

$$e_0=1,\quad e_1,e_2,\dots,e_n$$

with the following properties:

$$\begin{array}{ll}\text{(a)} & e_j^2=-1,\quad j=1,\dots,n,\\ \text{(b)} & e_ie_j=e_je_i=0,\quad i\neq j,\quad i,j=1,\dots,n,\\ \text{(c)} & (e_ie_j)e_k=e_i(e_je_k).\end{array}\tag{4.6}$$

Generators e_i are sometimes called *hyper-complex units* [62]. Obviously, each product of generators can be (up to a sign) transformed to one of the following expressions:

$$1,\ e_1,\dots,e_n,\ \ e_1e_2,e_1e_3,\dots,e_1e_2e_3,\dots,e_1e_2\cdots e_n. \tag{4.7}$$

These products are linearly independent and provide a basis for Cl_n and for this reason it is sometimes said that Cl_n is generated by hyper-complex units e_i.

Thus the matrices A_j, $j=1,\dots,n$, in the system 4.5, can be considered as a generating system of Cl_n because they satisfy the relations 4.6. Now we recall some notions and results from the theory of representations which can be found in [1], [62].

In order to formulate an explicit description of matrices A_i, recall that apart of the usual matrix sum and product two other operations over matrices are used in the theory of representations. We denote these operations by $\dot{+}$ and $\times$ respectively. Let A and B be two square matrices of dimension p and q respectively, then one defines:

$$A\dot{+}B=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix},\quad A\times B=\begin{pmatrix}a_{11}B & a_{12}B & \cdots & a_{1p}B\\ \multicolumn{4}{c}{\dotfill}\\ a_{p1}B & a_{p2}B & \cdots & a_{pp}B\end{pmatrix}.$$

The matrix $A\times B$ is called the *Kronecker product* of matrices A and B [1]. It is obtained by multiplying each element of A with each element of B. For the operations just defined the following computational rules are valid:

$$\begin{aligned}
&\text{(a)}\qquad (A_1\dot{+}A_2)+(B_1\dot{+}B_2)=(A_1+B_1)\dot{+}(A_2+B_2),\\
&\text{(b)}\qquad \lambda(A_1\dot{+}A_2)=\lambda A_1\dot{+}\lambda A_2,\\
&\text{(c)}\qquad (A_1\dot{+}A_2)(B_1\dot{+}B_2)=A_1B_1\dot{+}A_2B_2,\\
&\text{(d)}\qquad A\times(B_1+B_2)=A\times B_1+A\times B_2,\\
&\text{(e)}\qquad (A_1\times A_2)(B_1\times B_2)=A_1B_1\times A_2B_2.
\end{aligned}\tag{4.8}$$

For a finite dimensional vector space V denote by $L(V)$ the associative algebra of linear transformations of V. A *representation* of Cl_n is defined as any homomorphism of associative algebras $D: Cl_n\to L(V)$. After fixing a base in V each element $s\in Cl_n$ becomes represented by a matrix which we again denote by $D(s)$. Thus we do not distinguish between a representation and its matrix realization.

The dimension of the space V is called the *degree* of the representation. A representation is called *exact* if its kernel is trivial. Two representations $A(s)$ and $B(s)$ of the same degree are called *equivalent*, if there exists a non-singular $(m\times m)$ matrix P such that

$$A(s)=PE(s)P^{-1}\quad \forall s\in Cl_n.$$

In other words we are allowed to perform a coordinate transformation P in V under which the system of matrices $D(s)$ becomes $PD(s)P^{-1}$, which also defines a representation. Thus an equivalent system of matrices is nothing else than the same system of linear transformations expressed in another coordinate system. Representation D is said to be *reducible*, if there exists a proper subspace of V invariant under D, i.e., every vector of this subspace is mapped onto a vector of the same subspace by every linear transformation $D(s)$, for $s \in Cl_n$. Otherwise D is called *irreducible.*

Choosing the coordinate system suitably, a reducible representation can be brought to the form

$$D(s) = \begin{pmatrix} D_1(s) & K(s) \\ 0 & D_2(s) \end{pmatrix} .$$

If V is the direct sum of two invariant subspaces, then in an appropriate base we have

$$D(s) = \begin{pmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{pmatrix} = D_1(s)\dot{+}D_2(s).$$

$D_1(s)$ and $D_2(s)$ describe the transformations taking place in the both subspaces. We say that the system $D(s)$ *splits* .

A representation is called *completely reducible*, if it is irreducible or it splits into direct sum of several irreducible representations. As is well known that each representation of a Clifford algebra is completely reducible [62]. From these definitions it is clear that in order to completely determine a representation of Cl_n, it is sufficient to specify only the matrices representing the generators of Cl_n.

As is also well known, when considering the representations of Cl_n it is reasonable to distinguish two cases depending on whether n is even or odd.

Case (a): $n = 2k$

Define the following matrices:

$$\rho = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and for $j = 1, \dots, k$ put:

$$\begin{aligned} \rho_j &= i \cdot \tau \times \cdots \times \tau \times \rho \times \varepsilon \times \cdots \times \varepsilon, \\ \sigma_j &= i \cdot \tau \times \cdots \times \tau \times \sigma \times \varepsilon \times \cdots \times \varepsilon, \end{aligned} \tag{4.9}$$

where the factors ρ and σ appear at the jth position and the Kronecker product contains k factors. Using 4.8(e) one can easily prove that the $2k$ matrices 4.9 satisfy conditions 4.6. Therefore

$$e_{2j-1} \to \rho_j, \quad e_{2j} \to \sigma_j \tag{4.10}$$

is a representation of Cl_{2k} of degree 2^k. We have the following

Proposition 4.1. *The above representation of Cl_{2k} is irreducible and isomorphic to the matrix algebra M_{2^k}. Moreover, each irreducible representation of C_{2k} is equivalent to irreducible representation* (4.10) *so it is also* *isomorphic to M_{2^k}. Every representation of Cl_{2k} is exact and its degree is a multiple of 2^k.*

Case (b): $n = 2k+1$

Put

$$\tau_0 = i \cdot \tau \times \tau \times \cdots \times \tau \quad (k \text{ factors}),$$

and define the matrices representing generators by setting:

$$e_{2j-1} \to \rho_j, \quad e_{2j} \to \sigma_j, \quad w_n \to \tau_0, \quad j = 1, \ldots, k, \tag{4.11}$$

and

$$e_{2j-1} \to -\rho_j, \quad e_{2j} \to -\sigma_j, \quad w_n \to -\tau_0, \ j = 1, \ldots, k. \tag{4.12}$$

It is straightforward to see that both these assignments define representations of Cl_{2k+1}.

Proposition 4.2. *The above representations of Cl_{2k+1} are both irreducible and non-equivalent. Both the irreducible representations* (4.11) *and* (4.12) *are of degree 2^k, and Cl_{2k+1} is isomorphic to the direct sum of two matrix rings M_{2^k}. A representation of Cl_{2k+1} is exact if and only if its decomposition into a sum of irreducible representations contains both* (4.11) *and* (4.12) *at least once. The degree of each representation is a multiple of 2^k.*

Let us also explain how the matrices ρ_j, σ_j and τ_0 may be constructed recursively. Let ρ_j', σ_j' and τ_0' be the representation matrices of degree 2^{k-1}, then for matrices ρ_j, σ_j and τ_0 of degree 2^k, one has:

$$\begin{gathered} \rho_1 = \rho_1' \times \varepsilon = \begin{pmatrix} 0 & iE \\ iE & 0 \end{pmatrix}, \quad \rho_{j+1} = \tau \times \rho_j' = \begin{pmatrix} \rho_j' & 0 \\ 0 & -\rho_j' \end{pmatrix}, \\ \sigma_1 = \sigma_1' \times \varepsilon = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \quad \sigma_{j+1} = \tau \times \sigma_j' = \begin{pmatrix} \sigma_j' & 0 \\ 0 & -\sigma_j' \end{pmatrix}, \\ \tau_0 = \tau \times \tau_0' = \begin{pmatrix} \tau_o' & 0 \\ 0 & -\tau_0' \end{pmatrix}, \quad j = 1, \ldots, k-1. \end{gathered} \tag{4.13}$$

We note that

$$\rho_j^* = -\rho_j, \quad \sigma_j^* = -\sigma_j, \quad \tau_0^* = -\tau_0, \quad j = -1, \ldots, k, \tag{4.14}$$

i.e., all matrices are simultaneously unitary and skew-hermitian. The proof can easily be performed by induction over k using the recurrent representations (4.13).

At this point there naturally arises a question what is a way to find out how many times each of irreducible representations (4.11) and (4.12) is contained in any given representation of the Clifford algebra Cl_{2k+1} (the order of the irreducible representations in the direct sum is of course unimportant because a change of the order corresponds to an equivalent representation). An answer to this question will enable us to determine below the discrete invariants of a given GCRS. The solution can be given in terms of characters of representations [1]. Recall that the character of representation D is defined as the function

$$\chi(s) = \operatorname{trace} D(s).$$

As is well known, characters enable one to distinguish between non-equivalent representations [1]. Let us use characters to determine the number of irreducible components into which a representation of Cl_{2k+1} can be splitted.

Note first that the base element $e_1e_2\cdots e_n$ is mapped through (4.11) to the matrix

$$\rho_1\sigma_1\rho_2\sigma_2\cdots\rho_k\sigma_k\tau_0 = i^{k+1}E$$

and through (4.12) to the matrix

$$-\rho_1\sigma_1\rho_2\sigma_2\cdots\rho_k\sigma_k\tau_0 = -i^{k+1}E.$$

In the first case the trace equals $i^{k+1}\cdot 2^k$ and in the second $-i^{k+1}\cdot 2^k$. Now let D be a representation of C_{2k+1} of degree $m = 2^k \cdot l$. The representation generated by (4.11) may appear in $D(s)$ l_1-times and the one generated by (4.12) l_2-times. Then the following relations are valid for the unknown l_1 and l_2:

$$l = l_1 + l_2,$$
$$\operatorname{trace} D(e_1e_2\cdots e_n) = (l_1 - l_2)\cdot i^{k+1}\cdot 2^k.$$

From this we can directly obtain l_1 and l_2. Hence we see it is sufficient to compute the trace of a single matrix to identify the representation D of degree $2^k \cdot l$ of Cl_{2k+1}.

Thus we have specified all possible representations of the Clifford algebra Cl_n and the way to their identification. Therefore we now know in principle the explicit shapes of all generalized Cauchy–Riemann systems.

Before passing to boundary value problems let us state another characteristic property of generalized Cauchy–Riemann systems. About the classical two-dimensional Cauchy-Riemann system we know that it is invariant with respect to a rotation of the coordinate system, i.e., if $D = (d_{ij})$ is a proper orthogonal matrix and $w(x)$ represents a holomorphic function then $w(Dx)$ is also holomorphic. Generalized Cauchy-Riemann systems possess a similar property. If $w(x)$ is a solution and D an orthogonal transformation, then $Mw(Dx)$ is also a solution, where M is a certain non-singular matrix depending on D. This can be seen in the following way.

Let w be a solution of the system (4.5), which is defined in the whole space $\mathbb{R}^{n+1}$, i.e.,

$$\sum_{j=0}^{n} A_j\,\frac{\partial w(x)}{\partial x_j} = 0 \quad\text{with}\quad A_0 = E.$$

If $D = (d_{ij})$ is a proper orthogonal matrix, then for $y = Dx$ we have

$$0 = \sum_{j=0}^{n} A_j \frac{\partial w(y)}{\partial y_j} = \sum_{j,k} A_j \frac{\partial w(y)}{\partial x_k} d_{jk} = \sum_k \Big(\sum_j d_{jk} A_j \Big) \frac{\partial w(y)}{\partial x_k}$$

with $d_{jk} = \frac{\partial x_k}{\partial y_j}$.

Defining

$$\bar{A}_j = \begin{cases} E, & j = 0, \\ -A_j, & j = 1, \dots, n, \end{cases}$$

and multiplying the system of differential equations by $\sum\limits_j d_{j0} \bar{A}_j$ we obtain

$$\sum_{k=0}^{n} \widetilde{A}_k \frac{\partial w(Dx)}{\partial x_k} = 0$$

with

$$\widetilde{A}_k = \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big(\sum_j d_{jk} A_j \Big).$$

It is

$$\bar{A}_i A_j + \bar{A}_j A_i = 2\delta_{ij} E, \quad i,j = 0, 1, \dots, n, \tag{4.15}$$

and hence $\widetilde{A}_0 = E$. Let us show that matrices $\widetilde{A}_k$, $k = 1, \dots, n$, represent a generating system of the Clifford algebra C_n:

$$\begin{aligned} \widetilde{A}_p \widetilde{A}_q + \widetilde{A}_q \widetilde{A}_p &= \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big[\Big(\sum_j d_{jp} A_j \Big) \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big(\sum_j d_{jq} A_j \Big) + \\ &\quad + \Big(\sum_j d_{jq} A_j \Big) \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big(\sum_j d_{jp} A_j \Big) \Big] = \\ &= \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big[\sum_{i,j,k} d_{jp} d_{j0} d_{kq} (A_i \bar{A}_j A_k + A_k \bar{A}_j A_i) \Big]. \end{aligned}$$

Taking into account that

$$A_i \bar{A}_j A_k + A_k \bar{A}_j A_i = \begin{cases} 2A_k, & i = j, \\ 2A_i, & j = k, \\ 2A_j, & i = k,\, i \neq j, \\ 0, & \text{otherwise}, \end{cases}$$

we can conclude that

$$\begin{aligned} &\widetilde{A}_p \widetilde{A}_q + \widetilde{A}_q \widetilde{A}_p = \\ &= 2 \cdot \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big(\sum_{i,k} d_{ip} d_{i0} d_{kq} A_k + \sum_{\substack{i,j \\ i \neq j}} d_{ip} d_{j0} d_{jp} A_i - \sum_{\substack{i,j \\ i \neq j}} d_{ip} d_{j0} d_{iq} A_j \Big) = \\ &= 2 \cdot \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big(- \sum_j d_{ip} d_{j0} d_{jq} A_k - \sum_j d_{j0} \big(d_{pq} - d_{jp} d_{jq} \big) A_j \Big) = \end{aligned}$$

$$= -2 \cdot \Big(\sum_j d_{j0} \bar{A}_j \Big) \Big(\sum_j d_{j0} A_j \Big) \cdot d_{pq} = -2\delta_{pq} E.$$

Notice that only the orthogonality of the matrix (d_{ij}) was used in these transformations.

If n is even, we already obtain the desired result. Since in this case there exists only one representation of the algebra Cl_n of a given degree up to equivalence, there exists a non-singular matrix M such that

$$\widetilde{A}_k = M^{-1} A_k M, \quad k = 0, 1, \dots, n.$$

So we get

$$0 = \sum_{k=0}^{n} \widetilde{A}_k \frac{\partial w(Dx)}{\partial x_k} = M^{-1} \cdot \sum_{k=0}^{n} A_k \frac{\partial M\, w(Dx)}{\partial x_k}$$

and therefore

$$\sum_{k=0}^{n} A_k \frac{\partial M\, w(Dx)}{\partial x_k} = 0.$$

This means that $M\, w(Dx)$ is a solution of the same generalized Cauchy–Riemann system.

If n is odd, the equivalence of the two representations A_j and $\widetilde{A}_j$ can be proved using characters. By a direct computation one can verify that

$$\widetilde{A}_1 \widetilde{A}_2 \cdots \widetilde{A}_n = A_1 A_2 \cdots A_n.$$

Hence we get

$$\operatorname{trace}\big(\widetilde{A}_1 \widetilde{A}_2 \cdots \widetilde{A}_n\big) = \operatorname{trace}\big(A_1 A_2 \cdots A_n\big),$$

which implies the equivalence of both representations. Now the proof can be completed in the same way as in the case of even n.

Remark 4.3. E. M. Stein and G. Weiss defined generalized Cauchy–Riemann systems by the property of rotational invariance [150]. In this way they arrived to the same systems as presented here.

The development of functional-analytic methods for elliptic first order systems has shown that a big part of the classical two-dimensional theory can be generalized to higher-dimensional GCRS [31], [62]. Unfortunately this is no longer the case if one tries do study boundary value problems for such systems. In principle the Plemelj-Sokhotsky formulas can be used to transform boundary value problems into systems of singular integral equations but this does not lead to any effective criteria of solvability or Fredholmness.

Recently I.Stern [151] succeeded to obtain an explicit criterion of Fredholmness using some general results of the theory of elliptic boundary value problems in Sobolev spaces [162]. This enabled her to obtain an extensive list of GCRS possessing elliptic boundary value problems of Riemann–Hilbert type [152]. Recall that the problem of deciding which GCRS possess elliptic local boundary value problems attracted considerable attention in last three decades [17], [9]. The progress achieved by I.Stern enabled the present author to guess the topological mechanism which caused the existence of elliptic Riemann–Hilbert problems, which eventually led to further progress in this topic [90].

We present below the main results of [152] and [90] and outline their proofs. In those papers were basically studied boundary value problems which were slightly different from the linear conjugation problems which are in the focus of interest in the present paper (however they were still called Riemann–Hilbert problems). As was explained in [90], the linear conjugation problems for GCRS appear as a particular case of the boundary problems studied in [152]. Actually, the question about the existence of Fredholm boundary problems is much more interesting for general boundary value problems of Riemann–Hilbert type so we accept below the setting of [152], [90].

Let us now describe the boundary value problems we wish to deal with. To this end let us consider again a generalized Cauchy–Riemann system

$$E\,\frac{\partial w}{\partial x_0}+A_1\,\frac{\partial w}{\partial x_1}+\cdots+A_n\,\frac{\partial w}{\partial x_n}+Dw=f \tag{4.16}$$

in a domain G with complex $(m\times m)$-coefficient matrices and the A_j, $j=1,\ldots,n$, satisfying the relations

$$A_iA_j+A_jA_i=-2\delta_{ij}E,\quad i,j=1,\ldots,n.$$

Notice that in contrast to Definition 4.1 we have added a lower order term with a view to greater generality. This is reasonable because the function-analytic methods used in the sequel are related essentially with the principal part of the operator. From now on we consider the inhomogeneous system of differential equations because it is intrinsically involved in the definition of Fredholm property.

Together with the system of differential equations we investigate a boundary value problem arising by imposing boundary conditions of the form

$$(B_1B_2)\cdot w=g \quad \text{on} \quad \partial G, \tag{4.17}$$

where B_1 and B_2 are complex matrices of dimension $m/2$ depending on the boundary points, and g is a vector function with values in $\mathbb{C}^{m/2}$. This always makes sense since, as we have seen m is always even for a GCRS.

In line with terminology of [151], [152], and [90], the problem (4.16, 4.17) is called a Riemann–Hilbert problem for a given GCRS (4.16).

Notice that an equivalent problem could be formulated in the form:

$$\mathrm{Re}[Cw]=h \quad \text{on} \quad \partial G. \tag{4.18}$$

Here C denotes a complex $(m\times m)$-matrix, and h is a vector with values in $\mathbb{R}^m$. As is easy to see, both definitions of boundary value problem may be transformed to each other.

We will always assume that the rows of the matrix $(B_1(x)\,B_2(x))$ are linearly independent in every boundary point $x\in\partial G$. Then they can be orthonormalized by a formal application of the Schmidt algorithm, which does not change regularity (continuity, smoothness) of the matrix elements. Therefore we can always consider boundary conditions with orthonormal rows.

The following theorem gives an explicit criterion of Fredholmness for Riemann–Hilbert problems.

Theorem 4.4 ([152]). *Let $G \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain of class C^k, $k \geq 1$. Consider the Riemann–Hilbert boundary value problem*

$$E\,\frac{\partial w}{\partial x_0} + A_1\,\frac{\partial w}{\partial x_1} + \cdots + A_n\,\frac{\partial w}{\partial x_n} + Dw = f \quad \text{in} \quad G, \tag{4.19}$$

$$A_iA_j + A_jA_i = -2\delta_{ij}E, \quad i,j = 1,\ldots,n,$$

$$(B_1, B_2)\cdot w = g \quad \text{on} \quad \partial G, \tag{4.20}$$

and assume the $(m \times m)$-matrices A_j to be unitary:

$$A_j^* = -A_j, \quad j = 1,\ldots,n.$$

Let the rows of the $\frac{m}{2} \times m)$-matrix (B_1, B_2) be orthonormal. Furthermore, assume that $D \in C^{k-1}(\overline{G})$ and $B_1, B_2 \in C^k(\partial G)$.

Then the Riemann—Hilbert boundary value problem (4.19), (4.20) *is Fredholm for $w \in W_2^l(G)$, $f \in W_2^{l-1}(G)$, and $g \in W_2^{l-1/2}(\partial G)$, $1 \leq l \leq k$, $l \in \mathbb{Z}$, if and only if the following relation is valid for all $x \in \partial G$:*

$$\det\left[(B_1B_2)\Big(\sum_{j=0}^{n}\alpha_jA_j^*\Big)\Big(\sum_{j=0}^{n}t_jA_j\Big)\begin{pmatrix}B_1^*\\ B_2^*\end{pmatrix} - iE\right] \neq 0. \tag{4.21}$$

Here $\alpha = (\alpha_0,\ldots,\alpha_n)$ denotes the vector of the inner normal at point $x \in \partial G$ and $t = (t_0,\ldots,t_n)$ runs through all unit vectors which are tangent to ∂G at the same point x.

The proof makes use of the following lemma which was also established in [152]. For completeness we also include the proof of the lemma.

Lemma 4.1. *Let $(m \times m)$-matrices A_j, $j = 1,\ldots,n$, satisfy*

$$A_iA_j + A_jA_i = -2\delta_{ij}E, \quad i,j = 1,\ldots,n.$$

Then the matrix

$$i\cdot\sum_{j=1}^{n}\xi_jA_j \quad \text{with} \quad \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \setminus \{0\}$$

has exactly the eigenvalues $\lambda = \pm|\xi|$, and all eigenvectors with the eigenvalue $\pm|\xi|$ have the form

$$\nu = \Big(\pm|\xi| + i\sum_{j=1}^{n}\xi_jA_j\Big)\cdot c \quad \text{with} \quad c \in \mathbb{C}^m.$$

If c runs through all vectors in $\mathbb{C}^m$, then ν is either an eigenvector or it is the zero vector. For every eigenvalue there are exactly $m/2$ linearly independent eigenvectors.

Proof. Let

$$\det\left(i\sum_{j=1}^{n}\xi_j A_j - \lambda E\right) = 0,$$

then we have $\lambda = \pm|\xi|$. Indeed, it is

$$\left(i\sum_{j=1}^{n}\xi_j A_j + \lambda E\right)\left(i\sum_{j=1}^{n}\xi_j A_j - \lambda E\right) = \left(|\xi|^2 - \lambda^2\right)\cdot E$$

and the propositions is obvious.

Now we show that the matrices $(i\sum \xi_j A_j \pm |\xi|E)$ have the rank $\geq m/2$. By an equivalence transformation the matrices A_j take the form indicated above, i.e., the matrices $(i\sum \xi_j A_j \pm |\xi|E)$ are equivalent to matrices of block structure, where every block is of the form

$$\begin{pmatrix} \pm|\xi|E + i\cdot\sum\limits_{j=3}^{n}\xi_j A'_{j-2} & -(\xi_1 + i\xi_2)E \\ (-\xi_1 + i\xi_2)E & \pm|\xi|E - i\cdot\sum\limits_{j=3}^{n}\xi_j A'_{j-2} \end{pmatrix}$$

and the dimension 2^k ($m = 2^k\cdot l$, $n = 2k$ or $2k+1$). Obviously, the 2^{k-1}-dimensional submatrices $(-\xi_1 \pm i\xi_2)E$ are of maximal rank, if $\xi_1^2 + \xi_2^2 \neq 0$. Otherwise, i.e., if $\xi_1 = \xi_2 = 0$, we have to demonstrate that the other two submatrices both have the rank $\geq 2^{k-2}$. We restrict ourselves to the left upper submatrix, because the other is constructed quite analogously. Again using the recursive representation 4.13 gives us

$$\pm|\xi|E + i\cdot\sum_{j=3}^{n}\xi_j A'_{j-2} = \begin{pmatrix} \pm|\xi|E + i\cdot\sum\limits_{j=3}^{n}\xi_j A''_{j-4} & -(\xi_3 + i\xi_4)E \\ (-\xi_3 + i\xi_4)E & \pm|\xi|E - i\cdot\sum\limits_{j=3}^{n}\xi_j A''_{j-4} \end{pmatrix}.$$

If $\xi_3^2 + \xi_4^2 \neq 0$, then we see that the rank of this matrix is not less than 2^{k-2}. In the case of $\xi_2 = \xi_4 = 0$ we split up the submatrices again and prove that they have the rank not less that one half of the dimension.

This procedure either leads us to a pair of numbers (ξ_{2j-1}, ξ_{2j}) with $\xi_{2j-1}^2 + \xi_{2j}^2 \neq 0$, where the proposition is obvious, or finally we obtain the 2×2-matrices

$$\begin{pmatrix} \pm|\xi| & -\xi_{n-1} - i\xi_n \\ -\xi_{n-1} + i\xi_n & \pm|\xi| \end{pmatrix} \quad \text{for } n \text{ even}$$

and

$$\begin{pmatrix} \pm|\xi| - \xi_n & -\xi_{n-2} - i\xi_{n-1} \\ -\xi_{n-2} + i\xi_{n-1} & \pm|\xi| + \xi_n \end{pmatrix} \quad \text{for } n \text{ odd}$$

whose rank equals 1.

By this inductive procedure it follows that the block matrices of dimension 2^k have the rank $\geq 2^{k-1}$. So the whole matrix has the rank not less than 2^{l-1}, where $l = m/2$. Since

$$\Big(i\sum \xi_j A_j \mp |\xi|E\Big)\cdot\Big(i\sum \xi_j A_j \pm |\xi|E\Big)\cdot c = 0, \quad \forall c \in \mathbb{C}^m,$$

the vectors $(i\sum \xi_j A_j \pm |\xi|E)c \neq 0$ are eigenvectors of the eigenvalue $\lambda = \pm|\xi|$. So the rank of the matrices $(i\sum \xi_j A_j \pm |\xi|E)$ must be equal to $m/2$, because the number of linearly independent eigenvectors cannot exceed m. Furthermore, there cannot exist any other eigenvectors besides the ones indicated above. This obviously completes the proof of lemma. □

Proof of Theorem 4.4. As is well known, the fredholmness is equivalent to the ellipticity of the boundary value problem [162]. The ellipticity of the differential operator in question was already established above. So it remains to prove that the relation (4.21) is equivalent to the Shapiro-Lopatinski condition.

Let $x^0 \in \partial G$ be any boundary point. We choose a local coordinate system $X_1, X_1 \dots, X_n$ in the following way:

1. The origin lays in x^0.
2. The X_0-axis coincides with the direction of the inner normal.
3. The $X_1, \dots, X_n$-axis lay in the tangential plane.
4. The local $X_1, X_1, \dots, X_n$-system is obtained from the global $x_1, x_1, \dots, x_n$-system by a translation and a rotation, i.e.,

$$x_i = \sum_{j=0}^{n} d_{ij} X_j + x_i^0, \quad X_j = \sum_{j=0}^{n} d_{ij}(x_i - x_i^0),$$

with a proper orthogonal matrix $(d_{ij})_{i,j=0}^n$.

Therefore we have

$$\frac{\partial}{\partial x_i} = \sum_{j=0}^{n} d_{ij}\,\frac{\partial}{\partial X_j}$$

and

$$\sum_{i=0}^{n} A_i \frac{\partial w}{\partial x_i} = \sum_{i,j} A_i d_{ij} \frac{\partial w}{\partial X_j} = \sum_{j=0}^{n}\Big(\sum_{i=0}^{n} d_{ij} A_i\Big)\frac{\partial w}{\partial X_j}\,.$$

Because of the rotational invariance of the system of differential equations which was established above, by putting

$$\widetilde{A}_j = \Big(\sum_i d_{i0} A_i^*\Big)\Big(\sum_i d_{ij} A_i\Big)$$

we obtain a new generalized Cauchy–Riemann system

$$\sum_{j=0}^{n} \widetilde{A}_j \frac{\partial w}{\partial X_j} + \Big(\sum_i d_{i0} A_i^*\Big) Dw = \Big(\sum_i d_{i0} A_i^*\Big) f.$$

The matrices $\widetilde{A}_j$ are also unitary: $\widetilde{A}_j^* = -\widetilde{A}_j$, $j = 1, \dots, n$.

The principal part of this differential operator is

$$\sum_{j=0}^{n} \widetilde{A}_j \frac{\partial w}{\partial X_j}.$$

Let us now apply to it the Fourier transform with respect to the tangential coordinates $X_1, \ldots, X_n$. Putting $X_0 = t$, we have the corresponding homogeneous system of ordinary differential equations for $t \geq 0$:

$$\frac{d\nu(t)}{dt} + i \cdot \sum_{j=1}^{n} \xi_j \widetilde{A}_j \nu(t) = 0.$$

Freezing the coefficients of the boundary condition at the origin of the local coordinate system, the homogeneous boundary condition is transferred by the above Fourier transform into the homogeneous initial value condition

$$(B_1 B_2)\nu(0) = 0, \quad B_i = B_i(0), \quad i = 1, 2.$$

The Shapiro-Lopatinski condition at boundary point $x^0 \in \partial G$ can be formulated as the requirement that the initial value problem has only the trivial solution in the space of stable solutions for all $\xi = (\xi_1, \ldots, \xi_n) \neq 0$.

Making the substitution

$$\nu(t) = e^{\lambda t} \cdot c$$

we have

$$\left(\lambda E + i \cdot \sum_{j=1}^{n} \xi_j \widetilde{A}_j\right) \cdot c = 0.$$

In virtue of Lemma 4.1 there are exactly the two eigenvalues $\lambda = \pm|\xi|$. Only for $\lambda = -|\xi|$ do we get stable solutions the corresponding $m/2$ linearly independent eigenvalues are of the form

$$c = \left(|\xi| E + i \cdot \sum_{j=1}^{n} \xi_j \widetilde{A}_j\right) \cdot c', \quad c' \in \mathbb{C}^m.$$

So the space of stable solutions of the system of ordinary differential equations is constitutes by all function vectors

$$\nu(t) = \exp(-|\xi| t) \cdot \left(|\xi| E + i \cdot \sum_{j=1}^{n} \xi_j \widetilde{A}_j\right) \cdot c', \quad c' \in \mathbb{C}^m.$$

The matrix $i \cdot \sum \xi_j \widetilde{A}_j$ is hermitean, because the $\widetilde{A}_j$ are unitary. Hence we can perform a diagonal transformation

$$i \cdot \sum_{j=1}^{n} \xi_j \widetilde{A}_j = |\xi| P \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} P^*, \quad P = P(\xi, (d_{ij})),$$

with a unitary matrix P. We obtain

$$\nu(t) = 2|\xi| \exp(-|\xi| t) P \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} P^* c'$$

and by substituting

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \quad \dim P_j = m/2,$$

and

$$P^* c' = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x, y \in \mathbb{C}^{m/2},$$

we get

$$\nu(t) = 2|\xi| \exp(-|\xi| t) \begin{pmatrix} P_1 \\ P_3 \end{pmatrix} x, \quad x \in \mathbb{C}^{m/2} \text{ arbitrary},$$

for the stable solutions. The initial value condition is stated as

$$(B_1 B_2)\nu(0) \equiv 2|\xi|(B_1 B_2) \begin{pmatrix} P_1 \\ P_3 \end{pmatrix} x = 0.$$

Obviously, there is $\nu(t) \equiv 0$ if and only if $x = 0$. So demanding that the initial value problem has got only the trivial solution is equivalent to the condition

$$\det \left[(B_1 B_2) \begin{pmatrix} P_1 \\ P_3 \end{pmatrix} \right] \neq 0. \tag{4.22}$$

Put for short $(Q_1 Q_2) := (B_1 B_2) P$. Then (4.22) means

$$\det Q_1 \neq 0.$$

This is equivalent to

$$\det Q_1 A_1^* \neq 0.$$

Using the equation

$$Q_1 Q_1^* + Q_2 Q_2^* = E,$$

we have

$$\begin{aligned} 2 Q_1 Q_1^* &= (Q_1 Q_2) \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} Q_1^* \\ Q_2^* \end{pmatrix} + E = \\ &= (B_1 B_2) \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} P^* \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} + E = \\ &= (B_1 B_2) \frac{i}{|\xi|} \sum_{j=1}^{n} \xi_j \widetilde{A}_j \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} + E. \end{aligned}$$

Furthermore, there is

$$\sum_{j=1}^{n} \frac{\xi_j}{|\xi|} \widetilde{A}_j = \Big(\sum_{i=0}^{n} d_{i0} A_i^* \Big) \Big(\sum_{i,j} \frac{\xi_j}{|\xi|} d_{ij} A_i \Big).$$

The vector $(d_{i0})_{i=0}^{n}$ describes the inner normal $(\alpha_i)_{i=0}^{n}$ and the vector

$$\left(\sum_{j=1}^{n} \frac{\xi_j}{|\xi|} d_{ij} \right)_{i=0}^{n}$$

is a tangential vector $(t_i)_{i=0}^n$ of unit length. Therefore we write

$$\sum_{j=1}^{n} \frac{\xi_j}{|\xi|} \widetilde{A}_j = \Big(\sum_{j=0}^{n} \alpha_j A_j^* \Big) \Big(\sum_{j=0}^{n} t_j A_j \Big)$$

and obtain the relation equivalent to (4.22)

$$\det \left[(B_1 B_2) \Big(\sum_{j=0}^{n} \alpha_j A_j^* \Big) \Big(\sum_{j=0}^{n} t_j A_j \Big) \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} - iE \right] \neq 0,$$

which completes the proof of the theorem. □

The above results about the structure of GCRS and explicit criterion of fredholmness enabled I.Stern to show that certain GCRS do not possess any elliptic boundary value problems of Riemann–Hilbert type. We present here the main result of [152] without proof which is quite technical and lengthy. Actually, below we present a more precise result from [90] which is more relevant for describing the state of the art in this topic.

However we would like to add that up to our mind an amazing feature of the proof from [152] was that, using the above criterion of fredholmness, the desired conclusion was derived from a famous result of of J.Adams, P.Lax, and R.Phillips about the amount of matrices of given size all of whose real combinations are non-singular [2].

Theorem 4.5. *If the space dimension $n+1$ is even and the representation of Cl_n defined by the coefficient matrices A_i contains an odd number of irreducible components (i.e., its degree is of the form $m = 2^k l$ with l odd), then the corresponding GCRS does not possess any elliptic Riemann–Hilbert problems.*

I.Stern also proved that in $\mathbb{R}^4$ there exist no elliptic Riemann–Hilbert problems for GCRSs which contain only one of the two possible irreducible representations. On the other hand, it was shown in [152] that for an odd n elliptic Riemann–Hilbert problems exist if both irreducible representations appear in the same amount. It should be noted that these results covered all results of this type which were previously proved for various concrete systems (a review of those concrete results is contained in [152]).

Analyzing these results of I.Stern the present author noticed that they find a nice explanation in terms in the modern approach of K-homology developed by P.Baum, R.Douglas, and M.Taylor [10]. More precisely, it turned out that for GCRS one can compute the K-homological obstruction to the existence of elliptic boundary value problems suggested in [10] and determine all cases when it is vanishing. Combining this fact with some recent results of G.Gong [71] it became possible to show that these, and only these, GCRS possess elliptic Riemann–Hilbert problems.

The main results of [90] can be conveniently formulated in terms of the discrete invariants of GCRS which are yielded by a Clifford algebra interpretation described above. Recall that each such system is characterized by natural numbers n and m. One can equivalently substitute m by the number l of irreducible components in the associated representation of Clifford algebra Cl_n. For an odd $n = 2k+1$, there also appear multiplicities l_1, l_2 of each of the two irreducible representations of Cl_n (i.e., $m = 2^k l, l = l_1 + l_2$).

Theorem 4.6 ([90]). *If n is odd and $l_1 \neq l_2$ then there do not exist elliptic Riemann–Hilbert problems for a given GCRS. If $l_1 = l_2$ then there exist elliptic Riemann–Hilbert problems for the given system.*

Theorem 4.7 ([84]). *If n is even, $n \neq 4, 6$, then there exist elliptic boundary problems with a boundary condition of the form $(B \circ r)w = g$, where B is a zero order pseudo-differential operator between sections of appropriate Hermitian bundles on the boundary, and r is the usual trace (restriction to the boundary) map.*

It should be added that for even n we are not yet able to show the existence of elliptic Riemann–Hilbert problems as they were defined above. Besides being interesting by its own, this issue is important for our approach to *hyper-holomorphic cells* discussed in the last section. In fact, this can be shown for many concrete systems (in particular, for "self-conjugate systems" appearing in the Section 6) so the author's feeling is that elliptic Riemann—Hilbert problems exist for all even n.

As to linear conjugation problems for GCRS, here situation is much simpler and this can be established using the same ideas and techniques.

Theorem 4.8. *Elliptic linear conjugation problems exist for all generalized Cauchy-Riemann systems.*

We do not attempt to describe proofs of these results because they use rather delicate topological concepts and technical tools. Let us only mention that the existence of elliptic linear conjugation problems actually follows from the same considerations.

For example in the case when the domain is the unit ball $B \subset \mathbb{R}^{n+1}$, one just transforms a linear conjugation problem to a Riemann–Hilbert problem for a GCRS of double size. This can be done by means of substituting the outer component X_- by the vector function Y_+ which is the inversion of X_- in the unit sphere ∂B. Then it is easy to check that the new unknown vector (X_+, Y_+) satisfies a GCRS of the double size in the ball B. Since the new system is of a very special kind, it becomes possible to show that the Baum-Douglas obstruction vanishes for this system, hence it possesses elliptic Riemann–Hilbert problems which can be transformed backwards to give elliptic linear conjugation problems for the original GCRS.

5. Nonlinear Riemann–Hilbert Problems for Analytic Functions

In course of a long historical development, the Riemann–Hilbert problem became an "organizing center" for a number of important topics of complex analysis, differential equations, topology, operator theory, and nonlinear analysis. Most of these topics are developing quite actively and continue suggesting new interesting problems and interrelations. In particular, there exists vast literature devoted to nonlinear versions of Riemann–Hilbert problem (see, e.g., [159]).

In this section we discuss some nonlinear problems of such kind closely related to the concept of *analytic disc* which plays significant role in various modern topics of complex analysis and symplectic geometry [16], [54], [55], [72].

In light of our discussion the problem we wish to study may be considered as a direct generalization of a transmission problem (1.1). One looks again for two functions Φ_- and Φ_+ which are holomorphic in an interior and an exterior domain, respectively, but now they should satisfy a *nonlinear* coupling condition on the common boundary Γ of the two domains. More precisely, we admit nonlinear conditions of the form

$$\Phi_+(t) = G(t, \Phi_-(t)), \forall t \in \Gamma. \tag{5.1}$$

From the operator theoretic point of view, the linear transmission problem is related to Töplitz operators, i.e., to the interaction of *multiplication* with the Riesz projection, while the nonlinear problem concerns interaction of the Riesz projection with *superposition*. The operator theory approach appears helpful also for dealing with nonlinear transmission problems.

Following [160] we introduce a special class of nonlinear transmission problems and obtain a rather complete description of their solutions. Moreover, we discuss some relations between nonlinear transmission problems and the existence problem for so-called *attached analytic discs* [16]. Since this subject is a relatively recent one, we only present some sample results without trying to reach maximal possible generality.

More precisely, for a given continuously differentiable function $G\colon\ \mathbb{T} \times \mathbb{C} \to \mathbb{C}$, we consider a nonlinear transmission problem

$$\Phi_+(t) = G(t, \Phi_-(t)), \qquad \forall t \in \mathbb{T}. \tag{5.2}$$

It is supposed that the unknown functions Φ_+ and Φ_- extend holomorphically from the complex unit circle $\mathbb{T}$ into its interior $\mathbb{D}$ and its exterior $\mathbb{E}$, respectively, and that Φ_- vanishes at infinity.

If G is linear in z and $\overline{z}$, $G(.,z) = g_0 + g_1 z + g_2 \overline{z}$, we get a linear transmission problem with conjugation [61], [112]. The "holomorphic case" $\overline{\partial}_z G \equiv 0$ was studied by L. von Wolfersdorf [165].

The nonlinear problem 5.2 is said to be *elliptic* if

$$|\overline{\partial}_z G(t,z)| < |\partial_z G(t,z)|, \qquad \forall (t,z) \in \mathbb{T} \times \mathbb{C}. \tag{5.3}$$

Another case of particular interest corresponds to real-valued G, pertaining to the "parabolic case", since then $|\partial_z G| = |\overline{\partial}_z G|$. In this situation Φ_+ must be holomorphic in $\mathbb{D}$ and real-valued on $\mathbb{T}$ and hence (5.2) is equivalent to a scalar Riemann–Hilbert problem of the form

$$G(t, \Phi_-(t)) = const, \tag{5.4}$$

which is discussed in many places (see, e.g., [61], [112], [159]). It should be added that, in contrast to the general nonlinear transmission problem (5.2), there exists a rather complete geometric theory of Riemann–Hilbert problem (5.4) [159].

We say that (5.2) has a solution in W^1_r, if the functions Φ_- and Φ_+ have boundary functions in the Sobolev space $W^1_r(\mathbb{T})$. The following existence theorem which was established in [160]. Notice that it also covers the linear elliptic case with continuously differentiable coefficients and index zero.

Theorem 5.1. *Let $G\colon\ \mathbb{T}\times\mathbb{C}\to\mathbb{C}$ be continuously differentiable with uniformly bounded first derivatives.*

(i) *If there exist a positive constant δ and a smooth unimodular function $g\colon\ \mathbb{T}\to\mathbb{T}$ with winding number zero,* $\operatorname{wind} g=0$, *such that*

$$\big|\partial_z G(t,z)\big| - |\overline{\partial}_z G(t,z)| \ge \delta > 0 \quad \forall (t,z)\in\mathbb{T}\times\mathbb{C}, \tag{5.5}$$

$$\operatorname{Re}\,\big(g(t)\,\partial_z G(t,z)\big) \ge \delta > 0 \quad \forall (t,z)\in\mathbb{T}\times\mathbb{C}, \tag{5.6}$$

then the transmission problem (5.2) has a solution in W^1_r for each $r\in(1,\infty)$.

(ii) *The solution is unique if, in addition to the above assumptions,*

$$\operatorname{Re}\big(g(t)\,\partial_z G(t,z)\big) - |\overline{\partial}_z G(t,z)| \ge \delta > 0 \quad \forall (t,z)\in\mathbb{T}\times\mathbb{C}. \tag{5.7}$$

We reproduce here the proof from [160]. It is based on several observations. First of all, we remark that the function g in the condition 5.6 admits a factorization $g=g_H/g_R$, where g_R and g_H are smooth functions on $\mathbb{T}$, g_R is real and strictly positive and g_H extends to a holomorphic function in $\mathbb{D}$ without zeros. This allows to rewrite the boundary relation as

$$\widetilde{\Phi}_+ := g_H\cdot\Phi_+ = g_R\cdot g\cdot G(.,\Phi_-) =: \widetilde{G}(.,\Phi_-).$$

If G satisfies 5.5, 5.6 (and 5.7), then $\widetilde{G}$ satisfies the same conditions with $g\equiv 1$. Consequently we can assume that $g\equiv 1$.

The following constructions serve to transform the transmission problem 5.2 into a fixed-point equation for a compact operator K. The idea is to differentiate the boundary relation along $\mathbb{T}$ (cf. [166]), which gives rise to a quasi-linear transmission problem with conjugation. The main ingredient of the operator K is a primitive of an appropriate solution of this auxiliary problem.

Fix $s\in(1,\infty)$. For a given scalar complex valued function $\varphi\in W^1_s(\mathbb{T})$, we define

$$a(t) := \partial_z G\big(t,\varphi(t)\big), \quad b(t) := \overline{\partial}_z G\big(t,\varphi(t)\big), \quad c(t) := \mathrm{i}\,t\,\partial_t G\big(t,\varphi(t)\big), \tag{5.8}$$

where $\mathrm{i}\,t\,\partial_t \equiv \partial_\tau$ denotes the derivative with respect to the polar angle τ of $t\equiv \mathrm{e}^{\mathrm{i}\tau}\in\mathbb{T}$. Note that a, b, and c are continuous functions.

We denote by H^r_+ (resp. H^r_-) the Hardy spaces of functions φ which extend holomorphically into $\mathbb{D}$ (resp. in $\mathbb{E}$ with $\varphi(\infty)=0$), and let $H^r_\pm := H^r_+\times H^r_-$.

Lemma 5.1. *Let G be subject to the assumptions of Theorem* 5.1 *with* $g \equiv 1$, *fix* $r, s \in (1, \infty)$, *let* $\varphi \in W^1_s$, *and let* a, b, *and* c *be given by* (5.8).

(i) *For each* $\varphi \in W^1_s(\mathbb{T})$, *the linear transmission problem*

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c \tag{5.9}$$

has a unique solution $\widetilde{\Phi} := (\widetilde{\Phi}_+, \widetilde{\Phi}_-) \in H^r_\pm$.

(ii) *For each value of the constant* δ *in Theorem 5.1, there exists an* $r > 1$ *such that the* $H^r_\pm$*-norm of the solution* $\widetilde{\Phi} \equiv (\widetilde{\Phi}_+, \widetilde{\Phi}_-)$ *to* (5.9) *is bounded by a constant not depending on the choice of* φ.

Proof. 1. Existence and uniqueness of the solution follow, e.g., from [159].

2. In order to prove (ii), we derive a representation of the solutions which involves the inverses of a certain Toeplitz operator.

The function w defined on $\mathbb{T}$ by $w(t) := \big(\overline{\Phi_-(t)}/t, \Phi_+(t)\big)$ extends holomorphically into $\mathbb{D}$. With the definitions $f := -(\operatorname{Re} c, \operatorname{Im} c)$, and

$$A := \begin{bmatrix} \overline{a} + b & -1 \\ \mathrm{i}(\overline{a} - b) & \mathrm{i} \end{bmatrix} \cdot \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.10}$$

the problem 5.9 is equivalent to

$$Rw := \operatorname{Re} A w = f. \tag{5.11}$$

Let $P\colon\ L^r \to H^r_+$ denote the Riesz projection of $L^r(\mathbb{T})$ onto the Hardy space H^r_+ along H^r_-. We introduce the "adjoint Riemann–Hilbert operator"

$$S\colon\ L^r \to H^r_+,\ x \mapsto P\,\overline{t}\,\overline{A}^{-1}\operatorname{Re} x. \tag{5.12}$$

A straightforward verification shows that SR is a Toeplitz operator, $2\,SR = T := P\widetilde{B}P$. The symbol $\widetilde{B} := \overline{t}\,\overline{A}^{-1}A$ of T has the representation $\widetilde{B} = \frac{1}{a}\,J\,B$, where

$$J := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} |a|^2 - |b|^2 & \overline{bt} \\ -bt & 1 \end{bmatrix}. \tag{5.13}$$

Since $|a| > |b|$ and $\operatorname{wind} a = 0$, the Toeplitz operator T is invertible, which implies that the solution of 5.11 admits the representation

$$w = 2\,T^{-1}\,S\,f. \tag{5.14}$$

3. Remember that S and T depend on the choice of φ in 5.8. It is obvious that the norm of S is bounded by a constant not depending on the choice of φ in 5.8. In what follows, we prove that the norms of the inverse $T^{-1} \in L(H^{1/(r-1)}, H^r)$ are also uniformly bounded with respect to φ, provided that $r > 1$ is sufficiently small. Since J is constant, we can replace T by $\widehat{T} := P(\frac{1}{a})BP$.

4. Because $\operatorname{Re}(Bz, z) = \big(|a|^2 - |b|^2\big)\,|z_1|^2 + |z_2|^2 \geq m\,\|z\|^2$, for some positive number $m = m(\delta)$, Lemma 1 of [11] shows that the inverses of the Töplitz operators $T_B := PBP\colon\ H^2_+ \to H^2_+$ are uniformly bounded,

$\|T_B^{-1}\| \le 1/m$. The invertibility of T_B implies that the (continuous) symbol B admits a generalized Wiener-Hopf (Birkhoff) factorization (canonical factorization) $B = B_- B_+$, where B_-, B_+, B_-^{-1}, $B_+^{-1} \in L^p$ for each $p < \infty$ ([1], Section 5.5, see also [3], [7]). Since $T_B^{-1} = B_+^{-1} P B_-^{-1} P$ and $B_+^{-1} P B_-^{-1} = B_+^{-1} P B_-^{-1} P$ on L^2, the multiplication operator

$$f \mapsto B_+^{-1} P B_-^{-1} f \tag{5.15}$$

is bounded on L^2 (uniformly with respect to $\varphi \in W_s^1$).

5. The function a is continuous and its range lies in a compact subset of the right-half complex plane (independent of φ). Consequently a admits a Wiener-Hopf factorization $a = a_+ \cdot a_-$, where $a_+ = \exp(P \log a)$ and $a_- = \exp((I-P)\log a)$. Since $|\mathrm{Im}\, \log a| \le \gamma(\delta) < \pi/2$, Zygmund's lemma applied to estimate the norms of the factors a_+ and a_- in $L^{2+\varepsilon}$ (recall that $P = \frac{1}{2}(-\mathrm{i}H + I + P_0)$, where H denotes the conjugation operator). The result is

$$\|a_+\|_{2+\varepsilon} \le C(\delta), \qquad \|a_-\|_{2+\varepsilon} \le C(\delta), \tag{5.16}$$

for some sufficiently small positive $\varepsilon = \varepsilon(\delta)$.

6. So far we have the equality $(1/a)\, B = a_-^{-1} B_- B_+ a_+^{-1}$ almost everywhere on $\mathbb{T}$. Using 5.15 and 5.16, we get that the operator

$$H_+^{1/(r-1)} \to H_+^r\colon\ w \mapsto a_+ B_+^{-1} P B_-^{-1} a_- w \tag{5.17}$$

is bounded (uniformly with respect to φ). □

In order to prove that (5.17) is the inverse of $\widehat{T}$, we remark that $\widehat{T}w \equiv P a_-^{-1} B_- B_+ a_+^{-1} w = f$ is equivalent to $B_+ a_+^{-1} w = P B_-^{-1} a_- w$ (note that $B_+ a_+^{-1} w \in H_+^{2-\varepsilon}$ with $\varepsilon > 0$), which implies that $w = a_+ B_+^{-1} P B_-^{-1} a_- f$ almost everywhere on $\mathbb{T}$.

After establishing these technical facts we continue the construction of the fixed point equation. For any scalar complex valued function $\varphi \in W_s^1$, we denote by $\widetilde{\Phi}_+, \widetilde{\Phi}_-$ the solution of the associated transmission problem

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c, \tag{5.18}$$

with a, b, and c from (5.8). With

$$\widehat{\Phi}_-(\mathrm{e}^{\mathrm{i}\tau}) := \int_0^\tau \widetilde{\Phi}_-(\mathrm{e}^{\mathrm{i}\sigma})\, \mathrm{d}\sigma, \qquad P_0 \widehat{\Phi}_- := \frac{1}{2\pi} \int_0^{2\pi} \widehat{\Phi}_-(\mathrm{e}^{\mathrm{i}\sigma})\, \mathrm{d}\sigma,$$

the operator $K\colon\ W_s^1 \to W_r^1$ is given by $K\varphi := \widehat{\Phi}_- - P_0 \widehat{\Phi}_-$. The definition of $\widehat{\Phi}_-$ makes sense since $P_0 \widetilde{\Phi}_- = 0$.

Lemma 5.2.

(i) *The operator $K\colon\ W_s^1 \to W_r^1$ is compact for any $r, s \in (1, \infty)$.*

(ii) *The image of $K\colon\ W_r^1 \to W_r^1$ is bounded if $r > 1$ is sufficiently small.*

(iii) *The pair $(\Phi_+, \Phi_-) \in W_r^1$ is a solution of the transmission problem 5.2 if and only if $K\Phi_- = \Phi_-$ and $\Phi_+ = G(.,\Phi_-)$.*

Proof. 1. The embedding $W^1_s(\mathbb{T}) \to C(\mathbb{T})$ is compact, and hence (i) follows once it is shown that $K\colon\ C(\mathbb{T}) \to W^1_r(\mathbb{T})$ is continuous. The superposition operators $\varphi \mapsto a := \partial_z G(.,\varphi)$, $\varphi \mapsto b := \overline{\partial}_z G(.,\varphi)$, $\varphi \mapsto f := \mathrm{i}\, t \cdot \partial_t G(.,\varphi)$ are continuous in $C(\mathbb{T})$, and thus the associated Toeplitz operators $T := P\widetilde{B}P$ with $\widetilde{B} := \frac{1}{a}JB$ with J and B from 5.13, and the 'adjoint Riemann–Hilbert operators' S from 5.12 depend continuously on φ. Since all these operators are invertible, the solutions in $H^r_\pm$ to the transmission problems (5.9) also depend continuously on φ (cf. 5.14). Integrating these solutions along $\mathbb{T}$ proves the continuity of $K\colon\ C(\mathbb{T}) \to W^1_r(\mathbb{T})$.

2. If $r > 1$ is sufficiently small, then according to Lemma 5.1, the solutions $\widetilde{\Phi}_\pm$ of (5.9) are bounded in $H^r_\pm$ uniformly with respect to the choice of $\varphi \in W^1_r$, and hence the Kw are uniformly bounded in W^1_r.

3. Let $\Phi = (\Phi_+, \Phi_-) \in W^1_r$ be a solution of $\Phi_+ = G(.,\Phi_-)$. Differentiating this boundary relation with respect to the polar angle τ, we obtain that $\widetilde{\Phi} := \partial_\tau \Phi \equiv \mathrm{i}t\partial_t\Phi$ is a (unique) solution of the auxiliary transmission problem 5.9. Consequently,

$$K\Phi_-(\mathrm{e}^{\mathrm{i}\tau}) = \mathrm{const} + \int_0^\tau \widetilde{\Phi}_-(\mathrm{e}^{\mathrm{i}\sigma})\ \mathrm{d}\sigma =$$

$$= \mathrm{const} + \int_0^\tau \partial_\tau \Phi_-(\mathrm{e}^{\mathrm{i}\sigma})\,\mathrm{d}\sigma = \mathrm{const} + \Phi_-(\mathrm{e}^{\mathrm{i}\tau}).$$

The constant on the right-hand side vanishes, since $P_0\, K\Phi_- = 0$ and $P_0\Phi_- = 0$.

4. Conversely, let $\Phi_- \in W^1_r$, $K\Phi_- = \Phi_-$, and $\Phi_+ := G(.,\Phi_-)$. We prove that Φ_+ and Φ_- are holomorphic in $\mathbb{D}$ and $\mathbb{E}$, respectively, and $P_0\Phi_- = 0$.

First of all, $\partial_\tau\Phi_- = \widetilde{\Phi}_-$. Since $\widetilde{\Phi}_-$ is holomorphic in $\mathbb{E}$, so Φ_- is also holomorphic. Further, $P_0\Phi_- = P_0\, K\Phi_- = 0$. Inserting $\widetilde{\Phi}_- = \partial_\tau\Phi_-$ into (5.9) shows that

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c = \frac{\mathrm{d}}{\mathrm{d}\tau}\, G(.,\Phi_-) = \partial_\tau\Phi_+.$$

Consequently, Φ_+ is holomorphic in $\mathbb{D}$ and $\Phi := (\Phi_+,\Phi_-) \in W^1_r$ solves (2) which completes the proof of the lemma. □

By virtue of Lemma 5.2, the existence result (i) of Theorem 5.1 becomes a consequence of Schauder's fixed-point principle.

It remains to prove that the solution of (5.2) is unique under the assumption (5.7). Let $\Phi^{(1)}$, $\Phi^{(2)} \in H^\infty_\pm \cap W^r_1$ be two solutions of (5.2). The difference $\Delta\Phi \equiv (\Delta\Phi_+, \Delta\Phi_-) := \Phi^{(2)} - \Phi^{(1)}$ solves the homogeneous linear transmission problem

$$\Delta\Phi_+ = a \cdot \Delta\Phi_- + b \cdot \overline{\Delta\Phi_-}, \tag{5.19}$$

where

$$a := \int_0^1 \partial_z G\big(.,\lambda\Phi_-^{(1)} + (1-\lambda)\Phi_-^{(2)}\big)\mathrm{d}\lambda,\quad b := \int_0^1 \overline{\partial}_z G\big(.,\lambda\Phi_-^{(1)} + (1-\lambda)\Phi_-^{(2)}\big)\mathrm{d}\lambda,$$

and $\Delta\Phi_-(\infty) = 0$. The assumption (5.7) on $\partial_z G$ and $\overline{\partial}_z G$ (with $g \equiv 0$) ensures that

$$\operatorname{Re} a - |b| \geq \delta > 0 \quad \text{on} \quad \mathbb{T}, \tag{5.20}$$

and hence (5.19) has only the trivial solution. Thus the proof of uniqueness is also complete.

It is instructive to compare our formulation of the nonlinear transmission problem with the so-called *Bishop's problem* [16] which is related to a number of fundamental topics of multi-dimensional complex analysis [54], [55]. As is well-known, the Bishop's problem can be reformulated in terms of *analytic discs*.

Recall that an analytic disc in $\mathbb{C}^n$ is defined as a continuous (or smooth) mapping $\phi : D = \{|z| \leq 1\} \to \mathbb{C}^n$ which is holomorphic inside the unit disc D, i.e., it is defined by n functions $\phi_n \in A(D)$. If $M \subset \mathbb{C}^n$ is a submanifold and ϕ is an analytic disc with $\phi(\partial D) \subset M$ then ϕ is called an *analytic disc attached to* M.

In various problems of complex analysis one it is important to construct analytic discs attached to so-called *totally real submanifolds* and there exist some deep results on the existence and structure of such analytic discs [16], [54], [55], [66], [67], [72]. This topic is closely related to Riemann–Hilbert problem because structural properties (dimension, stability) of the family of analytic discs attached to M close to a given one ϕ_0, can be expressed in terms of the partial indices of matrix functions defining the pullback bundle $\phi_0^*(TM)$ [55], [72].

These results about attached analytic discs are widely used in complex analysis and exhibit a spectacular type of application of Riemann–Hilbert problems and Birkhoff factorization. Moreover, they can be visually interpreted in the case of nonlinear Riemann–Hilbert problem considered above so it seems appropriate to recall some related constructions and concepts.

Let M be a *maximal real* (i.e. totally real of maximal possible dimension n) submanifold of $\mathbb{C}^n$ and let f be an analytic disc attached to M. Assume that in a neighbourhood V of $f(\partial D)$ we have $M \cap V = \{x \in V : g(x) = 0\}$, where g is a smooth (it is sufficient to require that $g \in C^2$) function on V having no critical points on V. We wish to investigate the existence of nearby analytic discs attached to M. It turns out that this issue can be studied in terms of certain analytic invariants of $f(\partial D)$ in M.

For each $z \in T = \partial D$, let $T(z)$ be the tangent plane of M at $f(z)$. Since f is regular on $\overline{D}$ there is a smooth map $A : T \to GL(n, \mathbb{C})$ such that for each $z \in T$ the columns of $A(z)$ for m a basis of $T(z)$. Let $B(z) = A(z)\overline{A(z)^{-1}}$ where the bar denotes complex conjugation.

Since the spaces $T(z)$ are maximal real, the map B has some remarkable properties which justify the constructions to follow, so we now establish those properties of B.

Let L be a maximal real subspace of $\mathbb{C}^n$. Then of course $L \oplus \mathbf{i}L = \mathbb{C}^n$. To any such subspace L one can associate an $\mathbb{R}$-linear map R_L on $\mathbb{C}^n$, called the *reflection* in L, defined by $z = x + \mathbf{i}y \mapsto x - \mathbf{i}y$, where we use the above decomposition into direct sum.

The mapping R_L is also $\mathbb{C}$-antilinear, i.e., $R_L(\mathbf{i}v) = -\mathbf{i}R_L(v)$ for each $v \in \mathbb{C}^n$. Let us denote by R_0 the reflection about the maximal real subspace $\mathbb{R}^n \subset \mathbb{C}^n$. In the standard notation, R_0 is just the ordinary conjugation on $\mathbb{C}^n$ and for any $n \times n$ complex matrix A one has the identity $\overline{A} = R_0 A R_0$. Now it is easy to establish the following well known algebraic lemma which is crucial for next considerations.

Lemma 5.3. *Let L be a maximal real subspace of $\mathbb{C}^n$ and let $x_1, \dots, x_n$ be any basis of L. let A be the matrix whose columns are the given vectors $x_j, j = 1, \dots, n$ and let $B = A\overline{A^{-1}}$. Then $B = R_L R_0$, in particular, the matrix B does not depend on the basis of L. Moreover, $\overline{B} = B^{-1}$, $|\det B| = 1$.*

Proof. Obviously, A is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ which maps $\mathbb{R}^n$ onto L. Consider now the following composition of automorphisms of $\mathbb{C}^n$, $S = R_0 A^{-1} R_L A$. Then S is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ which coincides with the identity on $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a maximal real subspace of $\mathbb{C}^n$, S is the identity on $\mathbb{C}^n$. Hence $R_0 A^{-1} = A^{-1} R_L$. Since $B = A\overline{A^{-1}} = A R_0 A^{-1} R_0$, we obtain $B = AA^{-1} R_L R_0 = R_L R_0$, as claimed, and the rest becomes obvious. □

Thus the matrix B is uniquely determined by the bundle $T(z)$ and this enables one to extract from it the crucial invariants of this bundle using Birkhoff factorization theorem. Indeed, one can write

$$B(z) = B_+(z)\mathrm{diag}(z^{\mathbf{k}})B_-(z),$$

with the factors having the same meaning as in Section 1. It follows that the (left) partial indices k_i are determined by the bundle $T(z)$. They are called *partial indices of M along $f|T$* and their sum K is called the *total index of M along $f|T$* (or the *Maslov index of M along $f|T$* [54], [159]).

This construction becomes quite transparent for $n = 2$ and using a well known description of nearby analytic discs in terms of the *Bishop equation* [16] or nonlinear Riemann–Hilbert problems [159], one can obtain some conclusions about the structure of attached analytic discs. For example, in the situation considered above it is easy to see that all partial indices of M along the boundary of an attached analytic disc are equal to zero. At the same time, the boundaries of these discs give a foliation of M so the family of such analytic discs is locally two-dimensional.

Also, it was shown in [54] (cf. [159]) that if both partial indices of M along $f|T$ are nonnegative then the family of nearby analytic discs depends on $K + n$ real parameters and the same holds for small perturbations of M (or equivalently of its defining function g).

A general result on the structure of attached analytic discs was proved by J.Globevnik [67]. In order to give a precise formulation of that result we need to introduce some functional spaces adapted to the situation.

Denote by $C^2(V)$ the Banach space of all real valued functions of class C^2 with the standard *sup*-norm. Let $0 < s < 1$. Denote by H^s the Banach algebra of all real valued functions on T with finite Lipschitz norm of exponent s, i.e.,

$$||f||_s = \sup |f| + \sup \frac{|f(x) - f(y)|}{|x-y|^s}, x, y \in T,$$

by H^s_C the algebra of all complex valued functions on T with finite H^s-norm, and by A_s the closed subalgebra of all $\phi \in H^s_C$ which extend holomorphically to D. The Banach space A^n_s is the space of analytic discs convenient to work with. For each $r \in (C^2(V))^n$ sufficiently close to g put $M_r = \{x \in V : r(x) = 0\}$.

Assume now that $f \in A^n_s$ satisfies $f(T) \subset M$, that is $g(f(t)) = 0$ for each $t \in T$. If $U \subset A^n_s$ is a neighbourhood of f so small that $h(T) \subset V$ for each $h \in U$ the the analytic disc ϕ is attached to M if and only if $g(\phi(t)) = 0$ for each $t \in T$. Consider the map Q which sends $\phi \in U$ to the map $t \mapsto g(\phi(t))$. It is easy to see that Q is (at least) a C^1-map from U into $(H^s)^n$ and $X = \{\phi \in U : Q(\phi) = 0\}$ is precisely the set of all analytic discs in U attached to M. Our goal now is to understand its structure, and the first step is to find conditions when it is a finite dimensional smooth manifold because then one can hope to find a reasonable parametrization of nearby attached analytic discs.

Taking into account the implicit function theorem, we conclude that X will be a C^1-manifold if $DQ(f)$ maps A^n_s onto $(H^s)^n$ and if the kernel of *ker* $DQ(f)$ is complemented in A^n_s. As was established by J.Globevnik this is equivalent t requiring that all partial indices of M along $f|T$ are not less than -1. Moreover, one can explicitly compute its dimension.

Theorem 5.2 ([67]). *Let M, f, Q be as above. Let k_i be the partial indices of M along $f|T$ and let K be the total index. Then $DQ(f)$ is surjective if and only if $k_i \geq -1, i = 1, \dots, n$. If this is the case, then there exist a neighbourhood $P \subset (H^s)^n$ of g and a neighbourhood $W \subset A^n_s$ of f such that:*

1) *$\{(r, \phi) \in P \times W : \phi(T) \subset M_r\}$ is a C^1 submanifold of $P \times W$, and*

2) *for each $r \in P$ the set $\{\phi \in W : \phi(T) \subset M_r\}$ is a C^1 submanifold of W of dimension $K + n$.*

It is remarkable that this important result can be proved using only Birkhoff factorization and implicit function theorem in functional spaces. It seems also worthy of noting that in the situation of Theorem 5.1 the condition (i) implies that all partial indices of emerging attached analytic discs are equal to zero, so Theorem 5.2 guarantees that family of such discs is smooth and their boundaries cover the target manifold in the "Schlicht" manner (cf. [159]). In other words, the target manifold is foliated by the boundaries of attached analytic discs even without assumption (ii) of Theorem 5.1, which answers a question posed in [160].

The aforementioned Bishop's problem is concerned with constructing analytic discs attached to explicitly given submanifolds of complex Euclidean spaces. In the original formulation of [16] the target manifold is just a graph of a function $\mathbb{C} \to \mathbb{C}$. For simplicity we only discuss the case when $n = 2$ which corresponds to the original Bishop's problem and to the transmission problem studied above.

E.Bishop considered a differentiable function $f : \mathbb{C} \to \mathbb{C}$ and its graph $\Gamma_f \subset \mathbb{C}^2$ and looked for analytic discs in $\mathbb{C}^2$ attached to Γ_f. A minute thought shows that our Theorem 5.1 gives a solution to Bishop's problem for a class of functions f specified in its formulation. Notice that we have actually obtained a complete description of all attached analytic discs in this case.

Nowadays there exist a lot of papers devoted to solving the Bishop's problem for various classes of functions [56]. For example, E.Bishop himself established the existence of such analytic discs in a number of cases and described their structure in neighbourhoods of so-called *elliptic complex points* (we present the definition below) of the graph [16]. Since then, the problem of proving existence and counting complex points of various types attracted considerable attention (for an updated review of the topic see [56]).

The problem has a special flavour in the case when components of f are polynomials because one would like then to have some effective methods for counting complex points and establishing existence of attached analytic discs. An approach to this problem based on the so-called *signature technique for counting real roots* [94] was recently suggested in a joint paper of E.Wegert and the present author [96]. We describe below some essential ingredients of that approach since up to our mind it suggests some interesting viewpoints and perspectives.

Following [96] we prefer to treat the problem in terms of smooth transformations (self-mappings) of the plane $\mathbb{C} \cong \mathbb{R}^2$. We only consider transformations with polynomial components and call them *planar endomorphisms (plends)*.

We are interested in counting the complex points of such maps, in particular we would like to find the number of *elliptic complex points* [16].

In many situations it is sufficient to consider only *generic* planar endomorphisms which are proper and satisfy some additional transversality condition in the spirit of [76], [95]. More precisely, the Gauss map of the graph of such an endomorphism should be transversal to the subset G_C of all complex lines in $Gr_{\mathbb{R}}(2,4)$. This condition holds on a dense open subset of plends and it implies that the graph has only isolated complex points hence finite amount of those. For brevity such plends will be called excellent (by analogy with "excellent maps" of H.Whitney [70]).

It can be shown that an arbitrary plend admits excellent deformations and a well known paradigm of singularity theory [70] suggests that it is reasonable to count complex points in nearby excellent deformations of a given plend. Thus one comes to the problem of counting complex points of an excellent plend which is our main concern in the sequel. A local version of such strategy was applied in [76], [95] and it turned out that the so-called *signature formulae for topological invariants* [94] are quite effective in this context. In line with that we outline below how one can apply signature formulae in global setting.

Denote by $\mathbb{R}_2$ the algebra of real polynomials in two variables and by P_d the subset of polynomials of algebraic degree not exceeding d. Let $F = (f, g), f, g \in \mathbb{R}_2$ be a planar endomorphism (plend) with the components f and g.

Identify $\mathbb{R}^2 \times \mathbb{R}^2$ with $\mathbb{C}^2$ in the usual way then the graph Γ_F of F becomes a smooth (C^∞) two-dimensional surface in $\mathbb{C}^2$. This enables us to study Γ_F using some concepts of complex geometry. A natural step in this direction is to look at tangent planes to the graph.

Consider the Grassmanian $G = Gr_{\mathbb{R}}(2,4)$ of oriented two-planes in $\mathbb{R}^4$ identified with $\mathbb{C}^2$. As usual one can distinguish between *complex lines* and *totally real planes* [16].

The Gauss map $T : \Gamma_F \to G$ sends each point $q = (p, F(p)) \in \Gamma_F$ to the tangent plane $T_q\Gamma_F$. A well known strategy of singularity theory suggests to consider first objects satisfying appropriate transversality conditions.

Recall that a point p is called a complex point of F (and its graph Γ_F) if the tangent plane $T_p\Gamma_F$ at this point is a complex line [16]. Generically, in particular for excellent plends defined above, a complex point can be either elliptic or hyperbolic depending on the geometric properties of Γ_F at this point [16].

Denote by

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\Big)$$

the usual ∂-bar operator in the plane and write $F_{\overline{z}}$ for the plend obtained by applying $\overline{\partial}$ to both components of F. Then one can give a simple characterization of complex points in terms of $\overline{\partial}$.

Lemma 5.4. *A point p is a complex point of F if and only if $F_{\overline{z}}(p) = 0$.*

We now follow the pattern suggested by E.Bishop [16]. Namely, one first establishes the existence of elliptic complex points and then applies the fundamental result of Bishop about the existence of attached analytic discs in a vicinity of an elliptic complex point [16]. As is well known, this approach yields a lot of results about the existence of complex points on immersed compact surfaces which were basically established by topological methods [56]. In our setting, considerable information about the existence and amount of complex points can be obtained using the aforementioned signature formulae for topological invariants [94].

Notice that applying the *linking number construction* from [76] (cf. [95]) to sufficiently big circles in the plane one obtains a natural integer valued invariant of a proper plend which may be called its (global) Maslov index

$M(F)$. According to formulae from [76] and [95] one can calculate $M(F)$ by properly counting complex points of nearby excellent deformations of F. It turns out that the same can be done without examination of nearby deformations.

Theorem 5.3. *The Maslov index is algorithmically computable from the coefficients of a given planar endomorphism.*

This follows from an explicit algebraic formula for the topological degree [94] which implies effective computability of mapping degrees of explicitly given polynomial mappings. Notice that by virtue of the above lemma, the Maslov index is just the topological degree of the endomorphism defined by partial derivative $\partial F/\partial \overline{z}$.

The Maslov index alone is not sufficient for our purposes. So we consider also the numbers $C(F)$ and $C_e(F)$ of the complex points and elliptic complex points respectively. The total amount $C(F)$ of complex points can be also expressed in terms of mapping degree.

Theorem 5.4. *The total amount of complex points of a perfect planar endomorphism is algorithmically computable from the coefficients of its components.*

This follows from a general signature formula for the Euler characteristic of a compact algebraic subset ([94], Theorem 8.2). For a perfect plend F, the set of complex points is the finite algebraic subset of $\mathbb{R}^2$ defined as the zero-set of the polynomial system $\frac{\partial F}{\partial \overline{z}}(x, y) = 0$. Thus this number can be computed as the local topological degree of an auxiliary endomorphism of $\mathbb{R}^3$ which is given by simple explicit formulae [94].

Let us now explain how one can compute the exact amount of elliptic complex points. The trick is to represent the subset $C_e(F)$ as a semi-algebraic subset of the plane. Then it can be effectively computed using the results of Ch.9 of [94].

In other words, we only need to indicate explicit algebraic conditions which characterize elliptic complex points. To this end we use the geometric interpretation in terms of Gaussian curvature $K_p(\Gamma_F)$ [56]. Namely, from the normal form of a generic complex point [16] it follows that the elliptic points are exactly those complex points p where $K_p(\Gamma_F)$ is positive. Now it is not difficult to show that this condition can be expressed as a polynomial inequality.

Indeed one can use an explicit expression for the Gaussian curvature of a parameterized 2-surface in $\mathbb{R}^4$. It comes as no surprise that such an expression can be derived from general formulae for the first fundamental form and curvature tensor of a parameterized submanifold, which directly leads to the desired conclusion [96].

Proposition 5.1. *The set of elliptic complex points of a perfect plend F coincides with the finite semi-algebraic subset of the plane defined as*

$$\Big\{p = (x, y) \in \mathbb{R}^2 \,|\, \frac{\partial F}{\partial \overline{z}} = 0, K(x, y) > 0\Big\},$$

where $K(x,y)$ is the Gaussian curvature of Γ_F at the point $(x,y,f(x,y),g(x,y))$.

Combining this with the results of Chapter 9 of [94] it is not difficult to obtain an effective algorithm for computing the number of elliptic complex points.

Theorem 5.5. *The number of elliptic complex points of an excellent planar endomorophism can be effectively computed by a finite number of algebraic and logical operations over its coefficients.*

This follows from Theorem 9.1 of [94] which establishes the effective computability of the cardinality of a finite semi-algebraic subset.

It should be added that using a computer program for calculating topological degree developed by A.Lecki and Z.Szafraniec [107] one can easily count complex points for plends of not very high degree (everything depends just on the capacity of a computer at hand). After determining the bifurcation diagram along the standard lines of singularity theory [70], this in principle enables one to find all possible values of the above invariants for plends of fixed bi-degree, say, for elements of $(P_d)^2$. Using the results of [45], [96] it is also possible to calculate the average number of complex points of a random polynomial endomorphism of the plane with rotation invariant Gaussian distribution of coefficients introduced in [144].

6. Hyper-Holomorphic Cells and Riemann–Hilbert Problems

The classical boundary value problems (BVPs) for holomorphic functions, in particular the linear conjugation problem and the Riemann–Hilbert problem whose comprehensive theory owes so much to the works of N. Muskhelishvili and his followers (see, e.g., [122], [157], [158], [98], [100]), can be described by linear operators in appropriate function spaces (cf. the abstract operatorial setting developed in [20], [68], [135], [24]). Nowadays there also exist several interesting non-linear versions of these classical problems (see, e.g., [143], [159]).

One of the most spectacular generalizations of this kind was developed in the seminal paper of M. Gromov concerned with the pseudo-holomorphic curves [72]. Gromov's approach involves, in particular, two important new aspects: generalizing the equation satisfied by functions (which in in some sense equivalent to working with solutions of the Bers–Vekua equation [14], [157]) and consideration of non-linear boundary conditions in the spirit of "holomorphic discs attached to a totally real submanifold" [16].

All these results are concerned with functions which locally depend on two real variables and one may wonder if similar results can be obtained for functions of several real variables satisfying some elliptic system of equations. Such generalizations do not seem to have attracted much attention up to now, but the existing results about linear BVPs for elliptic systems (see [27], [151], [86]) suggested that some results of this type should be available for systems of Dirac type [62].

In this section we describe some steps in this direction. To be more precise, we discuss some geometric properties of families of solutions to certain elliptic first order systems of linear partial differential equations with constant coefficients [162] (cf. EES of Section 4; such systems are also called "canonical first order systems" (CFOS) [9]). Such systems were studied in many papers and they remain an object of a permanent interest (see [152] for a recent review of the topic).

An especially important class of such systems is provided by the so-called "generalized Cauchy–Riemann systems" (GCRS) [150] which were already discussed in Section 4. Solutions to generalized Cauchy–Riemann systems are often called *hyper-holomorphic mappings* [142] and in many problems it is necessary to consider images of some standard domains (e.g., balls) under such mappings. Standard examples of such systems in low dimensions are the classical Cauchy–Riemann system in the plane, the *Moisil–Theodoresco system* in $\mathbb{R}^3$ [17], and *Fueter system* for functions of one quaternionic variable [31]. There exists a vast literature devoted to such equations (see references in [17] and [31]). In particular, some important results about the so-called *generalized analytic vectors* were obtained by georgian mathematicians [17], [125]. Similar systems emerged in the theory of para-analytic functions developed by M. Frechet [57].

The main paradigm we follow in the sequel, has its origin in the theory of analytic (holomorphic) discs attached to a totally real submanifold [16] which was already discussed above. One takes a smooth bounded domain homeomorphic to a ball of appropriate dimension and considers its images under solutions to a given CFOS. Such images (we call them *elliptic cells*) are our main concern in this paper.

More concretely, inspired by the theory of attached analytic discs [16], [56] we consider elliptic cells with boundaries in a fixed submanifold M of the target space of the elliptic system in question. They are called *elliptic cells attached to M*. For our purposes it appears useful to regard them as solutions of non-linear boundary value problems of Riemann–Hilbert type [159]. Accepting terminology from [159], M will be called a *target manifold* (for attached elliptic cells).

Actually, we only consider hyper-holomorphic cells, i.e., those which are defined by solutions to a given GCRS. Notice that except the aforementioned theory of attached analytic discs [16], there also exist important generalizations of this classical example in the framework of symplectic geometry [72].

Recently, similar situations were discussed in mathematical physics in relation with so-called *D-branes*. *D*-branes have already found interesting applications in topological field theory and string theory [15], [60]. It is worthy of noting that in those physical contexts there also appear manifolds with boundaries attached to certain submanifolds. This confirms our trust that such objects deserve some attention by their own.

With this in mind, we investigate some situations where families of attached hyper-holomorphic cells can be locally described as kernels of some (non-linear) Fredholm operators. Such a phenomenon is well known in the case of analytic discs [6] and it plays an important role in M. Gromov's studies on pseudo-holomorphic curves [72]. In particular, one becomes able to use the well known concepts and techniques of Fredholm theory, which reveals some important topological aspects of the situation. We closely mimic Gromov's approach and establish some properties of emerging non-linear operators using the Fredholm theory of elliptic Riemann–Hilbert problems (RHPs) for GCRS discussed in the Section 4 (cf. also [151], [84]).

In particular, we show that, for certain values of dimensions n, k, and m, there exist non-compact k-submanifolds in affine m-space such that families of hyper-holomorphic n-cells attached to such submanifolds are locally described by Fredholm operators. Borrowing again terminology from [159], such submanifolds are called *admissible targets* (for a given GCRS). Existence of admissible targets and Fredholmness of arising non-linear operators are derived from the recent results on the existence of elliptic boundary value problems for GCRS [152], [86] (cf. also [142]).

Such aspects of generalized Cauchy–Riemann systems seem to have never been discussed in the literature, so we pursue but a modest goal of describing and illustrating the framework which naturally stems from our previous results on generalized Cauchy–Riemann systems. We proceed by presenting some notions from [150] and [162] in the form adjusted to our purposes.

Definition 6.1 ([150], cf. [62]). An elliptic system of first order with constant coefficients is called a generalized Cauchy–Riemann system (GCRS) if it is invariant under the natural action of the orthogonal group on the source space and all components of its differentiable solutions are harmonic functions. Solutions of such systems are called hyper-holomorphic (hh) mappings. For a given such system S, its solutions will be also called S-mappings.

As was explained above (cf. [151]), without loss of generality, one may always assume that such a system in $\mathbb{R}^{n+1}$ may be written in a canonical form described in the Section 4:

$$E\frac{\partial w}{\partial x_0} + A_1\frac{\partial w}{\partial x_1} + \cdots + A_n\frac{\partial w}{\partial x_n} + Dw = f, \tag{6.1}$$

where A_j, D are constant complex $(m \times m)$ matrices, $E = A_0$ is the identity matrix, and for all $i, j = 1, \ldots, n$, one has:

$$A_iA_j + A_jA_i = -2\delta_{ij}E. \tag{6.2}$$

We consider such a system in a smooth domain $U \Subset \mathbb{R}^{n+1}$ and assume that the unknown vector-function w belongs to class $C^1(U, \mathbb{C}^m)$.

As was also shown, system 6.1 is elliptic, in the usual sense [162], i.e.,

$$\det(t_0E + t_1A_1 + \cdots + t_nA_n) \neq 0,$$

for all $(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} - \{0\}$.

As we have seen above, such a system defines a representation of Clifford algebra Cl_n on $\mathbb{C}^m$ [62]. So the (complex) target dimension m, being the sum of dimensions of irreducible representations of Cl_n, is an integer multiple of $2^{[n/2]}$ [27]. If for a given system S this dimension is the minimal possible, $m(n) = 2^{[n/2]}$, we will say that system S is irreducible. In many situations it is sufficient to consider only irreducible GCRSs.

For the sake of visuality, we explicitly write down some examples of such systems in low dimensions. For $n = 1$, one has $m(1) = 1$ and the corresponding irreducible system is just the classical Cauchy–Riemann system for two real functions $u(x, y), v(x, y)$ of two real variables:

$$\begin{aligned} u_x - v_y &= 0, \\ u_y + v_x &= 0. \end{aligned}$$

For $n = 2$, one has $m(2) = 2$, and the corresponding irreducible systems for four real functions s, u, v, w of three real variables is the well-known Moisil-Theodoresco system which may be written using standard operators on vector-functions in $\mathbb{R}^3$ [62]:

$$\begin{aligned} \operatorname{div}(u, v, w) &= 0, \\ \operatorname{grad} s + \operatorname{rot}(u, v, w) &= 0. \end{aligned}$$

For $n = 3$, one has $m(3) = 2$, and the corresponding irreducible system is the so-called Fueter system for four real functions f_i of four real variables x_j [62]:

$$\begin{aligned} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0, \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} &= 0, \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} &= 0, \\ \frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} &= 0. \end{aligned}$$

As is well known, this system is a natural counterpart of Cauchy–Riemann system for a function of one quaternionic variable. Its solutions, called quaternionic-regular functions, have many interesting properties similar to those of usual holomorphic functions of one complex variable [62], [142].

The general theory of PDE yields that one can formulate various reasonable boundary value problems (BVPs) for such systems in bounded domains [162], [153]. For our purposes are relevant the classical local boundary value problems of Riemann–Hilbert type introduced in Section 4. In other words, one looks for solutions of (6.1) satisfying a boundary condition of the form:

$$(B_1 B_2) \cdot w = g, \tag{6.3}$$

where B_1, B_2 are continuous complex $(\frac{m}{2} \times \frac{m}{2})$-matrix-functions on bU such that the rows of $(\frac{m}{2} \times m)$-matrix-function (B_1, B_2) are linearly independent at every boundary point, and g is a continuous vector-function with values in $\mathbb{C}^m$.

For our purposes especially appropriate are those RHPs which are elliptic in the usual sense (i.e., satisfy Shapiro-Lopatinski condition [162]) because then the problem 6.1, 6.3 is described by a Fredholm operator in appropriate function spaces [162]. It is well-known that not all systems of the type 6.1 admit elliptic boundary conditions 6.3 [17], [162], so the first natural problem is to investigate which GCRSs possess elliptic RHPs. The answer to this question given in Section 4 enables us to indicate cases in which hyper-holomorphic cells are described by Fredholm operators.

Remark 6.1. An important class of GCRS is associated with Euclidean Dirac operators [27]. The corresponding systems of the first order are called *systems of Dirac type* and their solutions are called *monogenic mappings* [62].

For notational convenience, in the sequel we denote by V the target space $\mathbb{C}^m$ of system (6.1). We fix now a GCRS of the form (6.1) and denote by B a $(n+1)$-ball in its source space. We also take some submanifold M in V and refer to it as a target.

Definition 6.2. A hyper-holomorphic (hh) cell attached to M is defined as (the image of) a hyper-holomorphic mapping $u : \overline{B} \to V$ such that $u(bB) \in M$. For a fixed GCRS S, we will speak of S-cells attached to M.

The usual way of dealing with hh cells attached to a given submanifold is to consider families of cells attached at a given point. Such families may be described by certain non-linear operators in appropriate function spaces and if these operators appear to be Fredholm, then one may obtain a reasonable structural theory of such cells, as it happens, for example, for pseudo-holomorphic discs and curves [72], [6]. So it is natural to begin with looking for such situations where hh cells may be related to Fredholm operators. In order to make this idea precise, we need some constructions and definitions.

To this end, consider an irreducible GCRS S in $\mathbb{R}^{n+1}$ with values in V. Consider also some smooth (C^∞) submanifold M of V of the real dimension equal to the complex dimension $m(n)$ of V (in such case we speak of a *submanifold of middle dimension*). Let B denote an $(n+1)$-ball centered at the origin of the source space of S and let q be some fixed point on its boundary n-sphere bB. Furthermore, we fix a point $p \in M$ and a non-integer positive number $r > 1$, and let H^r denote the class of all maps such that all of $[r]$th order partial derivatives of their components belong to Hölder class (cf. [6]).

Let F be the space of H^{r+1} maps $f : (B, bB, q) \to (V, M, p)$ which are homotopic to the constant map $f_p = p$ in $\pi_{n+1}(V, M, p)$ (such maps will be called *homotopically trivial*). In a standard way one checks that F is a smooth Banach manifold (cf. [6]). Let G be the complex Banach space of all H^r maps $g : B \to V$. Define also a submanifold in $F \times G$ by putting

$$H = \{(f,g) \in F \times G : D(f) = g\}, \tag{6.4}$$

where by D is denoted the matricial partial differential operator defined by the left-hand-side of (6.1).

Then it is easy to see that H is a connected submanifold of $F \times G$ and one may define the projection map $L_p : H \to G$ given by $L_p(f, g) = g$. It is also easy to check that L_p is a differentiable map of H into Banach space G. Recall that a differentiable map $F : X \to Y$ between Banach manifolds X, Y is called a *Fredholm map* if, for each point p of the source X, the derivative T_pF at point p defines a Fredholm operator $T_pX \to T_{F(p)}Y$ between the tangent spaces [46].

Definition 6.3. Mapping L_p is called the Gromov's operator of pair (S, M) at point p. Manifold M is called an S-admissible target if Gromov's operator $L_p(S, M)$ is a (nonlinear) Fredholm operator for every $p \in M$.

Similar operators were introduced by M. Gromov for analytic discs [72] (cf. also [6]). General techniques of functional analysis (Fredholm theory, Sard-Smale theorem, theory of Fredholm structures) suggest that if this operator appears to be Fredholm, one may count for a reasonable structural theory for attached elliptic cells. In some sense this is the most natural way of formulating a version of Fredholm theory for elliptic cells.

We now present a typical result of this type available in our context. We are especially interested in those targets M for which L_p is Fredholm at any point $p \in M$, so we introduce a short-hand *admissible targets* for the targets possessing this property.

Recall that we are given an irreducible GCRS S in $\mathbb{R}^{n+1}$ with values in V. Construct another GCRS $D(S)$ with values in $W = V \times V$ which is a sort of a "double" of S. If n is even, than $D(S)$ simply consists of two identical copies of S. If n is odd, then one adds to S the canonical GCRS corresponding to the second irreducible representation of Cl_n. Thus the complex dimension of the target space of $D(S)$ is $2m(n)$.

Remark 6.2. Consideration of such "doubles" is suggested by the results of Section 4. From the viewpoint of K-theory this may be considered as a sort of "stabilization" and it is quite natural that this operation improves certain properties of the system (see [27]).

We construct now non-compact admissible targets M in W as images of appropriate embeddings of $\mathbb{R}^{2m(n)}$. We assume that all spaces of smooth mappings are endowed with Whitney topology [73].

Theorem 6.1. *There exists an open set of embeddings $f : \mathbb{R}^{2m(n)} \to W$ such that, for every point p of $M = f(\mathbb{R}^{2m(n)})$, Gromov's operator of the pair $(D(S), M)$ at point p is a (non-linear) Fredholm operator of index zero.*

In other words, non-compact admissible targets exist for systems of the form $D(S)$. At the moment we do not have any general existence results for compact admissible targets. Notice that for every (compact or non-compact) admissible target, fredholmness of Gromov's operators combined with a standard application of implicit function theorem for Banach spaces in the spirit of [73] implies that the homotopically trivial families of attached elliptic cells are locally finite-dimensional. Notice that here one need not restrict himself to systems of the form $D(S)$.

Corollary 6.1. *If M is a S-admissible target, then the family of homotopically trivial S-hyper-holomorphic cells attached to M at p is finite-dimensional.*

This result can be considered as a description of the subset of all hh cells attached to M which are close to a "degenerate" cell $f_p = p$. One obtains its natural generalization by considering the subset of all hh cells attached to M which are sufficiently close to an arbitrary given cell g attached to M. One need not even assume vanishing of the class of g in $\pi_{n+1}(V, M)$.

Corollary 6.2. *For a given S-cell g attached to an S-admissible target M, the set of all S-cells attached to M near g is finite-dimensional.*

In both these cases one encounters a natural problem of computing the "virtual dimension of nearby attached hh cells" (see [73]) in terms of S, M, and of the given cell g. Such formulae are available for (pseudo-)analytic discs (or Cauchy–Riemann cells, in our terminology) and they involve the notion of Maslov index of a curve along a totally real submanifold M [56], [72].

Up to the author's knowledge, in the general case this is an unsolved problem. As one can see from the discussion presented in the next section, progress in this problem depends on the availability of explicit index formulae for elliptic linear Riemann–Hilbert problems for GCRSs. Apparently in concrete cases one can successfully apply the analytic formulae for indices of elliptic boundary value problems in Euclidean space obtained by A.Dynin [44] and B.Fedosov [52].

We present below an outline of the proof of Theorem 6.1. Using a natural linearization procedure it can be derived from general results on existence of elliptic boundary value problems for GCRSs which were presented in Section 4. An examination of the proofs of those results shows that for GCRSs in spaces of odd dimension (in our notation this means that n should be even) the same result can be obtained without passing to doubles, which yields the second main result of this section.

Theorem 6.2. *For every irreducible GCRS on a space of odd dimension different from 5 and 7, there exists an open set of embeddings of $\mathbb{R}^{m(n)}$ into $\mathbb{C}^{m(n)}$ such that their images are admissible targets for attached S-cells.*

As was explained in Section 4, with every GCRS S there are associated integers m and l. For odd n, there also appear integers l_1, l_2 $(l_1 + l_2 = l)$ showing how many irreducible representations of each of the two possible non-isomorphic types do participate in the direct sum decomposition of the representation defined by system S. As was shown these discrete parameters completely determine existence of elliptic BVPs. For the purposes of this section we need stronger versions of the main results of Section 4. They follow easily from the results of Section 4 "by general principles".

Proposition 6.1. *Suppose that n is odd and $n \geq 3$. If l is also odd, then there exist no elliptic RHPs for the given GCRS. If l is even, then elliptic RHPs exist if and only if $l_1 = l_2$. In the latter case, the set of elliptic boundary conditions is open in the space of all local boundary conditions of the form* (6.3).

Proposition 6.2. *If $n \geq 2$ is even, $n \neq 4, 6$, then there always exist elliptic RHPs for GCRS of the form $D(S)$ and the set of elliptic boundary conditions is open in the space of all local boundary conditions of the form* (6.3).

Remark 6.3. The restriction that $n \neq 4, 6$ results from the method of proof used in [86]. It comes from the paper [71] and it is related to some delicate questions of K-theory. At the moment it still remains unclear for us if this restriction is essential indeed.

Taking into account these results we can now prove Theorem 6.1 and in course of proving it we will also see the way of generalizing it to arbitrary GCRSs on odd-dimensional spaces, which is the second main result of this section.

Proof of Theorem 6.1. Let us first determine the derivative (differential) of L_p at some point (f_0, g_0) and show that it may be interpreted as a boundary value problem 6.1,6.3 for system $D(S)$, i.e., that it is an RHP for the GCRS $D(S)$.

Using the standard description of the tangent space to a manifold of mappings in terms of vector fields along a given mapping, it is easy to see that $T_{(f_0,g_0)}H$ may be identified with the space

$$Z = \{f : B \to W : f \in H^{r+1}(B, W), f(x) \in T_{f_0(x)}M, \forall x \in bB, f(q) = p\}.$$

Granted that, it becomes clear that the derivative of L_p at (f_0, g_0) may be identified with the map $\delta : Z \to G$ given by $\delta(f) = Df$.

Let N_M denote the (geometric) normal bundle of M. This is a real vector bundle with fibre dimension $k = 2m(n)$. Consider its pull-back $E_0 = (f_0 \mid bB)^*(N_M)$. From the homotopy condition in the definition of F it follows that E_0 is a trivial bundle over bB, generated by k global sections, say, $p_1, \ldots, p_k$. Using P_j as rows we may form the $(k \times 2k)$-matrix-function $p \in H^{r+1}(bB)$. By the very construction of P, $T_{f_0(x)}M = \{w \in W : P(x)w = 0\}$ and one immediately observes that matrix P has exactly the same form as the matrix of boundary condition (6.3) for system $D(S)$.

Let us set $X = H^{r+1}(B, W), Y = H^r(B, W) \times H^{r+1}(bB, \mathbf{C}^{m(n)})$ and define a map $R : X \to Y$ by $R(f) = (D(f), (Pf)|bB)$. It is obvious that R is exactly the operator of a Riemann–Hilbert problem (6.1,6.3) for system $D(S)$.

Our next goal is to understand under which conditions one may guarantee that R is a Fredholm operator. Notice that if the corresponding RHP is elliptic (i.e., satisfies the Shapiro-Lopatinski condition [162]), then R is a Fredholm operator in virtue of the general theory of elliptic linear boundary value problems [162]. So we should only arrange that matrix P defines an elliptic boundary condition for $D(S)$.

Proposition 6.2 shows that in our situation (this is the crucial place where it is important that $D(S)$ is a "double") there are a plenty of elliptic boundary conditions. In particular, there exist constant matrices P_0 which define elliptic RHPs (6.1, 6.3). Let us embed $\mathbb{R}^{2m(n)}$ in W in such a way that the normal space of the image M is orthogonal to the subspace spanned by rows of such a P_0. For such target M, R is obviously Fredholm, so L_p is also Fredholm at any $p \in M$. Taking into account the stability of Fredholm property under small perturbations, we see that all sufficiently small perturbations of M will be admissible targets. The homotopy invariance of the Fredholm index implies that the index vanishes, which completes the proof. □

Remark 6.4. We used the fact that systems of the form $D(S)$ possess elliptic boundary conditions (6.3) defined by constant matrix-functions B_1, B_2. This fact is not self-evident but it follows from the results of [88]. The "raison d'être" of this result is the fact (see [88]) that, as was already mentioned, RHPs for systems of the form $D(S)$ are equivalent to linear conjugation problems for system S [88].

Existence of elliptic linear conjugation problems was established in [88]. For $n = 1$ this is just a trivial consequence of the classical theory of linear conjugation problems for holomorphic functions [122] (actually every non-degenerate constant matrix generates an elliptic conjugation problem because its partial indices obviously vanish). For irreducible systems with $n = 2$ (Moisil-Theodoresco system) and $n = 3$ (Fueter system), existence of constant elliptic transmission conditions follows from the criteria of fredholmness for such problems obtained in [142] (cf. [153]).

Remark 6.5. The situation with compact admissible targets remains unclear. It is well known that there might be topological obstructions to existence of such targets, which happens already for the classical Cauchy–Riemann system [56]. In order to clarify this issue it is necessary to achieve better understanding of geometric conditions on admissible targets, which can be hopefully done in terms of transversality to certain subspace of the Grassmanian $G(2m(n), 4m(n))$. This may be done in some simple cases, for example, the necessary "algebraic analysis" of the Moisil-Theodoresco system is presented in [140]. In the general case this seems to be quite difficult and it is even unclear what is the dimension of the subset of "prohibited" $2m(n)$-subspaces. This point of view is related to some other approaches to the construction of admissible targets which will be briefly discussed below.

Analyzing the proof of Theorem 6.1 and taking into account the previous remark, one sees that it may be extended to certain irreducible systems which leads to the desired result. We would like to point our that despite certain analogies with analytic discs, the situation with hh cells is much more subtle. In particular, the restriction to systems of the $D(S)$ type cannot be just omitted.

For example, the most straightforward generalization of analytic discs attached to totally real surfaces [56] would be to consider the Fueter system in $\mathbb{R}^4 \cong \mathbb{H}$ (quaternionic regular functions [62]) and try to construct Fueter cells attached to 4-dimensional submanifolds in $\mathbb{R}^8$. However, it turns out that in this way one cannot obtain a reasonable Fredholm theory for such cells, since in this situation Gromov's operator is never Fredholm. The latter fact follows directly from Proposition 6.1 because the resulting system has $2 = l_1 \neq l_2 = 0$.

We conclude the section by mentioning some geometric problems suggested by our approach.

A natural problem raised by our results is to understand how can one characterize admissible targets geometrically. Gromov's general approach to solving of under-determinate systems [73], suggests that this issue should be related to certain special subsets of appropriate Grassmanians. Indeed, some first steps in this direction can be done in a quite natural way and we proceed by a brief discussion of these matters.

Actually, a more comprehensive investigation of these connections shows that they may be conveniently described in terms of so-called *Grassmanian geometries* and *calibrations*, in the sense of [74]. We do not discuss relations to calibrated geometries but some of those ideas are implicitly present in the comments to follow.

For a given GCRS, it is also interesting to investigate what can be the minimal possible dimensions k of target manifolds for which one can derive Fredholmness of Gromov's operators. Gromov's h-principle suggests that admissible targets should satisfy some transversality condition with respect to certain special subset of Grassmanian $Gr(k, 2m)$ defined by the characteristic matrix of the system in question.

In order to make this idea more precise let us first reexamine the classical case of analytic discs. Results of Gromov [73] and Alexander [6] translated to our language mean that admissible targets for analytic discs are precisely totally real submanifolds of $\mathbb{C}^k$. For $k = 2$, the condition of total reality means that the image of Gauss mapping Γ_M of a submanifold M does not intersect the subset of complex lines in $Gr_{\mathbb{R}}(2,4)$. Since target M is in this case two-dimensional, this suggests to consider generic targets M such that Γ_M is transversal to the two-dimensional subset of complex lines $Gr_{\mathbb{C}}(1,2)$ in four-dimensional real Grassmanian $Gr_{\mathbb{R}}(2,4)$.

For such generic targets, their tangent planes can coincide with complex lines only at isolated points and one may wish to eliminate these points by deforming M. For compact M, it is well known [16] that the only obstruction for elimination of points with complex tangencies is given by the Euler characteristic $\chi(M)$. It may be also shown that, for non-compact contractible M homeomorphic to $\mathbb{R}^2$, the set of embeddings into $\mathbb{R}^4$ without complex tangencies is open and dense in the set of all such embeddings. The latter statement is exactly the special case of Theorem 6.1 for the classical Cauchy–Riemann system in $\mathbb{R}^2$.

Thus it becomes clear that admissible targets may admit characterization by some genericity conditions like transversality, and in order to find such conditions one should try to describe the subset of n-planes in $\mathbb{C}^k$ which can be represented as images of differentials of solutions to system S. Notice first that this is exactly what happens in the classical case, since for the usual Cauchy–Riemann systems these images are the complex lines.

Indeed, it is immediate to see that the most general form of a Jacobian matrix of a CR-solution (analytic disc) with values in $\mathbb{C}^2$ is:

$$\begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix},$$

where a, b, c, d are arbitrary real numbers. It is also clear that a vector expressed by the second column of this matrix is equal to $\imath$ times the vector expressed by the first column. So the image of the corresponding operator is a complex line and it is clear that every complex line may be obtained in this way. Of course the same holds for arbitrary value of the complex dimension k: the set of tangent planes to analytic discs coincides with the subset of complex lines in $Gr_{\mathbb{R}}(2, 2k)$.

Admissible targets in this case coincide with totally real ($2k$-dimensional) submanifolds. Notice that they are not generic $2k$-dimensional manifolds because those may have complex tangencies and actually homological properties of the set of complex tangencies are closely related to the topology of the target submanifold [56].

Similar considerations can be performed for the Fueter system. It is also instructive to have a look at the first irreducible system with non-equal (real) dimensions of the source and the target (i.e., $n + 1 \neq 2m$ in our notation). This is of course the Moisil-Theodoresco system ($n = 2$, $m = 2$). A simple calculation shows that tangents to its solutions are precisely the 3-dimensional subspaces in $\mathbb{R}^4$ generated by columns of matrices of the form

$$\begin{pmatrix} a & b & e-k \\ -l-r & e & f \\ k & l & a+q \\ b+f & q & r \end{pmatrix}.$$

Now it is quite simple to verify that every 3-dimensional subspace in $\mathbb{R}^4$ may be obtained as the tangent space of a MTS-solution. Obtaining a precise description of MTS-tangents in $\mathbb{R}^{4k}$ for $k \geq 2$ is a more delicate task. A further analysis of this problem shows that explicit description of this subset is closely related to certain homological problems. Similar problems were considered in [140] where Gröbner bases and syzygies for the Moisil-Theodoresco system are computed using computer programme

CoCoA (cf. [3] where the same problems are studied in the case of Fueter system).

As was already noticed, for $n = 2, 3$ one can indicate explicit geometric conditions on T_pM for a target manifold M to be admissible. This follows from explicit criteria of Fredholmness for RHPs for Moisil-Theodoresco and Fueter systems obtained in [142]. It would be interesting to obtain similar results for general GCRSs.

We would also like to mention the general problem of computing the index of an elliptic RHP for GCRS. In principle this is possible using general results of Atiyah and Bott, which should lead to explicit formulae of Dynin-Fedosov type [44], [52], but it does not seem that somebody have ever written down those explicit formulae in terms of the characteristic matrix and boundary condition. Thus it would be illuminating to obtain an exact recipe, or even an algorithm applicable in concrete situations. In low dimensions, some useful preparatory work for developing such an algorithm was done in [142].

We conclude the section by mentioning another promising perspective which emerges from the aforementioned connection between special Grassmanians and calibrated geometries. It is concerned with finding a proper calibration for a given GCRS. For the classical Cauchy–Riemann system this may be worked out in full detail and it turns out that the desired calibration is provided by the properly normalized Kähler form on the target space [74]. In fact, many properties of families of analytic discs may be derived directly from this interpretation, so one may hope that finding a proper calibration will be useful for achieving further progress in the theory of hyper-holomorphic cells.

7. Riemann–Hilbert Monodromy Problem

As was mentioned in the introduction, the Riemann–Hilbert transmission problem (or the linear conjugation problem for holomorphic functions) is closely related to another problem studied by the same two great mathematicians, namely, the so-called *Riemann–Hilbert monodromy problem* (or *Hilbert's* 21*st problem*, or else *Hilbert problem* 21) [7]. The latter problem is formulated in terms of systems of ordinary differential equations with the so-called *regular singular points* [36], [7]. For brevity, such systems will be called *regular systems.*

Regular systems are interesting from many points of view and their theory is rich with deep analytic and geometric results [36], [7], [34]. For example, there exist a lot of interesting results about the behaviour of solutions of such equations in a neighborhood of a singular point [36], [7]. Recently, intensive study has begun of global solutions of regular systems, i.e. solutions continued to the whole domain of definition of the equation. This domain is usually an algebraic or analytic variety whose topological invariants (e.g., homology or homotopy groups) influence the form of global solutions.

We only consider the case when the domain is a compact manifold of complex dimension one, which amounts to discussing regular systems on compact Riemann surfaces. This case is most closely related to the Riemann–Hilbert transmission problem and these interrelations were actively studied (see, e.g., [139], [36], [7]). It should be noted that some interesting geometric aspects appear already in the case of the Riemann sphere when solutions may be constructed in terms of Birkhoff factorization of piecewise constant matrix functions [139], [7], [149].

In the sequel we discuss some known results about the Riemann–Hilbert monodromy problem and their connections with Birkhoff factorization and transmission problems. Our presentation of the background and main results concerned with the Riemann–Hilbert monodromy problem is based on [63], [64], [65] and several useful discussions with G.Giorgadze. The point of view and setting is aimed at revealing new connections between Riemann–Hilbert transmission problem and the Riemann–Hilbert monodromy problem, especially the ones between the GRHP for compact Lie groups considered in Section 2 and regular G-systems studied by G.Giorgadze. It is the author's belief that further exploration of interrelations between these two topics may lead to new interesting developments in the spirit of geometric approach of preceding sections.

We recall first a precise formulation of the Riemann–Hilbert monodromy problem (RHMP). Fix a compact Riemann surface X together with a discrete subset S of it. Assume also that a representation of its fundamental group $\varrho : \pi_1(X \setminus S, z_0) \to \mathrm{GL}(n, \mathbb{C})$ is given. The problem then consists in constructing such a system $df = \omega f$ of differential equations on X whose singular set coincides with S, while the *monodromy group* induced by this system is $G = \mathrm{im}\ \varrho \subset \mathrm{GL}(n, \mathbb{C})$.

Some assumptions about the nature of singular points are usually made in this setting. The two most important cases are when all points of S are assumed to be *regular singular points* [7] or they are all assumed to be of the Fuchs type [7]. Correspondingly, one speaks of the monodromy problem for regular or Fuchsian systems.

We will be basically concerned with regular systems but it should be noted that Fuchsian systems have always been an object of special interest. One of the reasons is that by Lappo-Danilevsky theorem [106] such a system can be explicitly constructed from the monodromy matrices $M_1, M_2, \ldots, M_m \in \mathrm{GL}(n, \mathbb{C})$. It is also remarkable that Hilbert problem 21 in its original formulation refers to Fuchsian systems.

The monodromy representation ϱ enables one also to construct a holomorphic bundle $E' \to X \setminus S$ on the noncompact Riemann surface $X \setminus S$ for which $\nabla' = d - \omega$ is a holomorphic connection. There exists a well known construction [36], [7] which permits to extend the bundle (E', ∇') to a holomorphic bundle (E, ∇) with a regular connection on X. Such an extension is not unique, but there exists a so-called *canonical extension*

(E°, ∇°) whose holomorphic triviality for $X \cong \mathbb{C}P^1$ is a sufficient condition for the solvability of the Hilbert problem 21 [25].

Another sufficient condition for the existence of a system of Fuchs type is the irreducibility of representation ϱ [25]. Holomorphic classification of holomorphic bundles on compact Riemann surfaces has a long history, it arose in several contexts and after the works of G.Birkhoff, A.Grothendieck, H.Röhrl, M.Atiyah, D.Mumford, R.Narasimhan and T.Seshadri became a classical and well understood topic. In the case of the Riemann sphere, the classification is given by the following synthesis of theorems of Birkhoff and Grothendieck which is usually called the *Birkhoff-Grothendieck theorem.*

Birkhoff–Grothendieck theorem. *Each holomorphic bundle $E \to \mathbb{C}P^1$ on the Riemann sphere $\mathbb{C}P^1$ decomposes into a sum of line bundles: $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$, the integers $k_1 \geq \cdots \geq k_n$ being the Chern numbers of those line bundles.*

Classification of holomorphic bundles on Riemann surfaces of genus $g \geq 1$ has been accomplished with the aid of holomorphic connections by M. Atiyah, who assigned to each bundle $E \to X$ an element $b(E) \in H^1(X; \Omega^1)$ of the cohomology group $H^1(X; \Omega^1)$. Vanishing of this element $b(E)$ is necessary and sufficient for the existence of a holomorphic connection on $E \to X$.

D.Mumford determined an important subclass of holomorphic bundles $E \to X$, $g \geq 2$, the so-called *semistable bundles*, while R.Narasimhan and T.Seshadri showed that a bundle is semistable if and only if it is induced by an irreducible unitary representation $\varrho : \pi_1(X \setminus \{x_0\}; z_0) \to \mathrm{U}(n, \mathbb{C})$ of the fundamental group of the surface $X \setminus \{x_0\}$, where $x_0 \in X$ is some point. They also obtained a criterion of stability for such bundles. A new elegant proof of the latter result was given by S.Donaldson and we reproduce here the formulation of this criterion given in [40].

Stability criterion (Narasimhan, Seshadri, Donaldson). *An indecomposable holomorphic bundle $E \to X$ is semistable if and only if there exists a unitary connection ∇ on E having constant central curvature $*F_\nabla = -2\pi i \mu(E)\mathbf{1}$, where $\mu(E) = \mathrm{degree}(E)/\mathrm{rank}(E)$, $*$ is a Hodge operator, and $\mathbf{1}$ is the identity matrix.*

This result relates to the Riemann–Hilbert monodromy problem as follows: for a representation $\varrho : \pi_1(X \setminus \{x_0\}, z_0) \to \mathrm{U}(n)$ there exists a system $df = \omega f$ of differential equations on X for which x_0 is a regular singular point and its monodromy coincides with ϱ. Thus $\nabla = d - \omega$ will be a connection with a regular singularity on the holomorphic bundle $E_\varrho \to X$, and since $*F_\nabla$ is constant, one has $D_\nabla * F_\nabla = 0$, which means that ∇ is a Yang-Mills connection [8]. A wider class of Yang-Mills connections can be obtained from a linear elliptic system $\frac{\partial}{\partial \bar{z}} f(x) = A(z) f(z)$, where $\frac{\partial}{\partial \bar{z}}$ is the derivative in the Sobolev sense, $A(z)$ is a square matrix function of rank n with entries of class L_p [64], [65]. This obviously brings us in the situation of the Riemann–Hilbert transmission problem for elliptic system which was studied in previous sections.

As was already mentioned, the latter problem can be solved with the aid of the Birkhoff factorization of the coefficient matrix function $g(t)$, which means that $g(t)$ can be represented in the form $g(t) = g_+(t)d_K(t)g_-(t)$, where $g_\pm(t)$ are holomorphic, respectively, on $B_\pm$ and g_- vanishes (or satisfies a finiteness condition) at ∞. As usual, $d_K(t) = \mathrm{diag}(t^{k_1}, \dots, t^{k_n})$ is a diagonal matrix function with some integers $k_1 \geq k_2 \geq \cdots \geq k_n$.

An explicit connection between the Riemann–Hilbert monodromy problem and the Riemann–Hilbert transmission problem can be obtained by taking in the role of the coefficient $g(t)$, a piecewise constant function which relates to the monodromy matrices $M_1, \dots, M_m$ via the equality $g(t) = M_j \cdots M_1$, for t belonging to the arc $\langle s_j, s_{j+1} \rangle$, where $s_j \in S$, $j = 1, \dots, m$. Traditionally such a problem is reduced to a problem with the coefficient of the Hölder class and is then solved using the Birkhoff factorization [113], [149]. While dealing with the monodromy problem for the system $\frac{\partial}{\partial \bar{z}} f(x) = A(z)f(z)$, and replace the Birkhoff factorization by the so-called Φ-factorization [113], [65].

As was mentioned, according to Lappo-Danilevsky [106] it is possible to express analytically coefficients of a Fuchsian system by the monodromy matrices, provided these matrices satisfy certain conditions. Questions of analytic solvability of the corresponding differential equations appear to be related with the *differential Galois theory* [7], [146].

An algebraic version of the Riemann–Hilbert monodromy problem is known in the differential Galois theory under the name of *inverse problem* which is formulated as follows. Let k be a differential field with the field of constants C and $D(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y$ be a differential operator with coefficients in K. To the operator D one assigns the so-called *Picard-Vessiot* field K, whose automorphism group G is the *Galois group* of the equation $D(y) = 0$, isomorphic to some subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Inverse problem of differential Galois theory. Given a group G, find an extension of k with Galois group G.

This problem can be investigated by the methods similar to the ones used in the study of the Riemann–Hilbert monodromy problem [7]. As is well known this problem is closely related to the Riemann–Hilbert monodromy problem for Fuchsian equations. For the sake of completeness we present below the definitions of regular singular points and Fuchsian systems [36], [7].

Consider a system of ODE on a small disk $U \subset \mathbb{C}$ with center 0,

$$\frac{df}{dz} = A(z)f(z), \tag{7.1}$$

where $A(z)$ is a holomorphic matrix function on $U^* = U \backslash \{0\}$ and

$$f(z) = (f^1(z), \dots, f^n(z)) \in \mathbb{C}^n$$

is the unknown vector function.

Let $p : \widetilde{U^*} \to U^*$ be the universal covering of U^* and let ξ and z denote the local coordinate on $\widetilde{U^*}$ and U^*, respectively.

The system (7.1) has n linearly independent holomorphic solutions in a small neighbourhood of $z_0 \in U^*$. Denote the space of solutions by $\Re$. If $f \in \Re$, then f is a holomorphic function on $\widetilde{U^*}$.

Definition 7.1. A multivalued function f has *moderate growth* if for any sector $\Sigma = \{z | \theta_0 \le arg\, z \le \theta_1, 0 < |z| < \epsilon\}$, where ϵ is small, there is an integer $k > 0$ and a constant $C > 0$ such that

$$|f(z)| < C \frac{1}{|z|^k}$$

for all $z \in \Sigma$.

A point z_0 is called a regular singular point of a system of differential equation if in any punctured angular sector around z_0 the local holomorphic solutions have moderate growth. The system (7.1) is called Fuchsian if 0 is a pole of order one for the matrix valued function A(z).

Analogously one defines the regular singularity of the n-th order differential equation

$$f(z)^{(n)}(z) + a_1(z) f(z)^{(n-1)}(z) + \cdots + a_{n-1}(z) f(z)^{'}(z) + a_n(z) f(z) = 0. \quad (7.2)$$

Let Γ be the group of deck transformations of the covering

$$p : \widetilde{U^*} \to U^*.$$

If $\alpha \in \Gamma$, then α defines the automorphism $\alpha^* : \Re \to \Re$ of the solution space in such manner:

$$\alpha^* f = f \circ \alpha^{-1}, \text{ i.e. } (\alpha^* f)(\xi) = f(\alpha^{-1}\xi).$$

Clearly, $\alpha^* f$ is also a solution to (7.1) and therefore a map

$$\rho : \Gamma \to GL(n, \mathbf{C}), \quad \alpha \longmapsto \alpha^*. \quad (7.3)$$

can be defined. If $\beta \in \Gamma$ is another element, then $(\alpha\beta)^* = \alpha^* \beta^*$, i. e. the map (7.3) is a homomorphism. Thus

$$f = (f \circ \alpha)\rho(\alpha). \quad (7.4)$$

The homomorphism ρ is called the *monodromy representation* corresponding to the system (7.1).

Let $\Phi(z)$ be a fundamental system of solutions to (7.1) and let $\Phi_1(z)$ be any invertible matrix function satisfying the following matrix differential equation:

$$\frac{d\Phi_1}{dz} = A(z)\Phi_1(z).$$

Then $\Phi_1(z) = \Phi(z)G$ with some constant matrix $G \in GL(n, \mathbb{C})$. Instead of (7.4) we get then $\Phi_1(z) = (\Phi_1 \circ \alpha)\rho_1(\alpha)$ with some

$$\rho_1 : \Gamma \to GL(n, \mathbb{C}).$$

So

$$\Phi(z)G = (\Phi(z)G \circ \alpha)\rho_1(\alpha) = (\Phi \circ \alpha)G\rho_1(\alpha).$$

But $\Phi(z) = (\Phi(z) \circ \alpha)\rho(\alpha)$, thus $(\Phi(z) \circ \alpha)\rho(\alpha)G = (\Phi(z) \circ \alpha)G\rho_1(\alpha)$. Hence $\rho_1(\alpha) = G^{-1}\rho(\alpha)G$, where G is the same for all α. We see that to a system (7.1) there corresponds a class of mutually conjugate representations $\rho : \Gamma \to GL(n, \mathbb{C})$.

This conjugacy class is also called the *monodromy representation* or simply *monodromy*. The group of deck transformations Γ is now the infinite cyclic group generated by the deck transformation α which corresponds to one trip around 0 counterclockwise. Clearly, $\ln \xi$ is a holomorphic function on $\tilde{U}^*$ and $\ln(\alpha\xi) = \ln \xi + 2\pi i$. Let $G = \rho(\alpha^{-1})$ so that

$$\Phi(\alpha\xi) = \Phi(\xi)G. \tag{7.5}$$

Let $E = \frac{1}{2\pi i} \ln G$, so that if λ_j are eigenvalues of G and μ_j of E, then $\mu_j = \frac{1}{2\pi i} \ln \lambda_j$. Denote $\tau_j = Re\ \mu_j$ and normalize the choice of ln demanding that $0 \leq \tau_j < 1$.

Introduce the function $\xi^E = e^{E \ln \xi}$ (which is holomorphic on $\tilde{U}^*$):

$$(\alpha\xi)^E = e^{E(\ln \xi + 2\pi i)} = \xi^E G.$$

Then by (7.5)

$$\Phi(\alpha\xi)(\alpha\xi)^{-E} = \Phi(\xi)GG^{-1}\xi^{-E} = \Phi(\xi)\xi^{-E}.$$

Hence $\Phi(\xi)\xi^{-E}$ can be considered as a single-valued holomorphic function on U^*.

The following result was established by H.Poincaré (see [7]).

Theorem 7.1. *Let $f(z)$ be some solution of the system* (7.1). *Then $f(z)$ can be represented in the form*

$$f(\xi) = Z(z)\xi^E,$$

where Z is holomorphic on U^*.

Proposition 7.1 ([7]). *Every coordinate function $f_j(\xi)$ of a solution $f(\xi)$ has the form*

$$f_j(\xi) = \sum_{p,q} \xi^{\tau_p} h_{p,q}(z) \ln^{l_q} \xi, \tag{7.6}$$

$$0 \leq \text{Re } \tau_q < 1, l_q \in \mathbb{Z}, l_q \geq 0.$$

For a Fuchsian system (7.1) let

$$A = \text{Res}_{z=0} A(z),$$

then one has

$$\frac{df}{dz} = \frac{A}{z} f(z).$$

It is a classical observation that Fuchsian systems form a subclass of regular systems.

Proposition 7.2 (see, e.g., [139]). 1) *Every Fuchsian system is regular.*

2) *0 is a regular singular point for the equation* (7.2) *if and only if the functions $z^j a_j(z)$ are holomorphic at* 0.

Remark 7.1. The ordinary differential equation (7.2) is regular if and only if it is Fuchsian.

From the above proposition it follows that, for a Fuchsian equation, the coefficients $a_1(z), a_2(z), \dots, a_n(z)$ are holomorphic in some punctured neighborhood of 0 and $a_1(z)$ has there at most a pole of the 1-st order, ..., $a_i(z)$ at most a pole of the i-th order, ..., $a_n(z)$ at most a pole of the n-th order.

It turns out that (7.2) is regular at 0 if and only if the corresponding system for the vector

$$(f^1(z), \dots, f^n(z)) = \Big(x(z), \frac{dx(z)}{dz}, \dots, \frac{d^{n-1}x(z)}{dz^{n-1}}\Big),$$

i.e., the system

$$\frac{df(z)}{dz} = \begin{pmatrix} 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 \\ -a_n(z) & -a_{n-1}(z) & \dots & -a_1(z) \end{pmatrix} f(z)$$

is regular at 0.

The systems $\frac{df(z)}{dz} = A(z)f(z)$ and $\frac{dg(z)}{dz} = B(z)g(z)$ are holomorphically (meromorphically) equivalent if there exists a holomorphic (meromorphic) at 0 matrix function $H : V \to GL(n, \mathbb{C})$ such that the transformation $(z, f(z)) = (z, H(z)f(z))$ maps one equation to another, i.e.

$$B(z) = \frac{dH(z)}{dz} H^{-1}(z) + H(z)A(z)H^{-1}(z).$$

If two systems of equations are equivalent then their monodromy groups are conjugate. Moreover, if 0 is a regular singular point for the system, this system is equivalent to $\frac{df}{dz} = \frac{A}{z} f(z)$, where A is a constant matrix.

A *meromorphic connection* at the point 0 is, by definition, a pair (F, ∇), where F is an n-dimensional vector space over the field $K = K[\frac{1}{z}]$, whereas $\nabla : F \to F$ is an operator which satisfies the Leibniz rule

$$\nabla(h, s) = \frac{dh}{dz} s + h\nabla s,$$

for each function $f \in K$ and $s \in F$.

Let $e_1, e_2, \dots, e_n$ be a basis of F and let ∇e_i be expressed in this basis in the following manner:

$$\nabla e_i = -\sum_{j=1}^{n} \theta_{ij}(z) e_j,$$

where $\theta = (\theta_{ij}(z)) \in \mathrm{End}(n, K)$. Then for $s = \sum_{j=1}^{n} s_j(z) e_i$ we obtain

$$\nabla s = \sum_{i=1}^{n} \Big(\frac{ds_i(z)}{dz} - \sum_{j=1}^{n} \theta_{ij}(z) e_j(z)\Big) e_i.$$

From the last formula it follows that $\nabla s = 0$ is equivalent to $\frac{ds}{dz} = \theta s$, or $(d - \theta)s = 0$. Let us denote the matrix-valued 1-form θdz by ω, then the system will be $(d-\omega)=0$ and the connection will be $\nabla = d-\omega$. Two connections are gauge equivalent if and only if corresponding systems of equations are equivalent.

Let X be a Riemann surface of genus g and $S=\{s_1,s_2,\dots,s_m\}$ be a set of marked points on X. Denote by $X_m = X\setminus S$. Let $\tilde{X}\to X_m$ be the universal covering map of X_m. It is a bundle with fibre $\pi_1(X_m,z_0)$, where $z_0\in X_m$. Also, $\pi_1(X_m,z_0)$ is isomorphic to the group of deck transformations of this covering and therefore acts on $\tilde{X}$.

Let

$$\rho:\pi_1(X_m,z_0)\to GL(n,\mathbb{C}) \tag{4.1}$$

be some representation.

Consider a trivial principal bundle $\tilde{X}\times GL(n,\mathbb{C})\to\tilde{X}$ (or vector bundle $\tilde{X}\times\mathbb{C}^n\to\tilde{X}$). The quotient space $\tilde{X}\times GL(n,\mathbb{C})/\sim$ gives a locally trivial bundle on X_m, where $\sim$ is an equivalence relation identifying the pairs $(\tilde{x},g)$ and $(\sigma\tilde{x},\rho(\sigma)g)$,for every $\tilde{x}\in\tilde{X}, g\in GL(n,\mathbb{C})$ (or $g\in\mathbb{C}^n$). Denote the obtained bundle by $\mathbf{P}_\rho\to X_m$ (or $\mathbf{E}_\rho\to X_m$) and call it the bundle associated with the representation ρ. In obvious form this bundle according to the transformation functions may be constructed in the following manner.

Let $\{U_\alpha\}$ be a simple covering of X_m, i. e. every intersection $U_{\alpha_1}\cap U_{\alpha_2}\cap\cdots\cap U_{\alpha_k}$ is connected and simply connected. For each U_α, we choose a point $z_\alpha\in U_\alpha$ and join z_0 and z_α by a γ_α starting at z_0 and ending at z_α. For a point $z\in U_\alpha\cap U_\beta$ we choose a path $\tau_\alpha\subset U_\alpha$ which starts at z_α and ends at z. It is well known that one can obtain a *cocycle* [139] by setting

$$g_{\alpha\beta}(z)=\rho\left(\gamma_\alpha\tau_\alpha(z)\,\tau_\beta^{-1}(z)\,\gamma_\beta^{-1}\right). \tag{4.2}$$

Indeed, it is immediate that

$$g_{\alpha\gamma}(z)=g_{\beta\alpha}(z)$$

and

$$g_{\alpha\beta}g_{\beta\gamma}(z)=g_{\alpha\gamma}(z)$$

on $U_\alpha\cap U_\beta\cap U_\gamma$.

The cocycle $\{g_{\alpha\beta}(z)\}$ does not depend on the choice of z. Hence from this cocycle we obtain a *flat* vector (or principal) bundle [36], [7], which is denoted by $\mathbf{E}'_\rho$ ($\mathbf{P}'_\rho$). Let $\{t_\alpha(z)\}$ be a trivialization of our bundle, i. e.

$$t_\alpha : p^{-1}(U_\alpha)\to GL(n,\mathbb{C})$$

is a holomorphic mapping. Consider the matrix valued 1-form $\{\omega_\alpha\}$:

$$\omega_\alpha=-t_\alpha^{-1}dt_\alpha.$$

$\{g_{\alpha\beta}(z)\}$ are constant on the intersection $U_\alpha\cap U_\beta$ and $g_{\alpha\beta}(z)t_\beta(z)=t_\alpha(z)$, so on $U_\alpha\cap U_\beta$ the identity $\omega_\alpha=\omega_\beta$ holds. Indeed, replacing t_β by $t_\beta^{-1}g_{\alpha\beta}$ in the expression $\omega_\beta=-t_\beta^{-1}dt_\beta$, we obtain

$$\omega_\beta=-t_\alpha^{-1}g_{\alpha\beta}(z)\,dt_\alpha g_{\alpha\beta}^{-1}(z)=-t_\alpha^{-1}dt_\alpha.$$

So, $\omega = \{\omega_\alpha\}$ is a holomorphic 1-form on X_m and therefore is a connection form of the bundle $\mathbf{P}'_\rho \to X_m$. The corresponding connection is denoted by ∇'. One can now extend the pair $(\mathbf{P}'_\rho, \nabla')$ to X. As the required construction is of local character, we extend $\mathbf{P}'_\rho \to X_m$ to the bundle $\mathbf{P}''_\rho \to X_m \cup \{s_i\}$, where $s_i \in S$.

First consider the extension of the principal bundle $\mathbf{P}'_\rho \to X_m$. Let a neighbourhood V_i of the point s_i meet $U_{\alpha_1}, U_{\alpha_2}, \dots U_{\alpha_k}$. As we noted when constructing the bundle from transition functions (4.1) only one of them is different from identity. Let us denote it by g_{1k}, then $g_{1k} = G_i$, where G_i is the monodromy which corresponds to the singular point s_i. Mark a branch of the multi-valued function $(\tilde{z} - s_i)^{E_i}$ containing the point $\tilde{s}_i \in \tilde{U}_i$ (where $E_i = \frac{1}{2\pi i} \ln G_i$). Thus the marked branch defines a function

$$g_{01} = (z - s_i)^{E_i}.$$

Denote by g_{02} the extension of g_{01} along the path which goes around s_i counterclockwise, and similarly for other points. At last on $U_i \cap U_{\alpha_k} \cap U_{\alpha_1}$ we shall have:

$$g_{0k}(z) = g_{01}(z) G_i = g_{01}(z) g_{0k}(z).$$

The function $g_{0k} : V_i \to GL(n, \mathbf{C})$ is the one defined at the point s_i, and takes there value coinciding with the monodromy matrix. It means, that we made extension of the bundle to the point s_i. In a neighbourhood of s_i one has

$$\omega_i = dg_{0k} g_{0k}^{-1} = E_i \frac{dz}{z - s_i}.$$

Thus we obtain the holomorphic principal bundle $\mathbf{P}_\rho \to X$ on surface X. The vector bundle associated to $\mathbf{P}_\rho \to X$, which we denote by $\mathbf{E}_\rho \to X$ and call *canonical*, need not be topologically trivial. Denote by ∇ a connection on this bundle. The holomorphic sections of $\mathbf{E}_\rho$ are solutions of the equation

$$\nabla f = 0 \iff df = \omega f.$$

Theorem 7.2 ([139]). 1) *The constructed system has regular singularities at points* $s_1, s_2, \dots, s_m$.

2) *The Chern number* $c_1(\mathbf{E}_\rho)$ *of* $\mathbf{E}_\rho \to X$ *is equal to*

$$c_1(\mathbf{E}_\rho) = \sum_{i=1}^{m} tr(E_i).$$

The triple (X, S, ρ) is called *Riemann data*, where X is a Riemann surface, $S \subset X$ denotes a finite subset of X, $\rho : \pi_1(X \setminus S, z_0) \to GL(n, \mathbb{C})$ is any representation with trivial kernel.

We can now formulate the general monodromy problem for Riemann surfaces.

Riemann–Hilbert monodromy problem for Riemann surfaces

It is required to construct a system of ODE

$$df = \omega f,$$

on a Riemann surface X for the given Riemann data (X, S, ρ), where S is the set of regular singular points of the system and its monodromy representation coincides with ρ.

The following fundamental result shows that RHMP is solvable in the class of regular systems.

Theorem 7.3 ([139], [36]). *For every Riemann data there exists a solution of the Riemann–Hilbert monodromy problem for ODE's with regular singularity.*

It seems appropriate to add that in the case of Riemann sphere this result can be derived from the theory of Riemann–Hilbert transmission problem [139]. Correspondingly, there emerges a possibility to translate the geometric constructions of Section 1 into the language of regular systems. There already exist some developments in this direction [63] but this possibility is not yet sufficiently explored.

The situation with solvability of RHMP for Fuchsian systems is much more subtle. Let us present a precise formulation of RHMP in this case.

Hilbert's 21st problem.

Let $s_1, \dots, s_m \in \mathbb{C}P^1$ be some points, with no ∞ among them, and let $\varrho : \pi_1(\mathbb{C}P^1 \setminus \{s_1, \dots, s_m\}, z_0) \to \mathrm{GL}(n, \mathbb{C})$ be a representation. For the representation ϱ, one looks for a Fuchsian system

$$df = \left(\sum_{j=1}^{m} \frac{A_j}{z - s_j} dz \right) f, \tag{7.7}$$

whose monodromy representation coincides with ϱ. In (7.7), the A_j are constant matrices satisfying the condition $\sum_{j=1}^{m} A_j = 0$.

Results of A. Bolibruch revealed that there exist representations ϱ which are not realizable for a Fuchs system in the above way (see, e.g., [25], [26], [7]). However it is known that any irreducible representation ϱ is realizable by a Fuchsian system [26]. Moreover, there exists a special class of reducible representations, which are realizable by Fuchsian systems. More precisely, a representation ϱ is called a *simple* representation if the Jordan normal forms of all the monodromy matrices G_j consist of a single Jordan cell.

Theorem 7.4 ([26]). *A simple representation is realizable as a monodromy representation of a Fuchsian system if and only if the splitting type of the canonical bundle induced by it (i.e. the one obtained by canonical extension) is the vector* $(k, \dots, k)$.

We now concentrate on the case of a Riemann surface of genus zero, i.e., the Riemann sphere $\mathbb{C}P^1$.

As was already mentioned, a precise relation between the Riemann–Hilbert monodromy problem and the Riemann–Hilbert transmission problem can be established in the following way [139]. Let $s_1, \dots, s_m$ be points of $\mathbb{C}P^1$ different from ∞. Let us connect these points consecutively by arcs:

s_1 with s_2, s_2 with s_3, etc., s_{m-1} with s_m, and s_m with s_1. We will obtain a closed contour, on which we will define the piecewise constant matrix function $a(t)$ which equals the matrix $M_j \cdots M_1$ on the interval $\langle s_j, s_{j+1}\rangle$, where M_j is the monodromy matrix corresponding to the singular point s_j in the Riemann–Hilbert monodromy problem.

Consider now the Riemann–Hilbert transmission problem for the piecewise constant matrix function $a(t)$. Traditionally this problem is reduced [157] to the Riemann–Hilbert problem in the Hölder class, which is solved by methods developed in the theory of singular integral equations. Specifically, the Birkhoff factorization of Hölder class matrix function was used to obtain the solution. Correspondingly, it is important to know partial indices of the coefficient. To this end one can use the formulae and algorithm described in Section 1. Thus we can conclude that the theories of RHTP and RHMP in many aspects may be developed in parallel. We would also like to point out that the RHMP has natural counterparts in higher dimensions which can be described as follows.

Consider a matrix of holomorphic differential forms

$$\omega = \sum_{1\le i<j\le n} \Omega_{ij} d\log(z_i - z_j) \tag{7.8}$$

where the Ω_{ij}, $1 \le i < j \le n$ are $m \times m$-matrices with entries in $\mathbb{C}$.

Let

$$E \to X_n$$

be a trivial vector bundle of rank m, with a frame $(e_1, \dots, e_m)$. Denote, respectively, by $\mathcal{O}(E)$ and Ω^1 the space of holomorphic sections of E and the sheaf of holomorphic 1-forms on X_n. Let

$$\nabla : \mathcal{O}(E) \to \Omega^1 \otimes \mathcal{O}(E)$$

be the connection defined on the vectors e_i, $i = 1, \dots, m$, in the following way:

$$\nabla(e_i) = -\sum_{j=1}^{m} \omega_{ji} \otimes e_j, \qquad \omega_{ij} \in \Omega^1,$$

then the sections of the bundle $E \to X_n$ horizontal with respect to ∇ will satisfy the Fuchs type equation on X_n:

$$df = \omega f,$$

where ω has the form (7.8).

Proposition 7.3. *The system of differential equations*

$$df = \left(\sum_{1\le i<j\le n} \frac{\Omega_{ij}}{z_i - z_j} d(z_i - z_j) \right) f \tag{7.9}$$

is integrable if and only if the matrices Ω_{ij} satisfy the following conditions:

$$[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = [\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] \quad \textit{for} \quad i < j < k, \tag{7.10}$$

$$[\Omega_{ij}, \Omega_{kl}] = 0 \quad \textit{for all} \quad i,j,k,l \textit{ with } i \neq j \neq k \neq l. \tag{7.11}$$

Let F be the fundamental matrix of solutions of system (7.9). Then for any $\gamma \in \pi_1(X_n)$ one has for the analytic continuation of F along γ

$$\gamma^* F = F \cdot \varrho(\gamma), \qquad \text{where } \varrho(\gamma) \in \mathrm{GL}(m, \mathbf{C}).$$

One thus obtains a representation of the braid group

$$\varrho : P_n \to \mathrm{GL}(m, \mathbb{C}), \tag{7.12}$$

which is given by the Chen's iterated integral

$$\varrho(\gamma) = \mathbf{1} + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots .$$

One can now formulate the multi-dimensional analogue of RHMP.

Riemann–Hilbert monodromy problem.

For a given representation (7.12), does there exist an integrable Fuchs system of type (7.9) whose monodromy representation coincides with (7.12)?

The answer to this question is positive [99]. The corresponding result can be formulated as follows. Let us introduce notation

$$\gamma_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}, \qquad 1 \leq i < j \leq n.$$

Let $\varrho : P_n \to \mathrm{GL}(m, \mathbb{C})$ be a representation such that $\varrho(\gamma_{ij})$ is sufficiently close to $\mathbf{1}$ for all $1 \leq i < j \leq n$. Then there exist matrices Ω_{ij}, $1 \leq i < j \leq n$, sufficiently close to 0, satisfying the conditions (7.10), (7.11) and moreover the monodromy matrix of the system (7.9) coincides with ϱ.

This result was formulated in [99], its complete proof is given in [108]. It would be very interesting to establish some connections between the above problem and multidimensional Riemann–Hilbert transmission problems considered in Section 4. Up to the author's knowledge no such relations were ever mentioned in the literature.

Another type of generalization of RHMP is related to the problem of explicit construction of a system of differential equations with a given monodromy [106]. As was shown in [106], if the monodromy matrices $M_1, \ldots, M_m$ are close to the identity matrix, then coefficients A_j of the system of differential equations of the desired Fuchs type $\frac{df}{dz} = \left(\sum_{j=1}^m \frac{A_j}{z - s_j} \right) f$ are expressed by the singular points s_j and monodromy matrices M_j via noncommutative power series

$$A_j = \frac{1}{2\pi i} \tilde{M}_j + \sum_{1 \leq k,l \leq n} \xi_{kl}(s) \tilde{M}_k \tilde{M}_l + \cdots ,$$

where ξ_{kl} is a function depending on the singular points which can be given explicitly expressed through $s \in S$, and $\tilde{M}_j = M_j - \mathbf{1}$.

Analogous problems are considered in the differential Galois theory [146]. Recall that the inverse problem over a differential field K, with algebraically closed field of constants C can be formulated as a question: which linear algebraic groups over C are the differential Galois groups of linear differential equations over K? The answer for $\mathbb{C}(x)$ is given by the following theorem.

Theorem 7.5 ([146]). *For a linear algebraic group G over the field of complex numbers $\mathbb{C}$, there exists a regular differential equation $df = \omega f$ over $\mathbb{C}(x)$ with differential Galois group G.*

In the case of $\mathbb{C}$ there exists a close connection between the inverse problem of differential Galois theory and the Riemann–Hilbert monodromy problem. In particular the differential Galois group of the regular system can be easily found from its monodromy group.

Theorem 7.6 ([146]). *The differential Galois group of the regular system $df = \omega f$ coincides with the Zariski closure of the monodromy group of this system.*

Moreover, it is known that a linear algebraic group G is the differential Galois group of a system of differential equation $df = \omega f$, where ω is defined on Riemann surface X of genus $g > 1$ and has a given singular set $S = \{s_1, \ldots, s_m\}$, if and only if there exists a homomorphism $\varrho : \pi_1(X - S) \to G$ such that its image is dense in G for the Zariski topology. In particular, if $m > 0$ then G is a differential Galois group for this system if and only if G contains Zariski dense subgroup H generated by at most $2g + m - 1$ elements.

Thus we see that there is a number of interesting connections between the classical RHTP and RHMP. It is now natural to wonder if these connections can be extended to the case of Riemann–Hilbert transmission problems with coefficients in a compact Lie group G considered in Section 2. It is quite natural that such connections should involve principal G-bundles over Riemann surfaces. It seems remarkable that there really exist some results in this direction and we wish to present some of them in the rest of this section.

We have seen that on many holomorphic vector bundles there exist connections which have regular singularities at given points. This result can be generalized for holomorphic principal G-bundles. In order to explain that, we consider systems of the form $Df = \alpha f$, where α there is $\mathfrak{g}$-valued 1-form defined on the manifold M, and $f :\to G$ is G-valued unknown function.

Expression of the form $Df = \alpha f$ is called a *G-system* [63]. Regular singular points and monodromy representations of such systems can be defined by analogy with the classical case corresponding to $G = U(n)$ (cf. [63]).

Having all these concepts at hand, we can eventually formulate the monodromy problem for G-systems in the form which was suggested in [63].

Riemann–Hilbert monodromy problem for G-systems.

For a given homomorphism $\rho : \pi_1(M - S) \to G$, where S is a finite subset of a Riemann surface M, find a G-system with regular singularities on S whose monodromy coincides with ρ.

This problem appears to be non-trivial because solution does not always exist. We only present two simplest positive results concerning this problem. It seems interesting that in the formulations one has to take into account some topological properties of the group G. Let M be any connected Riemann surface (compact or not) and let $\rho : \pi_1(M) \to G_{\mathbb{C}}$ be a given homomorphism.

Theorem 7.7 ([65]). 1) *If $\pi_1(M)$ is a free group and $G_{\mathbb{C}}$ is connected then ρ is the monodromy homomorphism of certain G-system.*

2) *If $\pi_1(M)$ is free abelian and G is a connected compact Lie group with torsion free cohomology, and if $\operatorname{im}\rho \subset G$, then ρ is the monodromy homomorphism for some G-system.*

This theorem gives a positive solution to the RHMP in the case when $M = X - \{x_0\}$. The next result is concerned with the stable principal bundles. Recall that the notion of stability of a holomorphic principal G-bundles can be introduced by analogy with the case $G = U(n)$ [136].

Theorem 7.8 ([65]). *For a connected compact Lie group G, each stable holomorphic principal G-bundle has a connection with regular singularity at a given point.*

Thus we see that for G-systems there exists a reasonable theory parallel to the classical theory of Riemann–Hilbert monodromy problem. It would be interesting to interpret the results for G-systems in the language of Riemann–Hilbert transmission problem for group G studied in Section 2.

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