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# ON A CLASS OF GENERALIZED VARIATIONAL INEQUALITIES

**Abstract.** Generalized variational inequalities of a new class are proved. The best constants in some classical variational inequalities are obtained.

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The Perm Seminar on functional differential equations has elaborated new methods of investigation of non-classical variational problems (see, for example, [1, 3]). These methods allow to prove many known variational inequalities in a more effective way. Moreover, we now can obtain new variational inequalities, in particular, for functionals with deviating arguments. Here we give some of such results. Generalized variational inequalities of a new class are proved. The best constants in some classical variational inequalities are obtained.

Let $\mathbf{L}_2$ denote the space of real square integrable on $[0,1]$ functions, $\mathbf{W}_0$ be the space of real absolutely continuous on $[0,1]$ functions with the derivative from $\mathbf{L}_2$ and such that $x(0)=0$.

Our methods of the investigation of variational inequalities are based on the reduction to the problem on a minimum for the quadratic functional in the space $\mathbf{L}_2$:

$$\int_0^1 (z(t) - (Kz)(t))z(t)\,dt \to \min, \tag{1}$$

where $K : \mathbf{L}_2 \to \mathbf{L}_2$ is a self-adjoint bounded operator.

It is known (see, for example, [1]) that the problem (1) is solvable if and only if the maximum of the spectrum of the operator $K : \mathbf{L}_2 \to \mathbf{L}_2$ is not greater than 1. Moreover, the following assertion is valid ([4]).

**Theorem 1.** *Let* $\int_0^1 (Kz)(t)z(t)\,dt \geqslant 0$ *for every non-negative function* $z \in \mathbf{L}_2$. *Then the problem* (1) *is solvable if and only if the norm of the operator* $K$ *is less than or equal to* 1.

The conditions of Theorem 1 will be always fulfilled in this paper. So, the question on the solvability of the problem (1) is reduced to computing or estimating the spectral radius, which is equal to the norm, of the self-adjoint operator $K$.

§ **1.** In the paper [9] W. Troy estimated the values $\gamma$ for which the inequality

$$\int_0^1 \dot{x}^2(t)\,dt \geqslant \gamma \int_0^1 t^p\,|\dot{x}(t)x(t)|\,dt, \quad \gamma > 0, \quad p > -1, \tag{2}$$

holds for all continuously differentiable functions $x$ such that $x(0)=0$. Only for the case $p=0$ the best constant was known: $\gamma = 2$ (P. R. Beesack [2]).

Note that the functional of the variational problem, which corresponds to the inequality (2), is non-quadratic and non-differentiable at zero.

We consider the generalized inequality

$$\int_0^1 \dot{x}^2(t)\,dt \geqslant \gamma \int_0^1 t^p\,|\dot{x}(t)x(h(t))|\,dt, \qquad x(0)=0, \quad x(\xi)=0 \text{ if } \xi \notin [0,1], \tag{3}$$

where the function $h : [0,1] \to (-\infty, \infty)$ is measurable.

Define the function $\bar{h} : [0,1] \to [0,1]$ by

$$\bar{h}(t) = \begin{cases} h(t) & \text{if } h(t) \in [0,1], \\ 0 & \text{if } h(t) \notin [0,1]. \end{cases}$$

By $\bar{\gamma}_p$ denote the the maximal constant $\gamma$ for which the inequality (3) is valid for any $x \in \mathbf{W}_0$. In the case $h(t) \equiv t$ we can compute $\bar{\gamma}_p$, and in the general case we will estimate $\bar{\gamma}_p$.

**Lemma 1.** *The inequality* (3) *holds for every* $x \in \mathbf{W}_0$ *if and only if the variational problem*

$$\mathcal{J}(x) = \int_0^1 \Big(\dot{x}^2(t) - \gamma\, t^p \dot{x}(t)x(\bar{h}(t))\Big)\,dt \to \min, \qquad x(0)=0 \tag{4}$$

*has the solution* $x \equiv 0$.

*Proof.* If $x = 0$ is not a solution to the variational problem (4), then the functional $\mathcal{J}$ is negative for some function $x_0 \in \mathbf{W}_0$. Therefore

$$\int_0^1 (\dot{x}_0^2(t) - \gamma t^p|\dot{x}_0(t)x_0(\bar{h}(t))|)dt \leqslant \int_0^1 (\dot{x}_0^2(t) - \gamma t^p \dot{x}_0(t)x_0(\bar{h}(t))\,dt < 0$$

and the inequality (3) is not fulfilled for the function $x_0$.

If for some function $x_1 \in \mathbf{W}_0$ the inequality (3) does not hold, then $\mathcal{J}(x_2) < 0$ for the function $x_2(t) = \int_0^t |\dot{x}_1(s)|\,ds$. Indeed, $\dot{x}_2(t) = |\dot{x}_1(t)|$, $x_2(t) \geqslant |x_1(t)|$, $x_2(\bar{h}(t)) \geqslant |x_1(\bar{h}(t))|$, $t \in [0,1]$, and therefore

$$\mathcal{J}(x_2) = \int_0^1 \left(\dot{x}_2^2(t) - \gamma t^p|\dot{x}_2(t)x_2(\bar{h}(t))|\right)\,dt \leqslant \leqslant \int_0^1 \left(\dot{x}_1^2(t) - \gamma t^p|\dot{x}_1(t)x_1(\bar{h}(t))|\right)\,dt < 0. \quad \square$$

Thus the inequality (3) is valid if and only if the variational problem (4) is solvable.

The substitution $x(t) = \int_0^t z(s)\,ds$ reduces the problem (4) to the problem (1), where the kernel of the self-adjoint integral operator $K : \mathbf{L}_2 \to \mathbf{L}_2$ admits the representation

$$K(t,s) = \frac{\gamma}{2}(\chi(t,s)t^p + \chi(s,t)s^p),$$
$$\chi(t,s) = \begin{cases} 1 & \text{if } 0 \leqslant s \leqslant h(t) \leqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For $p > -1$ the operator $K$ is completely continuous, and its norm is equal to the maximum module of the eigenvalues. By theorem 1 it follows

**Theorem 2.** *Let $p > -1$. The inequality (3) is valid for any function $x \in \mathbf{W}_0$ if and only if $\gamma \leqslant \bar{\gamma}_p$, where $\bar{\gamma}_p$ is the smallest value $\gamma$ for which the Cauchy problem*

$$\dot{x}(t) = \frac{\gamma}{2}(t^p x(\bar{h}(t)) + \int_0^1 \chi(s,t)s^p\dot{x}(s)\,ds), \tag{6}$$
$$x(0) = 0$$

*has a non-zero solution.*

*Proof.* After the substitution $x = \int_0^1 z(s)\,ds$ we see that $\frac{\gamma}{\bar{\gamma}_p}$ is the maximal eigenvalue of the completely continuous operator $K$. Then by Theorem 1 the problem (1) is solvable if and only if $\gamma \leqslant \bar{\gamma}_p$. □

We can solve the problem (6) and find the exact constant $\bar{\gamma}_p$ only in single cases. In the general case we can, in particular, use the well-known estimate of the norm for an integral operator $K$ in the space $\mathbf{L}_2$:

$$\|K\| \leqslant \left(\int_0^1\int_0^1 (K^{(n)}(t,s))^2\,ds\,dt\right)^{\frac{1}{2n}}, \tag{7}$$

$n = 1, 2, \ldots$, where $K^{(n)}(t,s)$ is the kernel of the integral operator $K^n$.

Moreover, for any completely continuous self-adjoint integral operator $K$ in $\mathbf{L}_2$ with a non-negative kernel the following assertion is valid (see, for example, [5]):

$$\text{if } v(t) > 0 \text{ and } \mu v(t) \geqslant (Kv)(t),\ t \in [0,1], \text{ then } \|K\| \leqslant \mu. \tag{8}$$

The inequality (7) for $n = 1$ and Theorem 1 give the following results.

**Corollary 1.** *Let $e(t) = \{s \in [0,1] : s \leqslant \bar{h}(t)$ and $t \leqslant \bar{h}(s)\}$. Then for any function $x \in \mathbf{W}_0$ the inequality* (3) *holds if*

$$\gamma \leqslant \frac{\sqrt{2}}{\sqrt{\int\limits_0^1 (t^{2p}\bar{h}(t) + t^p \int\limits_{e(t)} s^p \, ds) dt}}.$$

In the Volterra case $h(t) \leqslant t$ on $[0,1]$ the measure of the set $e(t)$ is equal to zero for every $t \in [0,1]$. So, we have

**Corollary 2.** *Let $h(t) \leqslant t$ on $[0,1]$ and*

$$\gamma \leqslant \frac{\sqrt{2}}{\sqrt{\int\limits_0^1 t^{2p}\bar{h}(t) dt}}.$$

*Then for any function $x \in \mathbf{W}_0$ the inequality* (3) *is valid.*

Note that in the case $h(t) \equiv t$ Corollary 2 implies the estimate

$$\bar{\gamma}_p \geqslant 2\sqrt{p+1}, \tag{9}$$

which was obtained in [9].

**Corollary 3.** *Let*

$$\gamma \leqslant \frac{\sqrt{2}}{\sqrt{\int\limits_0^1 \bar{h}(t) t^p (t^p + \frac{1}{p+1}\bar{h}^p(t)) \, dt}}.$$

*Then for any function $x \in \mathbf{W}_0$ the inequality* (3) *is valid.*

Now consider the case $h(t) \equiv t$ in detail. We show that the best constant $\bar{\gamma}_p$ is expressed by the first zero of some Bessel function.

Let $p > -1$.

For $p > 0$ let $\theta_p$ be the smallest positive root of the modified Bessel function $I_{-\frac{1+2p}{1+p}}$.

For $p < 0$ let $\theta_p$ be the smallest positive root of the Bessel function $J_{-\frac{1+2p}{1+p}}$.

**Theorem 3.** *Let $h(t) \equiv t$, $p \neq 0$. The inequality* (2) *holds for all $x \in \mathbf{W}_0$ if and only if*

$$\gamma \leqslant \bar{\gamma}_p \equiv \frac{(p+1)^2\theta_p^2}{2|p|}.$$

*Proof.* In this case the problem (6) has the form

$$\dot{x}(t) = \frac{\gamma}{2}\Big(t^p x(t) + \int_t^1 s^p \dot{x}(s)\, ds\Big), \quad t \in [0,1], \quad x(0) = 0. \tag{10}$$

Therefore

$$\ddot{x}(t) = \frac{\gamma p}{2} t^{p-1} x(t), \quad t \in [0,1],$$
$$x(0) = 0.$$

The general solution to this problem is the function

$$c\sqrt{t} I_{\frac{1}{p+1}}\left(\frac{\sqrt{2\gamma p}}{p+1} t^{\frac{p+1}{2}}\right)$$

if $p > 0$, and the function

$$c\sqrt{t} J_{\frac{1}{p+1}}\left(\frac{\sqrt{-2\gamma p}}{p+1} t^{\frac{p+1}{2}}\right)$$

if $p < 0$, where $c$ is an arbitrary constant. This solution is a solution to the problem (10) if and only if $\dot{x}(1) = \frac{\gamma}{2} x(1)$. After elementary transformations we see that the smallest constant $\gamma$ such that there exists a non-zero solution to the problem (10) is equal to $\bar{\gamma}_p$. So, the assertion of the Theorem follows from Theorem 2. □

Thus

$$\bar{\gamma}_p = \frac{(p+1)^2 \theta_p^2}{2|p|}$$

is the unimprovable estimate of $\gamma$ in the inequality (2).

In some cases we are able to compute $\theta_p$ in the explicit form:

$\theta_1$ is the positive solution to the equation

$$e^{2\theta} = \frac{\theta + 1}{\theta - 1}, \tag{11}$$

which implies $\bar{\gamma}_1 = 2.878457679781$ to the last decimal place;

$\theta_{-\frac{1}{3}}$ is the smallest positive solution to the equation

$$\cos(\theta) = 0,$$

hence $\bar{\gamma}_{-\frac{1}{3}} = \frac{\pi^2}{6}$;

$\theta_{-\frac{3}{5}}$ is the smallest positive solution to the equation

$$\sin(\theta) = 0,$$

hence $\bar{\gamma}_{-\frac{3}{5}} = \frac{2\pi^2}{15}$;

$\theta_{-\frac{5}{7}}$ is the smallest positive solution to the equation

$$\mathrm{tg}(\theta) = \theta,$$

hence $\bar{\gamma}_{-\frac{5}{7}} = \frac{2}{35}\theta^2_{-\frac{5}{7}} = 1.153755918$ to the last decimal place.

Since the Bessel functions can be expressed by the elementary functions only in the case $p+1 = \frac{2}{2m+1}$, where $m$ is an integer, we can conclude that only in exceptional cases is it possible to obtain the exact values of the zeros $\theta_p$. Therefore, the question on estimates of $\bar{\gamma}_p$ arises.

Estimating $\|K\|$ by (7) for $n = 1,2,3,4$, we have respectively:

1) $\bar{\gamma}_p \geqslant 2\sqrt{p+1} \overset{\text{def}}{=} \gamma_p^{(1)}$ (it coincides with W.Troy's estimate (9));

2) $\bar{\gamma}_p \geqslant 2\left(\dfrac{(p+1)(p+2)(2p+3)}{p+6}\right)^{1/4} \overset{\text{def}}{=} \gamma_p^{(2)}$;

3) $\bar{\gamma}_p \geqslant 2\left(\dfrac{(4+3p)(5+4p)(2p+3)(p+2)^2(p+1)}{6p^3+100p^2+292p+240}\right)^{1/6} \overset{\text{def}}{=} \gamma_p^{(3)}$;

4) $\bar{\gamma}_p \geqslant \gamma_p^{(4)} \overset{\text{def}}{=}$

$$\overset{\text{def}}{=} 2\left(\frac{(7+6p)(6+5p)(p+1)(p+2)^2(2p+3)^2(5+4p)(4+3p)}{180p^5+5431p^4+31882p^3+74652p^2+77832p+30240}\right)^{1/8}.$$

Estimating $\|K\|$ with (8) for

$$v(t) = \left(1 - \frac{\mu}{p+2}\right)\frac{1}{1-\mu} + t^{p+1}, \quad t \in [0,1],$$

where

$$\mu = 4\,\frac{(p+1)(p+2)}{\sqrt{8(p+1)^3 + 9(p+1)^2 - 2p - 1} + 3p + 4},$$

we obtain

5) $\bar{\gamma}_p \geqslant \gamma_p^{(5)} \equiv 2\mu$ for $p \geqslant 0$.

Estimating $\|K\|$ with (8) for

$$v(t) = t^{-1/2+\varepsilon}, \quad t \in [0,1],$$

where $\varepsilon > 0$ is small enough, we get

6) $\bar{\gamma}_p \geqslant \gamma_p^{(6)} \equiv \dfrac{1}{2}\dfrac{1}{|p|}$ for $p \in (-1,-1/2]$.

We have

$$\gamma_p^{(1)} < \gamma_p^{(2)} < \gamma_p^{(3)} < \gamma_p^{(4)}$$

for all $p > -1$ except zero. So, the estimate 1), obtained in [9], is not exact anywhere besides the known case $p = 0$ [2].

Now, in particular, for $p = 1$ the exact constants are known. The above obtained estimates give

$$\gamma_1^{(1)} = 2.8284271;$$
$$\gamma_1^{(2)} = 2.8776356;$$
$$\gamma_1^{(3)} = 2.8784392;$$
$$\gamma_1^{(4)} = 2.8784572;$$
$$\bar{\gamma}_1 = 2.8784577$$

within the last digit.

Note that

a) $\gamma_p^{(4)} < \gamma_p^{(5)}$ for $p > 50.1$, hence the estimate $\gamma_p^{(5)}$ gives the best result for large values of $p$;

b) estimate 6) is the best one for all small $p$.

In the singular case $p = -1$ the operator $K$ is not completely continuous, but from the just now obtained estimate 6) we can conclude that

$$\int_0^1 \dot{x}^2(t)\,dt \geqslant \frac{1}{2}\int_0^1 \frac{1}{t}\,|\dot{x}(t)x(t)|\,dt,$$

for any $x \in \mathbf{W}_0$.

Moreover, we have

$$\bar{\gamma}_{-1} = \frac{1}{2}.$$

§ **2.** In [6] the variational inequality

$$\int_0^1 \dot{x}^2(t)\,dt \geqslant \delta \int_0^1 t^p x^2(t)\,dt,$$
$$x(0) = 0$$

was investigated for integer $p \geqslant -2$.

For all real $p > -2$ we consider the generalized inequality

$$\begin{gathered}\int_0^1 \dot{x}^2(t)\,dt \geqslant \delta \int_0^1 t^p |x(t)x(h(t))|\,dt, \\ x(0) = 0, \quad x(\xi) = 0 \text{ if } \xi \notin [0,1],\end{gathered} \tag{12}$$

where $h : [0,1] \to (-\infty, \infty)$ is a measurable function.

Note that the inequality (12) is valid for any $x \in \mathbf{W}_0$ if and only if the inequality

$$\int_0^1 \dot{x}^2(t)\,dt \geqslant \delta \int_0^1 t^p x(t)x(h(t))\,dt,$$

is valid for any $x \in \mathbf{W}_0$.

By $\bar{\delta}_p$ denote the maximal constant $\delta$ for which the inequality (12) is valid for any $x \in \mathbf{W}_0$. In the case $h(t) \equiv t$ we can compute $\bar{\delta}_p$, and in the general case we estimate $\bar{\delta}_p$.

The substitution $x(t) = \int\limits_0^t z(s)\,ds$ reduces the inequality (12) to the problem (1), where the kernel of the self-adjoint integral operator $K : \mathbf{L}_2 \to \mathbf{L}_2$ admits the representation

$$K(t,s) = \frac{\delta}{2}\Big(\int_t^1 \chi(\tau,s)\tau^p\,d\tau + \int_s^1 \chi(\tau,t)\tau^p\,d\tau\Big),$$

where the function $\chi$ is defined by (5).

Similarly to section 1 we have

**Theorem 4.** *The inequality* (12) *is valid for any function $x \in \mathbf{W}_0$ if and only if $\delta \leqslant \bar{\delta}_p$, where $\bar{\delta}_p$ is the smallest value $\delta$ for which the Cauchy problem*

$$\begin{aligned} \dot{x}(t) &= \frac{\delta}{2}\Big(\int_t^1 s^p x\big(\bar{h}(s)\big)\,ds + \int_0^1 \chi(s,t)s^p x(s)\,ds\Big), \\ x(0) &= 0 \end{aligned} \tag{13}$$

*has a non-zero solution.*

From this it follows

**Corollary 4.** *Let*

$$\delta \leqslant \frac{2}{\int\limits_0^1\Big(\int\limits_0^1 \chi(\tau,s)\tau^p\,d\tau + \int\limits_s^1 \chi(\tau,0)\tau^p\,d\tau\Big)ds}.$$

*Then for any function $x \in \mathbf{W}_0$ the inequality* (12) *holds.*

Note that for $h(t) \equiv t$ from this Corollary we have $\bar{\delta}_p \geqslant p+2$.

Let $g : [0,1] \to [0,1]$ be a continuous function such that if $\chi(t,s) = 1$, then $0 \leqslant g(s) \leqslant t \leqslant 1$.

**Corollary 5.** *Let*

$$\delta \leqslant \frac{2(p+1)(p+2)}{2p+3-(p+2)\int\limits_0^1 g^{p+1}(s)\,ds-(p+1)g^{p+2}(0)}.$$

*Then for any function $x \in \mathbf{W}_0$ the inequality* (12) *holds.*

Estimating the norm of the operator $K$ by (7) for $n=1$, we obtain

**Corollary 6.** *Let $p > -\frac{3}{2}$ and*

$$\delta^2 \leqslant \frac{2(2p+3)(p+2)^2}{4p+7}.$$

*Then for any function $x \in \mathbf{W}_0$ the inequality* (12) *holds.*

Consider the classical case $h(t) \equiv t$ in detail. Similarly to Theorem 3 we have.

By $\vartheta_p$ denote the minimal positive root of the Bessel function of the first kind $J_{-\frac{p+1}{p+2}}$.

**Theorem 5.** *Let $h(t) \equiv t$. The inequality* (12) *is valid for any function $x \in \mathbf{W}_0$ if and only if*

$$\delta \leqslant \bar{\delta}_p = \frac{(p+2)^2\vartheta_p^2}{4}.$$

**Corollary 7.** *Let $h(t) \equiv t$. Then*

$$\bar{\delta}_0 = \frac{\pi^2}{4}, \quad \bar{\delta}_{-\frac{4}{3}} = \frac{\pi^2}{9}, \quad \bar{\delta}_{-\frac{8}{5}} = \frac{\vartheta_{\frac{3}{2}}^2}{9},$$

*where $\vartheta_{\frac{3}{2}}$ is equal to the smallest positive solution to the equation $\operatorname{tg}\theta = \theta$.*

Estimating the norm of the operator $K$ by (7) for $n=1$, we obtain

**Corollary 8.** *Let $h(t) \equiv t$ and*

$$\delta \leqslant \sqrt{(2+p)(3+p)} \stackrel{def}{=} \delta_p^{(1)}. \tag{14}$$

*Then the inequality* (12) *is valid for any function $x \in \mathbf{W}_0$.*

Estimating the norm of the operator $K$ by (7) for $n=2$, we obtain

**Corollary 9.** *Let $h(t) \equiv t$ and*

$$\delta \leqslant \sqrt[4]{\frac{(2+p)(3+p)^2(5+2p)(7+3p)}{17+6p}} \stackrel{def}{=} \delta_p^{(2)}. \tag{15}$$

*Then the inequality* (12) *is valid for any function $x \in \mathbf{W}_0$.*

Let us remark that the inequality (14) follows from (15). For example, to the last decimal place we have

$$\begin{aligned}\delta^{(1)}_{-4/3} &= 1.0540,\\ \delta^{(2)}_{-4/3} &= 1.0955,\\ \bar{\delta}_{-4/3} &= 1.0966;\\ \delta^{(1)}_{0} &= 2.4495,\\ \delta^{(2)}_{0} &= 2.4673,\\ \bar{\delta}_{0} &= 2.4674.\end{aligned}$$

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