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PSEUDO-DIFFERENTIAL EQUATIONS IN ANISOTROPIC WEIGHTED BESSEL POTENTIAL SPACES WITH ASYMPTOTICS

Abstract. In the present paper we investigate the asymptotics of solutions of pseudo-differential equations in anisotropic weighted Bessel potential spaces on manifolds with smooth boundary. Using the method suggested by Duduchava and Chkadua we obtain complete asymptotic expansion of a solution near the boundary, when the given data has a certain discrete asymptotic type.

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Introduction

For studying the crack-type, screen-type and mixed problems of mathematical physics by the potential method, we need to investigate elliptic pseudo-differential operators on manifolds (surfaces) with boundary. It is well-known that solutions of such problems have singularities on the boundary regardless smootness properties of the given data. To characterize the behavior of solutions near the boundary it is adequate to talk in terms of asymptotics. The general analysis on manifolds with geometric singularities (Kondrat'ev [13], Schulze [18], [19], Dauge [4], etc.) gives us a possibility to analyze a local asymptotic expansion near the boundary as well as near conical points, edges or corners, see also [11], [12], [15], [16], [17].

From the point of view of applications we prefer an alternative approach based on the Wiener-Hopf method (cf. Eskin [7]), because it affords more efficient formulae for the exponents and coefficients of the asymptotic expansion. Moreover, it provides rather explicit results concerning the Fredholm criteria and solvability of pseudo-differential equations on manifolds with boundary. The method based on the factorisation of symbols was successfully applied by Chkadua and Duduchava [2] to derive full asymptotic expansion of a solution demanding additional smoothness of the given data.

The main goal of this paper is to invent an analogue of the spaces with asymptotics from Schulze [18], [19] for the case of anisotropic weighted Bessel potential spaces (L_p-theory) and to write full (discrete) asymptotics of a solution when the given data have a certain (discrete) asymptotic type. The result in this paper gives us a transparent rule on how exponents of the expansion depend on the symbol of pseudo-differential operator.

In this paper we restrict ourselves to the case of Hörmander symbol classes, though the results are true for the extended classes of symbols of pseudo-differential operators which are relevant with respect to the Wiener-Hopf factorisation ([2]).

1. Basic Notation

We use the following notation:

$\mathbb{N}$ – the set of non-negative integers;

$\mathbb{R}$ – the set of real numbers;

$\mathbb{R}_+$ – the set of positive reals;

$\overline{\mathbb{R}}_+$ – the set of non-negative reals;

$\mathbb{C}$ – the set of complex numbers.

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, then $|x| = \big(\sum_{j=1}^n x_j^2\big)^{\frac{1}{2}}$, $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$, $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}$, $D_x^\alpha = i^{|\alpha|}\partial_x^\alpha$, where i is the imaginary unit. For $\xi \in \mathbb{R}^n$ we set $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

$\mathcal{S}(\mathbb{R}^n)$ – the space of all rapidly decreasing functions.

$\mathcal{S}'(\mathbb{R}^n)$ – the space of temperate distributions.

$L_p(\mathbb{R}^n)$ – the space of all p-integrable functions on $\mathbb{R}^n$, $1 < p < \infty$.

The Fourier transform

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int\limits_{\mathbb{R}^n} e^{ix\xi}\varphi(x)\,\mathrm{d}\,x, \qquad x,\xi \in \mathbb{R}^n,$$

and its inverse

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}\varphi(\xi)\,\mathrm{d}\,\xi, \qquad x,\xi \in \mathbb{R}^n,$$

are bounded operators in both spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

$H^s_p(\mathbb{R}^n)$ – the Bessel potential space of smoothness $s \in \mathbb{R}$, $1 < p < \infty$, is defined as the subspace of all $u \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\|u\| = \left\{ \int\limits_{\mathbb{R}^n} |\mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}u(x)|^p\,\mathrm{d}\,x \right\}^{\frac{1}{p}}$$

is finite, cf. Triebel [21]. Note that $H^0_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

$\widetilde{H}^s_p(\mathbb{R}^n_+)$ – the subspace of $H^s_p(\mathbb{R}^n)$ consisting of those functions $\varphi \in H^s_p(\mathbb{R}^n)$, which are supported in the half space: supp $\varphi \subset \overline{\mathbb{R}}^n_+ := \mathbb{R}^{n-1}\times\overline{\mathbb{R}}_+$.

$H^s_p(\mathbb{R}^n_+)$ – the quotient space $H^s_p(\mathbb{R}^n_+) = H^s_p(\mathbb{R}^n)/\widetilde{H}^s_p(\mathbb{R}^n\backslash\overline{\mathbb{R}}^n_+)$; it can be identified with the space of distributions φ on $\mathbb{R}^n_+$ which admit an extension $\mathrm{e}^+\varphi \in H^s(\mathbb{R}^n)$. Therefore $\mathrm{r}^+ H^s(\mathbb{R}^n) = H^s(\mathbb{R}^n_+)$. Here

r^+ – the restricting operator $\mathrm{r}^+\varphi = \varphi\big|_{\mathbb{R}^n_+}$,

e^+ – an extending operator (the right inverse to r^+), $\mathrm{e}^+\varphi(x) = \varphi(x)$ if $x \in \mathbb{R}^n_+$ and $\mathrm{e}^+\varphi(x) = 0$ if $x \in \mathbb{R}^n\backslash\mathbb{R}^n_+$ for $\varphi(x)$ defined on $\mathbb{R}^n_+$.

A function $\omega \in C^\infty(\overline{\mathbb{R}}_+)$ is called a cut-off function if it is real-valued, equals 1 near 0 and has a bounded support.

A function $\chi \in C^\infty(\mathbb{R}^n)$ is called an excision function if it vanishes in a neigbourhood of 0 and equals 1 near infinity.

$[\xi]$ – any strictly positive function in $C^\infty(\mathbb{R}^n)$ which coincides with absolute value of ξ near infinity. Then we have $c_1[\xi] \le \langle\xi\rangle \le c_2[\xi]$ for suitable constants $c_1, c_2 > 0$.

Assume that a Fréchet space E is a (left) module over an algebra A. We then define for any $a \in A$ the space

$$[a]E = \{\text{completion of } \{ae : e \in E\} \text{ in } E\}.$$

Let $q \in \mathbb{C}$, $\tau, \varepsilon \in \mathbb{R}$, $l \in \mathbb{N}$ and fix the branch of the logarithm $\log(\tau+i\varepsilon) = \log|\tau+i\varepsilon| + i\arg(\tau+i\varepsilon)$, such that $\arg(\tau+i\varepsilon) = 0$ if $\tau > 0$, $\varepsilon = 0$. For each $\tau > 0$, the functions $(\tau+i\varepsilon)^q\log^l(\tau+i\varepsilon)$ and $(\tau+i0)^q\log^l(\tau+i0)$ for $\mathrm{Re}\, q > 1$ define regular functionals on $\mathcal{S}(R)$. For more details cf. Eskin [8].

Set

$$t^q_+ = \begin{cases} e^{q\log t} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \le 0, \end{cases}$$

then

$$
\begin{aligned}
(\tau + i\varepsilon)^q \log^l(\tau + i\varepsilon) &= \sum_{k=0}^{l} c_{lk}(y)\mathcal{F}_{t\to\tau}(t_+^{-q-1}\log^k t_+ e^{-\varepsilon t}), \quad \operatorname{Re} q < 0, \\
(\tau + i0)^q \log^l(\tau + i0) &= \sum_{k=0}^{l} c_{lk}(y)\mathcal{F}_{t\to\tau}(t_+^{-q-1}\log^k t_+), \quad \operatorname{Re} q < 0,
\end{aligned} \tag{1}
$$

where

$$
c_{lk}(q) = \frac{(-1)^k l!}{k!(l-k)!}\Big(\frac{\mathrm{d}}{\mathrm{d}\,q}\Big)^{l-k}\frac{e^{i\frac{\pi}{2}q}}{\Gamma(-q)}, \quad 0 \le k \le l;
$$

here $\Gamma(-q)$ is the Gamma-function, cf. [8], formulae (2.36), (2.37) and (2.39). It is easy to show that for $\operatorname{Re} q < 0$

$$
\begin{aligned}
t_+^{-q-1}\log^k t_+ e^{-\varepsilon t} &= \sum_{j=0}^{k} b_{kj}(q)\mathcal{F}^{-1}_{\tau\to t}((\tau + i\varepsilon)\log^j(\tau + i\varepsilon)), \\
t_+^{-q-1}\log^k t_+ &= \sum_{j=0}^{k} b_{kj}(q)\mathcal{F}^{-1}_{\tau\to t}((\tau + i0)\log^j(\tau + i0)),
\end{aligned} \tag{2}
$$

where $b_{kj}(q)$ are defined by the recurrence relation $b_{kk}(q) = \frac{1}{c_{kk}(q)}$ and

$$
b_{kj}(q) = -\frac{1}{c_{kk}(q)}\sum_{m=j}^{k-1} c_{km}(q)b_{mj}(q), \quad \text{for} \quad 0 \le j < k.
$$

Note that for $t \in \overline{\mathbb{R}}_+$ we often write t instead of t_+.

For the following proposition cf. [2], Lemma 2.8. A similar assertion can be found in [8], Remark 10.3 and in [1].

Proposition 1.1. *For a given constant $q \in \mathbb{C}$ and given functions $\{a_k(y, \operatorname{sgn} t)\}_0^m$, $a_k(., \pm 1) \in C^\infty(\mathbb{R}^{n-1})$, $k = 0, 1, \ldots, m$, the following representation holds*

$$
\begin{aligned}
\sum_{k=0}^{m} a_k(y, \operatorname{sgn} t)|t|^q \log^k |t| &= \sum_{k=0}^{m+\zeta(q)} b_k(y)(t - i0)^q \log^k(t - i0) + \\
&+ \sum_{k=\zeta(q)}^{m+\zeta(q)} c_k(y)(t + i0)^q \log^k(t + i0),
\end{aligned} \tag{3}
$$

where $y \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, $b_k, c_k \in C^\infty(\mathbb{R}^{n-1})$, $k = 0, \ldots, m + \zeta(q)$,

$$
\zeta(q) = \begin{cases} 0 & \text{if } \ q \notin \mathbb{Z}, \\ 1 & \text{if } \ q \in \mathbb{Z}. \end{cases}
$$

The representation (3) is unique.

Let $S^\mu(U \times \mathbb{R}^n)$ for $\mu \in \mathbb{R}$ and $U \subset \mathbb{R}^m$ open denote the space of all $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ that satisfy the symbol estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq c\langle\xi\rangle^{\mu-|\beta|}$$

for all $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$ and all $x \in K$ for arbitrary $K \Subset U$, $\xi \in \mathbb{R}^n$, with constants $c = c(\alpha,\beta,K) > 0$. The best constants in these symbol estimates form a semi-norm system in which $S^\mu(U \times \mathbb{R}^n)$ is a Fréchet space. We then have

$$S^{-\infty}(U \times \mathbb{R}^n) := \bigcap_{\mu \in \mathbb{R}} S^\mu(U \times \mathbb{R}^n) = C^\infty(U, \mathcal{S}(\mathbb{R}^n)).$$

Let $S^{(\mu)}(U \times (\mathbb{R}^n \backslash 0))$ be the space of all $f \in C^\infty(U \times (\mathbb{R}^n \backslash 0))$ with the property $f(x,\lambda\xi) = \lambda^\mu f(x,\xi)$ for all $\lambda \in \mathbb{R}_+$, $(x,\xi) \in U \times (\mathbb{R}^n \backslash 0)$. Then $\chi(\xi)S^{(\mu)}(U \times (\mathbb{R}^n \backslash 0)) \subset S^\mu(U \times \mathbb{R}^n)$ for any excision function $\chi(\xi)$. We then define $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$ to be the subspace of all $a(x,\xi) \in S^\mu(U \times \mathbb{R}^n)$ such that there are elements $a_{(\mu-j)}(x,\xi) \in S^{\mu-j}(U \times (\mathbb{R}^n \backslash 0))$, $j \in \mathbb{N}$, with

$$a(x,\xi) - \sum_{j=0}^{N} \chi(\xi) a_{(\mu-j)}(x,\xi) \in S^{\mu-(N+1)}(U \times \mathbb{R}^n)$$

for all $N \in \mathbb{N}$. Symbols in $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$ are called classical. The functions $a(x,\xi)$ (uniquely determined by a) are called homogeneous components of a of order $\mu - j$, and we call

$$\sigma_\psi^\mu(a)(x,\xi) := a_{(\mu)}(x,\xi)$$

the homogeneous principal symbol of order μ. We do not repeat here all known properties of symbol spaces, such as the relevant Fréchet topologies, asymptotic sums, etc., but we tacitly use them. For details we refer to Hörmander [9]. Finally, if a relation holds for symbols in S^μ and S^μ_{cl}, we shortly write $S^\mu_{(\mathrm{cl})}$.

When $m = 2n$, $U = \Omega \times \Omega$ for an open $\Omega \subset \mathbb{R}^n$, symbols in this case are also denoted by $a(x,x',\xi)$, $(x,x') \in \Omega \times \Omega$. The Leibnitz product the between symbols $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, $b(x,\xi) \in S^\nu(\Omega \times \mathbb{R}^n)$ is denoted by #, i.e.,

$$a(x,\xi)\#b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(D_\xi^\alpha a(x,\xi))\partial_x^\alpha b(x,\xi).$$

We define the space of classical or non-classical pseudo-differential operators to be

$$L^\mu_{(\mathrm{cl})}(\Omega) = \{\mathrm{Op}(a) : a(x,x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)\}.$$

Here, Op is the pseudo-differential action, based on the Fourier transform in $\mathbb{R}^n$, i.e.,

$$\mathrm{Op}(a)u(x) = \iint e^{i(x'-x)\xi} a(x,x',\xi)u(x')\,\mathrm{d}\,x'\,đ\xi,$$

$đ\xi = (2\pi)^{-n}\,d\,\xi$. As usual, this is interpreted in the sense of oscillatory integrals, first for $u \in C_0^\infty(\Omega)$, and then extended to more general distribution spaces. Note that $L^{-\infty}(\Omega) = \cap_{\mu\in\mathbb{R}} L^\mu(\Omega)$ coincides with the space of all integral operators in Ω with kernel in $C^\infty(\Omega \times \Omega)$, cf. [9]. Let $S^\mu_{(\mathrm{cl})}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n) = \{a = \tilde{a}\ |_{\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^n}\colon \tilde{a}(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R} \times \mathbb{R}^n)\}$, where $\Omega \subseteq \mathbb{R}^{n-1}$ is an open set. Then pseudo-differential operators with symbols $a \in S^\mu_{(\mathrm{cl})}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ are defined by the rule

$$\mathrm{Op}^+(a)u(x) = \mathrm{r}^+ \,\mathrm{Op}(\tilde{a})\mathrm{e}^+ u(x), \tag{4}$$

where $\tilde{a} \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is any extension of a to $\Omega \times \mathbb{R}$.

Define $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\mathrm{tr}}$ for $\mu \in \mathbb{Z}$ to be the subspace of all $a(x,\xi) \in S^\mu_{\mathrm{cl}}(\Omega_y \times \mathbb{R}_t \times \mathbb{R}^n_\xi)$ such that

$$D_t^k D_\eta^\alpha\{a_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j} a_{(\mu-j)}(y,t,-\eta,-\tau)\} = 0$$

on the set $\{(x,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : y \in \Omega, t = 0, \eta = 0, \tau \in \mathbb{R}\backslash\{0\}\}$, for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-1}$ and all $j \in \mathbb{N}$. Moreover, set $S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\mathrm{tr}} = \{a = \widetilde{a}|_{\Omega\times\overline{\mathbb{R}}_+\times\mathbb{R}^n} : \widetilde{a}(x,\zeta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\mathrm{tr}}\}$.

2. Anisotropic Weighted Bessel Potential Spaces

We now introduce the anisotropic weighted Bessel potential spaces $H_p^{(r,s),m}(\mathbb{R}^n)$ as a generalization of the spaces $H_p^s(\mathbb{R}^n)$.

Definition 2.1. By $H_p^{(r,s),m}(\mathbb{R}^n)$, for $r, s \in \mathbb{R}$, $m \in \mathbb{N}$, $1 < p < \infty$ we denote the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ endowed with the norm

$$\|u\| = \|u|H_p^{(r,s),m}(\mathbb{R}^n)\| := \sum_{k=0}^{m} \|\langle D_y\rangle^r \langle D\rangle^{s+k} t^k u|L_p(\mathbb{R}^n)\|,$$

where $x = (y,t) \in \mathbb{R}^n$, $y = (x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}$, $\xi = (\eta,\tau) \in \mathbb{R}^n$, $\eta = (\xi_1,\dots,\xi_{n-1}) \in \mathbb{R}^{n-1}$ and $\langle D\rangle^{s_1} := \mathrm{Op}(\langle\xi\rangle^{s_1})$ and $\langle D_y\rangle^{r_1} := \mathrm{Op}_y(\langle\eta\rangle^{r_1})$ (the pseudo-differential action with respect to the Fourier transform in y-variables).

Note that $H_p^{(0,s),0}(\mathbb{R}^n) = H_p^s(\mathbb{R}^n)$ and the corresponding "comp" and "loc" versions are denoted by $H^{(r,s),m}_{p,\mathrm{comp}}(\Omega)$ and $H^{(r,s),m}_{p,\mathrm{loc}}(\Omega)$, respectively.

In the following remark we collect some properties of $H_p^{(r,s),m}$ spaces.

Remark 2.2. (i) There are canonical continuous embeddings

$$H_p^{(r_1,s_1),m_1}(\mathbb{R}^n) \hookrightarrow H_p^{(r,s),m}(\mathbb{R}^n)$$

for $r_1 \geq r$, $s_1 \geq s$, $m_1 \geq m$.

(ii) The operator of multiplication by a function t^l induces a continuous operator

$$t^l I : H_p^{(r,s),m}(\mathbb{R}^n) \longrightarrow H_p^{(r,s+l),m-l}(\mathbb{R}^n)$$

for every $l \in \mathbb{N}$, $l \leq m$ and $r, s \in \mathbb{R}$.

(iii) The operator

$$\langle D_y\rangle^{r_1}\langle D\rangle^{s_1} : H_p^{(r,s),m}(\mathbb{R}^n) \longrightarrow H_p^{(r-r_1,s-s_1),m}(\mathbb{R}^n),$$

induces an isomorphism of spaces for arbitrary $r_1, s_1 \in \mathbb{R}$ and the inverse operator reads $\langle D_y\rangle^{-r_1}\langle D\rangle^{-s_1}$, cf. Chkadua and Duduchava [2].

(iv) The operator

$$\langle D_t\rangle^{s_1} : H_p^{(\infty,s),\infty}(\mathbb{R}^n) \longrightarrow H_p^{(\infty,s-s_1),\infty}(\mathbb{R}^n)$$

is bounded for all $s_1 \in \mathbb{R}^n$, where

$$\langle D_t\rangle^{s_1} := \mathrm{Op}(\langle\tau\rangle^{s_1})$$

and

$$H_p^{(\infty,s),m}(\mathbb{R}^n) := \bigcap_{r\in\mathbb{R}} H_p^{(r,s),m}(\mathbb{R}^n),$$

cf. [2].

Theorem 2.3. *Let $m \in \mathbb{N}, 1 < p < \infty$. If a symbol $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n \times \mathbb{R}^n)$ is independent of x for large x, then the operator*

$$a(x,D) := \mathrm{Op}(a) : H_p^{(r,s),m}(\mathbb{R}^n) \longrightarrow H_p^{(r,s-\mu),m}(\mathbb{R}^n)$$

is continuous for arbitrary $r, s \in \mathbb{R}$.

Proof. Indeed,

$$\|a(x,D)u|H_p^{(r,s-\mu),m}(\mathbb{R}^n)\| = \sum_{k=0}^{m} \|t^k a(x,D)u|H_p^{(r,s+k-\mu)}(\mathbb{R}^n)\| \leq$$
$$\sum_{k=0}^{m}\sum_{l=0}^{k} \frac{k!}{l!(k-l)!}\|(\partial_\tau^l a)(x,D)t^{k-l}u|H_p^{(r,s+k-\mu)}(\mathbb{R}^n)\| \leq$$
$$C_0\sum_{j=0}^{m} \|t^j u|H_p^{(r,s+j)}(\mathbb{R}^n)\| \leq C\|u|H_p^{(r,s),m}(\mathbb{R}^n)\|. \qquad \square$$

Let $\Omega \subset \mathbb{R}^{n-1}_y$, $\tilde\Omega \subset \mathbb{R}^{n-1}_{\tilde y}$ be open sets and $\varkappa : \Omega \to \tilde\Omega$ be a diffeomorphism. Then the pull-back

$$\varkappa^* : C_0^\infty(\tilde\Omega \times \mathbb{R}) \to C_0^\infty(\Omega \times \mathbb{R})$$

defined by $\varkappa^* u(y,t) := u(\varkappa(y),t),\ u(y,t) \in C_0^\infty(\tilde\Omega\times\mathbb{R})$, for all $(y,t) \in \Omega\times\mathbb{R}$, extends to an isomorphism

$$\varkappa^* : H^{(r,s),m}_{p,\mathrm{comp}}(\Omega \times \mathbb{R}) \to H^{(r,s),m}_{p,\mathrm{comp}}(\tilde\Omega \times \mathbb{R}).$$

The corresponding result on conordinate invariance is true for the loc-spaces.

Clearly, because of the specific choice of the diffeomorphism $\varkappa$ (i.e., it does not touch t-variable) there is no essential difference between the arguments for the case of the anisotropic Bessel potential spaces and $H^s_{p,\mathrm{comp}}$ spaces, so the proof of this assertion is left to the reader.

As in Section 1, we set

$$\widetilde{H}^{(r,s),m}_p(\mathbb{R}^n_+) = \{u : u \in H^{(r,s),m}_p(\mathbb{R}^n), \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+\},$$
$$H^{(r,s),m}_p(\mathbb{R}^n_+) = H^{(r,s),m}_p(\mathbb{R}^n)/\widetilde{H}^{(r,s),m}(\mathbb{R}^n \backslash \overline{\mathbb{R}}^n_+)$$

and

$$H^{(\infty,s),\infty}_p(\mathbb{R}^n) := \bigcap_{m\in\mathbb{N}} H^{(m,s),m}_p(\mathbb{R}^n).$$

Proposition 2.4. *The multiplication operator by the Heaviside function*

$$\Theta_+ I : H^{(r,s),m}_p(\mathbb{R}^n) \longrightarrow H^{(r,s),m}_p(\mathbb{R}^n),$$
$$\Theta_+(x) = \frac{1}{2}(1 + \operatorname{sgn} t), \quad x = (x_1, \dots, x_{n-1}, t),$$

is bounded provided

$$\frac{1}{p} - 1 < s < \frac{1}{p}, \quad 1 < p < \infty, \ \ r \in \mathbb{R}, \ \ m \in \mathbb{N}.$$

In particular, under the asserted conditions the space $\widetilde{H}^{(r,s),m}_p(\mathbb{R}^n_+)$ *and* $H^{(r,s),m}_p(\mathbb{R}^n_+)$ *can be identified.*

For the proof of this result cf. [2], Lemma 1.8.

Introduce the notation

$$\widetilde{H}^{(\infty,s),\infty}_p(\mathbb{R}^n_+) := \bigcap_{m\in\mathbb{N}} \widetilde{H}^{(m,s),m}_p(\mathbb{R}^n_+), \ H^{(\infty,s),\infty}_p(\mathbb{R}^n_+) := \bigcap_{m\in\mathbb{N}} H^{(m,s),m}_p(\mathbb{R}^n_+).$$

Lemma 2.5. *Let* $q \in \mathbb{C}, s \in \mathbb{R}$ *with* $\operatorname{Re} q < \frac{1}{p} - s$, $k \in \mathbb{N}$, *and fix a cut-off function* $\omega(t)$ *and* $\varepsilon \in \mathbb{R}_+$. *Then*

$$\omega(t) t^{-q} \log^k t \, v(y) \in \widetilde{H}^{(\infty,s),\infty}_p(\mathbb{R}^n_+)$$

and

$$t^{-q} \log^k t \, e^{-\varepsilon t} v(y) \in \widetilde{H}^{(\infty,s),\infty}_p(\mathbb{R}^n_+),$$

for all $1 < p < \infty$ *and* $v(y) \in C^\infty(\mathbb{R}^{n-1})$, $x = (y, t) \in \mathbb{R}^n_+$.

The easy proof is left to the reader.

3. Spaces with Asymptotics

In this sequel we introduce the anisotropic weighted Bessel potential spaces with asymptotics (cf. Definition 3.6 below). In order to see the relations to the wedge Sobolev spaces with asymptotics we recall some necessary material from the analysis on manifolds with edges, cf. Schulze [18], [19].

Let

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\,\mathrm{d}\,t$$

be the Mellin transform on the half-axis $\mathbb{R}_+ \ni t$, first defined for functions $u \in C_0^\infty(\mathbb{R}_+)$. The Mellin covariable $z \in \mathbb{C}$ varies on $\Gamma_\beta = \{z \in \mathbb{C} : \mathrm{Re} z = \beta\}$ for some $\beta \in \mathbb{R}$. The standard function and distribution spaces on $\mathbb{R}$ can be transformed to Γ_β by bijection $\Gamma_\beta \to \mathbb{R}$, $z \to \mathrm{Im}\, z$; in particular, we get the spaces $\mathcal{S}(\Gamma_\beta)$, $\mathcal{S}'(\Gamma_\beta)$, $L_2(\Gamma_\beta)$, $H^s(\Gamma_\beta)$, $S^\mu_{(\mathrm{cl})}(\Gamma_\beta), \ldots$. Then the weighted Mellin transform with the weight $\gamma \in \mathbb{R}$

$$(M_\gamma u)(z) = Mu(z)\big|_{\Gamma_{\frac{1}{2}-\gamma}}$$

induces an isomorphism

$$M_\gamma : t^\gamma L_2(\mathbb{R}_+) \longrightarrow L_2(\Gamma_{\frac{1}{2}-\gamma})$$

with the inverse

$$(M_\gamma^{-1} h)(t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} h(z)\,\mathrm{d}\,z.$$

Let $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ for $s, \gamma \in \mathbb{R}$ be the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\|\langle z\rangle^s (Mu)(z)\big|_{\Gamma_{\frac{1}{2}-\gamma}}\|_{L_2(\Gamma_{\frac{1}{2}-\gamma})},$$

where $\langle z\rangle = (1 + |z|^2)^{\frac{1}{2}}$. Clearly, $\mathcal{H}^{0,0}(\mathbb{R}_+) = L_2(\mathbb{R}_+)$ and $\mathcal{H}^{0,\gamma}(\mathbb{R}_+) = t^\gamma L_2(\mathbb{R}_+)$.

Definition 3.1. For $s, \gamma \in \mathbb{R}$ and fixed cut-off function $\omega(t)$, let

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+), v \in H_2^s(\mathbb{R}_+)\}.$$

The definition is independent of the choice of ω. The topology of $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ is the non-direct sum of the Hilbert spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ and $H_2^s(\mathbb{R}_+)$. Concerning a choice of the Hilbert space structure in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and other simple properties of these spaces, cf. e.g. Schulze [19], or [18].

Lemma 3.2. *The norm in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, is equivalent to*

$$\Big\{ \sum_{j=0}^{s} \int \Big| \Big(\frac{t}{\langle t\rangle}\Big)^{-\gamma} \Big(\frac{t}{\langle t\rangle}\Big)^{j} \partial_t^j u(t) \Big|^2 \mathrm{d}\,t \Big\}^{\frac{1}{2}}. \qquad (5)$$

Proof. First we have an equivalent description of the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+) = \{u(t) \in t^\gamma L_2(\mathbb{R}_+) : (t\partial_t)^j u(t) \in t^\gamma L_2(\mathbb{R}_+) \text{ for all } j \le s\}$, while the space $H_2^s(\mathbb{R}_+)$, $s \in \mathbb{N}$, coincides with

$$\{v(x) \in L_2(\mathbb{R}_+) : \partial_t^j u(t) \in L_2(\mathbb{R}_+) \text{ for all } j \le s\}. \tag{6}$$

Since $(t\partial_t)^k = \sum_{l=0}^k S_{kl} t^l \partial_t^l$, where S_{kl} are Stirling numbers of the second kind, cf. Dorschfeldt [5], then for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$ we have

$$\|u\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R})} = \sum_{j \le s} \int |t^{-\gamma} t^j \partial_t^j u(t)|^2 \, \mathrm{d}\, t$$

and

$$\|u\|^2_{H^s(\mathbb{R})} = \sum_{j \le s} \int |\partial_t^j u(t)|^2 \, \mathrm{d}\, t$$

(here and below equalities of norms are understood in the sense of equivalencies). Moreover, we have

$$\|u\|^2_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)} = \\ = \sum_{j \le s} \int \Big| \Big(\frac{t}{\langle t \rangle}\Big)^{-\gamma+j} \partial_t^j \omega(t) u(t) \Big|^2 \mathrm{d}\, t + \sum_{j \le s} \int |\partial_t^j (1-\omega(t)) u(t)|^2 \, \mathrm{d}\, t. \tag{7}$$

Now it is clear that (5) and (7) give equivalent norms. □

Lemma 3.3. *If $u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, then for all $k \in \mathbb{Z}$ we get $\Big(\frac{t}{\langle t \rangle}\Big)^k u \in \mathcal{K}^{s,\gamma+k}(\mathbb{R}_+)$.*

Proof. This result is a consequence of the following direct calculation, where Lemma 3.2 is used

$$\begin{aligned}
&\sum_{j=0}^{s} \int \Big| \Big(\frac{t}{\langle t \rangle}\Big)^{-(\gamma+k)} \Big(\frac{t}{\langle t \rangle}\Big)^{j} \partial_t^j \Big(\Big(\frac{t}{\langle t \rangle}\Big)^{k} u(t) \Big) \Big|^2 \mathrm{d}\, t \le \\
&\le \sum_{j=0}^{s} \int \Big| \Big(\frac{t}{\langle t \rangle}\Big)^{-(\gamma+k)} \Big(\frac{t}{\langle t \rangle}\Big)^{j} \sum_{l=0}^{j} \partial_t^l \Big(\frac{t}{\langle t \rangle}\Big)^{k} \partial_t^{j-l} u(t) \Big|^2 \mathrm{d}\, t \le \\
&\le \sum_{j=0}^{s} \sum_{l=0}^{j} \int \Big| \Big(\frac{t}{\langle t \rangle}\Big)^{-(\gamma+k)} \Big(\frac{t}{\langle t \rangle}\Big)^{k} \Big(\frac{t}{\langle t \rangle}\Big)^{j-l} \partial_t^{j-l} u(t) \Big|^2 \mathrm{d}\, t \le \\
&\le \tilde{c} \sum_{j=0}^{s} \int \Big| \Big(\frac{t}{\langle t \rangle}\Big)^{-\gamma} \Big(\frac{t}{\langle t \rangle}\Big)^{j} \partial_t^j u(t) \Big|^2 \mathrm{d}\, t \le C \|u\|^2_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)}. \quad \square
\end{aligned}$$

It can be shown, cf. [18], Theorem 1.1.23, that

$$\mathcal{K}^{s,s}(\mathbb{R}_+) = \widetilde{H}_2^s(\mathbb{R}_+) \tag{8}$$

for all $s \in \mathbb{R}_+$, $s > -\frac{1}{2}$.

Definition 3.4. Let R_j be a finite sequence $\{r_{j\kappa}, m_{j\kappa}\}_{\kappa=1}^{N_j} \subset \mathbb{C} \times \mathbb{N}$ with $N_j = N_j(R_j) \in \mathbb{N}$ and

$$\frac{1}{p} - s - (M+1) < \mathrm{Re}\; r_{j\kappa} < \frac{1}{p} - s, \text{ for all } \kappa = 0, \dots, N_j,\; j = 1, \dots, N, \quad (9)$$

where $1 < p < \infty$, $s \in \mathbb{R}$, $M, N \in \mathbb{N}$. By $\mathbf{As}_N(s,p,M)$ we denote the set of all $\boldsymbol{R} = (R_1, \dots, R_N)$ vector valued (discrete) asymptotic types associated with weight data (s,p,M). The elements of $\mathbf{As}_N(s,p,M)$ we denote by $\boldsymbol{R} = \{(\boldsymbol{r}_\kappa, \boldsymbol{m}_\kappa)\}_{\kappa=0}^{\boldsymbol{N}(\boldsymbol{R})}$, with $\boldsymbol{r}_\kappa = (r_{1\kappa}, \dots, r_{N\kappa})$, $\boldsymbol{m}_\kappa = (m_{1\kappa}, \dots, m_{N\kappa})$, $\boldsymbol{N}(\boldsymbol{R}) = (N_1, \dots, N_N)$.

Note that the set $\mathrm{As}_1(s,2,M)$ coincides with the set of discrete asymptotic types introduced by Schulze, cf. [18], Section 1.1.2 or [19], Section 2.3.1. In this case by $\mathcal{K}_R^{s,\gamma}(\mathbb{R}_+)$ we denote the subspace of all $u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ such that there are coefficients $c_{jk} = c_{jk}(u)$, $0 \le k \le m_j$, $0 \le j \le N(R)$ such that

$$u(t) - \omega(t) \sum_{j=0}^{N(R)} \sum_{k=0}^{m_j} c_{jk} t^{-r_j} \log^k t \in \bigcap_{\varepsilon > 0} \mathcal{K}^{s,\gamma+M-\varepsilon}(\mathbb{R}_+),$$

for some fixed cut-off function $\omega(t)$. The space $\mathcal{K}_R^{s,\gamma}(\mathbb{R}_+)$ is Fréchet in a canonical way.

Definition 3.5. Set $\boldsymbol{R}^0 = (R_1^0, \dots, R_N^0)$, where R_j^0 is a finite sequence of $\{r_{j\kappa}\}_{\kappa=1}^{N_j}$ satisfying (9). We will say that $\boldsymbol{R} \in \mathbf{As}_N(s,p,M)$ is shadow invariant asymptotic type with generator $\boldsymbol{R}^0$ if

$$r \in R_j^0 \Rightarrow (r - l, m(l)) \in R_j, \quad j = 1, \dots, N,$$

for certain $m(l)$ and all $l \in \mathbb{N}$, provided $\frac{1}{p} - s - (M+1) < \mathrm{Re}\; r - l$.

For example, $R = \{(-j,0)\}_{j=0}^{M} \in \mathbf{As}_1(0,2,M)$, $M \in N$, is shadow invariant asymptotic type with generator $R^0 = \{0\} \in \mathbf{As}_1(0,2,M)$ with $m(l) = 0$, $l = 1, \dots, M$.

Definition 3.6. If $\boldsymbol{R} = \{(\boldsymbol{r}_\kappa, \boldsymbol{m}_\kappa)\}_{\kappa=0}^{\boldsymbol{N}(\boldsymbol{R})} \in \mathbf{As}_N(s,p,M)$, $1 < p < \infty$, $s, M, N \in \mathbb{N}$, then $H_{p,\boldsymbol{R}}^{(\infty,s),\infty}(\mathbb{R}_+^n, \mathbb{C}^N)$ is defined to be the space of those functions $\boldsymbol{f}(y,t) = (f_1(y,t), \dots, f_N(y,t)) \in H_p^{(\infty,s),\infty}(\mathbb{R}_+^n, \mathbb{C}^N)$ for which

$$f_j(y,t) - \sum_{\kappa=0}^{N_j} \sum_{l=0}^{m_{j\kappa}} \omega(t) t^{-r_{j\kappa}} \log^l t\, v_{j\kappa l}(y) \in H_p^{(\infty,s+M+1),\infty}(\mathbb{R}_+^n),\; j = 1, \dots, N,$$

for some $v_{j\kappa l}(y) \in C^\infty(\mathbb{R}_+^{n-1})$, $0 \le l \le m_{j\kappa}$, $\kappa = 0, \dots, N_j$, $j = 1, \dots, N$.

The space $\widetilde{H}^{(\infty,s),\infty}_{p,\boldsymbol{R}}(\mathbb{R}^n_+,\mathbb{C}^N)$ is defined as follows

$$\widetilde{H}^{(\infty,s),\infty}_{p,\boldsymbol{R}}(\mathbb{R}^n_+,\mathbb{C}^N)=\Big\{\boldsymbol{f}=(f_1,\dots,f_N)\in\widetilde{H}^{(\infty,s),\infty}_{p}(\mathbb{R}^n_+,\mathbb{C}^N):$$

$$f_j(y,t)-\sum_{\kappa=0}^{N_j}\sum_{l=0}^{m_{j\kappa}}\omega(t)t^{-r_{j\kappa}}\log^l t\,\tilde{v}_{j\kappa l}(y)\in\widetilde{H}^{(\infty,s+M+1),\infty}_{p}(\mathbb{R}^n_+),\ j=1,\dots,N\Big\}$$

for some $\tilde{v}_{j\kappa l}(y)\in C^\infty(\mathbb{R}^{n-1}_+)$, $0\le l\le m_{j\kappa}$, $\kappa=0,\dots,N_j$, $j=1,\dots,N$, and a fixed cut-off function ω.

Note that if $\varkappa$ is a diffeomorphism as in Section 2, then we have a coordinate invariance of the anisotropic weighted Bessel potential spaces with asymptotics.

For $N=1$ we also use the notation $\mathrm{As}(s,p,M)$, $H^{(\infty,s),\infty}_{p,R}(\mathbb{R}^n_+)$, $\widetilde{H}^{(\infty,s),\infty}_{p,R}(\mathbb{R}^n_+)$ instead of $\mathbf{As}_1(s,p,M)$, $H^{(\infty,s),\infty}_{p,\boldsymbol{R}}(\mathbb{R}^n_+,\mathbb{C})$, and $\widetilde{H}^{(\infty,s),\infty}_{p,\boldsymbol{R}}(\mathbb{R}^n_+,\mathbb{C})$, respectively.

Let us now pass to wedge Sobolev spaces with an anisotropic reformulation of the spaces $H^s_2(\mathbb{R}^{1+q})$, $s\in\mathbb{R}$.

Lemma 3.7. *For every $s\in\mathbb{R}$ we have*

$$\|u\|_{H^s_2(\mathbb{R}^{1+q})}=\Big\{\int[\eta]^{2s}\|\kappa^{-1}(\eta)(\mathcal{F}_{y\to\eta}u)(\eta)\|^2_{H^s_2(\mathbb{R})}\,\mathrm{d}\,\eta\Big\}^{\frac12}. \tag{10}$$

Here

$$(\kappa_\lambda f)(t)=\lambda^{\frac12}f(\lambda t),\quad \lambda\in\mathbb{R}_+,$$

$f\in H^s_2(\mathbb{R}^{1+q})$, and $\kappa(\eta)=\kappa_{[\eta]}$.

Proof. Since $\|u\|_{H^s_2(\mathbb{R}^{1+q})}$ is equivalent to $\{\iint(|\tau|^2+[\eta]^2)^s|(\mathcal{F}u)(\tau,\eta)|^2\,\mathrm{d}\tau\,\mathrm{d}\,\eta\}^{\frac12}$, we get

$$\begin{aligned}
&\iint(|\tau|^2+[\eta]^2)^s|(\mathcal{F}u)(\tau,\eta)|^2\,\mathrm{d}\,\tau\,\mathrm{d}\,\eta=\\
&=\iint[\eta]^{2s}\Big(1+\Big(\frac{|\tau|}{[\eta]}\Big)^2\Big)^s|(\mathcal{F}u)(\tau,\eta)|^2\,\mathrm{d}\,\tau\,\mathrm{d}\,\eta=\\
&=\iint[\eta]^{2s}(1+|\tau|^2)^s|(\mathcal{F}u)([\eta]\tau,\eta)|^2[\eta]\,\mathrm{d}\,\tau\,\mathrm{d}\,\eta=\\
&=\int[\eta]^{2s}\int(1+|\tau|^2)^s|\kappa(\eta)(\mathcal{F}u)(\tau,\eta)|^2\,\mathrm{d}\,\tau\,\mathrm{d}\,\eta=\\
&=\int[\eta]^{2s}\|\kappa^{-1}(\eta)(\mathcal{F}_{y\to\eta}u)(\eta)\|^2_{H^s_2(\mathbb{R})}\,\mathrm{d}\,\eta.\quad\square
\end{aligned}$$

Definition 3.8. Let E be a Hilbert space equipped with a group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \to E$, strongly continuous in $\lambda \in \mathbb{R}_+$. Then $\mathcal{W}^s(\mathbb{R}^q, E)$ for $s \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\left\{ \int [\eta]^{2s} \|\kappa^{-1}_{[\eta]} (\mathcal{F}_{y\to\eta} u)(\eta)\|^2_E \, \mathrm{d}\,\eta \right\}^{\frac{1}{2}}.$$

This definition directly extends to the case of Fréchet spaces $E = \operatorname{proj\,lim}\{E^j : j \in \mathbb{N}\}$, where $(E^j)_{j\in\mathbb{N}}$ is a sequence of Hilbert spaces with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j, (we assume that $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, first given on E^0, restricts to strongly continuous groups of isomorphisms on E^j for all j. Whenever a Fréchet space E can be written as such a projective limit with $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ being given in the described way, we will say that $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action on E.) Then we get continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{j+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^j)$ for all j, and we set

$$\mathcal{W}^s(\mathbb{R}^q, E) = \operatorname{proj\,lim}\{\mathcal{W}^s(\mathbb{R}^q, E^j) : j \in \mathbb{N}\}. \tag{11}$$

Due to Lemma 3.7 we get

$$H^s_2(\mathbb{R}^n) = \mathcal{W}^s(\mathbb{R}^{n-1}, H^s_2(\mathbb{R})).$$

Moreover, we have

$$H^s_2(\mathbb{R}^n_+) = \mathcal{W}^s(\mathbb{R}^{n-1}, H^s_2(\mathbb{R}_+)), \quad \widetilde{H}^s_2(\mathbb{R}^n_+) = \mathcal{W}^s(\mathbb{R}^{n-1}, \widetilde{H}^s_2(\mathbb{R}_+)). \tag{12}$$

Setting $E = \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ we obtain so-called weighted wedge Sobolev spaces, while for $E = \mathcal{K}^{s,\gamma}_R(\mathbb{R}_+)$, $R \in \mathrm{As}_1(s, 2, M)$ we get wedge Sobolev spaces with discrete asymptotics. For more details we refer to Schulze [18], [19]. Here we use the spaces $\mathcal{K}^{\infty,\gamma}(\mathbb{R}_+)$ and $\mathcal{K}^{\infty,\gamma}_R(\mathbb{R}_+)$.

Lemma 3.9. *There are continuous embeddings*

$$\widetilde{H}^{(\infty,s),\infty}_2(\mathbb{R}^n_+) \hookrightarrow \mathcal{W}^\infty(\mathbb{R}^{n-1}, \mathcal{K}^{\infty,s}(\mathbb{R}_+))$$

for all $s \in \mathbb{R}$, $s > -\frac{1}{2}$.

Proof. If $u \in \widetilde{H}^{(\infty,s),\infty}_2(\mathbb{R}^n_+)$, then by definition we have $\langle D\rangle^{s+k} t^k u \in L_2(\mathbb{R}^n_+)$, $k \in \mathbb{N}$, i.e., $t^k u \in \widetilde{H}^{s+k}_2(\mathbb{R}^n_+)$. This gives that $(t\langle t\rangle^{-1})^k u \in \langle t\rangle^{-k} \widetilde{H}^{s+k}_2(\mathbb{R}^n_+) \hookrightarrow \widetilde{H}^{s+k}_2(\mathbb{R}^n_+)$. Since

$$\widetilde{H}^{s+k}_2(\mathbb{R}^n_+) = \mathcal{W}^{s+k}(\mathbb{R}^{n-1}, \mathcal{K}^{s+k,s+k}(\mathbb{R}_+))$$

(cf. (8), (12)), due to Lemma 3.3 we obtain $u \in \mathcal{W}^{s+k}(\mathbb{R}^{n-1}, \mathcal{K}^{s+k,s}(\mathbb{R}_+))$ for all $k \in \mathbb{N}$, i.e., $u \in \mathcal{W}^\infty(\mathbb{R}^{n-1}, \mathcal{K}^{\infty,s}(\mathbb{R}_+))$. □

Remark 3.10. For $s \in \mathbb{R}$, $s > -\frac{1}{2}$, $R \in \mathrm{As}(s, 2, M)$

$$\widetilde{H}^{(\infty,s),\infty}_{2,R}(\mathbb{R}^n_+) \hookrightarrow \mathcal{W}^\infty(\mathbb{R}^{n-1}, \mathcal{K}^{\infty,s}_R(\mathbb{R}_+)).$$

This result is an immediate consequence of Proposition 3.1.33 from [19] and Lemma 3.9.

4. Pseudo-Differential Operators, Ellipticity and Fredholm Property

Let $\mathcal{M}$ be a closed, compact, smooth n-dimensional manifold with smooth boundary $\partial\mathcal{M} = \Gamma$. According to a well-known theorem there is a collar neighbourhood V of Γ in $\mathcal{M}$ and a diffeomorphism

$$\varkappa : V \longrightarrow \Gamma \times [0, 1).$$

Moreover, we use a special local coordinate system $x = (y, t)$ defined as in [8], i.e., in any V with $V \cap \Gamma \neq \emptyset$ the variable $t \in [0, 1)$ is the direct distance to the boundary, whereas the tangential variables $y = (y_1, \dots, y_{n-1})$ are a local coordinate system on Γ. We fix Riemannian metrics on $\mathcal{M}$ and Γ and the product metric on $\Gamma \times [0, 1)$ such that $\varkappa$ is an isometry in a neighbourhood of $t = 0$.

To $\mathcal{M}$ we can associate a smooth manifold $2\mathcal{M}$ (the double of $\mathcal{M}$), the closed compact manifold that is obtained by gluing together two copies of $\mathcal{M}$ along the common boundary Γ by the identity map on Γ. Using a standard procedure we can define all our function spaces first on $2\mathcal{M}$ and then restrict to $\mathcal{M}$, e.g.,

$$H_p^{(r,s),m}(\mathcal{M}) = \{u|_{\operatorname{int}\mathcal{M}} : u \in H_p^{(r,s),m}(2\mathcal{M})\}.$$

Definition 4.1. An operator

$$\mathcal{A} : \widetilde{H}_p^{(r,s),m}(\mathcal{M}) \longrightarrow H_p^{(r,s-\mu),m}(\mathcal{M}) \tag{13}$$

is called a (classical) pseudo-differential operator with the symbol $a \in S^{\mu}_{(\mathrm{cl})}(T^*\mathcal{M})$, if:

(i) for every $\varphi, \psi \in C_0^\infty(M)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, an operator

$$\varphi\mathcal{A}\psi\mathcal{I} : H_p^{(r,s-\mu),m}(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$$

is continuous,

(ii) for every $\varphi, \psi \in C_0^\infty(M)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi \neq \emptyset$, and their supports are in one of the coordinate neighbourhoods X_j, then the transformed operator

$$(\chi_j^{-1})^*\varphi\mathcal{A}\psi\chi_j^*$$

is a pseudo-differential operator on $\mathbb{R}^n$ or $\mathbb{R}^n_+$ and the corresponding symbol is an element of $S^{\mu}_{(\mathrm{cl})}(\tilde{X}_j \times \mathbb{R}^n)$. Here, $\chi_j : X_j \to \widetilde{X}_j$ ($\widetilde{X}_j \subset \mathbb{R}^n_+$ if $X_j \cap \Gamma \neq \emptyset$ and $\widetilde{X}_j \subset \mathbb{R}^n$ if $X_j \cap \Gamma = \emptyset$) is a local diffeomorphism and $(\chi_j)^*$ and $(\chi_j^{-1})^*$ denote the push-forward under

χ_j and $\chi_j^{(-1)}$, respectively, i.e.,

$$(\chi_j^* u)(x) = \begin{cases} u(\chi_j(x)), & \text{when } x \in U_j \\ 0, & \text{when } x \notin U_j; \end{cases}$$

$$((\chi_j^{-1})^* f)(\tilde{x}) = \begin{cases} f(\chi_j^{-1}(\tilde{x})), & \text{when } \chi^{-1}(\tilde{x}) \in U_j \\ 0, & \text{when } \chi^{-1}(\tilde{x}) \notin U_j. \end{cases}$$

Note that for the classical case the principal symbol of $\mathcal{A}$ is correctly defined as an element of the quotient space $S^\mu(T^*\mathcal{M})/S^{\mu-1}(T^*\mathcal{M})$, cf. Shubin [20] or Egorov and Schulze [7]. Further, by $S^\mu_{\mathrm{cl}}(T^*\mathcal{M})_{\mathrm{tr}}$ we denote the space of all classical sympols of order $\mu \in \mathbb{Z}$ with the transmission properts (see Section 1).

Definition 4.1 extends to the case of pseudo-differential operators acting between the corresponding spaces of sections of vector bundles over $\mathcal{M}$, but here we consider only the systems of pseudo-differential operators and the adequate definition in this case is obvious.

Now we recall some necessary results from [2]. Let us consider $N \times N$ system of pseudo-differential equations on $\mathcal{M}$

$$\mathrm{r}^+ a(x, D)\boldsymbol{u}(x) = \boldsymbol{f}(x), \qquad x \in \mathcal{M}, \tag{14}$$

with a symbol $a(x, \xi) \in \boldsymbol{S}^\mu_{\mathrm{cl}}(T^*\mathcal{M})$ $(:= S^\mu_{\mathrm{cl}}(T^*\mathcal{M}) \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$.

Proposition 4.2. *If $a(x, \xi) \in \boldsymbol{S}^\mu_{\mathrm{cl}}(T^*\mathcal{M})$, then the operator*

$$\mathrm{r}^+ \breve{a}(x, D) :=$$
$$= \mathrm{r}^+(a(x, D) - \chi(D)a(x, D)) : \widetilde{H}^{(r,s),m}_p(\mathcal{M}, \mathbb{C}^N) \longrightarrow H^{(r,s),m}_p(\mathcal{M}, \mathbb{C}^N)$$

is bounded for all $r, s \in \mathbb{R}$, $m \in \mathbb{N}$ and $1 < p < \infty$. Here $\chi(D)$ is a pseudo-differential operator with symbol $\chi(\xi) \in C_0^\infty(\mathbb{R})$, such that $1 - \chi(\xi)$ is a excision function. Moreover, the operator

$$\mathrm{r}^+ \chi(D)a(x, D) : \widetilde{H}^{(\infty,s),m}_p(\mathcal{M}, \mathbb{C}^N) \longrightarrow C^\infty(\mathcal{M}, \mathbb{C}^N)$$

is continuous for all $s \in \mathbb{R}$, $m \in \mathbb{N}$.

Definition 4.3. An operator $a(x, D)$ in (14) is called elliptic if the corresponding symbol $a(x, \xi)$ is elliptic, i.e.,

$$\inf\{|\det \sigma_\psi(a)(x, \xi)| : x \in \mathcal{M}, |\xi| = 1\} > 0. \tag{15}$$

Suppose that $a(x, \xi)$ is elliptic and let $\lambda_1(y), \dots, \lambda_l(y)$ be the eigenvalues of the matrix-function

$$A = A(y) := [\sigma_\psi(a)(y, 0, 0, +1)]^{-1}\sigma_\psi(a)(y, 0, 0, -1), \quad x = (y, t) \in \mathcal{M}, \tag{16}$$

of algebraic multiplicity $m_1, \dots, m_l$, respectively $(m_1 + \cdots + m_l = N)$.

Note that for all $\eta \in \mathbb{R}^{n-1}$

$$\sigma_\psi(a)(y, 0, 0, \pm 1) = \lim_{\tau \to \pm\infty} |\tau|^{-\mu}\sigma_\psi(a)(y, 0, \eta, \tau).$$

Then we set

$$\delta_j := \delta_j(y) = \frac{1}{2\pi i} \log \lambda_j(y), \ \frac{1}{p} - 1 < s - \mathrm{Re}\ \delta_j - \frac{\mu}{2} \le \frac{1}{p}, \ j = 1, \dots, l. \quad (17)$$

Proposition 4.4. *If (and only if) the matrix $A(y)$ in (16) is normal (i.e., commutes with its adjoint matrix):*

$$A^*(y)A(y) = A(y)A^*(y),$$

then it has no generalized associated eigenvalues, $l = N$ and is unitarily similar

$$A(y) = \mathcal{K}(y)\Lambda(y)\mathcal{K}^*(y) \quad (18)$$

with

$$\Lambda(y) = \mathrm{diag}(\lambda_1(y), \dots, \lambda_N(y)), \quad \det \mathcal{K}(y) \neq 0, \quad \mathcal{K}(y) \in C^\infty(\partial M).$$

If $\sigma_\psi(a)(x,\xi)$ is positive definite on $\partial\mathcal{M}$ i.e.,

$$(\sigma_\psi(a)(y,0,\xi)c,c) \ge M|\xi|^\mu |c|^2, \quad \textit{for all} \quad y \in \partial\mathcal{M}, \ \ \xi \in \mathbb{R}^n, \ \ c \in \mathbb{C}^N,$$

with some constant $M > 0$, then $\sigma_\psi(a)(y, \pm 1)$ has no associated eigenvalues (i.e., $l = N$ and the representation (18) holds with $\mathcal{K}^{-1}(y)$ instead of $\mathcal{K}^*(y)$) and

$$\mathrm{Re}\ \delta_j(y) = 0$$

for all $j = 1, \dots, N$.

Proposition 4.5. *Let a symbol $a(x,\xi) \in \boldsymbol{S}^\mu_{\mathrm{cl}}(T^*\mathcal{M})$ be elliptic and strongly elliptic on the boundary*

$$\mathrm{Re}\,(\sigma_\psi(a)(y,0,\xi)c,c) \ge M|\xi|^\mu |c|^2$$

for all $y \in \partial\mathcal{M}$, $\xi \in \mathbb{R}^n$ and $c \in \mathbb{C}^N$, with constant $M > 0$. Then the operator

$$\mathrm{r}^+ a(x,D) : \widetilde{H}^{(r,s),m}_p(\mathcal{M}, \mathbb{C}^N) \longrightarrow H^{(r,s-\mu),m}_p(\mathcal{M}, \mathbb{C}^N) \quad (19)$$

is Fredholm if and only if

$$\mathrm{Re}\ \delta_j(y) \neq s - \frac{1}{p} - \frac{\mu}{2}, \quad j = 1, \dots, l. \quad (20)$$

Moreover, if for each interior point $x \in \mathcal{M}$ there exists $\alpha_x \in (0, 2\pi]$ such that the numerical range of the matrix symbol $\sigma_\psi(a)(x,\xi)$, i.e.,

$$R_x(a) := \{(\sigma_\psi(a)(x,\xi)c,c) : \xi \in \mathbb{R}^n, \ c \in \mathbb{C}^N, \ |\xi| = |c| = 1\}$$

does not intersect with the ray $\{z \in \mathbb{C} : \mathrm{Arg}\ z = \alpha_x\}$, then the index of the operator (19) is zero, $\mathrm{ind}\ \mathrm{r}^+ a(x,D) = 0$.

If, in addition, the homogeneous equation $\mathrm{r}^+ a(x,D)\boldsymbol{u} = 0$ has only the trivial solution $\boldsymbol{u} = 0$ in one of the spaces $\widetilde{H}^{(r,s),m}_p(\mathcal{M}, \mathbb{C}^N)$, where s and p satisfy conditions (20), then (14) has unique solutions in all these spaces.

If the conditions (20) *hold, then* (19) *has the same kernel in all the spaces* $\widetilde{H}_p^{(r,s),m}(\mathcal{M},\mathbb{C}^N)$, $m\in\mathbb{N}$, $r\in\mathbb{R}$. *In particular,* $\boldsymbol{u}\in\widetilde{H}_p^{(\infty,s),\infty}(\mathcal{M},\mathbb{C}^N)$ *provided* $\boldsymbol{f}\in\widetilde{H}_p^{(\infty,s-\mu),\infty}(\mathcal{M},\mathbb{C}^N)$.

Note that the Fredholm properties, the index and the kernel of (19) are independent of the parameters $m\in\mathbb{N}$ and $r\in\mathbb{R}$.

The proof of these results and Proposition 4.6 below are based on the factorisation of the elliptic symbol. Here we give only some necessary notation for further purposes. For more details and precise formulation of factorisation theorem we refer to Chkadua and Duduchava [2], Section 1.5.

Let $A(y)$ be the matrix-function from (16). Then according to Jordan's theorem, $A(y)$ has the following representation

$$A(y)=\mathcal{K}(y)\Lambda(y)B^0_{\sigma_\psi}(1)\mathcal{K}^{-1}(y),\quad \det\mathcal{K}(y)\neq 0,\ y\in\partial\mathcal{M},$$

where

$$\begin{aligned} B^0_{\sigma_\psi}(z)&:=\operatorname{diag}\{B_{m_1}(z),\dots,B_{m_l}(z)\},\quad z\in\mathbb{C},\\ \Lambda(y)&:=\operatorname{diag}\{\lambda_1(y)I_{m_1},\dots,\lambda_l(y)I_{m_l}\},\end{aligned}\tag{21}$$

I_m is the $m\times m$ identity matrix and

$$B_m(z):=|||B_{jk}(z)|||_{m\times m},\quad B_{jk}(z)=\begin{cases}0, & k<j\\ 1, & k=j\\ \frac{z^{k-j}}{(k-j)!}, & k>j\end{cases},\quad z\in\mathbb{C}.$$

Setting

$$B_\pm(\tau):=B^0_{\sigma_\psi}\Big(\frac{1}{2\pi i}\log(\tau\pm i)\Big),\tag{22}$$

where the branch of the logarithm is fixed in the complex plane cut along the ray $\{z\in\mathbb{C}:\arg z=-\pi\}$, i.e., $\log z:=\log|z|+i\arg z$, $-\pi<\arg z<\pi$, then

$$B^0_{\sigma_\psi}\Big(\frac{1}{2\pi i}\log\frac{\tau-i}{\tau+i}\Big)=B_-(\tau)B_+^{-1}(\tau).\tag{23}$$

Proposition 4.6. *Let* $\mathcal{M}_+:=\partial\mathcal{M}\times\mathbb{R}_+$, $b,c\in C^\infty(\partial\mathcal{M})$, $a\in\boldsymbol{S}^\mu_{\rm cl}(T^*\mathcal{M}_+)$,

$$a^\infty(y,\tau)=c(y)\langle\tau\rangle^\mu\Big(\frac{\tau-i}{\tau+i}\Big)^{-\Delta}B^0_{\sigma_\psi}\Big(\frac{1}{2\pi i}\log\frac{\tau+i}{\tau-i}\Big)b(y),\ y\in\partial\mathcal{M},\ \tau\in\mathbb{R}.$$

with

$$\Delta(y)=\operatorname{diag}\{\underbrace{\delta_1(y),\dots,\delta_1(y)}_{m_1-times},\dots,\underbrace{\delta_l(y),\dots,\delta_l(y)}_{m_l-times}\}.\tag{24}$$

Then the corresponding pseudo-differential operator

$$\mathrm{r}^+\,a^\infty(y,D_t):\widetilde{H}_p^{(\infty,s),m}(\mathcal{M}_+,\mathbb{C}^N)\longrightarrow H_p^{(\infty,s-\mu),m}(\mathcal{M}_+,\mathbb{C}^N)$$

with the symbol $a^\infty(y,\tau)$ *is bounded for all* $s,\mu\in\mathbb{R}$, $m\in\mathbb{N}$, $1<p<\infty$.

The equation

$$\mathrm{r}^+ a^\infty(y, D_t)\boldsymbol{u} = \boldsymbol{f}, \quad \boldsymbol{f} \in H_p^{(\infty, s-\mu), m}(\mathcal{M}_+, \mathbb{C}^N),$$

has a unique solution $u \in \widetilde{H}_p^{(\infty,s),m}(\mathcal{M}_+, \mathbb{C}^N)$ *for all* $m \in \mathbb{N}$ *provided* b, c *are non-degenerate matrices and the conditions*

$$\frac{1}{p} - 1 < s - \mathrm{Re}\ \delta_j - \frac{\mu}{2} < \frac{1}{p}, \quad j = 1, \dots, l,$$

hold. The solution reads

$$\boldsymbol{u} = b^{-1}(y)(D_t + i)^{-\Delta - \frac{\mu}{2}} B_+^{-1}(D_t)\Theta_+(D_t - i)^{\Delta - \frac{\mu}{2}} B_-(D_t) a^{-1}(y)\boldsymbol{f}.$$

Proposition 4.7. *Let* $a \in \boldsymbol{S}^\mu_{\mathrm{cl}}(T^*\mathcal{M}_+)$. *Then*

$$\mathrm{r}^+ \chi(D)[a_{(\mu)}(x, D) - a^\infty(y, D_t)] :$$
$$H_p^{(\infty,s),\infty}(\mathcal{M}_+, \mathbb{C}^N) \longrightarrow H_p^{(\infty, s-\mu+1),\infty}(\mathcal{M}_+, \mathbb{C}^N)$$

is a continuous operator for all $s, \mu \in \mathbb{R}$.

5. Asymptotics

In this sequel we always assume that the Jordan block dimensions in the block-diagonal matrix $B^0_{\sigma_\psi}(1)$ are stable, i.e., for all $j = 1, \dots, l$ the multiplicities $m_j(y)$ are independent of $y \in \partial\mathcal{M}$. Then Theorem 5.3 below allows us to write full asymptotics of a solution when the given data belong to anisotropic weighted Bessel potential spaces with (discrete) asymptotics. For the proof of this result we use the technique of Bennish [1], and Chkadua and Duduchava [2].

Theorem 5.1. *Let* $\mathcal{M}_+ := \partial\mathcal{M} \times \mathbb{R}_+$, $s, \mu \in \mathbb{R}$, $1 < p < \infty$, $a \in \boldsymbol{S}^\mu_{\mathrm{cl}}(T^*\mathcal{M}_+)$,

$$a^\infty(y, \tau) = \langle\tau\rangle^\mu \sigma_\psi(a)(y, +1)\Big(\frac{\tau + i}{\tau - i}\Big)^{\frac{1}{2\pi i}\log A(y)},$$

conditions of Proposition 4.5 *be fulfilled* (*a is elliptic and strongly elliptic on the boundary* $\partial\mathcal{M}$) *and*

$$\frac{1}{p} - s - 1 < -\frac{\mu}{2} - \mathrm{Re}\ \delta_j(y) < \frac{1}{p} - s$$

for all $j = 1, \dots, l$, $y \in \partial\mathcal{M}$. *Then for any*

$$\boldsymbol{f}(y, t) \in H^{(\infty, s-\mu),\infty}_{p,\boldsymbol{Q}}(\mathcal{M}_+, \mathbb{C}^N) \tag{25}$$

with $\boldsymbol{Q} = \{(\boldsymbol{q}_\kappa, \boldsymbol{n}_\kappa)\}_{\kappa=0}^{\boldsymbol{N}(\boldsymbol{Q})} \in \mathbf{As}_N(s - \mu, p, M)$ *and*

$$\mathrm{Re}\ q_{j\kappa} < 1 + \mu, \quad \mathrm{Re}\ q_{j\kappa} < 1 \quad \textit{for all} \quad \kappa = 0, \dots, N_j,\ j = 1, \dots, N,$$

the equation

$$\mathrm{r}^+ a^\infty(y, D_t)\boldsymbol{u} = \boldsymbol{f} \tag{26}$$

has a unique solution in $\widetilde{H}_p^{(\infty,s),\infty}(\mathcal{M}_+,\mathbb{C}^N)$ *represented by the formulae*

$$\begin{aligned} \boldsymbol{u} &= \mathcal{K}(y)a_+^{-1}(D_t)\Theta_+ a_-(D_t)\mathcal{K}^{-1}(y)[\sigma_\psi(y,+1)]^{-1}\boldsymbol{f}, \\ a_+^{-1}(D_t) &:= B_+^{-1}(D_t)(D_t+i)^{-\frac{\boldsymbol{\mu}}{2}-\Delta(y)}, \\ a_-(D_t) &:= B_-(D_t)(D_t-i)^{-\frac{\boldsymbol{\mu}}{2}+\Delta(y)}, \end{aligned} \tag{27}$$

(*cf.* (21) – (23) *and Proposition* 4.6)*. Moreover, the solution has the form*

$$\boldsymbol{u}(y,t) = B^0_{\sigma_\psi}\Big(\frac{1}{\pi i}\log t\Big)\boldsymbol{u}^1(y,t), \qquad \boldsymbol{u}^1(y,t) \in \widetilde{H}^{(\infty,s),\infty}_{p,\boldsymbol{R}}(\mathcal{M}_+,\mathbb{C}^N) \tag{28}$$

where $\boldsymbol{R} \in \mathbf{As}_N(s,p,M)$ *is the shadow invariant asymptotic type with generator*

$$\boldsymbol{R}^0 = \boldsymbol{R}^0_1 \cup \boldsymbol{R}^0_2; \tag{29}$$

here

$$\boldsymbol{R}^0_1 = \{\boldsymbol{q}_\kappa - \boldsymbol{\mu}\}_{\kappa=0}^{\boldsymbol{N}(\boldsymbol{R}^0_1)}, \quad \textit{with} \quad m^1_{j\kappa}(l) = n_{j\kappa} + \zeta_{j\kappa}, \tag{30}$$

$$\zeta_{j\kappa} = \zeta_{j\kappa}(y) = \begin{cases} 1 & \textit{if} \quad -\frac{\mu}{2} + \delta_j(y) + q_{j\kappa} \in \mathbb{Z}, \\ 0 & \textit{if} \quad -\frac{\mu}{2} + \delta_j(y) + q_{j\kappa} \notin \mathbb{Z} \end{cases} \tag{31}$$

and

$$\boldsymbol{R}^0_1 = \Big\{ -\frac{\boldsymbol{\mu}}{2} - \Delta(y)\Big\} \quad \textit{with} \quad m^2_{j\kappa}(l) = 0, \quad l \in N, \tag{32}$$

for $j = 1,\dots,N$ (*cf. Definition* 3.5)*. Here* $\boldsymbol{\mu} = (\mu,\dots,\mu) \in \mathbb{C}^N$ *and* Δ *is written as* $\Delta = \operatorname{diag}\{\delta_1,\dots,\delta_N\}$ (*compare with* (24))*.*

Proof. Assertions about solvability of the equation (26) and the solution formula (27) follow from Proposition 4.6. For simplicity we prove (28)–(32) for the scalar case, i.e., we assume $N = 1$. It is known from [2], Lemma 2.6, that if f additionally belongs to the space $H_p^{(\infty,s-\mu+M+1),\infty}(\mathcal{M}_+)$, then the solution can be represented as $u = B^0_{\sigma_\psi}(\frac{1}{\pi i}\log t)u^1$ with $u^1 \in \widetilde{H}^{(\infty,s+M+1),\infty}_{p,R}(\mathcal{M}_+)$, where R is the shadow invariant asymptotic type with generator $R^0_2 = \big\{-\frac{\mu}{2} - \delta(y)\big\}$ see (32). Therefore it is suffices to consider a case $f(y,t) = \omega(t)t^{-q}\log^m t\, v(t)$, with

$$\frac{1}{p} - (s-\mu) - (M+1) < \operatorname{Re} q < \frac{1}{p} - (s-\mu), \quad \operatorname{Re} q < 1+\mu, \quad \operatorname{Re} q < 1.$$

Moreover, since $\lim_{\varepsilon\to 0}(\partial_t^j(e^{-\varepsilon t} - \omega(t))) = 0$, $t > 0$ for all $j \in \mathbb{N}$ in a sufficiently small neighbourhood of $t = 0$, we can first calculate the asymptotic expansion of $(a^\infty(y,D_t))^{-1}(t_+^{-q}\log^n t_+\, e^{-\varepsilon t})v(y) \in \widetilde{H}_p^{(\infty,s-\mu),\infty}(\mathcal{M}_+)$ and then we obtain “precise” asymptotic expansion when $\varepsilon \to 0$.

Homogeneous symbols and kernels of corresponding pseudo-differential operators with negative order have singularities at 0 and multiplying them by a function $(1-\chi(\tau))$ we cut-off the singularity. Here χ is an excision function. Proposition 4.2 shows that the perturbation operator is smoothing

$\chi(D_t)\psi \in C^\infty(\mathcal{M})$ for arbitrary $\psi \in H_p^{(\infty,s),\infty}(\mathcal{M}_+)$ and we can ignore it. Although we will not write the function $1-\chi$, we suppose its presence and can forget about singularities of symbols at $\tau = 0$.

The formula (2) with the condition Re $q < 1$ gives us

$$\mathcal{F}_{t\to\tau}(t_+^{-q} \log^n t_+ \, e^{-\varepsilon t}) = \sum_{j=0}^{n} b_{nj}(q)(\tau + i\varepsilon)^{q-1} \log^j(\tau + i\varepsilon).$$

By the Taylor formula we replace the summands $(\tau + i\varepsilon)^{q-1}\log^j(\tau+i\varepsilon)$ by $(\tau + i0)^{q-1}\log^j(\tau + i0)$, other terms which have the factor ε will disappear when $\varepsilon \to 0$. Therefore they do not affect the final result. To avoid complicated expressions we do not include such kind of terms in equalities. Further, using

$$\begin{aligned}(\tau + i0)^{q-1} &= (\Theta_+(\tau) + e^{i(q-1)\pi}\Theta_-(\tau)|\tau|^{q-1}),\\ \log(\tau + i0) &= \log|\tau| + i\pi\Theta(-\tau),\end{aligned} \tag{33}$$

cf. [8], we get

$$\mathcal{F}_{t\to\tau}(t_+^{-q} \log^n t_+ \, e^{-\varepsilon t}) = \sum_{j=0}^{n} b_{nj}(q, \operatorname{sgn}\tau)|\tau|^{q-1} \log^j|\tau|, \tag{34}$$

with certain coefficients $b_{nj}(q, \operatorname{sgn}\tau)$. Analogously, applying the Taylor formula and (33), we get

$$\begin{aligned}(\tau - i)^{-\frac{\mu}{2}+\delta}\Big(\frac{1}{2\pi i}\log(\tau - i)\Big)^{m_-} &= \sum_{l=0}^{\tilde N} c_j \partial_\varepsilon^l (\tau - i\varepsilon)^{-\frac{\mu}{2}+\delta}\log^{m_-}(\tau - i\varepsilon)\big|_{\varepsilon=0} +\\ +\tilde f_{\tilde N+1}(\tau) &= \sum_{l=0}^{\tilde N}\sum_{k=0}^{m_-} c_{lk}(\tau - i0)^{-\frac{\mu}{2}+\delta-l}\log^k(\tau - i0) + \tilde f_{\tilde N+1}(\tau) =\\ &= \sum_{l=0}^{\tilde N}\sum_{k=0}^{m_-} \tilde c_{lk}(\operatorname{sgn}\tau)|\tau|^{-\frac{\mu}{2}+\delta-l}\log^k|\tau| + \tilde f_{\tilde N+1}(\tau),\end{aligned}$$

with certain coefficients $\tilde c_{lk}(\operatorname{sgn}\tau)$, $0 < l < \tilde N$, $0 < k < m_-$, and sufficiently $\tilde N > M + 1$ such that

$$\tilde f_{\tilde N+1}(D_t)(t_+^{-q}\log^n t_+ \, e^{-\varepsilon t}) \in H_p^{(\infty, s+M+1-\frac{\mu}{2}-\mathrm{Re}\,\delta),\infty}(\mathcal{M}_+).$$

The last term can also be dropped since $a_+^{-1}(D_t)\tilde f_{\tilde N+1}(D_t)(t_+^{-q}\log^n t_+ e^{-\varepsilon t}) \in H_p^{(\infty,s+M+1),\infty}(\mathcal{M}_+)$. Thus we have

$$\begin{aligned}(\tau - i)^{-\frac{\mu}{2}+\delta}\Big(\frac{1}{2\pi i}\log(\tau - i)\Big)^{m_-}\mathcal{F}_{t\to\tau}(t_+^{-q}\log^n t_+ \, e^{-\varepsilon t}) =\\ = \sum_{l=0}^{\tilde N}\sum_{j=0}^{m_-+n} b_{nlj}(q, \operatorname{sgn}\tau)|\tau|^{-\frac{\mu}{2}+\delta+q-1-l}\log^j|\tau|\end{aligned}$$

with some coefficients $b_{nlj}(q, \text{sgn}\ \tau)$.

Using Proposition 1.1 we proceed as follows: we replace $|\tau|^{-\frac{\mu}{2}+\delta+q-1-l} \times \log^j|\tau|$ by $(\tau \pm i0)^{-\frac{\mu}{2}+\delta+q-1-l}\log^j(\tau \pm i0)$, this operation increasing the exponents of logarithms by 1 if $\frac{\mu}{2}+\delta+q \in \mathbb{Z}$. The summands $(\tau - i0)^{-\frac{\mu}{2}+\delta+q-1-l}\log^j(\tau - i0)$ will be canceled by the projection $\mathcal{F}_{t\to\tau}\Theta_+\mathcal{F}^{-1}_{\tau\to t}$. Thus we have

$$\mathcal{F}_{t\to\tau}\Theta_+(D_t-i)^{-\frac{\mu}{2}+\delta}\Big(\frac{1}{2\pi i}\log(D_t-i)\Big)^{m_-}\times$$
$$\times\mathcal{K}^{-1}(y)[\sigma_\psi(y,+1)]^{-1}(t_+^{-q}\log^n t_+\, e^{-\varepsilon t}v(t)) =$$
$$=\sum_{l=0}^{\tilde{N}}\sum_{j=\zeta}^{m_-+n+\zeta}\tilde{C}_{lj}(q,y)(\tau+i0)^{-\frac{\mu}{2}+\delta+q-1-l}\log^j(\tau+i0)\mathcal{K}^{-1}(y),$$

cf. (31), with $\tilde{C}_j(q,y) \in C^\infty(\partial\mathcal{M})$, provided $\text{Re}(-\frac{\mu}{2}+\delta+q) < 1$. But this condition is a consequence of $\frac{1}{p}-1 < s - \text{Re}\ \delta - \frac{\mu}{2}$ and $\text{Re}\ q < \frac{1}{p}-(s-\mu)$.

Arguing as above, we use Taylor formula now for $\big(-\frac{1}{2\pi i}\log(D_t+i)\big)^{m_+}(D_t+i)^{-\frac{\mu}{2}-\delta}$. Then applying (1) ($\text{Re}\ q < 1+\mu$), we obtain

$$u(y,t)=\sum_{l=0}^{\tilde{N}}\sum_{k=0}^{m_++m_-+n+\zeta}\mathcal{K}(y)\omega(t)t^{-q+\mu+l}\log^k t\, v_{lk}(y)\mathcal{K}^{-1}(y)+\tilde{u}_1(y,t), \quad (35)$$

where $\tilde{u}_1(y,t) \in \widetilde{H}_p^{(\infty,s+M+1),\infty}(\mathcal{M}_+)$. Since the summands with $l > M+1$ belong to $\widetilde{H}_p^{(\infty,s+M+1),\infty}(\mathcal{M}_+)$, we can write $M+1$ instead of $\tilde{N}$.

The sum m_-+m_+ in (35) shows that for the general case $N>1$ we obtain $B^0_{\sigma_\psi}(\frac{1}{2\pi i}\log t) \times B^0_{\sigma_\psi}(\frac{1}{2\pi i}\log t) = B^0_{\sigma_\psi}(\frac{1}{\pi i}\log t)$, i.e., we get (28), (29). □

Remark 5.2. In other words, the solution of (26), (25) has the following asymptotic expansion

$$u_j(y,t) = \mathcal{K}(y)\omega(t)B\Big(\frac{1}{\pi i}\log\ t\Big)\mathcal{K}^{-1}(y)\times$$
$$\times\Big[\sum_{\kappa=0}^{N_j}\sum_{l=0}^{\widetilde{M}_{j\kappa}}\sum_{k=0}^{n_{j\kappa}+\zeta_{j\kappa}} t^{-q_{j\kappa}+\mu+l}\log^k t\, v_{k\kappa l}(y) + \sum_{l=0}^{M} t^{\frac{\mu}{2}+\delta_j(y)+l}v_l(y)\Big]+$$
$$+u_{M+1}(y,t), \quad u_{M+1} \in H_p^{(\infty,s+M+1),\infty}(\mathcal{M}_+) \quad (36)$$

(cf. Definition 3.6), for $v_{k\kappa l}, v_l \in C^\infty(\partial\mathcal{M})$, $\widetilde{M}_{j\kappa} \in \mathbb{N}$ such that $\text{Re}\ q_{j\kappa} - \mu - (\widetilde{M}_{j\kappa}+1) < \frac{1}{p} - s - (M+1) < \text{Re}\ q_{j\kappa} - \mu - \widetilde{M}_{j\kappa}$ and a suitable cut-off function ω.

As we see (for fixed y and j), the leading term in the asymptotic expansion (36) is determined by $t^{\frac{\mu}{2}+\delta_j(y)}v_0(y)$ if $\frac{\mu}{2}+\text{Re}\ \delta_j \le -\text{Re}\ q_{j\kappa}+\mu$ for all $\kappa = 1,\dots,N_j$ or by $\sum_{k=0}^{n_{j\kappa}+\zeta_{j\kappa}} t^{-q_{j\tilde{\kappa}}+\mu}\log^k t\, v_{k\tilde{\kappa}0}(y)$ if the real part of $-q_{j\tilde{\kappa}}+\mu := \min\{(-q_{j\kappa}+\mu)_{\kappa=1}^{N_j}\}$ is less than $\frac{\mu}{2}+\text{Re}\ \delta_j$.

Theorem 5.3. *Let $\mathcal{M}$ be a compact, smooth manifold with smooth boundary $\partial\mathcal{M}$, $s,\mu \in \mathbb{R}$, $N \in \mathbb{N}$, $a \in \boldsymbol{S}^{\mu}_{\mathrm{cl}}(T^*\mathcal{M})$ and $N \times N$ system of pseudo-differential equations*

$$\mathrm{r}^+\, a(x,D)\boldsymbol{u} = \boldsymbol{f} \tag{37}$$

has a unique solution $\boldsymbol{u}\in\widetilde{H}^{(\infty,s),\infty}_p(\mathcal{M},\mathbb{C}^N)$ for each given $\boldsymbol{f}\in H^{(\infty,s-\mu),\infty}_p(\mathcal{M}, \mathbb{C}^N)$. Then

$$\frac{1}{p} - 1 < s - \frac{\mu}{2} - \mathrm{Re}\ \delta_j(y) < \frac{1}{p} \quad \textit{for all} \quad j = 1,\dots,l. \tag{38}$$

Let further $\frac{\mu}{2} + \mathrm{Re}\ \delta_j(y) > -1$ for all $j = 1,\dots,l$, $M \in \mathbb{N}$, $\delta_1,\dots,\delta_l \in C^\infty(\partial\mathcal{M})$, $\mathcal{K}(y) \in C^\infty(\partial\mathcal{M}) \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ and

$$\boldsymbol{f} \in H^{(\infty,s-\mu),\infty}_{p,\boldsymbol{Q}}(\mathcal{M},\mathbb{C}^N)$$

for some $\boldsymbol{Q} = \{(\boldsymbol{q}_\kappa, \boldsymbol{n}_\kappa)\}^{\boldsymbol{N}(\boldsymbol{Q})}_{\kappa=1} \in \mathbf{As}_N(s-\mu,p,M)$, with $\mathrm{Re}\ q_{j\kappa} < 1+\mu$, $\mathrm{Re}\ q_{j\kappa} < 1$ for all $\kappa = 1,\dots,N_j(Q)$, $j = 1,\dots,N$. Then the solution admits the representation

$$\boldsymbol{u} = B^0_{\sigma_\psi}\Big(\frac{1}{\pi i}\log t\Big)\boldsymbol{u}^1, \qquad \boldsymbol{u}^1 \in \widetilde{H}^{(\infty,s),\infty}_{p,\boldsymbol{R}}(\mathcal{M},\mathbb{C}^N), \tag{39}$$

where $\boldsymbol{R} \in \mathbf{As}_N(s,p,M)$ is the shadow invariant asymptotic type with generator $\boldsymbol{R}^0 = \boldsymbol{R}^0_1 \cup \boldsymbol{R}^0_2$; here,

$$\boldsymbol{R}^0_1 = \{\boldsymbol{q}_\kappa - \boldsymbol{\mu},\}^{\boldsymbol{N}(\boldsymbol{R}^0_1)}_{\kappa=0}, \tag{40}$$

with

$$m^1_{j\kappa}(0) = n_{j\kappa} + \zeta_{j\kappa}, \quad m^1_{j\kappa}(l) = m^1_{j\kappa}(0) + l\zeta_{j\kappa},\ l \in \mathbb{N}\backslash\{0\}, \tag{41}$$

where $\zeta_{j\kappa}$ are defined by (31), and

$$\boldsymbol{R}^2 = \Big\{-\frac{\boldsymbol{\mu}}{2} - \Delta(y)\Big\} \quad \textit{with} \quad m^2_{j\kappa}(l) = l, \quad l \in N, \tag{42}$$

for $j = 1,\dots,N$.

Proof. Due to Theorem 4.5 we have the conditions (20), but from the invertibility of $a(x,D)$ we get solvability conditions (38) of the equation (37), cf. the proof of Theorem 1.12 in [2]. Since we are interested in asymptotic expansion of the solution near $\partial\mathcal{M}$, we suppose that $\mathcal{M}$ is $\mathcal{M}_+$, but the functions are compactly supported. We will apply iteration starting with the case $M = 0$. Arguing as in the beginning of the proof of Theorem 5.1, we assume $N = 1$ and we can "forget" about singularities of homogeneous symbols with negative order at $\tau = 0$. The equation (37) can be written in the following equivalent form

$$\mathrm{r}^+\, a^\infty(y,D_t)u(y,t) =$$
$$= f(y,t) - \mathrm{r}^+\, a_1(x,D)u(y,t) - \mathrm{r}^+[a_{(\mu)}(x,D) - a^\infty(y,D_t)]u(y,t),$$

where $a_1 = a - a_{(\mu)}$, and $a_1(x,D)u \in H_p^{(\infty,s-\mu+1),\infty}(\mathcal{M}_+)$ and using Proposition 4.7 we get $\mathrm{r}^+ a_1(x,D)u(y,t) - \mathrm{r}^+[a_{(\mu)}(x,D) - a^\infty(y,D_t)]u \in H_p^{(\infty,s-\mu+1),\infty}(\mathcal{M}_+)$, i.e.,

$$f(y,t) - \mathrm{r}^+ a_1(x,D)u(y,t) - \mathrm{r}^+[a_{(\mu)}(x,D) - \\ -a^\infty(y,D_t)]u(y,t) \in H_{p,Q}^{(\infty,s-\mu+1),\infty}(\mathcal{M}_+).$$

Then Theorem 5.1 proves (39) for $M = 0$.

Now let $M \geq 1$ and suppose we have proved the assertion for $M - 1$. Then u can be written in the form

$$u = \sum_{k=0}^{M-1} u_k + \tilde{u}_M, \qquad u_k \in \widetilde{H}_p^{(\infty,s+k),\infty}(\mathcal{M}_+).$$

The equation (37) can be represented as follows

$$\mathrm{r}^+ a^\infty(y,D_t)u = f - \sum_{k=0}^{M-1}\sum_{j=1}^{M-k} \mathrm{r}^+ a_{(\mu-j)}(x,D)u_k - \\ - \sum_{k=0}^{M-1} \mathrm{r}^+[a_{(\mu)}(x,D) - a^\infty(y,D_t)]u_k + \tilde{f}_{M+1}, \tag{43}$$

where $\tilde{f}_{M+1} = -\mathrm{r}^+ a_{M+1}(x,D)u - \sum_{k=0}^{M-1}\sum_{j=M-k+1}^{M} \mathrm{r}^+ a_{(\mu-j)}(x,D)u_k - \mathrm{r}^+[a_{(\mu)}(x,D) - a^\infty(y,D_t)]\tilde{u}_M \in H_p^{(\infty,s-\mu+M+1),\infty}(\mathcal{M}_+)$, cf. Proposition 4.7.

The Taylor formula at $t = 0$, and then at $\frac{\eta}{|\tau|} = 0$, gives:

$$a_{(\mu-j)}(y,t,\eta,\tau) = \sum_{m=0}^{M-k-j} \frac{t^m}{m!}(\partial_t^m a_{(\mu-j)})(y,0,|\tau|^{-1}\eta, \operatorname{sgn}\tau)|\tau|^{\mu-j} + \\ + t^{M-k-j+1} a_{(\mu-j)}^{M-k-j+1}(y,t,\xi) = \\ = \sum_{m=0}^{M-k-j} \frac{t^m}{m!}\Bigg(\sum_{|\alpha'|\leq M-k-j-m} \frac{\eta^{\alpha'}}{\alpha'!|\tau|^{|\alpha'|}}(\partial_\eta^{\alpha'}\partial_t^m a_{(\mu-j)})(y,0,0,\operatorname{sgn}\tau) + \\ + |\tau|^{-(M-k-j-m+1)}|\tau|^{\mu-j}\tilde{a}_{(\mu-j)}^{M-k-j-m+1}(y,0,\xi)\Bigg) = \\ = \sum_{m=0}^{M-k-j} \frac{t^m}{m!} \sum_{l=0}^{M-k-j-m} |\tau|^{\mu-j-l} \sum_{|\alpha'|=l} \frac{\eta^{\alpha'}}{\alpha'!}(\partial_t^m\partial_\eta^{\alpha'} a_{(\mu-j)})(y,0,0,\operatorname{sgn}\tau) + \\ + a_{(\mu-j)}^{M-k-j+1,M-k-j-m+1}(x,\xi), \tag{44}$$

where

$$a_{(\mu-j)}^{M-k-j+1,M-k-j-m+1}(x,\xi) = t^{M-k-j+1} a_{(\mu-j)}^{M-k-j+1}(y,t,\eta,\tau) +$$

$$+\sum_{m=0}^{M-k-j}|\tau|^{\mu-j-(M-k-j-m+1)}t^m\tilde{a}_{(\mu-j)}^{M-k-j-m+1}(y,0,\xi).$$

It is clear that

$$a_{(\mu-j)}^{M-k-j+1,M-k-j-m+1}(x,D):[\omega]\widetilde{H}_p^{(\infty,s+k),\infty}(\mathcal{M}_+)\to \\ \to H_p^{(\infty,s-\mu+M+1),\infty}(\mathcal{M}_+)$$

is bounded for all $j\in\mathbb{N}$.

Similarly,

$$a^\infty(y,\tau)=|\tau|^\mu\langle\tau^{-1}\rangle^\mu\sigma_\psi(y,+1)\Big(\frac{1+i\tau^{-1}}{1-i\tau^{-1}}\Big)^{b(y)}= \\ =\sum_{l=0}^{M-k}|\tau|^{\mu-l}a_l(y,\mathrm{sgn}\;\tau)+\tilde{a}_{M-k+1}^\infty(y,\tau), \tag{45}$$

where an operator

$$\tilde{a}_{M-k+1}^\infty(y,D_t):[\omega]\widetilde{H}_p^{(\infty,s+k),\infty}(\mathcal{M}_+)\longrightarrow H_p^{(\infty,s-\mu+M+1),\infty}(\mathcal{M}_+)$$

is bounded.

Therefore (43), (44), (45) yield

$$\mathrm{r}^+\,a^\infty(y,D_t)u=f(y,t)+\sum_{k=0}^{M-1}\sum_{j=1}^{M-k}\sum_{m=0}^{M-k-j}\sum_{\iota=0}^{M-k-j-m}t^m a_{jm\iota}(y,D)u_k+\tilde{f}_{M+1}^1,$$

with

$$a_{jm\iota}(y,\tau):=\sum_{|\alpha'|=\iota}a_{jm\alpha'}^1(y,\mathrm{sgn}\;\tau)|\tau|^{\mu-j-\iota}\eta^{\alpha'},$$

where

$$a_{jm\alpha'}^1(y,\mathrm{sgn}\;\tau)=[a_{jm\alpha'}(y,\mathrm{sgn}\;\tau)-\delta_{j+m,0}\,\delta_{|\alpha'|,0}a_\iota(y,\mathrm{sgn}\;\tau)],$$

$\delta_{j\iota}$ are Kronecker's delta, $\tilde{f}_{M+1}^1\in H_p^{(\infty,s-\mu+M+1),\infty}(\mathcal{M}_+)$ and $a_{jm\alpha'}(y,\pm1)\in C^\infty(\partial\mathcal{M})$.

Due to Theorem 5.1, we find

$$u=(a^\infty(y,D_t))^{-1}f+ \\ +\sum_{k=0}^{M-1}\sum_{j=1}^{M-k}\sum_{m=0}^{M-k-j}\sum_{l=0}^{M-k-j-m}(a^\infty(y,D_t))^{-1}t^m a_{jm\iota}(y,D)u_k+ \\ +(a^\infty(y,D_t))^{-1}f_{M+1}^1. \tag{46}$$

Assume for a moment that

$$u_k(y,t)=\omega(t)t^{-r}\log^n t\,v(y)\in\widetilde{H}_p^{(\infty,s+k),\infty}(\mathcal{M}_+)$$

and take $\frac{1}{2\pi i}\log(D_t+i)^{m_+}$, $\frac{1}{2\pi i}\log(D_t-i)^{m_-}$ instead of the operators $B_+^{-1}(D_t)$, $B_-(D_t)$ (see also the proof of Theorem 5.1). Then

$$
\begin{aligned}
t^m a_{jm\iota}(y,D)u_k &= \sum_{|\alpha'|=\iota} t^m \mathcal{F}^{-1}_{\tau\to t} a^1_{jm\iota}(y,\operatorname{sgn}\tau)|\tau|^{\mu-j-\iota}(i\partial_y)^{\alpha'}\mathcal{F}_{t\to\tau}u_k = \\
&= \sum_{|\alpha'|=\iota} \mathcal{F}^{-1}_{\tau\to t}(i\partial\tau)^m a^1_{jm\iota}(y,\operatorname{sgn}\tau)|\tau|^{\mu-j-\iota}(i\partial_y)^{\alpha'}\mathcal{F}_{t\to\tau}u_k = \\
&= \sum_{\kappa=0}^{n} a^2_{jm\iota\kappa}(y,\operatorname{sgn}\tau)|\tau|^{\mu+r-j-\iota-m-1}\log^\kappa|\tau|. \qquad (47)
\end{aligned}
$$

Setting $q=\mu+r-j-\iota-m$, we proceed analogously as in the proof of Theorem 5.1 (compare (47) with (34)). Now it is easy to see that finally we obtain

$$
\begin{aligned}
&\sum_{j=1}^{M-k}\sum_{m=0}^{M-k-j}\sum_{\iota=0}^{M-k-j-m}(a^\infty(y,D_t))^{-1}t^m a_{jm\iota}(y,D)t^{-r}\log^n t\, v(y) = \\
&= \sum_{l=1}^{M-k}\sum_{\kappa=0}^{n+m_-+m_++\zeta}\mathcal{K}(y)t^{-r+l}\log^\kappa t\, v_{l\kappa}(y)\mathcal{K}^{-1}(y)+\tilde{u}_{k,M+1}(y,t) \qquad (48)
\end{aligned}
$$

for certain $v_{l\kappa}(y)\in C^\infty(\partial\mathcal{M})$, $\tilde{u}_{k,M+1}\in \widetilde{H}_p^{(\infty,s+M+1),\infty}(\mathcal{M}_+)$. Here we have replaced summations with respect to j,m,ι and l from (35) by one sum with respect to l and $j+\iota+m+l$ (exponents of t) by l, respectively. Note that since $j\geq 1$, we have started summation from $l=1$. To find ζ, we take $r=q-\mu$ (the exponent of the leading term of the asymptotics) and as in Theorem 5.1 we get (31), while $r=-\frac{\mu}{2}-\delta$ gives us $\zeta=1$ because of $-\frac{\mu}{2}+\delta+\mu-\frac{\mu}{2}-\delta-j-\iota-m\in\mathbb{Z}$, (cf. (47)). It is clear that the terms in (48) do not generate the leading term of asymptotics since $l\geq 1$. The leading term is determined by $(a^\infty(y,D_t))^{-1}f$ or $(a^\infty(y,D_t))^{-1}f^1_{M+1}$. For the general case $N>1$ we may drop m_-+m_+ and write $B^0_{\sigma_\psi}(a)(\frac{1}{\pi i}\log t)$ as a factor of terms in the formula corresponding to (48) for $N>1$. Moreover, (48) and (46) give us a possibility to find the exponents of logarithms knowing them in the previous step of iteration i.e., if we take $t^{-r+l}\log^{m(l,M-1)}t$, for $l\in\mathbb{N}$ (not including the factors $B^0_{\sigma_\psi}$

in $m(l)$), $M-1$ indicating the $M-1$-th step of iteration, then for $m(l,M)$ we get

$$
\begin{aligned}
m(l,M)=\max\{&m(l-1,M-1)+\zeta,\ldots,m(1,M-1)+\zeta,\\
&m(0,M-1)+\zeta,m(0)\}.
\end{aligned} \qquad (49)
$$

As it was shown above, the exponent of the logarithm in the leading term does not depend on the iteration and we have $m(0,M)=m(0)$ for all $M\in\mathbb{N}$. Using (49), we obtain $m(1,M-1)=\max\{m(0)+\zeta,m(0)\}=m(0)+\zeta$ and etc. This argument shows that the exponents of the logaritms do not depend on M and we have $m(l)=m(0)+l\zeta$. For $r=-\frac{\mu}{2}-\delta$, as it was mentioned above, $\zeta = 1$ and $m(0) = 0$ for $t^{-r}\log^{m(0)} t$. Therefore $m(l) = l$ and we get (42). Analogously we obtain (41), (40). □

Remark 5.4. If $\boldsymbol{f} \in H_p^{(\infty,s-\mu+M+1),\infty}(\mathcal{M},\mathbb{C}^N)$, $M \in \mathbb{N}$ in (37) and conditions of Theorem 5.3 are fulfilled, then we obtain the result from Chkadua and Duduchava (cf. [2], Theorem 2.1), i.e. the solution of (37) locally has the following asymptotic expansion

$$\boldsymbol{u}(y,t) = \mathcal{K}(y)\omega(t)t^{\frac{\mu}{2}+\Delta(y)}B^0_{\sigma_\psi}\Big(\frac{1}{\pi i}\log t\Big)\mathcal{K}^{-1}(y)[\boldsymbol{v}_0(y) +$$

$$+\sum_{l=1}^{M} t^l \sum_{j=0}^{l} \log^j t\, \boldsymbol{v}_{lj}(y)] + \boldsymbol{u}_{M+1}(y,t), \quad \boldsymbol{u}_{M+1} \in \widetilde{H}_p^{(\infty,s+M+1),\infty}(\mathcal{M},\mathbb{C}^N),$$

for $\boldsymbol{v}_{lj}, \boldsymbol{v}_0 \in C^\infty(\partial\mathcal{M},\mathbb{C}^N)$.

Remark 5.5. Let the conditions of Theorem 5.3 be fulfilled and $N = 1$, $f(y,t) = \omega(t)t^{-q}\log^n t\, v(y) \in H_{p,Q}^{(\infty,s-\mu),\infty}(\mathcal{M})$, where $Q = \{(q,n)\} \in As(s-\mu,p,M)$ with $\operatorname{Re} q < 1+\mu$, $\operatorname{Re} q < 1$. Then the solution of (37) locally has the following asymptotic expansion

$$u(y,t) = \sum_{l=0}^{\widetilde{M}} \sum_{j=0}^{n+(l+1)\zeta} \omega(t)t^{-q+\mu+l}\log^j t\, v_{jl}(y) +$$

$$+\sum_{l=0}^{M}\sum_{j=0}^{l} \omega(t)t^{\frac{\mu}{2}+\delta_1+l}\log^j t\, \tilde{v}_{jl}(y) + u_{M+1}(y,t),$$

where $v_{jl}, \tilde{v}_{jl} \in C^\infty(\partial\mathcal{M})$, $u_{M+1} \in \widetilde{H}_p^{(\infty,s+M+1),\infty}(\mathcal{M})$ and $\widetilde{M} \in \mathbb{N}$ is such that

$$\operatorname{Re} q - \mu - (\widetilde{M}+1) \le \frac{1}{p} - s - (M+1) < \operatorname{Re} q - \mu - \widetilde{M},$$

ζ is defined as follows

$$\zeta = \begin{cases} 1 & \text{if} \quad -\frac{\mu}{2} + \delta_1 + q \in \mathbb{Z}, \\ 0 & \text{if} \quad -\frac{\mu}{2} + \delta_1 + q \notin \mathbb{Z}. \end{cases}$$

For applications it is very important to know that the asymptotic expansion contains neither oscillatory terms (i.e., non real exponents of t) nor logarithmic terms (i.e., $\log^k t$ with $k \ge 1$). Concerning oscilations, Theorem 5.3 shows that the solution $\mathbf{u}$ never oscillates provided $\delta_1, \dots, \delta_N$, $q_{j\kappa} \in \mathbb{R}$, for all κ. Concerning logarithms, in the following theorem we give a sufficient condition for absence of logaritms as in the leading term as in the further terms.

Theorem 5.6. *Let conditions of Theorem* 5.3 *be fulfilled, and* $a \in S^\mu_{\mathrm{cl}}(T^*\mathcal{M})_{\mathrm{tr}}$, $\mu \in \mathbb{Z}$, $\mu > -2$ *and*

$$f \in H_{p,\mathbf{Q}}^{(\infty,s-\mu),\infty}(M,\mathbb{C}^N)$$

for some $\mathbf{Q} = \{(\boldsymbol{q}_\kappa, 0)\}_{\kappa=1}^{N(\mathbf{Q})} \in \mathbf{As}_N(s-\mu, p, M)$, *with* $\operatorname{Re} q_{j\kappa} < 1+\mu$, $\operatorname{Re} q_{j\kappa} < 1$, *and* $-\frac{\mu}{2} + q_{j\kappa} \notin \mathbb{Z}$ *for all* $\kappa = 1, \dots, N_j(\boldsymbol{Q})$, $j = 1, \dots, N$. *Then* $\mathbf{u} \in H_{p,\boldsymbol{R}}^{(\infty,s),\infty}(M, \mathbb{C}^N)$, *where* $\boldsymbol{R} \in \mathbf{As}_N(s, p, M)$ *is the shadow invariant asymptotic type with generator* $\boldsymbol{R}^0 = \boldsymbol{R}_1^0 \cup \boldsymbol{R}_2^0$*; hare*

$$\boldsymbol{R}_1^0 = \{\boldsymbol{q}_\kappa - \boldsymbol{\mu}\}_{\kappa=1}^{N(\boldsymbol{R}_1^0)} \quad \textit{with} \quad m_{j\kappa}^1(l) = 0, \quad l \in \mathbb{N},$$

and

$$\boldsymbol{R}_2^0 = \Big\{ -\frac{\boldsymbol{\mu}}{2} \Big\} \quad \textit{with} \quad m_{j\kappa}^2(l) = 0, \quad l \in \mathbb{N}.$$

The proof is based on Theorem 5.3 and on the result obtained by M. Costabel, M. Dauge and R. Duduchava [3].

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