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ON UNIQUE SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let $-\infty < a < b < +\infty$, $I = [a,b]$, $p : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\ell : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ be linear bounded operators, $q \in L(I;\mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$. On the basis of the results from [5], in the present paper we establish new sufficient conditions for solvability of the boundary value problem

$$\frac{dx(t)}{dt} = p(x)(t) + q(t), \tag{1}$$

$$\ell(x) = c_0, \tag{2}$$

which supplement the results of [1-4, 6-9].

Throughout the paper, the following notation will be used.

$\mathbb{R} =]-\infty,\infty[$, $\mathbb{R}_+ = [0,\infty[$;

χ_I is the characteristic function of the interval I, i.e.,

$$\chi_I(t) = \begin{cases} 1 & \text{for } t \in I \\ 0 & \text{for } t \notin I \end{cases};$$

$\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the elements $x_i \in \mathbb{R}$ $(i = 1,\dots,n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is the space of $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with the elements $x_{ik} \in \mathbb{R}$ $(i,k = 1,\dots,n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$$\mathbb{R}_+^n = \Big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_i \geq 0 \ (i = 1,\dots,n)\Big\};$$

$$\mathbb{R}_+^{n\times n} = \Big\{(x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n} : \ x_{ik} \geq 0 \ (i,k = 1,\dots,n)\Big\};$$

if x, $y \in \mathbb{R}^n$ and X, $Y \in \mathbb{R}^{n\times n}$, then

$$x \leq y \Longleftrightarrow y - x \in \mathbb{R}_+^n, \quad X \leq Y \Longleftrightarrow Y - X \in \mathbb{R}_+^{n\times n};$$

if $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ and $X = (x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \ |X| = (|x_{ik}|)_{i,k=1}^n;$$

$\det(X)$ is the determinant of the matrix X;

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X^{-1} is the inverse matrix to X;
$r(X)$ is the spectral radius of the matrix X;
E is the unit matrix;
Θ is the zero matrix;

$$\operatorname{diag}(x_1,\dots,x_n)=\begin{pmatrix} x_1 & 0 & \dots & 0 & 0\\ 0 & x_2 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0 & x_n\end{pmatrix};$$

if $x=(x_i)_{i=1}^n$, then $\operatorname{Sgn}(x)=\operatorname{diag}(\operatorname{sgn}x_1,\dots,\operatorname{sgn}x_n)$;
$C(I;\mathbb{R}^n)$ is the space of continuous vector functions $x:I\to\mathbb{R}^n$ with the norm

$$\|x\|_C=\max\Big\{\|x(t)\|:\ t\in I\Big\};$$

$C(I;\mathbb{R}^n_+)=\Big\{x\in C(I;\mathbb{R}^n):\ x(t)\in\mathbb{R}^n_+ \text{ for } t\in I\Big\}$;
$\widetilde{C}(I;\mathbb{R}^n)$ is the space of absolutely continuous vector functions $x:I\to\mathbb{R}^n$;
$L(I;\mathbb{R}^n)$ is the space of integrable vector functions $x:I\to\mathbb{R}^n$ with the norm

$$\|x\|_L=\int\limits_a^b\|x(t)\|\,dt;$$

$L(I;\mathbb{R}^n_+)=\Big\{x\in L(I;\mathbb{R}^n):\ x(t)\in\mathbb{R}^n_+ \text{ for almost all } t\in I\Big\}$;
$L(I;\mathbb{R}^{n\times n})$ is the space of integrable matrix functions $X:I\to\mathbb{R}^{n\times n}$;
if $Z\in C(I;\mathbb{R}^{n\times n})$ is a matrix function with the columns $z_1,\dots,z_n$ and $g:C(I;\mathbb{R}^n)\to L(I;\mathbb{R}^n)$ is a linear operator, then $g(Z)$ stands for the matrix function with columns $g(z_1),\dots,g(z_n)$.

Below we will assume that $p:C(I;\mathbb{R}^n)\to L(I;\mathbb{R}^n)$ is a strongly bounded operator, i.e., there exists $\eta\in L(I;\mathbb{R}_+)$ such that

$$\|p(x)(t)\|\le\eta(t)\|x\|_C \text{ for } t\in I,\ \ x\in C(I;\mathbb{R}^n).$$

Definition 1. A vector function $x\in\widetilde{C}(I;\mathbb{R}^n)$ is said to be a **solution of the system (1)** if it satisfies this system almost everywhere on I. A solution x of the system (1) is said to be a **solution of the problem (1), (2)** if it satisfies the condition (2).

Definition 2. A linear operator $v:C(I;\mathbb{R}^n)\to L(I;\mathbb{R}^n)$ ($v_0:C(I;\mathbb{R}^n)\to\mathbb{R}^n$) is called **positive** if

$$v(x)\in L(I;\mathbb{R}^n_+)\quad\Big(v_0(x)\in\mathbb{R}^n_+\Big) \text{ for } x\in C(I;\mathbb{R}^n_+).$$

Along with (1), (2) we consider the problems

$$\frac{dx(t)}{dt}=p_0(x)(t)+q(t), \tag{3}$$

$$\ell_0(x)=c_0; \tag{4}$$

$$\frac{dx(t)}{dt}=p_0(x)(t), \tag{3_0}$$

$$\ell_0(x)=0. \tag{4_0}$$

Introduce

Definition 3. Let $\sigma_i:I\to\mathbb{R}$ $(i=1,\dots,n)$ be measurable functions such that $\sigma_i(t)\in\{-1,1\}$ $(i=1,\dots,n)$ for almost all $t\in I$. We say that a pair (p_0,ℓ_0), where $p_0:C(I;\mathbb{R}^n)\to L(I;\mathbb{R}^n)$ is a linear strongly bounded operator and $\ell_0:C(I;\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded operator, belongs to the set $M_I^{\sigma_1,\dots,\sigma_n}$ if the homogeneous problem

$(3_0), (4_0)$ has only the trivial solution, and for any $c_0 \in \mathbb{R}^n_+$ and $q \in L(I; \mathbb{R}^n)$ satisfying the condition

$$\operatorname{diag}(\sigma_1, \dots, \sigma_n) q \in L(I; \mathbb{R}^n_+),$$

the solution x of the problem $(3), (4)$ is nonnegative, i.e., $x(t) \in \mathbb{R}^n_+$ for $t \in I$.

Theorems 1.1–1.3 and Corollaries 1.1–1.2 from [5] contain the necessary and sufficient conditions for the validity of the inclusion $(p_0, \ell_0) \in M_I^{\sigma_1,\dots,\sigma_n}$.

By $X_{p,\ell}$ we denote the space of solutions of the homogeneous problem

$$\frac{dx(t)}{dt} = p(x)(t), \quad \ell(x) = 0.$$

Theorem 1. *Let there exist measurable functions $\sigma_i : I \to \{-1, 1\}$ $(i = 1, \dots, n)$, a linear bounded operator $\ell_0 : C(I; \mathbb{R}^n) \to \mathbb{R}^n$ and a strongly bounded linear operator $p_0 : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}^n)$ such that*

$$(p_0, \ell_0) \in M_I^{\sigma_1,\dots,\sigma_n}, \tag{5}$$

$$\operatorname{diag}\Big(\sigma_1(t), \dots, \sigma_n(t)\Big)\Big[\operatorname{Sgn}(x(t)) p(x)(t) - p_0(|x|)(t)\Big] \le 0 \ \ \text{for } t \in I, \ \ x \in X_{p,\ell}, \tag{6}$$

and

$$\ell_0(|x|) \le 0 \ \ \text{for } x \in X_{p,\ell}. \tag{7}$$

Then the problem $(1), (2)$ has a unique solution.

Proof. Let $x \in X_{p,\ell}$. Set

$$y(t) = |x(t)|.$$

Then according to (6) and (7) we obtain

$$\operatorname{diag}\Big(\sigma_1(t), \dots, \sigma_n(t)\Big)\left[\frac{dy(t)}{dt} - p_0(y)(t)\right] \le 0.$$

Hence by Proposition 1.2 from [5] and the condition (5) we have $y(t) \le 0$ for $t \in I$. Consequently, $x(t) \equiv 0$. If now we apply Theorem 1.1 from [9], then the validity of Theorem 1 becomes evident. □

Corollary 1. *Let there exist numbers $t_i \in I$, $s_k \in I$, $\gamma_{ik} \in \mathbb{R}$ $(i = 1, \dots, n;\ k = 1, \dots, m)$, linear positive operators $\overline{\ell} : C(I; \mathbb{R}^n) \to \mathbb{R}^n$ and $\overline{p} : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}^n)$ and a matrix $A \in \mathbb{R}^{n\times n}_+$ such that $r(A) < 1$,*

$$\begin{aligned} &\operatorname{diag}\Big(\operatorname{sgn}(t - t_1), \dots, \operatorname{sgn}(t - t_n)\Big) \times \\ &\times \Big[\operatorname{Sgn}(x(t)) p(x)(t) - \operatorname{diag}\Big(p_1(t), \dots, p_n(t)\Big)|x(t)|\Big] \le \overline{p}(|x|)(t) \\ &\qquad\qquad \text{for } t \in I, \ \ x \in X_{p,\ell}, \end{aligned} \tag{8}$$

$$\left|\ell(x) - \Big(x_i(t_i) - \sum_{k=1}^m \gamma_{ik} x_i(s_k)\Big)_{i=1}^n\right| \le \overline{\ell}(|x|) \ \ \text{for } x \in C(I; \mathbb{R}^n), \tag{9}$$

$$\gamma_i = \exp\Big(\int_a^{t_i} p_i(s)\, ds\Big) - \sum_{k=1}^m |\gamma_{ik}| \exp\Big(\int_a^{s_k} p_i(s)\, ds\Big) > 0 \ \ (i = 1, \dots, n) \tag{10}$$

and

$$Y_0(t)\overline{\ell}(E) + \int_a^b |G_0(t, s)| \overline{p}(E)(s)\, ds \le A \ \ \text{for } t \in I, \tag{11}$$

where

$$Y_0(t) = \operatorname{diag}\Big(\exp\Big(\int_a^t p_1(s)\,ds\Big),\dots,\exp\Big(\int_a^t p_n(s)\,ds\Big)\Big),$$

$$G_0(t,s) = \operatorname{diag}\big(g_1(t,s),\dots,g_n(t,s)\big),$$

$$g_i(t,s) = \frac{1}{\gamma_i}\big(\chi_{[a,t]}(s) - \chi_{[a,t_i]}(s)\big)\exp\Big(\int_s^t p_i(\xi)\,d\xi + \int_a^{t_i} p_i(\xi)\,d\xi\Big) -$$

$$-\sum_{k=1}^m \frac{|\gamma_{ik}|}{\gamma_i}\big(\chi_{[a,t]}(s) - \chi_{[a,s_k]}(s)\big)\exp\Big(\int_s^t p_i(\xi)\,d\xi + \int_a^{s_k} p_i(\xi)\,d\xi\Big) \tag{12}$$

$$(i = 1,\dots,n).$$

Then the problem (1), (2) *has a unique solution.*

Proof. From (8) and (9) the inequalities (6) and (7) follow, where $\sigma_i(t) = \operatorname{sgn}(t - t_i)$ $(i = 1,\dots,n)$,

$$p_0(y)(t) = \operatorname{diag}\big(p_1(t),\dots,p_n(t)\big)y(t) + \operatorname{diag}\big(\sigma_1(t),\dots,\sigma_n(t)\big)\overline{p}(y)(t),$$

$$\ell_0(y) = \Big(y_i(t_i) - \sum_{k=1}^m |\gamma_{ik}|y_i(s_k)\Big)_{i=1}^n - \overline{\ell}(y).$$

On the other hand, by Theorem 1.2 from [5] the inequalities (11) and $r(A) < 1$ guarantee the validity of the inclusion (5). Therefore all the conditions of Theorem 1 are fulfilled. □

Corollary 2. *Let there exist numbers $t_i \in I$, $s_k \in I$, $\gamma_{ik} \in \mathbb{R}$ $(i = 1,\dots,n;\ k = 1,\dots,m)$ and linear positive operators $\overline{\ell} : C(I;\mathbb{R}^n) \to \mathbb{R}^n$, $\overline{p} : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ such that*

$$\gamma_i = 1 - \sum_{k=1}^m |\gamma_{ik}| > 0 \ \ (i = 1,\dots,n),$$

$$r\Big(\overline{\ell}(E) + \operatorname{diag}\Big(\frac{1}{\gamma_1},\dots,\frac{1}{\gamma_n}\Big)\int_a^b \overline{p}(E)(s)\,ds\Big) < 1,$$

$$\operatorname{diag}(t - t_1,\dots,t - t_n)\operatorname{Sgn}\big(x(t)\big)p(x)(t) \le \overline{p}(|x|)(t) \ \ \text{for } t \in I,\ \ x \in X_{p,\ell}$$

and the inequality (9) *holds. Then the problem* (1), (2) *has a unique solution.*

This corollary follows from Corollary 1 in the case $p_i(t) \equiv 0$ $(i = 1,\dots,n)$.

Consider now the problem

$$\frac{dx(t)}{dt} = P(t)x(\tau(t)) + q_0(t), \tag{13}$$

$$x(t) = u(t) \ \text{ for } t \notin I,\ \ \ell(x) = c_0, \tag{14}$$

where $P \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $\tau : I \to \mathbb{R}$ is a measurable function and $u : \mathbb{R} \to \mathbb{R}^n$ is a continuous and bounded vector function*. This problem can be reduced to the

*If $\tau(t) \in I$ for almost all $t \in I$, then the condition $x(t) = u(t)$ for $t \notin I$ is to be dropped.

problem (1), (2). To see this, set

$$\tau_0(t) = \begin{cases} a & \text{for } \tau(t) < a \\ \tau(t) & \text{for } a \le \tau(t) \le b \\ b & \text{for } \tau(t) > b \end{cases},$$

$$p(x)(t) = \chi_I(\tau(t))P(t)x(\tau_0(t)), \tag{15}$$

and

$$q(t) = \Big(1 - \chi_I(\tau(t))\Big)P(t)u(\tau(t)) + q_0(t).$$

Theorem 2. *Let there exist numbers $t_i \in I$, $s_k \in I$, $\gamma_{ik} \in \mathbb{R}$ $(i = 1, \dots, n;\ k = 1, \dots, m)$, functions $p_i \in L(I;\mathbb{R})$ $(i = 1, \dots, n)$, a linear positive operator $\overline{\ell} : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ and a matrix $A \in \mathbb{R}_+^{n\times n}$ such that along with* (9) *and* (10) *the following conditions*

$$\Big(\chi_I(\tau(t))p_{ii}(t) - p_i(t)\Big)\operatorname{sgn}(t - t_i) \le 0 \ \text{ for } t \in I \ \ (i = 1, \dots, n), \tag{16}$$

$$Y_0(t)\overline{\ell}(E) + \int_a^b |G_0(t,s)| \times$$

$$\times\Big[|\mathcal{P}_0(s)|\,\Big|\int_s^{\tau_0(s)} |\mathcal{P}(\xi)|\,d\xi\Big| + \Big|\mathcal{P}(s) - \mathcal{P}_0(s)\Big|\Big]\chi_I(\tau(s))\,ds \le A \ \text{ for } t \in I, \tag{17}$$

and $r(A) < 1$ hold, where

$$Y_0(t) = \operatorname{diag}\Big(\exp\Big(\int_a^t p_1(s)\,ds\Big), \dots, \Big(\int_a^t p_n(s)\,ds\Big)\Big),$$

$$\mathcal{P}_0(t) = \operatorname{diag}\Big(p_{11}(t), \dots, p_{nn}(t)\Big), \quad G_0(t,s) = \operatorname{diag}\Big(g_1(t,s), \dots, g_n(t,s)\Big)$$

and g_i $(i = 1, \dots, n)$ are the functions given by the equalities (12). *Then the problem* (13), (14) *has a unique solution.*

Proof. Let p be the operator defined by (15) and $x \in X_{p,\ell}$. Then

$$x'(t) = \chi_I(\tau(t))\mathcal{P}_0(t)x(t) + \chi_I(\tau(t))\mathcal{P}_0(t)\int_t^{\tau_0(t)} x'(s)\,ds +$$

$$+ \chi_I(\tau(t))\Big[\mathcal{P}(t) - \mathcal{P}_0(t)\Big]x(\tau_0(t)) =$$

$$= \chi_I(\tau(t))\mathcal{P}_0(t)x(t) + \chi_I(\tau(t))\mathcal{P}_0(t)\int_t^{\tau_0(t)} \chi_I(\tau(s))\mathcal{P}(s)x(\tau_0(s))\,ds +$$

$$+ \chi_I(\tau(t))\Big[\mathcal{P}(t) - \mathcal{P}_0(t)\Big]x(\tau_0(t)).$$

From this equality, by (16) and (17), we get the inequalities (8) and (9), where

$$\overline{p}(y)(t) = |\mathcal{P}_0(t)|\Big(\int_t^{\tau_0(t)} |\mathcal{P}(\xi)|y(\tau_0(\xi))\,d\xi\Big)\operatorname{sgn}\Big(\tau_0(t) - t\Big) +$$

$$+ \chi_I(\tau(t))\Big|\mathcal{P}(t) - \mathcal{P}_0(t)\Big|y(\tau_0(t)).$$

Therefore all the assumptions of Corollary 1 are satisfied. □

In the case $p_i(t) \equiv 0$, Theorem 2 yields

Corollary 3. *Let there exist numbers* $t_i \in I$, $s_k \in I$, $\gamma_{ik} \in \mathbb{R}$ $(i = 1, \dots, n;\ k = 1, \dots, m)$ *and a linear positive operator* $\overline{\ell} : C(I; \mathbb{R}^n) \to \mathbb{R}^n$ *such that along with* (9) *the following conditions*

$$\chi_I(\tau(t)) p_{ii}(t) \operatorname{sgn}(t - t_i) \le 0 \ \text{for}\ t \in I \ \ (i = 1, \dots, n),$$

$$\gamma_i = 1 - \sum_{k=1}^{m} |\gamma_{ik}| > 0 \ \ (i = 1, \dots, n), \quad r(A) < 1$$

hold, where

$$A = \overline{\ell}(E) +$$

$$+ \operatorname{diag}\left(\frac{1}{\gamma_1}, \dots, \frac{1}{\gamma_n}\right) \int_a^b \Big[|\mathcal{P}_0(s)| \, \Big| \int_s^{\tau_0(s)} |\mathcal{P}(\xi)| \, d\xi \Big| + \Big| \mathcal{P}(s) - \mathcal{P}_0(s) \Big| \Big] \chi_I(\tau(s)) \, ds.$$

Then the problem (13), (14) *has a unique solution.*

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