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NONCLASSICAL BIHARMONIC BOUNDARY VALUE PROBLEMS DESCRIBING BENDING OF FINITE AND INFINITE PLATES WITH INCLUSIONS

Abstract. Contact problems of the theory of elasticity on bending of finite and infinite, isotropic or anisotropic plates with an elastic inclusion of variable bending rigidity are considered. The problems are reduced to an integro-differential equation with a variable coefficient. When the coefficient turns to zero of higher order at the ends of the interval of integration, the equation is out of the framework of cases already studied. Such equations are studied, exact or approximate solutions are obtained, the bahaviour of unknown contact stresses at the ends of the line of contact is established.

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Introduction

The contact problems on interaction of thin-shelled elements (stringers or inclisions) of various geometric forms with massive deformable bodies belong to the at present extensive field of the theory of contact and mixed problems of mechanics of deformable rigid bodies. Interest in such type of problems is motivated by the fact that the investigations in this area will make it possible to solve a number of questions connected with problems of engineering industry, shipbuilding, as well as with designing aircraft and other thin-shelled constructions.

Stringers and inclusions, like stamps and cuts, concentrate stresses, therefore it is of great theoretic and practic importance to investigate the influence exerted by the inclusion on the stress-strained state of deformable bodies, to study questions on the concentration of stresses in such problems and to elaborate methods for their lowering. Taking into account thin-shellness in different assumptions and theories, we arrive at new statements of the contact problem of deformable bodies which substantially differ from those of classical contact problems of elasticity, and, as a result, there arises a class of new problems of solid mechanics with displaced boundary conditions.

A vast number of works are devoted to problems of bending of plates with thin inclusions differing by the conditions imposed on the inclusion (see, e.g., [1–5]). The problems are reduced to systems of integral equations whose the characteristic part has in general the form

$$\int_{-1}^{1} \frac{(t-\tau)^2}{2}\Big[\frac{a\,\mathrm{sgn}(t-\tau)}{2}+\frac{b}{\pi i}\ln\frac{1}{|t-\tau|}\Big]\varphi(\tau)d\tau = f(t), \quad |t|<1. \qquad (0.1)$$

A solution of this equation is sought in a class of functions with non-integrable singularities by using the method of regularization of diverging integrals [6]. The exact solution of the equation is constructed by means of the method of analytic functions [7], while an approximate solution is constructed by the method of orthogonal polynomials.

In the problems we consider, the inclusions either are thin, absolutely rigid elements or elements with constant bending rigidity.

The distinctive feature of the investigation carried out in the present paper is that we have managed to establish the dependence of the bahaviour of contact stresses on the law of variation of bending rigidity of the inclusion. In our statement the problems are reduced to the solution of integro-differential equation whose characteristic part is the Prandtl integro-differential equation which under certain conditions has been studied in [8–11].

1. Bending of a Circular Plate

We consider the problem on bending of a circular plate of unit radius, supported along the segment: $y = 0$, $|x| < a$ $(a < 1)$ by a thin elastic inclusion of variable bending rigidity. A normal load of constant intensity q is applied to the plate, and the inclusion is free of the load.

Introduce the notation: $\Omega = \{(x, y)|x^2+y^2 < 1\}$, $\Gamma = \partial\Omega$, $S = \Omega\backslash(-a, a)$. It is required to find contact interaction stresses between the inclusion and the plate. The problem posed is equivalent to the finding of a solution of nonhomogeneous biharmonic equation

$$D\Delta\Delta\omega(x, y) = q, \quad (x, y) \in S, \tag{1.1}$$

satisfying the boundary conditions

$$\omega = 0, \quad \frac{\partial\omega}{\partial n} = 0, \quad (x, y) \in \Gamma, \tag{1.2}$$

and also the conditions

$$\langle\omega\rangle = \langle\omega'_y\rangle = \langle M_y\rangle = 0, \quad \langle N_y\rangle = \mu(x), \quad |x| < a, \quad |y| = 0 \tag{1.3}$$

imposed on the inclusion.

We use here the notation: $\langle f\rangle = f(x, -0) - f(x, +0)$, $f(x, -0) \equiv f^-$, $f(x, +0) \equiv f^+$, ω is the plate deflection, D is cylindrical rigidity of the plate, ω'_y, M_y and N_y are, respectively, the angle of rotation, the bending moment and the generalized transversal force in the plate, $\mu(x)$ is an unknown contact stress of interaction between the inclusion and the plate (note that $\mu(x) \equiv 0$ for $|x| > a$), and n is the normal external to Γ.

Assuming the ends of the plate to be free, for the plate deflection $\omega_0(x)$ we obtain the following conditions:

$$\frac{d^2}{dx^2}\, D_0(x)\, \frac{d^2\omega_0(x)}{dx^2} = -\mu(x), \quad |x| < a, \tag{1.4}$$

$$D_0(x)\omega''_0(x)|_{x=\pm a} = 0, \quad \big[D_0(x)\omega''_0(x)\big]'\big|_{x=\pm a} = 0, \tag{1.5}$$

where $D_0(x) = \frac{E_0(x)h_0^3(x)}{12}$ is the bending rigidity of the inclusion, $E_0(x)$ is the elasticity modulus of its material and $h_0(x)$ is its thickness.

The conditions (1.5) are equivalent to the usual statical conditions of equilibrium of the inclusion:

$$\int_{-a}^{a} \mu(t)dt = 0, \quad \int_{-a}^{a} t\mu(t)dt = 0. \tag{1.6}$$

On the interval $[-a, a]$ of contact between the inclusion and the plate the condition

$$\omega(x, 0) = \omega_0(x) \tag{1.7}$$

must be satisfied.

A solution of the boundary value problem (1.1)–(1.5) will be sought in the Banach space $W(\Omega)$ of the functions $\omega(x,y)$ satisfying the conditions (1.2) and having summable second derivatives with the norm

$$\|\omega\|_W = \Big(\iint\limits_{\Omega}\Big[(\Delta\omega)^2 - 2(1-\sigma)\Big(\frac{\partial^2\omega}{\partial x^2}\,\frac{\partial^2\omega}{\partial y^2} - \Big(\frac{\partial^2\omega}{\partial x\partial y}\Big)^2\Big)\Big]d\Omega\Big)^{1/2},$$

where σ are the Poisson coefficients, $0<\sigma<\frac{1}{2}$.

From the mechanical point of view, this space describes a class of deflection functions for which the potential energy of the plate bending is positive and finite.

Theorem 1. *If the above-formulated problem* (1.1)–(1.5) *has a solution, then the solution is unique.*

Indeed, suppose that the problem admits two solutions. Let $\omega_1^0(x,y)$ be the plate deflection corresponding to the first of the solutions and $\omega_2^0(x,y)$ to the second one. We make up the "difference" of these solutions, i.e., we put

$$\omega_0(x,y) = \omega_1^0(x,y) - \omega_2^0(x,y).$$

It is obvious that $\omega_0(x,y)$ satisfies the basic equations when the external forces are absent, i.e., $q\equiv 0$.

By Ostrogradsky-Green's formula we have

$$\iint\limits_{S'}(X_{n'}u + Y_{n'}v + Z_{n'}\omega)ds =$$

$$= \iiint\limits_{V}[\lambda(e_{xx}+e_{yy}+e_{zz})^2 + 2\mu(e_{xx}^2+e_{yy}^2+e_{zz}^2+2e_{xy}^2+2e_{xz}^2+2e_{yz}^2)]dxdydz, \quad (1.8)$$

where $X_{n'}$, $Y_{n'}$, $Z_{n'}$ are the components of the stress vector acting on the surface S' with the normal n', u, v, ω are the displacement components, e_{xx}, $e_{yy},\dots,e_{yz}$ are the deformation components, λ, μ are the Lamé parameters, and V is the domain occupied by the plate.

Under the conditions of our problem, using formulas (1.4)–(1.7) for the "difference" of two solutions, the potential energy of deformation accumulated in the system "plate-inclusion" can be represented in the form

$$\int\limits_{-h/2}^{h/2} dz \int\limits_{-a}^{a} \langle Z_n\rangle \omega_0(x,0)dx = h\int\limits_{-a}^{a}\langle N_y\rangle\omega_0(x,0)dx =$$

$$= h\int\limits_{-a}^{a}\mu(x)\omega_0(x,0)dx = h\int\limits_{-a}^{a}\omega_0(x,0)d\Big(\int\limits_{-a}^{x}\mu(t)dt\Big) =$$

$$= h\omega_0(x,0)\Big(\int\limits_{-a}^{x}\mu(t)dt\Big)\Big|_{-a}^{a} - h\int\limits_{-a}^{a}\omega_0'(x,0)\Big(\int\limits_{-a}^{x}\mu(t)dt\Big)dx =$$

$$= -h\int\limits_{-a}^{a}\omega_0'(x,0)d\Big(\int\limits_{-a}^{x}dt\int\limits_{-a}^{t}\mu(\tau)d\tau\Big) =$$

$$= -h\omega_0'(x,0)\Big(\int\limits_{-a}^{x}dt\int\limits_{-a}^{t}\mu(\tau)d\tau\Big)\Big|_{-a}^{a} -$$

$$-h\int\limits_{-a}^{a}\omega_0''^2(x,0)D_0(x)dx = -h\int\limits_{-a}^{a}\omega_0''^2(x,0)D_0(x)dx,$$

where h is the plate thickness.

The potential energy of deformation in the formula (1.8) is the positive square form of the components of deformation, and therefore from the last representation we conclude that $\omega_0''(x,0) = 0$, $|x| < a$. In its turn it means that in the absence of external forces

$$\mu(x) = 0, \quad |x| < a. \tag{1.9}$$

As is known, for any biharmonic function satisfying certain conditions of regularization in the vicinity of L of the domain S, the formula

$$\iint\limits_{S}\Big\{(\Delta\omega_0)^2 - (1-\sigma)\Big[\frac{\partial^2\omega_0}{\partial x^2}\frac{\partial^2\omega_0}{\partial y^2} - \Big(\frac{\partial^2\omega_0}{\partial x\partial y}\Big)^2\Big]\Big\}dxdy +$$

$$+\int\limits_{L}\Big(\omega_0 N\omega_0 - \frac{d\omega_0}{dn}M\omega_0\Big)ds = 0 \tag{1.10}$$

is valid, where

$$M\omega_0 = \sigma\Delta\omega_0 + (1-\sigma)\Big[\cos^2\theta\frac{\partial^2\omega_0}{\partial x^2} + \sin^2\theta\frac{\partial^2\omega_0}{\partial y^2} + \sin 2\theta\frac{\partial^2\omega_0}{\partial x\partial y}\Big],$$

$$N\omega_0 = \frac{d\Delta\omega_0}{dn} + (1-\sigma)\frac{d}{ds}\Big[\cos 2\theta\frac{\partial^2\omega_0}{\partial x\partial y} + \frac{1}{2}\sin 2\theta\Big(\frac{\partial^2\omega_0}{\partial x^2} - \frac{\partial^2\omega_0}{\partial y^2}\Big)\Big]$$

θ is the angle formed by n and the ox-axis, $L = \Gamma \cup [-a,a]$.

With regard for the conditions (1.2), transforming the integral

$$\int\limits_{-a}^{a}\Big(\omega_0^+ N^+\omega_0 - \frac{d\omega_0^+}{dn}M^+\omega_0 - \omega_0^- N^-\omega_0 + \frac{d\omega_0^-}{dn}M^-\omega_0\Big)dx =$$

$$= \int\limits_{-a}^{a}\Big[(\omega_0^+ - \omega_0^-)N^+\omega_0 + \omega_0^-(N^+\omega_0 - N^-\omega_0) -$$

$$-M\omega_0^+\Big(\frac{d\omega_0^+}{dy}-\frac{d\omega_0^-}{dy}\Big)-\frac{d\omega_0^-}{dy}(M^+\omega_0-M\omega_0^-)\Big]dx,$$

and taking into account the conditions (1.3) and (1.9), we conclude that the last integral in the formula (1.10) is equal to zero.

Since the expression

$$(\Delta\omega_0)^2-(1-\sigma)\Big[\frac{\partial^2\omega_0}{\partial x^2}\frac{\partial^2\omega_0}{\partial y^2}-\Big(\frac{\partial^2\omega_0}{\partial x\partial y}\Big)^2\Big]=$$
$$=\Big(\frac{\partial^2\omega_0}{\partial x^2}\Big)^2+\Big(\frac{\partial^2\omega_0}{\partial y^2}\Big)^2+(1+\sigma)\frac{\partial^2\omega_0}{\partial x^2}\frac{\partial^2\omega_0}{\partial y^2}+(1-\sigma)\Big(\frac{\partial^2\omega_0}{\partial x\partial y}\Big)^2$$

represents the positive definite square form of the second derivatives of the function $\omega_0(x,y)$, it follows from (1.10) that all the second partial derivatives of the function $\omega_0(x,y)$ are equal to zero, and hence $\omega_0(x,y)$ is the linear function of its arguments, but as far as $\omega_0=\frac{d\omega_0}{dn}=0$ on Γ, we can easily see that $\omega_0(x,y)=0$ everywhere on S, and hence everywhere in Ω.

Thus the uniqueness theorem for the above-formulated problem is proved.

The general solution of the equation (1.1) can be represented as

$$\omega(x,y)=\omega_1(x,y)+\omega_2(x,y),$$

where $\omega_1(x,y)$ is the partial solution, for example, $\omega_1(x,y)=\frac{q}{64D}(x^4+2x^2y^2+y^4)$, and $\omega_2(x,y)$ satisfies the biharmonic equation $\Delta\Delta\omega_2(x,y)=0$ with inhomogeneous boundary conditions

$$\omega_2=-\omega_1,\quad \frac{\partial\omega_2}{\partial n}=-\frac{\partial\omega_1}{\partial n},\quad (x,y)\in\Gamma. \tag{1.11}$$

A solution of the biharmonic equation is representable by the well known Goursat formula

$$\omega_2(x,y)=2\operatorname{Re}[\overline{z}\varphi(z)+\chi(z)], \tag{1.12}$$

where $\varphi(z)$ and $\chi(z)$ are functions of the complex variable $z=x+iy$, holomorphic in S. For the bending moments M_x and M_y, the twisting moment H_{xy} and for the cutting forces N_x and N_y, we have the formulas

$$\begin{aligned} M_y-M_x+2iH_{xy}&=4(1-\sigma)D[\overline{z}\varphi''(z)+\psi'(z)],\\ M_x+M_y&=-8(1+\sigma)D\operatorname{Re}\varphi'(z),\\ N_x-iN_y&=-8D\varphi''(z), \end{aligned} \tag{1.13}$$

where $\psi(z)=\chi'(z)$.

Let us introduce into consideration a new function $\Omega_0(z)$ by the equality

$$\Omega_0(z)=z\varphi'(z)+\psi(z).$$

Then on the basis of the formula (1.12), the formula

$$\frac{\partial\omega_2}{\partial x}+i\frac{\partial\omega_2}{\partial y}=\varphi(z)+\overline{\Omega_0(z)}+(z-\overline{z})\overline{\varphi'(z)}$$

is valid.

From the first two conditions (1.3) we get

$$[\varphi(t)-\overline{\Omega}_0(t)]^- - [\varphi(t)-\overline{\Omega}_0(t)]^+ = 0, \quad t\in(-a,a),$$

whence

$$\varphi(z)-\overline{\Omega_0(\overline{z})} = F_{01}(z), \quad z\in\Omega, \tag{1.14}$$

where $F_{01}(z)$ is holomorphic in the domain Ω.

Substituting the expression for the function $\psi(z)$ from the last equality into (1.13), we can write

$$\begin{aligned} M_y &= 2(1-\sigma)D\,\mathrm{Re}[\overline{\varphi}'(z)-\varphi'(z)-(z-\overline{z})\varphi''(z)-\overline{F}'_{01}(z)]-4(1+\sigma)D\,\mathrm{Re}\,\varphi'(z),\\ N_y &= 8D\,\mathrm{Im}\,\varphi''(z). \end{aligned}$$

From the last two conditions (1.3), we have

$$\begin{aligned} &[\varphi''(t)+\overline{\varphi''(t)}]^- - [\varphi''(t)+\overline{\varphi''(t)}]^+ = 0,\\ &[\varphi''(t)-\overline{\varphi''(t)}]^- - [\varphi''(t)-\overline{\varphi''(t)}]^+ = \frac{i\mu(t)}{4D}, \quad |t|<a. \end{aligned}$$

Summing up the above conditions, we obtain

$$\varphi''^{+}(t)-\varphi''^{-}(t) = -\frac{i\mu(t)}{8D}, \quad |t|<a. \tag{1.15}$$

The function $\mu(t)$ may have non-integrable singularities on the segment $[-a,a]$. Taking into account the proof given in [7] on the transfer of the results of the monograph [15] to the regularized values of diverging integrals [6], the solution of the boundary value problem (1.15) is given by the formula

$$\varphi''(z) = -\frac{1}{16\pi D}\int_{-a}^{a}\frac{\mu(t)dt}{t-z} + F_{02}(z), \quad z\in\Omega, \tag{1.16}$$

where $F_{02}(z)$ is a function holomorphic in Ω.

Then on the basis of the formulas (1.14) and (1.16), the functions $\varphi(z)$ and $\psi(z)$ are represented as follows:

$$\begin{aligned} \varphi(z) &= -\frac{1}{16\pi D}\int_{-a}^{a}(t-z)\ln(t-z)\mu(t)dt + F_1(z),\\ \psi(z) &= -\frac{1}{16\pi D}\int_{-a}^{a}t\ln(t-z)\mu(t)dt + F_2(z), \quad z\in\Omega, \end{aligned}$$

where $F_1(z)$ and $F_2(z)$ are holomorphic in Ω functions to be defined. To define these functions, on the boundary of the circle we obtain by virtue of (1.11) the following boundary value problem:

$$F_1(t)+t\overline{F_1'(t)}+\overline{F_2(t)}=-f_1(t)-t\overline{f_1'(t)}-\overline{f_2(t)}-\frac{\partial\omega_1}{\partial x}-i\frac{\partial\omega_1}{\partial y},\qquad(1.17)$$

where $f_1(z)=-\frac{1}{16\pi D}\int_{-a}^{a}(t-z)\ln(t-z)\mu(t)dt$, $f_2(z)=-\frac{1}{16\pi D}\int_{-a}^{a}t\ln(t-z)\mu(t)dt$ are analytic functions in the plane cut on the segment $[-a,a]$.

Multiplying the equality (1.17) by $\frac{1}{2\pi i}\frac{dt}{t-z}$, where $z\in\Omega$, and integrating with respect to Γ, we get

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{F_1(t)dt}{t-z}+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{t\overline{F_1'(t)}dt}{t-z}+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\overline{F_2(t)}dt}{t-z}=$$
$$=-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f_1(t)dt}{t-z}-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{t\overline{f_1'(t)}dt}{t-z}-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\overline{f_2(t)}dt}{t-z}-$$
$$-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(t)dt}{t-z},\quad\text{where}\quad g(t)=\frac{\partial\omega_1}{\partial x}+i\frac{\partial\omega_1}{\partial y}=\frac{q}{16D}t,\ \ t\in\Gamma.\qquad(1.18)$$

Consider now the decomposition of the functions $F_1(z)$ and $F_2(z)$ and write out only its first three terms

$$F_1(z)=a_0+a_1z+a_2z^2+\cdots,$$
$$F_2(z)=a_0'+a_1'z+a_2'z^2+\cdots.$$

Consequently, by the Cauchy formula [15], the first integral in the left-hand side of (1.18) is equal to $F_1(z)$, while the second and the third ones are equal to $\overline{a}_1z+2\overline{a}_2$ and $\overline{a}_2'$, respectively. Since the function $f_1(z)$ is holomorphic outside of Γ, and the functions $t\overline{f_1'(t)}$ and $\overline{f_2(t)}$ are the boundary values of the functions $z\overline{f_1'\left(\frac{1}{\overline{z}}\right)}$ and $\overline{f_2\left(\frac{1}{\overline{z}}\right)}$, holomorphic in Γ, from (18) we finally obtain

$$F_1(z)+\overline{a}_1z+2\overline{a}_2+\overline{a}_0'=-z\overline{f_1'\left(\frac{1}{\overline{z}}\right)}-\overline{f_2\left(\frac{1}{\overline{z}}\right)}-\frac{q}{16D}z.\qquad(1.19)$$

Passing to conjugate values in (1.17), similarly to the previous reasoning, we have

$$\overline{a}_0+\frac{F_1'(z)-a_1}{z}+F_2(z)=-\overline{f_1\left(\frac{1}{\overline{z}}\right)}.\qquad(1.20)$$

Regarding the constants appearing in the formulas (1.19)–(1.20), we note that

$$a_0+2\overline{a}_2+a_0'=0,$$

$$a_1+\overline{a}_1=-\frac{1}{16\pi D}\int\limits_{-a}^{a}t^2\mu(t)dt-\frac{q}{16D},$$

$$2a_2=\frac{1}{16\pi D}\int\limits_{-a}^{a}t^3\mu(t)dt.$$

The above formulas show that if the function $\mu(t)$ is found, then the constants a_2 and $\operatorname{Re}a_1$ are defined, and therefore the function $\varphi(z)$, as it was to be expected, is defined to within the expression $Ciz+\gamma$, where C and γ are arbitrary constants C being real and γ being comples, and the function $\psi(z)$ is defined to within a complex constant γ'.

From (1.19) and (1.20) we define the unknown functions $F_1(z)$ and $F_2(z)$:

$$F_1(z)=-\overline{a}_1z-2\overline{a}_2-\overline{a}_0'-z\overline{f}_1'\Big(\frac{1}{z}\Big)-\overline{f}_2\Big(\frac{1}{z}\Big)-\frac{q}{16D}z,$$

$$F_2(z)=-\overline{a}_0+\frac{a_1+\overline{a}_1}{z}-\overline{f}_1\Big(\frac{1}{z}\Big)-\frac{1}{z}\overline{f}_1'\Big(\frac{1}{z}\Big)-$$
$$-\frac{1}{z^2}\overline{f}_1''\Big(\frac{1}{z}\Big)-\frac{1}{z^3}\overline{f}_2'\Big(\frac{1}{z}\Big)+\frac{q}{16D}\frac{1}{z}.$$

It is obvious that $\frac{\partial\omega_2(x,0)}{\partial x}=\operatorname{Re}[\varphi(x)+x\overline{\varphi'(x)}+\overline{\psi(x)}]$, where from we can get

$$\frac{\partial^2\omega_2(x,0)}{\partial x^2}=\operatorname{Re}[f_1'(x)+\overline{f}_1'(x)+x\overline{f}_1''(x)+\overline{f}_2'(x)]+$$
$$+\operatorname{Re}[F_1'(x)+\overline{F}_1'(x)+x\overline{F}_1''(x)+\overline{F}_2'(x)]=$$
$$=\frac{1}{8\pi D}\int\limits_{-a}^{a}\ln|t-x|\mu(t)dt-\frac{x}{16\pi D}\int\limits_{-a}^{a}\frac{\mu(t)dt}{t-x}+\frac{1}{16\pi D}\int\limits_{-a}^{a}\frac{t\mu(t)dt}{t-x}+$$
$$+2\operatorname{Re}\Big[-\overline{a}_1-\overline{f}_1'\Big(\frac{1}{x}\Big)+\frac{1}{x}\overline{f}_1''\Big(\frac{1}{x}\Big)+\frac{1}{x^2}\overline{f}_2'\Big(\frac{1}{x}\Big)-\frac{q}{16D}x\Big]-$$
$$-\operatorname{Re}\Big[\frac{1}{x^2}\overline{f}_1'''\Big(\frac{1}{x}\Big)+\frac{2}{x^2}\overline{f}_2\Big(\frac{1}{x}\Big)+\frac{1}{x^3}\overline{f}_2''\Big(\frac{1}{x}\Big)\Big]+$$
$$+\operatorname{Re}\Big[-\frac{a_1+\overline{a}_1}{x^2}+\frac{1}{x^3}\overline{f}_1''\Big(\frac{1}{x}\Big)+$$
$$+\frac{1}{x^4}\overline{f}_1'''\Big(\frac{1}{x}\Big)+\frac{3}{x^4}\overline{f}_2'\Big(\frac{1}{x}\Big)+\frac{1}{x^5}\overline{f}_2''\Big(\frac{1}{x}\Big)-\frac{q}{16D}\frac{1}{x^2}\Big].\qquad(1.21)$$

Taking into account that $f_1(x)$ and $f_2(x)$ are real functions of real variables, using the conditions (1.6), then performing the transformations

$$-2f_1'\Big(\frac{1}{x}\Big)+\frac{2}{x}f_1''\Big(\frac{1}{x}\Big)-\frac{1}{x^2}f_1'''\Big(\frac{1}{x}\Big)+\frac{1}{x^3}f_1''\Big(\frac{1}{x}\Big)+\frac{1}{x^4}f_1'''\Big(\frac{1}{x}\Big)=$$
$$=-\frac{1}{8\pi D}\int\limits_{-a}^{a}\ln\Big|t-\frac{1}{x}\Big|\mu(t)dt-\Big(\frac{2}{x}+\frac{1}{x^3}\Big)\frac{1}{16\pi D}\int\limits_{-a}^{a}\frac{\mu(t)dt}{t-\frac{1}{x}}+$$

$$+\Big(\frac{1}{x^2}-\frac{1}{x^4}\Big)\frac{1}{16\pi D}\int\limits_{-a}^{a}\frac{\mu(t)dt}{(t-\frac{1}{x})^2}=$$

$$=-\frac{1}{8\pi D}\int\limits_{-a}^{a}\ln|tx-1|\mu(t)dt-\frac{(2x^2+1)}{16\pi D}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{tx-1}+$$

$$+\frac{3(x^2-1)}{16\pi D}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{(tx-1)^2}-\frac{x^3-x}{8\pi D}\int\limits_{-a}^{a}\frac{t^3\mu(t)dt}{(tx-1)^2};$$

$$\frac{1}{x^3}f_2''\Big(\frac{1}{x}\Big)=\frac{1}{16\pi Dx^3}\int\limits_{-a}^{a}\frac{t\mu(t)dt}{(t-\frac{1}{x})^2}=$$

$$=\frac{1}{16\pi Dx}\int\limits_{-a}^{a}\frac{t\mu(t)dt}{(tx-1)^2}=\frac{1}{16\pi Dx}\int\limits_{-a}^{a}\Big(\frac{1}{(tx-1)^2}-1\Big)t\mu(t)dt=$$

$$=\frac{1}{8\pi D}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{(tx-1)^2}-\frac{x}{16\pi D}\int\frac{t^3\mu(t)dt}{(tx-1)^2};$$

$$-\frac{a_1+\overline{a}_1+\frac{q}{16D}}{x^2}+\frac{3}{x^4}f_2'\Big(\frac{1}{x}\Big)+\frac{1}{x^5}f_2''\Big(\frac{1}{x}\Big)=$$

$$=\frac{1}{16\pi D}\Big\{\frac{1}{x^2}\int\limits_{-a}^{a}t^2\mu(t)dt+\frac{3}{x^3}\int\limits_{-a}^{a}\frac{t\mu(t)dt}{tx-1}+\frac{1}{x^3}\int\limits_{-a}^{a}\frac{t\mu(t)dt}{(tx-1)^2}\Big\}=$$

$$=\frac{1}{16\pi D}\int\limits_{-a}^{a}\frac{t^4\mu(t)dt}{(tx-1)^2},$$

and substituting in (1.21), we obtain

$$\frac{\partial^2\omega_2(x,0)}{\partial x^2}=\frac{1}{8\pi D}\int\limits_{-a}^{a}\ln|t-x|\mu(t)dt-\frac{1}{8\pi D}\int\limits_{-a}^{a}\ln|tx-1|\mu(t)dt-$$

$$-\frac{2x^2-1}{16\pi D}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{tx-1}+\frac{3x^2-1}{16\pi D}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{(tx-1)^2}+$$

$$+\frac{x-2x^3}{16\pi D}\int\limits_{-a}^{a}\frac{t^3\mu(t)dt}{(tx-1)^2}+\frac{1}{16\pi D}\int\limits_{-a}^{a}\frac{t^4\mu(t)dt}{(tx-1)^2}-2\,\mathrm{Re}\,a_1-\frac{q}{8D}.$$

With regard for the contact condition (1.7), the differential equation (1.4) for the inclusion bending takes the form

$$\frac{1}{8\pi D}\Bigg[\int\limits_{-a}^{a}\ln|t-x|\mu(t)dt-\int\limits_{-a}^{a}\ln|tx-1|\mu(t)dt-\frac{2x^2-1}{2}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{tx-1}+$$

$$+\frac{3x^2-1}{2}\int\limits_{-a}^{a}\frac{t^2\mu(t)dt}{(tx-1)^2}+\frac{x-2x^3}{2}\int\limits_{-a}^{a}\frac{t^3\mu(t)dt}{(tx-1)^2}+\frac{1}{2}\int\limits_{-a}^{a}\frac{t^4\mu(t)dt}{(tx-1)^2}\Bigg]-$$

$$-\frac{q}{8D}-2\operatorname{Re}a_1=-\frac{1}{D_0(x)}\int\limits_{-a}^{x}dt\int\limits_{-a}^{t}\mu(\tau)d\tau.$$

Introducing the notation $\lambda_(x)\equiv\int_{-a}^{x}dt\int_{-a}^{t}\mu(\tau)d\tau$, we arrive at the integral differential equation

$$\lambda_1(x)-\frac{D_0(x)}{8\pi D}\int\limits_{-a}^{a}\frac{\lambda_1'(t)dt}{t-x}+$$

$$+\frac{D_0(x)}{8\pi D}\int\limits_{-a}^{a}R_1(x,t)\lambda_1(t)dt=f_1D_0(x),\quad |x|<a, \tag{1.22}$$

where

$$R_1(x,t)=-\frac{\partial\overset{\circ}{R}(t,x)}{\partial t},\quad \overset{\circ}{R}(t,x)=\frac{x}{xt-1}+\frac{(2x^2-1)t^2x}{2(tx-1)^2}+$$

$$+\frac{2(3x^2-1)t-(x-2x^3)(t^3x-3t^2)-2t^3(tx-2)}{2(tx-1)^3}-2t,$$

$$f_1=\frac{q}{16D}$$

provided

$$\lambda_1(\pm a)=0,\quad \lambda_1'(\pm a)=0. \tag{1.23}$$

2. Bending of a Rectangular Plate

Consider a rectangular ($|x|<\frac{c}{2}, 0\le y\le b$) hinged supporteel plate with an inclusion along the segment: $y=\frac{b}{2}$, $|x|<a$ ($2a<c$), which causes discontinuity of principal quantities in the general case. But from the symmetry of the problem with respect to the straight line $y=\frac{b}{2}$ and from the assumption that the inclusion shifts vertically under the action of the load $q(x)$, it follows that the conditions (1.3) are fulfilled. To simplify our reasoning, we assume that the deflection of the plate $\omega(x,y)$ caused by bending of the inclusion is even with respect to x.

Then the function ω, satisfying the equation $D\Delta\Delta\omega(x,y)=0$ for $y\neq\frac{b}{2}$ and the boundary condition

$$\begin{aligned}\omega = M_x = 0 \quad \text{for} \quad x=\pm\frac{c}{2},\\ \omega = M_y = 0 \quad \text{for} \quad y=0,b\end{aligned} \tag{2.1}$$

can be represented in the form

$$\omega(x,y)=\sum_{k=1,3,\dots}^{\infty}\cos\alpha_k x Y_k(y), \quad \alpha_k=\frac{\pi k}{c}, \tag{2.2}$$

where

$$\begin{aligned}Y_k(y)&=A_k\operatorname{sh}\alpha_k y+\alpha_k B_k y\operatorname{ch}\alpha_k y, \quad 0\le y<\frac{b}{2},\\ Y_k(y)&=C_k\operatorname{sh}\alpha_k(b-y)+D_k\alpha_k(b-y)\operatorname{ch}\alpha_k(b-y), \quad \frac{b}{2}<y\le b.\end{aligned} \tag{2.3}$$

The deflection of the inclusion $\omega_0(x)$ satisfies the conditions

$$\begin{aligned}\frac{d^2}{dx^2}D_0(x)\frac{d^2\omega_0(x)}{dx^2}=q(x)-\mu(x), \quad |x|<a,\\ D_0(x)\omega_0''(x)|_{x=\pm a}=0, \quad [D_0(x)\omega_0''(x)]'|_{x=\pm a}=0,\end{aligned} \tag{2.4}$$

and the equations of equilibrium of the inclusion are of the form

$$\int_{-a}^{a}(\mu(t)-q(t))dt=0, \quad \int_{-a}^{a}t(\mu(t)-q(t))dt=0. \tag{2.5}$$

Realization of the conditions (1.3) leads to the fact that the constants A_k, B_k, C_k, D_k are expressed through $\mu(x)$, and we obtain that $A_k=C_k$, $B_k=D_k$ and

$$\sum_{k=1,3,\dots}^{\infty}\cos\alpha_k x\Big[\alpha_k\operatorname{ch}\frac{\alpha_k b}{2}A_k+\Big(\alpha_k\operatorname{ch}\frac{\alpha_k b}{2}+\frac{\alpha_k^2 b}{2}\operatorname{sh}\frac{\alpha_k b}{2}\Big)B_k\Big]=0,$$

$$\sum_{k=1,3,\dots}^{\infty}\cos\alpha_k x\Big[\alpha_k^3\operatorname{ch}\frac{\alpha_k b}{2}A_k+\Big(3\alpha_k^3\operatorname{ch}\frac{\alpha_k b}{2}+\frac{\alpha_k^4 b}{2}\operatorname{sh}\frac{\alpha_k b}{2}\Big)B_k\Big]=\frac{1}{2D}\mu(x).$$

The above equalities result in a system of equations with respect to the coefficients A_k and B_k:

$$\alpha_k\operatorname{ch}\frac{\alpha_k b}{2}A_k+\Big(\alpha_k\operatorname{ch}\frac{\alpha_k b}{2}+\frac{\alpha_k^2 b}{2}\operatorname{sh}\frac{\alpha_k b}{2}\Big)B_k=0,$$

$$\alpha_k^3\operatorname{ch}\frac{\alpha_k b}{2}A_k+\Big(3\alpha_k^3\operatorname{ch}\frac{\alpha_k b}{2}+\frac{\alpha_k^4 b}{2}\operatorname{sh}\frac{\alpha_k b}{2}\Big)B_k=\frac{1}{2Dc}\int_{-a}^{a}\mu(\zeta)\cos\alpha_k\zeta d\zeta.$$

Solving this system, we find that

$$A_k = -\frac{\operatorname{ch}\frac{\alpha_k b}{2} + \frac{\alpha_k b}{2}\operatorname{sh}\frac{\alpha_k b}{2}}{2\alpha_k^3 \operatorname{ch}^2\frac{\alpha_k b}{2}} \cdot \frac{1}{2Dc}\int_{-a}^{a} \mu(\zeta)\cos\alpha_k\zeta d\zeta,$$

$$B_k = -\frac{1}{2\alpha_k^3 \operatorname{ch}\frac{\alpha_k b}{2}} \cdot \frac{1}{2Dc}\int_{-a}^{a} \mu(\zeta)\cos\alpha_k\zeta d\zeta.$$

Substituting now these expressions in the representation (2.2), the limiting value of the function $\omega(x,y)$ for $y=\frac{b}{2}$ takes the form

$$\omega\Big(x,\frac{b}{2}\Big) =$$

$$= \sum_{k=1,3,\dots}^{\infty} \cos\alpha_k x\Big\{\frac{(-\operatorname{ch}\frac{\alpha_k b}{2} - \frac{\alpha_k b}{2}\operatorname{sh}\frac{\alpha_k b}{2})\operatorname{sh}\frac{\alpha_k b}{2} + \frac{\alpha_k b}{2}\operatorname{ch}^2\frac{\alpha_k b}{2}}{2\alpha_k^3\operatorname{ch}^2\frac{\alpha_k b}{2}}\Big\}\times$$

$$\times\frac{1}{2cD}\int_{-a}^{a}\mu(\zeta)\cos\alpha_k\zeta d\zeta = \frac{1}{2cD}\sum_{k=1,3,\dots}^{\infty}\int_{-a}^{a}\frac{\cos\alpha_k(x-\zeta)}{\alpha_k^3}\rho_k\mu(\zeta)d\zeta, \quad (2.6)$$

where $\rho_k = th\frac{\alpha_k b}{2} - \frac{\alpha_k b}{2\operatorname{ch}^2\frac{\alpha_k b}{2}}$.

In (2.6) we separate the principal part of the obtained integral operator, for which we take into account the asymptotics $\rho_k = 1 + O(\alpha_k e^{-\alpha_k})$, and use the formula

$$\sum_{k=1,3,\dots}^{\infty}\frac{\cos kt}{k^3} = -\frac{t^2}{4}\ln\frac{1}{|t|} + B_0(t), \quad |t|<\pi, \quad (2.7)$$

where $B_0(t) = -\frac{t^2}{4}\Big(\frac{3}{2}+\ln 2\Big) + \sum\limits_{k=1,3,\dots}^{\infty}\frac{1}{k^3} + \sum\limits_{n=1}^{\infty}\frac{2^{2n-1}-1}{(2n+2)!}B_n t^{2n+2}$, and B_n are Bernoulli numbers.

Taking now (2.7) into account, the formula (2.6) takes the form

$$\omega\Big(x,\frac{b}{2}\Big) = -\frac{1}{2\pi D}\int_{-a}^{a}\frac{(x-\zeta)^2}{4}\ln\frac{1}{|x-\zeta|}\mu(\zeta)d\zeta + \int_{-a}^{a}\overset{\circ}{R}_2(x,\zeta)\mu(\zeta)d\zeta, \quad (2.8)$$

where $\overset{\circ}{R}_2(x,\zeta) = \frac{1}{2Dc}B_0(x-\zeta) + \frac{1}{2Dc}\sum_{1,3,\dots}^{\infty}\frac{\cos\alpha_k(x-\zeta)}{\alpha_k^3}(\rho_k - 1)$ is an infinitely differentiable kernel. Note that variation of the boundary values as well as of the shape of the plate itself leads to the variation of $\overset{\circ}{R}_2(x,\zeta)$ only. In the case of an infinite plate strengthened by an elastic inclusion along the line $y=0$, $|x|<a$ and loaded at infinity by the bending moment $M_x^\infty = M$, $M_y^\infty = 0$, the above-formulated contact problem is solved by the methods of the theory of analytic functions,

and as a result the limiting value of the function $\frac{\partial^2\omega(x,y)}{\partial x}$ has the form $\frac{\partial^2\omega(x,\frac{b}{2})}{\partial x^2} = -\frac{1}{8\pi D}\int_{-a}^{a}(t-x)\ln|t-x|\mu(t)dt - \frac{M}{4D(1-\sigma^2)}$ [16].

Realizing of the contact condition between the inclusion and the plate, by integrating (2.4) twice and introducing the notation $\lambda_2(x)=\int_{-a}^{x}dt\int_{-a}^{t}(q(\tau)-\mu(\tau))d\tau$, we get the following integro-differential equation

$$\lambda_2(x) - \frac{D_0(x)}{4\pi D}\int\limits_{-a}^{a}\frac{\lambda_2'(t)dt}{t-x}+$$
$$+D_0(x)\int\limits_{-a}^{a}R_2(x,t)\lambda_2(t)dt = f_2(x)D_0(x), \quad |x|<a, \tag{2.9}$$

where

$$f_2(x) = \int\limits_{-a}^{a}\Big[\overset{\circ}{R}{}_2''(x,\zeta) - \frac{1}{8\pi D}\Big(2\ln\frac{1}{|x-\zeta|}+3\Big)\Big]q(\zeta)d\zeta, \quad R_2(x,t) = \overset{\circ}{R}{}_2''^{\cdot}(x,t),$$

and the prime denotes the derivative with respect to the first variable, while the point the derivative with respect to the second variable.

The unknown function must satisfy the conditions

$$\lambda_2(\pm a) = 0, \quad \lambda_2'(\pm a) = 0. \tag{2.10}$$

The uniqueness theorem for the above-posed problem is proved in the same way as the corresponding theorem in the previous section.

Remark. The problem for an infinite plate is reduced to the integro-differential equation of Prandtl [16].

3. Solution of the Characteristic Equation

Solution of the integral equations (1.22)–(1.23) and (2.9)–(2.10) allows one to define the jump of crosscutting forces along the segment of the inclusion. This function $\mu(x)$ may be of a class of functions with nonintegrable singularities at the ends of the contact interval. This singularity may at least be of the type $O((a^2-x^2)^{-\frac{3}{2}})$ as $x \to \pm a$; note that the second derivatives of the deflection $\omega(x,y)$ behave as $r^{-\frac{1}{2}}$ when approaching at the points $x=\pm a$, $y=0$, and the energy integral converges like the improper which this makes it possible to investigate the question on the uniqueness of a solution of the problems under consideration.

The characteristic equation corresponding to the integral equations (1.22) and (2.9) is the Prandtl integro-differential equation in that principal case when the coefficient of the singular operator turns to zero of higher order at the ends of the interval of integration:

$$\lambda(x) - D_0(x)\frac{\lambda_0}{\pi}\int_{-a}^{a}\frac{\lambda'(t)dt}{t-x} = D_0(x)g(x), \quad |x| < a, \tag{3.1}$$

provided

$$\lambda(\pm a) = \lambda'(\pm a) = 0, \tag{3.2}$$

where $D_0(x) = d_0(a^2 - x^2)^{n+\frac{1}{2}}$, $d_0 = \text{const} > 0$, $\lambda_0 = \text{const} > 0$, n is a nonnegative integer, $g(x)$ is the given even function on the interval $[-a, a]$ satisfying the Hölder condition, and $\lambda(x)$ is the unknown function from the same class, though its derivative may have an integrable singularity of integrable of the interval, i.e., $\lambda'(x) = O(a^2 - x^2)^{-\alpha}$, $0 \le \alpha < 1$ as $x \to \pm a$.

Consider the Cauchy type integral

$$\phi(z) = \frac{1}{2\pi i}\int_{-a}^{a}\frac{\lambda(t)dt}{t-z} \tag{$*$}$$

with the density $\lambda(x)$ which, obviously, represents the function holomorphic everywhere on the plane, except for the segment $[-a, a]$. Passing to the limit, on the basis of the well-known properties of Cauchy type integrals [15] we obtain

$$\begin{gathered}\lambda(x) = \phi_+(x) - \phi_-(x),\\ \int_{-a}^{a}\frac{\lambda'(t)dt}{t-x} = \pi i[\phi'_+(x) + \phi'_-(x)],\end{gathered} \tag{3.3}$$

where $\phi_+(x)$ and $\phi_-(x)$ are the limiting values of the function $\phi(z)$ defined in the neighborhood of the segment $[-a, a]$, when the point z approaches the point x of that segment respectively from the upper and the lower half-plane.

The function $D_0(z)$ is holomorphic in the plane cut along the segment $[-a, a]$. It should be noted that in the sequel under$D_0(z)$ it will be ment the branch of the function which satisfies the condition

$$D_{0+}(x) = -D_{0-}(x) \equiv D_0(x) > 0, \quad |x| < a.$$

Then by virtue of (3.3), the equation (3.1) can be represented in the form

$$\left[\phi'_+(x) + \frac{i}{\lambda_0 D_{0+}(x)}\phi_+(x)\right] + \left[\phi'_-(x) + \frac{i}{\lambda_0 D_{0-}(x)}\phi_-(x)\right] = \frac{g(x)i}{\lambda_0}. \tag{3.4}$$

If we introduce a new function

$$F(z) = \left[\phi'(z) + \frac{i}{\lambda_0 D_0(z)}\phi(z)\right]\sqrt{a^2 - z^2}, \tag{3.5}$$

then the equation (3.4) will take the form

$$F_+(x) - F_-(x) = \frac{i\sqrt{a^2 - x^2}g(x)}{\lambda_0}. \tag{3.6}$$

Note that by introducing the multiplier $\chi(z) = \sqrt{a^2 - z^2}$ we pass from the Riemann problem (3.4) with the coefficient $G(t) = -1$ to the jump problem (3.6). This multiplier is connected with the factorization $G(t) = \frac{\chi^+(t)}{\chi^-(t)}$.

The function $F(z)$, given by the formula (3.5), is holomorphic everywhere on the plane, cut at the segment $[-a, a]$, except for the points $z = \pm a$, where it has the poles of the multiplicity n; it vanishes at infinity and is continuously extendable to the interior points of the segment both from the upper and from the lower half-plane. Then the solution of the boundary value problem (3.6) is given by the formula

$$F(z) = \frac{1}{2\pi\lambda_0}\int_{-a}^{a} \frac{g(t)\sqrt{a^2 - t^2}}{t - z}dt + \sum_{k=1}^{n} A_k\Big(\frac{1}{(a-z)^k} - \frac{1}{(a+z)^k}\Big), \tag{3.7}$$

where A_k $(k = 1, 2, \dots, n)$ are arbitrary constants to be defined.

The solution of the first order differential equation (3.5) with respect to the function $\phi(z)$ is given by the formula

$$\phi(z) = e^{-iQ(z)}\left[\phi(0) + \int_0^z \frac{F(t)e^{iQ(z)}}{\sqrt{a^2 - t^2}}dt\right], \tag{3.8}$$

where the function $F(z)$ is representable in the form (3.7), $Q(z) = \frac{1}{\lambda_0}\int_0^z \frac{dt}{D_0(t)}$. The formula (3.8) can be transformed by integration by parts as follows:

$$\phi(z) = -i\lambda_0 A(z) + e^{-iQ(z)}\left[\phi(0) + \frac{\lambda_0 i}{a}F(0)D_0(0) + \lambda_0 i\int_0^z A'(t)e^{iQ(t)}dt\right], \tag{3.9}$$

where $A(z) = d_0 F(z)(a^2 - z^2)^n$.

The points $z = \pm a$ are transcendental branch points of the finite order for the solution of the homogeneous differential equation which corresponds to the equation (3.5). If we divide the neighbourhood of these points into segments by rays $\operatorname{Im} Q(z) = 0$, then the values of the function $e^{-iQ(z)}$ in each of these segments will coincide with the corresponding values of one of the branches.

Lemma 1. *The number of rays in the neighbourhood of the points* $z = \pm a$ *at which* $\operatorname{Im} Q(z) = 0$*, is equal to* $2n - 1$.

Proof. In the neighbourhood of the point $z = -a$ the function $Q(z)$ is represented in the form

$$Q(z) = \frac{1}{\lambda_0}\int\limits_0^z \frac{\sum_{k=0}^{\infty} b_k(t+a)^k}{(t+a)^{n+\frac{1}{2}}}dt = \frac{1}{\lambda_0}\int\limits_0^z \sum_{k=0}^{\infty}\frac{b_k}{(t+a)^{n+\frac{1}{2}-k}}dt =$$

$$= -\frac{1}{\lambda_0}\sum_{k=0}^{\infty}\frac{b_k}{(n-\frac{1}{2}-k)}\Big[\frac{1}{(z+a)^{n-\frac{1}{2}-k}} - \frac{1}{a^{n-\frac{1}{2}-k}}\Big],$$

where the coefficients of expansion in power series b_k are real numbers, and moreover, $b_0 > 0$.

If we assume that $z + a = \rho e^{i\alpha}$, where $\rho = |z+a|$, $\alpha = \arg(z+a)$, then we obtain

$$Q(z) = -\frac{b_0 e^{i\alpha(n-\frac{1}{2})}}{\lambda_0(n-\frac{1}{2})\rho^{n-\frac{1}{2}}}[1+\rho(p+iq)].$$

In the sequel we will not use the expressions p and q, therefore they are not written out here. We can choose the number ρ such that $1 + \rho p > 0$, and then from the equality $\operatorname{Im} Q(z) = 0$ we have

$$(1+\rho p)\sin\Big(n-\frac{1}{2}\Big)\alpha - \rho q\cos\Big(n-\frac{1}{2}\Big)\alpha = 0, \quad \text{i.e.,} \quad \operatorname{tg}\Big(n-\frac{1}{2}\Big)\alpha = \frac{\rho q}{1+\rho p},$$

which yields that $\alpha \to \frac{2k\pi}{2n-1}$, $k = 1, 2, \dots, 2n-1$ as $\rho \to 0$.

Urder our assumptions it is obvious that $-z - a = \rho e^{i(\alpha-\pi)}$, therefore for the point $z = a$ we analogously get $\alpha = \pi + \frac{2k\pi}{2n-1}$, which was to be proved.

Thus, as $z \to -a$ along one of the rays $\operatorname{Im} Q(z) = 0$ which makes with the ox-axis the angle α_0, then $-z \to a$ along the ray for which the corresponding angle is equal to $\pi + \alpha_0$ (see Fig. I, $n = 2$).

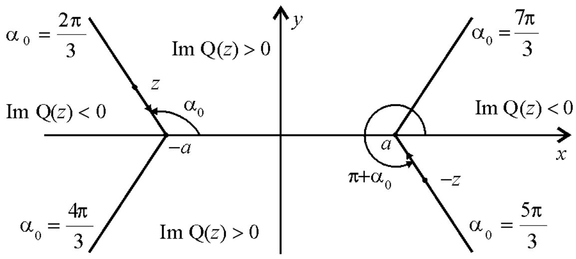

Fig. I

The number of sectors where $\operatorname{Im} Q(z) > 0$ can be defined by solving the inequality $\sin\Big(n-\frac{1}{2}\Big)\alpha > 0$, which yields

$$\frac{2\pi(2j-2)}{2n-1} < \alpha < \frac{2\pi(2j-1)}{2n-1}, \quad j = 1, 2, \dots, n,$$

i.e., the number of such sectors is equal to n, and of those with $\operatorname{Im} Q(z) < 0$ is equal to $n-1$.

As $z \to \pm a$, in the sectors, where $\operatorname{Im} Q(z) < 0$, the expression in the brackets in the formula (3.9) tends to infinity, and the solution of the homogeneous equation corresponding to the equation (3.5), $e^{-iQ(z)}$, vanishes.

Revealing indeterminacy, we find that

$$\lim_{z\to\pm a}\{\phi(z) + i\lambda_0 A(z)\} =$$

$$= \lim_{z\to\pm a}\left\{\frac{\phi(0) + \frac{\lambda_0 i}{a}F(0)D_0(0) + \lambda_0 i\int_0^z A'(t)e^{iQ(t)}dt}{e^{iQ(z)}}\right\} =$$

$$= \lim_{z\to\pm a}\frac{\lambda_0 iA'(z)e^{iQ(z)}}{iQ'(z)e^{iQ(z)}} = \lambda_0^2\lim_{z\to\pm a}D_0(z)A'(z) = 0,$$

$$\text{i.e.,}\quad \phi(z) + i\lambda_0 A(z) = O\big((a^2-z^2)^{n+\frac{1}{2}}\big) \quad \text{as} \quad z\to\pm a,$$

and hence taking into account the formula (3.7), we get

$$\phi(z) = O(1), \quad \text{as} \quad z\to\pm a. \tag{3.10}$$

In the sectors, where $\operatorname{Im} Q(z) > 0$ as $z \to -a$, from the condition of tending to zero of the expression inthe brackets in (3.9) we obtain the following system of linear algebraic equations for determining the constants A_k $(k = 1, 2, \ldots, n)$

$$\lim_{|z+a|\to 0}\left\{\phi(0) + \frac{\lambda_0 i}{a}F(0)D_0(0) + \lambda_0 i\int_0^z A'(t)e^{iQ(t)}dt\right\} = 0 \tag{3.11}$$

$$\frac{2\pi(2j-2)}{2n-1} \le \arg(z+a) \le \frac{2\pi(2j-1)}{2n-1}, \quad j = 1, 2, \ldots, n.$$

This system can be obtained also as $|z-a| \to 0$ in the corresponding sectors. The determinant of the system differs from zero which follows from the the uniqueness of the above-posed problem.

The estimate (3.10) is also valid as $z \to \pm a$ in the sectors, where $\operatorname{Im} Q(z) > 0$. The requirement $\phi(\infty) = 0$ allows one to determine the constant $\phi(0)$. If $\operatorname{Im} Q(z) = 0$, by (3.11) we have

$$\lim_{z\to\pm a}\{\phi(z) + i\lambda_0 A(z)\} =$$

$$= \lim_{z\to\pm a}(a\mp z)\left\{\frac{\phi(0) + \frac{\lambda_0 i}{a}F(0)D_0(0) + \lambda_0 i\int_0^z A'(t)e^{iQ(t)}dt}{(a\mp z)e^{iQ(z)}}\right\} =$$

$$= \lim_{z\to\pm a}(a\mp z)\lim_{z\to\pm a}\frac{\lambda_0 iA'(z)e^{iQ(z)}}{e^{iQ(z)}[\mp 1 + (a\mp z)iQ'(z)]} =$$

$$= \lim_{z\to\pm a}(a\mp z)\lim_{z\to\pm a}\frac{\lambda_0 d_0(a\mp z)^{n-\frac{1}{2}}(a\pm z)^{n+\frac{1}{2}}A'(z)}{1\pm i\lambda_0 d_0(a\mp z)^{n-\frac{1}{2}}(a\pm z)^{n+\frac{1}{2}}} =$$

$$= \lim_{z\to\pm a}(a\mp z)^{n+\frac{1}{2}}l(z)=0, \quad \text{as} \quad l(z)=O(1), \quad z\to\pm a.$$

Hence in this case the estimate (3.10) is valid.

By virtue of (3.5), for the boundary values of the function $\phi(z)$ we obtain the following differential equations:

$$\phi'_+(x)+\frac{i}{\lambda_0 D_0(x)}\phi_+(x)=\frac{F_+(x)}{\sqrt{a^2-x^2}},$$
$$\phi'_-(x)-\frac{i}{\lambda_0 D_0(x)}\phi_-(x)=-\frac{F(x)}{\sqrt{a^2-x^2}}.$$

Integrating them, we get

$$\phi_+(x)=\frac{d_0\lambda_0}{i}F_+(x)(a^2-x^2)^n+$$
$$+e^{-iQ(x)}\left[\phi_+(0)-\frac{d_0\lambda_0}{i}F_+(0)a^{2n}-\frac{\lambda_0}{i}\int_0^x A'_+(t)e^{iQ(t)}dt\right],$$
$$\phi_-(x)=\frac{d_0\lambda_0}{i}F_-(x)(a^2-x^2)^n+$$
$$+e^{iQ(x)}\left[\phi_-(0)-\frac{d_0\lambda_0}{i}F_-(0)a^{2n}-\frac{\lambda_0}{i}\int_0^x A'_-(t)e^{-iQ(t)}dt\right],$$
$$Q(-x)=-Q(x), \quad |x|<a.$$

Since the function $\lambda(x)$ is even, from $(*)$ we find that $\phi_+(0)=-\phi_-(0)=\frac{1}{2}\lambda(0)$ and $\phi(0)=-\lim\limits_{|z|\to\infty}\int_0^z \frac{F(t)e^{iQ(t)}dt}{\sqrt{a^2-t^2}}$, where the integrand is single-valued and order $\frac{c}{z^2}$ at infinity.

The unknown function $\lambda(x)$ is representable as

$$\lambda(x)=\phi_+(x)-\phi_-(x)=\frac{d_0\lambda_0}{i}(a^2-x^2)^n[F_+(x)-F_-(x)]+$$
$$+[\lambda(0)-d_0a^{2n+1}g(0)]\cos Q(x)+d_0\lambda_0a^{2n}[F_+(0)+F_-(0)]\sin Q(x)+$$
$$+\lambda_0 i\int_0^x[A'_+(t)-A'_-(t)]\cos(Q(t)-Q(x))dt-$$
$$-\lambda_0\int_0^x[A'_+(t)+A'_-(t)]\sin(Q(t)-Q(x))dt=$$
$$=d_0(a^2-x^2)^{n+\frac{1}{2}}g(x)+[\lambda(0)-d_0a^{2n+1}g(0)]\cos Q(x)-$$
$$-\int_0^x B'_1(t)\cos(Q(t)-Q(x))dt-\lambda_0\int_0^x B'_2(t)\sin(Q(t)-Q(x))dt, \qquad (3.12)$$

where $B_1(t) = (a^2 - t^2)^{n+\frac{1}{2}} g(t)$,

$$B_2(t) = (a^2 - t^2)^n \left[\frac{1}{\pi\lambda_0} \int_{-a}^{a} \frac{g(\tau)\sqrt{a^2 - \tau^2}}{\tau - t} d\tau + \sum_{k=1}^{n} A_k \left(\frac{1}{(a-t)^k} - \frac{1}{(a+t)^k} \right) \right]. \ \square$$

Theorem 2. *The solution of the integro-differential equation* (3.1) *represented by formula* (3.12) *admits the estimate*

$$\lambda(x) = O((a^2 - x^2)^{n+\frac{1}{2}}) \quad as \quad x \to \pm a. \tag{3.13}$$

Indeed, satisfying the conditions (3.2), in the framework of the previous lemma and also of the system of algebraic equations (3.11) the equalities

$$\lim_{x \to \pm a} \left[\lambda(0) - d_0 a^{2n+1} g(0) - \int_0^x B_1'(t) \cos Q(t) dt - \int_0^x B_2'(t) \sin Q(t) dt \right] = 0,$$

$$\lim_{x \to \pm a} \left[\int_0^x B_1'(t) \sin Q(t) dt - \int_0^x B_2'(t) \cos Q(t) dt \right] = 0$$

hold.

Integrating by parts, the expression in the left-hand side of the last equalities can be transformed as follows:

$$\lambda(0) - d_0 a^{2n+1} g(0) - \int_0^x B_1'(t) \cos Q(t) dt + \int_0^x B_2'(t) \sin Q(t) dt =$$

$$= \lambda(0) - d_0 a^{2n+1} g(0) - \lambda_0 \int_0^x D_0(t) B_1'(t) d \sin Q(t) +$$

$$+ \lambda_0 \int_0^x D_0(t) B_2'(t) d \cos Q(t) = \lambda(0) - d_0 a^{2n+1} g(0) -$$

$$- \lambda_0 D_0(x) B_1'(x) \sin Q(x) + \lambda_0 \int_0^x [D_0(t) B_1'(t)]' \sin Q(t) dt +$$

$$+ \lambda_0 D_0(x) B_2'(x) \cos Q(x) -$$

$$- \lambda_0 D_0(0) B_2'(0) - \lambda_0 \int_0^x [D_0(t) B_2'(t)]' \cos Q(t) dt,$$

$$\int_0^x B_1'(t) \sin Q(t) dt - \int_0^x B_2'(t) \cos Q(t) dt =$$

$$= -\lambda_0 \int_0^x D_0(t) B_1'(t) d \cos Q(t) - \lambda_0 \int_0^x D_0(t) B_2'(t) d \sin Q(t) =$$

$$= -\lambda_0 D_0(x) B_1'(x) \cos Q(x) + \lambda_0 D_0(0) B_1'(0) + \lambda_0 \int\limits_0^x [D_0(t) B_1'(t)]' \cos Q(t) dt -$$

$$-\lambda_0 D_0(x) B_2'(x) \sin Q(x) + \lambda_0 \int\limits_0^x [D_0(t) B_2'(t)]' \sin Q(t) dt.$$

Substituting these transformations in the formula (3.12), we obtain

$$\lambda(x) = d_0 (a^2 - x^2)^{n+\frac{1}{2}} g(x) - \widetilde{B}_2(x) +$$
$$+ \left[\lambda(0) - d_0 a^{2n+1} g(0) + \widetilde{B}_2(0) - \int\limits_0^x \widetilde{B}_1'(t) \sin Q(t) dt + \int\limits_0^x \widetilde{B}_2'(t) \cos Q(t) dt \right] \times$$
$$\times \cos Q(x) + \left[\widetilde{B}_1(0) + \int\limits_0^x \widetilde{B}_1'(t) \cos Q(t) dt + \int\limits_0^x \widetilde{B}_2'(t) \sin Q(t) dt \right] \sin Q(x),$$

where $\widetilde{B}_1(x) = \lambda_0 D_0(x) B_1'(x)$, $\widetilde{B}_2(x) = \lambda_0 D_0(x) B_2'(x)$.

Taking into account the expressions of the functions $B_2(x)$ and $\widetilde{B}_2(x)$, we may state that the expressions in the brackets in the latter formula vanish at the points $x = \pm a$. Performing analogous transformations and introducing the notation $\widetilde{\widetilde{B}}_1(x) = \lambda_0 D_0(x) \widetilde{B}_1'(x)$ and $\widetilde{\widetilde{B}}_2(x) = \lambda_0 D_0(x) \widetilde{B}_2'(x)$, we obtain

$$\lambda(x) = d_0 (a^2 - x^2)^{n+\frac{1}{2}} g(x) - \widetilde{B}_2(x) + \widetilde{\widetilde{B}}_2(x) +$$
$$+ \left[\lambda(0) - d_0 a^{2n+1} g(0) - \widetilde{B}_2(0) + \widetilde{\widetilde{B}}_2(0) + \right.$$
$$\left. + \int\limits_0^x \widetilde{\widetilde{B}}_1'(t) \cos Q(t) dt + \int\limits_0^x \widetilde{\widetilde{B}}_2'(t) \sin Q(t) dt \right] \cos Q(x) +$$
$$+ \left[\widetilde{B}_1(0) + \widetilde{\widetilde{B}}_1(0) - \int\limits_0^x \widetilde{\widetilde{B}}_1'(t) \sin Q(t) dt + \right.$$
$$\left. + \int\limits_0^x \widetilde{\widetilde{B}}_2'(t) \cos Q(t) dt \right] \sin Q(x). \tag{3.14}$$

Thus the following relations are valid:

$$\lim_{x \to \pm a} \frac{\mu_0 + \int_0^x \widetilde{\widetilde{B}}_1'(t) \cos Q(t) dt + \int_0^x \widetilde{\widetilde{B}}_2'(t) \sin Q(t) dt}{(a^2 - x^2)^{n+\frac{1}{2}}} =$$

$$= \lim_{x\to\pm a} \frac{\widetilde{\widetilde{B}}{}'_1(x)\cos Q(x) + \widetilde{\widetilde{B}}{}'_2(x)\sin Q(x)}{-(2n+1)x(a^2-x^2)^{n-\frac{1}{2}}} = 0,$$

$$\lim_{x\to\pm a} \frac{\mu_1 - \int_0^x \widetilde{\widetilde{B}}{}'_1(t)\sin Q(t)dt + \int_0^x \widetilde{\widetilde{B}}{}'_2(t)\cos Q(t)dt}{(a^2-x^2)^{n+\frac{1}{2}}} = \tag{3.15}$$

$$= \lim_{x\to\pm a} \frac{-\widetilde{\widetilde{B}}{}'_1(x)\sin Q(x) + \widetilde{\widetilde{B}}{}'_2(x)\cos Q(x)}{-(2n+1)x(a^2-x^2)^{n-\frac{1}{2}}} = 0,$$

where $\mu_0 = \lambda(0) - d_0 a^{2n+1} g(0) - \widetilde{B}_2(0) + \widetilde{\widetilde{B}}_2(0)$, $\mu_1 = \widetilde{B}_1(0) + \widetilde{\widetilde{B}}_1(0)$. The above relations show that the expressions in the brackets in the formula (3.13) are infinitely small values as $x \to \pm a$, of higher order than $(a^2 - x^2)^{n+\frac{1}{2}}$. With regard for the functions $\widetilde{B}_2(x)$ and $\widetilde{\widetilde{B}}_2(x)$, from (3.14) follows the validity of our theorem.

Corollary. *For the functions $\lambda_1(x)$ and $\lambda_2(x)$ having certain physical meaning (see Section 1 and Section 2), the following estimates are valid:*

$$\lambda'(x) = O((a^2-x^2)^{n-\frac{1}{2}}), \lambda''(x) = O((a^2-x^2)^{n-\frac{3}{2}}), \quad \text{as} \quad x \to \pm a. \tag{3.16}$$

Indeed, from (3.14) we have

$$\lambda'(x) = -d_0(2n+1)x(a^2-x^2)^{n-\frac{1}{2}} g(x) + d_0(a^2-x^2)^{n+\frac{1}{2}} g'(x) -$$
$$-\widetilde{B}'_2(x) + \widetilde{\widetilde{B}}{}'_2(x) - \widetilde{\widetilde{B}}_1{}'(x) -$$
$$-\left[\mu_0 + \int_0^x \widetilde{\widetilde{B}}{}'_1(t)\cos Q(t)dt + \int_0^x \widetilde{\widetilde{B}}{}'_2(t)\sin Q(t)dt\right] \frac{\sin Q(x)}{\lambda_0 D_0(x)} +$$
$$+\left[\mu_1 - \int_0^x \widetilde{\widetilde{B}}{}'_1(t)\sin Q(t)dt + \int_0^x \widetilde{\widetilde{B}}{}'_2(t)\cos Q(t)dt\right] \times \frac{\cos Q(x)}{\lambda_0 D_0(x)}, \tag{3.17}$$

and (1.15) yields

$$\frac{\mu_0 + \int_0^x \widetilde{\widetilde{B}}{}'_1(t)\cos Q(t)dt + \int_0^x \widetilde{\widetilde{B}}{}'_2(t)\sin Q(t)dt}{D_0(x)} = O((a^2-x^2)^{n-\frac{1}{2}}),$$

$$\frac{\mu_1 - \int_0^x \widetilde{\widetilde{B}}{}'_1(t)\sin Q(t)dt + \int_0^x \widetilde{\widetilde{B}}{}'_2(t)\cos Q(t)dt}{D_0(x)} = O((a^2-x^2)^{n-\frac{1}{2}}) \text{ as } x \to \pm a.$$

Then (3.17) implies that the first estimate in (3.16) is valid.

To obtain the second estimate in (3.16), we have to perform some transformations in (3.17),

$$\lambda'(x) = G(x) -$$

$$-\Big[\mu_0+\lambda_0\int\limits_0^x D_0(t)\tilde{\tilde{B}}_1'(t)d\sin Q(t)-\lambda_0\int\limits_0^x D_0(t)\tilde{\tilde{B}}_2'(t)d\cos Q(t)\Big]\frac{\sin Q(x)}{\lambda_0 D_0(x)}+$$

$$+\Big[\mu_1+\lambda_0\int\limits_0^x D_0(t)\tilde{\tilde{B}}_1'(t)d\cos Q(t)+\lambda_0\int\limits_0^x D_0(t)\tilde{\tilde{B}}_2'(t)d\sin Q(t)\Big]\frac{\cos Q(x)}{\lambda_0 D_0(x)}=$$

$$=G(x)+\tilde{\tilde{B}}_1'(x)-$$

$$-\Big[\mu_0-\lambda_0\tilde{\tilde{\tilde{B}}}_2(0)-\int\limits_0^x\tilde{\tilde{\tilde{B}}}_1'(t)\sin Q(t)dt+\int\limits_0^x\tilde{\tilde{\tilde{B}}}_2'(t)\cos Q(t)dt\Big]\frac{\sin Q(x)}{\lambda_0 D_0(x)}+$$

$$+\Big[\mu_1-\lambda_0\tilde{\tilde{\tilde{B}}}_1'(0)-\int\limits_0^x\tilde{\tilde{\tilde{B}}}_1'(t)\cos Q(t)dt-\int\limits_0^x\tilde{\tilde{\tilde{B}}}_2'(t)\sin Q(t)dt\Big]\frac{\cos Q(x)}{\lambda_0 D_0(x)}.$$

where $G(x)=-d_0(2n+1)x(a^2-x^2)^{n-\frac{1}{2}}g(x)+d_0(a^2-x^2)^{n+\frac{1}{2}}g'(x)-\tilde{B}_2'(x)+\tilde{\tilde{B}}_2'(x)-\tilde{\tilde{B}}_1'(x)$, $\tilde{\tilde{\tilde{B}}}_1(x)=\lambda_0 D_0(x)\tilde{\tilde{B}}_1'(x)$, $\tilde{\tilde{\tilde{B}}}_2(x)=\lambda_0 D_0(x)\tilde{\tilde{B}}_2'(x)$. Then the function $\lambda''(x)$ can be represented as

$$\lambda''(x)=G'(x)+\tilde{\tilde{B}}_1''(x)-\frac{\tilde{\tilde{\tilde{B}}}_1'(x)}{\lambda_0 D_0(x)}-$$

$$-\Big[\mu_0-\lambda_0\tilde{\tilde{\tilde{B}}}_2(0)-\int\limits_0^x\tilde{\tilde{\tilde{B}}}_1'(t)\sin Q(t)dt+\int\limits_0^x\tilde{\tilde{\tilde{B}}}_2'(t)\cos Q(t)dt\Big]\times$$

$$\times\frac{\cos Q(x)-\lambda_0 D_0'(x)\sin Q(x)}{\lambda_0^2 D_0^2(x)}-$$

$$-\Big[\mu_1-\lambda_0\tilde{\tilde{\tilde{B}}}_1(0)-\int\limits_0^x\tilde{\tilde{\tilde{B}}}_1'(t)\cos Q(t)dt-\int\limits_0^x\tilde{\tilde{\tilde{B}}}_2'(t)\sin Q(t)dt\Big]\times$$

$$\times\frac{\sin Q(x)+\lambda_0 D_0'(x)\cos Q(x)}{\lambda_0^2 D_0^2(x)}.$$

Here the expressions in the brackets vanish as $x\to\pm a$. Taking now into account the expressions of the functions $\tilde{\tilde{\tilde{B}}}_1(x)$ and $\tilde{\tilde{\tilde{B}}}_2(x)$ and uncovering the indeterminacy, we get

$$\frac{\mu_0-\lambda_0\tilde{\tilde{\tilde{B}}}_2(0)-\int_0^x\tilde{\tilde{\tilde{B}}}_1'(t)\sin Q(t)dt+\int_0^x\tilde{\tilde{\tilde{B}}}_2'(t)\cos Q(t)dt}{D_0^2(x)}=$$

$$=O\Big((a^2-x^2)^{n-\frac{3}{2}}\Big),$$

$$\frac{\mu_1-\lambda_0\tilde{\tilde{\tilde{B}}}_1(0)-\int_0^x\tilde{\tilde{\tilde{B}}}_1'(t)\cos Q(t)dt-\int_0^x\tilde{\tilde{\tilde{B}}}_2'(t)\sin Q(t)dt}{D_0^2(x)}=$$

$$= O\Big((a^2 - x^2)^{n-\frac{3}{2}}\Big) \quad \text{as} \quad x \to \pm a,$$

which proves the validity of the second estimate in (3.16).

4. Reduction of the Integro-Differential Equations to the Fredholm Integral Equation of Second Kind

As is shown in Sections 1 and 2, the boundary value problems (1.1)–(1.5) and (2.1)–(2.5) are reduced to finding a solution of the integro-differential equations (1.22)–)1.23) and (2.9)–(2.10), respectively. We write them as follows:

$$K\lambda \equiv \Pi\lambda + R\lambda = f, \tag{4.1}$$

where Π is the characteristic part of the operator K, i.e.,

$$\Pi\lambda \equiv \frac{\lambda(x)}{D_0(x)} - \frac{\lambda_0}{\pi} \int\limits_{-a}^{a} \frac{\lambda'(t)dt}{t-x}, \quad |x| < a,$$

$R\lambda \equiv \int_{-a}^{a} R(x,t)\lambda(t)dt$, $R(x,t) = R_1(x,t)$, $f(x) = f_1$, $\lambda_0 = \frac{1}{8D}$ for the problem (1.1)–(1.5) and $R(x,t) = R_2(x,t)$, $f(x) = f_2(x)$, $\lambda_0 = \frac{1}{4D}$ for the problem (2.1)–(2.5).

Write (4.1) in the form

$$\Pi\lambda = f - R\lambda \tag{4.2}$$

and solve the previous equation as if its right-hand side were a given function. As is shown in Section 3, the latter equation has a unique solution representable explicitly for coefficients $D_0(x)$ from a sufficiently wide class. In particular, we can take $D_0(x) = d_0(a^2 - x^2)^{n+\frac{1}{2}}$, $d_0 = \text{const}$, $n \geq 1$ is a natural number, bearing in mind that the method of construction of the characteristic equation works also in case $D_0(x) = (a^2 - x^2)^{n+\frac{1}{2}}P(x)$, where $P(x)$ is a polynomial or a rational function [16]. Moreover, we note that for $n = 0$ we can consider $D_0(x) = \sqrt{a^2 - x^2}d(x)$, where $d(x)$ is any continuous function in the segment $[-a, a]$, $d(x) > 0$. [1]

On the basis of the results obtained in the previous section, we have

$$\lambda(x) + K^*R\lambda = K^*f + \lambda(0)\cos Q(x) +$$

$$+ \int\limits_{0}^{x} \frac{cos(Q(t) - Q(x))}{\sqrt{a^2 - t^2}} \sum_{k=1}^{n} A_k\Big(\frac{1}{(a-t)^k} - \frac{1}{(a+t)^k}\Big)dt, \tag{4.3}$$

where

$$K^*f = -\frac{1}{\lambda_0} \int\limits_{0}^{x} \sin(Q(t) - Q(x))f(t)dt +$$

[1] This version will be considered at the end of this section.

$$+\frac{1}{\lambda_0\pi}\int\limits_0^x \frac{\cos(Q(t)-Q(x))}{\sqrt{a^2-t^2}}\Bigg[\int\limits_{-a}^{a}\frac{f(\tau)\sqrt{a^2-\tau^2}}{t-\tau}d\tau\Bigg]dt;$$

here A_k $(k=1,2,\dots n)$ are unknown constants, $Q(x)=\frac{1}{\lambda_0}\int_0^x\frac{dt}{D_0(t)}$.

To the equation (4.3) we join the following system of equations:

$$\lim_{|z+a|\to 0}\Bigg\{\frac{\lambda_0 i}{a}F_\lambda(0)D_0(0)+\lambda_0 i\int\limits_0^z A'_\lambda(t)e^{iQ(t)}dt\Bigg\}=$$

$$=\lim_{|z+a|\to 0}\Bigg\{\phi(0)+\frac{\lambda_0 i}{a}D_0(0)F_f(0)+\lambda_0 i\int\limits_0^z A'_f(t)e^{iQ(t)}dt\Bigg\},$$

$$\frac{2\pi(2j-2)}{2n-1}\le\arg(z+a)\le\frac{2\pi(2j-1)}{2n-1},\quad j=1,2,\dots,n,\tag{4.4}$$

where

$$\phi(0)=-\lim_{|z|\to\infty}\int\limits_0^z\frac{F_f(t)-F_\lambda(t)}{\sqrt{a^2-t^2}}e^{iQ(t)}dt,\tag{4.4$_0$}$$

$$F_f(z)=\frac{1}{2\pi\lambda_0}\int\limits_{-a}^{a}\frac{f(t)\sqrt{a^2-t^2}dt}{t-z}+\sum_{k=1}^{n}A_k\Big(\frac{1}{(a-z)^k}-\frac{1}{(a+z)^k}\Big),$$

$$F_\lambda(z)=\frac{1}{2\pi\lambda_0}\int\limits_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-z}\Bigg[\int\limits_{-a}^{a}R(t,s)\lambda(s)ds\Bigg]dt,$$

$$A_f(z)=d_0(a^2-z^2)^nF_f(z),\quad A_\lambda(z)=d_0(a^2-z^2)^nF_\lambda(z),$$

$$\lambda(0)=2\phi_+(0),\tag{4.5}$$

$$\phi(z)=e^{-iQ(z)}\Bigg[\phi(0)+\int\limits_0^z\frac{F_f(t)-F_\lambda(t)}{\sqrt{a^2-t^2}}e^{iQ(t)}dt\Bigg].$$

Thus the original integro-differential equation (4.1) is equivalent to the union of the equations (4.3), (4.4) and (4.5). The equation (4.3) is the Fredholm equation of second kind. It sould be noted, however, that for $n\ge 1$ we obtain, besides the Fredholm equation of second kind, some additional equations (4.4) and (4.5), i.e., a system of linear algebraic equations, but this is not important because the problem is reduced to the solution of Fredholm's equation (4.3). For $n=0$, the integro-differential equation (4.3) is equivalent to a Fredholm equation of second kind only.

The equation (4.3) can be transformed as follows:

$$\lambda(x)=d_0(a^2-x^2)^{n+\frac12}f(x)-d_0(a^2-x^2)^{n+\frac12}\int\limits_{-a}^{a}R(x,s)\lambda(s)ds+$$

$$+\Big[\lambda(0)-d_0a^{2n+1}f(0)-\int\limits_0^x \overset{\circ}{B}{}_1'(t)\cos Q(t)dt-\lambda_0\int\limits_0^x \overset{\circ}{B}{}_2'(t)\sin Q(t)dt+$$

$$+d_0a^{2n+1}\int\limits_{-a}^{a} R(0,s)\lambda(s)ds+d_0\int\limits_{-a}^{a}\widetilde{R}(x,s)\lambda(s)ds+$$

$$+\frac{d_0}{\pi}\int\limits_{-a}^{a}\widetilde{L}(x,s)\lambda(s)ds\Big]\cos Q(x)+\Big[-\int\limits_0^x \overset{\circ}{B}{}_1'(t)\sin Q(t)dt+$$

$$+\lambda_0\int\limits_0^x \overset{\circ}{B}{}_2'(t)\cos Q(t)dt-\frac{d_0a^{2n}}{\pi}\int\limits_{-a}^{a} L(0,s)\lambda(s)ds+$$

$$+d_0\int\limits_{-a}^{a}\widetilde{\widetilde{R}}(x,s)\lambda(s)ds-\frac{d_0}{\pi}\int\limits_{-a}^{a}\widetilde{\widetilde{L}}(x,s)\lambda(s)ds\Big]\sin Q(x), \tag{4.6}$$

where

$$\overset{\circ}{B}_1(t)=(a^2-t^2)^{n+\frac{1}{2}}f(t),$$

$$\overset{\circ}{B}_2(t)=(a^2-t^2)^n\Big[\frac{1}{\pi\lambda_0}\int\limits_{-a}^{a}\frac{f(\tau)\sqrt{a^2-\tau^2}}{\tau-t}dt+\sum_{k=1}^{n}A_k\Big(\frac{1}{(a-t)^k}-\frac{1}{(a+t)^k}\Big)\Big],$$

$$\widetilde{R}(x,s)=\int\limits_0^x\big[(a^2-t^2)^{n+\frac{1}{2}}R(t,s)\big]_t'\cos Q(t)dt,$$

$$\widetilde{\widetilde{R}}(x,s)=\int\limits_0^x\big[(a^2-t^2)^{n+\frac{1}{2}}R(t,s)\big]_t'\sin Q(t)dt,$$

$$\widetilde{L}(x,s)=\int\limits_0^x\big[(a^2-t^2)^{n}L(t,s)\big]_t'\sin Q(t)dt,$$

$$\widetilde{\widetilde{L}}(x,s)=\int\limits_0^x\big[(a^2-t^2)^{n}L(t,s)\big]_t'\cos Q(t)dt,$$

$$L(t,s)=\int\limits_{-a}^{a}\frac{\sqrt{a^2-\tau^2}}{t-\tau}R(\tau,s)d\tau.$$

For $z=x$, $x\in(-a,a)$, the system (4.4) shows that the expressions in the square brackets in (4.6) vanish as $x\to\pm a$. Moreover, on the basis of the results obtained in Section 3, we

$$\lambda(0) - d_0 a^{2n+1} f(0) - \int\limits_0^x \overset{\circ}{B}{}'_1(t)\cos Q(t)dt - \lambda_0 \int\limits_0^x \overset{\circ}{B}{}'_2(t)\sin Q(t)dt +$$
$$+ d_0 a^{2n+1} \int\limits_{-a}^{a} R(0,s)\lambda(s)ds + d_0 \int\limits_{-a}^{a} \widetilde{R}(x,s)\lambda(s)ds +$$
$$+ \frac{d_0}{\pi} \int\limits_{-a}^{a} \widetilde{L}(x,s)\lambda(s)ds = O\Big((a^2-x^2)^{n+\frac{1}{2}}\Big),$$

$$-\int\limits_0^x \overset{\circ}{B}{}'_1(t)\sin Q(t)dt + \lambda_0 \int\limits_0^x \overset{\circ}{B}{}'_2(t)\cos Q(t)dt - \frac{d_0 a^{2n}}{\pi}\int\limits_{-a}^{a} L(0,s)\lambda(s)ds +$$
$$+ d_0 \int\limits_{-a}^{a} \widetilde{\widetilde{R}}(x,s)\lambda(s)ds - \frac{d_0}{\pi}\int\limits_{-a}^{a} \widetilde{\widetilde{L}}(x,s)\lambda(s)ds = O\Big((a^2-x^2)^{n+\frac{1}{2}}\Big) \text{ as } x \to \pm a.$$

Thus we have achieved regularization of the initial equation, and what is very important, the obtained integral equation has the form

$$\lambda(x) - d_0(a^2-x^2)^{n+\frac{1}{2}} \int\limits_{-a}^{a} D(x,t)\lambda(t)dt = d_0(a^2-x^2)^{n+\frac{1}{2}} \widetilde{g}(x), \quad |x| < a, \tag{4.7}$$

where

$$D(x,t) = \frac{[a^{2n+1}R(0,t) + \widetilde{R}(x,t) + \frac{1}{\pi}\widetilde{L}(x,t)]\cos Q(x)}{(a^2-x^2)^{n+\frac{1}{2}}} -$$
$$- \frac{[\frac{a^{2n}}{\pi}L(0,t) - \widetilde{\widetilde{R}}(x,t) + \frac{1}{\pi}\widetilde{\widetilde{L}}(x,t)]\sin Q(x)}{(a^2-x^2)^{n+\frac{1}{2}}}$$

$$\widetilde{g}(x) = f(x) + \frac{\lambda(0) - d_0 a^{2n+1} f(0) - \int_0^x \overset{\circ}{B}{}'_1(t)\cos Q(t)dt}{d_0(a^2-x^2)^{n+\frac{1}{2}}} \cos Q(x) -$$
$$- \frac{\lambda_0 \int_0^x \overset{\circ}{B}{}'_2(t)\sin Q(t)dt}{d_0(a^2-x^2)^{n+\frac{1}{2}}} \cos Q(x) +$$
$$+ \frac{[-\int_0^x \overset{\circ}{B}{}'_1(t)\sin Q(t)dt + \lambda_0 \int_0^x \overset{\circ}{B}{}'_2(t)\cos Q(t)dt]\sin Q(x)}{d_0(a^2-x^2)^{n+\frac{1}{2}}},$$

$D(x,t)$ is at least twice differentiable in the square $-a \le (x,t) \le a$, while the function $\widetilde{g}(x)$ has the same property on the interval $[-a,a]$.

Corollary. *By Fredholm's alternatives, the integral equation* (4.7) *with the system of equations* (4.4)–(4.5) *has a unique solution satisfying Hölder's condition on the segment* $[-a, a]$.

Passing to the interval $(-1, 1)$ and replacing $x = ay$ by $t = a\tau$, from (4.6) we get the equation

$$\lambda_0(y) - \nu_0(1-y^2)^{n+\frac{1}{2}} \int\limits_{-1}^{1} D_0(y,\tau)\lambda_0(\tau)d\tau =$$
$$= \nu_0(1-y^2)^{n+\frac{1}{2}} g_0(y), \quad |y| < 1, \tag{4.8}$$

in which the following notation is adopted:

$$\lambda_0(y) \equiv \lambda(ay), \quad D_0(y,\tau) \equiv aD(ay, a\tau), \quad g_0(y) \equiv \widetilde{g}(ay), \quad \nu_0 = d_0 a^{2n+1}.$$

Now let us consider the method of reducing the last integral equation to the equivalent infinite system of linear algebraic equations. Towards this end, on the basis of the asymptotic behaviour of the solution of the characteristic equation, we represent the solution in the form of an infinite series

$$\lambda_0(y) = (1-y^2)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} a_{2k} P_{2k}^{(n+\frac{1}{2}, n+\frac{1}{2})}(y) \tag{4.9}$$

with unknown coefficients a_{2k} $(k = 1, 2, \dots)$ from the space of bounded number sequences, where $P_{2k}^{\alpha,\alpha}(y)$ are the Jacobi polynomials.

It is known that $P_{2k}^{(\alpha,\alpha)}(y) = P_{2k}^{(\alpha,\alpha)}(-y)$ and $P_{2k+1}^{(\alpha,\alpha)}(y) = -P_{2k+1}^{(\alpha,\alpha)}(-y)$, therefore the representation (4.8) yields an even function.

Further, we substitute (4.9) in (4.8), and as a result, we obtain the equality

$$(1-y^2)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} a_{2k} P_{2k}^{(n+\frac{1}{2}, n+\frac{1}{2})}(y) -$$
$$-\nu_0(1-y^2)^{n+\frac{1}{2}} \int\limits_{-1}^{1} D_0(y,\tau)(1-\tau^2)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} a_{2k} P_{2k}^{(n+\frac{1}{2}, n+\frac{1}{2})}(\tau)d\tau =$$
$$= \nu_0(1-y^2)^{n+\frac{1}{2}} g_0(y), \quad |y| < 1.$$

Multiplying both parts by $P_{2m}^{(n+\frac{1}{2}, n+\frac{1}{2})}(y)$, integrating from -1 to 1 and taking into account the orthogonality of the Jacobi polynomials [17], we get

$$\int\limits_{-1}^{1} (1-y^2)^{n+\frac{1}{2}} P_{2k}^{(n+\frac{1}{2}, n+\frac{1}{2})}(y) P_{2m}^{(n+\frac{1}{2}, n+\frac{1}{2})}(y)dy =$$

$$
= \begin{cases} 0, & m \neq k, \\ \frac{2^{n+1}\Gamma^2(2m+n+\frac{3}{2})}{(2m)!(2m+n+1)\Gamma(2m+2n+2)}, & m = k. \end{cases}
$$

Simple operations for determining unknown coefficients a_{2k} result in an infinite system of linear equations

$$
\frac{2^{n+1}\Gamma^2(2m+n+\frac{3}{2})}{(2m)!(2m+n+1)\Gamma(2m+2n+2)} a_{2m} -
-\nu_0 \sum_{k=0}^{\infty} D_{2k,2m} a_{2k} = g_{2m}, \quad m = 0, 1, \dots, \tag{4.10}
$$

where we have introduced the notation

$$
D_{2k,2m} = \int_{-1}^{1} (1-y^2)^{n+\frac{1}{2}} P_{2m}^{(n+\frac{1}{2},n+\frac{1}{2})}(y) \times
\times \left(\int_{-1}^{1} (1-\tau^2)^{n+\frac{1}{2}} P_{2k}^{(n+\frac{1}{2},n+\frac{1}{2})}(\tau) D_0(y,\tau) d\tau \right) dy \tag{4.11}
$$

$$
g_{2m} = \nu_0 \int_{-1}^{1} (1-y^2)^{n+\frac{1}{2}} P_{2m}^{(n+\frac{1}{2},n+\frac{1}{2})}(y) g_0(y) dy. \tag{4.12}
$$

Using Stirling's formula for asymptotic behaviour of Gamma-function, we conclude that

$$
\frac{\Gamma^2(2m+n+\frac{3}{2})}{(2m)!\Gamma(2m+2n+2)} \to 1 \quad \text{as} \quad m \to \infty. \tag{4.13}
$$

The interior integral in (4.11) we integrate twice by parts with respect the to the variable τ and the outer integral twice with respect to the variable y. Using the Rodrigues formula [17] for the Jacobi polynomials, we get

$$
D_{2k,2m} = \frac{1}{2^4 \cdot 2k(2k-1)2m(2m-1)} \int_{-1}^{1} (1-y^2)^{n+\frac{5}{2}} P_{2m-2}^{(n+\frac{5}{2},n+\frac{5}{2})}(y) \times
\times \left(\int_{-1}^{1} (1-\tau^2)^{n+\frac{5}{2}} P_{2k-2}^{(n+\frac{5}{2},n+\frac{5}{2})}(\tau) \frac{\partial^4 D_0(y,\tau)}{\partial y^2 \partial \tau^2} d\tau \right) dy. \tag{4.14}
$$

Analogously to (4.12), we have

$$
g_{2m} = \frac{\nu_0}{2^2 \cdot 2m(2m-1)} \int_{-1}^{1} (1-y^2)^{n+\frac{5}{2}} P_{2m-2}^{(n+\frac{5}{2},n+\frac{5}{2})}(y) \frac{\partial^2 g_0(y)}{\partial y^2} dy.
$$

We will now proceed to the investigation for regularity of the infinite system (4.10). Towards this end, we rewrite it in the form

$$a_{2m} - \nu_0 \sum_{k=0}^{\infty} \widetilde{D}_{2k,2m} a_{2m} = \widetilde{g}_{2m}, \quad m = 0, 1, 2, \dots, \tag{4.15}$$

and introduce the notation

$$\widetilde{D}_{2m,2k} = \frac{(2m)!(2m+n+1)\Gamma(2m+2n+2)}{2^{2n+1}\Gamma^2(2m+n+\frac{3}{2})} D_{2m,2k},$$
$$\widetilde{g}_{2m} = \frac{(2m)!(2m+n+1)\Gamma(2m+2n+2)}{2^{2n+1}\Gamma^2(2m+n+\frac{3}{2})} g_{2m}.$$

Taking into account (4.13) and (4.14), for the system (4.5) we obtain

$$\sum_{k,m=0}^{\infty} \widetilde{D}^2_{2m,2k} < \infty, \quad \sum_{m=0}^{\infty} \widetilde{g}^2_{2m} < \infty. \tag{4.16}$$

These estimates allow one to state that the infinite system (4.15) is quasi-completely regular for any ν_0 ($|\nu_0| < \infty$) [18].

Although under the conditions (4.16) the matrix $D = \{D_{2k,2m}\}_{k,m=0}^{\infty}$ in the space l_2 defines a linear continuous operator, the existence theorem of the is, generally speaking, inapplicable here [19], but the investigation of the system (4.15) under the conditions (4.16) is reduced to that of a finite system, and this makes it possible to point out the following condition: if the homogeneous system corresponding to (4.15) has in l_2 the unique (obviously, zero) solution, then the given system (4.15) has a unique solution for any sequence in the right-hand sides is.

The right-hand side of the system (4.15) can be represented as

$$\widetilde{g}_{2m} = \widetilde{g}^f_{2m} + \lambda(0)\widetilde{g}^0_{2m} + \sum_{i=1}^{n} \widetilde{g}^i_{2m} A_i, \quad m = 0, 1, 2, \dots.$$

We denote a solution of the infinite system for the right-hand side equal to $\widetilde{g}^f_{2m}$ by a^f_{2m}, for that equal to $\widetilde{g}^0_{2m}$ by a^0_{2m} and for that equal to $\widetilde{g}^i_{2m}$ by a^i_{2m}. Then

$$a_{2m} = a^f_{2m} + \lambda(0)a^0_{2m} + \sum_{i=1}^{n} a^i_{2m} A_i, \quad m = 0, 1, \dots. \tag{4.17}$$

Substituting the latter in (4.4), (4.4^0) and (4.5), we obtain a finite system of algebraic equations with respect to the constants $\lambda(0)$, $\phi(0)$, A_i ($i = 1, \dots, n$).

The uniqueness theorem for the above-posed problems and the equivalence between the initial integral differential equation and the infinite system of linear algebraic equations allows one to make the following

Conclusion. *The homogeneous system of equations corresponding to the system* (4.4), (4.4^0), (4.5) *and* (4.15) *has in* l_2 *only zero solution, while the inhomogeneous system has only unique solution.*

For the case $D_0(x) = \sqrt{a^2-x^2}d(x)$, where $d(x)$ is any continuous on $[-a,a]$ function, $d(x) > 0$, we present another way of reducing integro-differential equation (4.1) to the Fredholm integral equation of second kind.

Using the well-known Cauchy type inversion formula, from (4.1) we get

$$\lambda'(s) = -\frac{1}{\sqrt{a^2-s^2}}\frac{1}{\pi\lambda_0}\int\limits_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-s}\frac{\lambda(t)}{D_0(t)}dt+$$

$$+\frac{1}{\sqrt{a^2-s^2}}\frac{1}{\pi\lambda_0}\int\limits_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-s}f(t)dt-$$

$$-\frac{1}{\sqrt{a^2-s^2}}\frac{1}{\pi\lambda_0}\int\limits_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-s}dt\int\limits_{-a}^{a}R(\tau,t)\lambda(\tau)d\tau+\frac{c}{\sqrt{a^2-s^2}}, \quad |s|<a$$

where c is an arbitrary constant. Integrating both parts of the last equality and taking into account the integral expression

$$\int\frac{dx}{\sqrt{1-x^2}(x-t)} = -\frac{1}{2\sqrt{1-t^2}}\ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}},$$

we obtain

$$\lambda_0(x) = -\frac{a}{2\pi\lambda_0}\int\limits_{-1}^{1}\ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}}\frac{\lambda_0(t)}{\widetilde{D}_0(t)}dt+$$

$$+\frac{a}{2\pi\lambda_0}\int\limits_{-1}^{1}\ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}}\times$$

$$\times\left[f_0(t)-\int\limits_{-1}^{1}R_0(\tau,t)\lambda_0(\tau)d\tau\right]dt+c\arcsin x+c_1,$$

where $\lambda_0(x) = \lambda(ax)$, $\widetilde{D}_0(x) = D_0(ax)$, $f_0(x) = f(ax)$, $R_0(\tau,t) = aR(a\tau, at)$. If we satisfy the boundary conditions $\lambda_0(\pm 1) = 0$, we can find that $c_1 = c = 0$.

Consequently the integro-differential equation (4.1) under the boundary conditions $\lambda_0(\pm 1) = 0$ is equivalent to the following Fredholm integral equation of second kind:

$$\lambda_0(x)+\frac{a}{2\pi\lambda_0}\int\limits_{-1}^{1}\frac{L(x,t)}{\widetilde{D}_0(t)}\lambda_0(t)dt = r(x), \tag{4.18}$$

where

$$L(x,t) = \ln \frac{1 - xt + \sqrt{(1-x^2)(1-t^2)}}{1 - xt - \sqrt{(1-x^2)(1-t^2)}} +$$

$$+\widetilde{D}_0(t) \int_{-1}^{1} \ln \frac{1 - x\tau + \sqrt{(1-x^2)(1-\tau^2)}}{1 - x\tau - \sqrt{(1-x^2)(1-\tau^2)}} R_0(t,\tau)d\tau,$$

$$r(x) = \frac{a}{2\pi\lambda_0} \int_{-1}^{1} \ln \frac{1 - xt + \sqrt{(1-x^2)(1-t^2)}}{1 - xt - \sqrt{(1-x^2)(1-t^2)}} f_0(t)dt.$$

It is obvious that the equation (4.18) can be considered in the space $L_2(-1,1)$ with the weight $1/\widetilde{D}_0(t)$, in which it is a quasi-regular integral equation. According to the well-known results [20], on the basis of Banach's fixed point theorem, under the condition $\lambda_0 > \frac{a}{2\pi}\|L\|$, with

$$\|L\| = \left\{ \int_{-1}^{1}\int_{-1}^{1} \frac{L^2(x,t)}{\widetilde{D}_0(t)\widetilde{D}_0(x)} dxdt \right\}^{\frac{1}{2}},$$

the solution of the equation can be constructed by the method of successive approximations.

Note that by the method suggested in [10–11] we can obtain a new regular integral equation, equivalent to the integro-differential equation (4.1).

We will now cite the method of reducing the equation (4.1) to the equivalent system of linear algebraic equations. Towards this end, on the basis of the equation (3.15) we represent a solution of the equation (4.1) by an infinite series

$$\lambda'(x) = \frac{1}{\sqrt{1-x^2}} \sum_{m=0}^{\infty} a_m T_m(x), \quad |x| < 1, \tag{4.19}$$

with unknown coefficients a_m $(n = 0, 1, 2, \dots)$.

Satisfying the boundary conditions, the equation (4.19) results in

$$\lambda(x) = -\sqrt{1-x^2} \sum_{m=1}^{\infty} \frac{a_m}{m} U_{m-1}(x), \quad |x| \le 1, \tag{4.20}$$

where $T_m(x)$ and $U_{m-1}(x)$ are Chebyshev's polynomials of the first and second Kind, respectively.

Substituting the equations (4.19) and (4.20) in (4.1) and using the relation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_m(t)dt}{(t-x)\sqrt{1-t^2}} = U_{m-1}(x), \quad |x| < 1,$$

we arrive at the equality

$$-\frac{1}{d(x)}\sum_{m=1}^{\infty}\frac{a_m}{m}U_{m-1}(x)-\lambda_0\sum_{m=1}^{\infty}a_mU_{m-1}(x)-$$

$$-\sum_{m=1}^{\infty}\frac{a_m}{m}\int\limits_{-1}^{1}R(t,x)\sqrt{1-t^2}U_{m-1}(t)dt=f(x),\quad |x|<1.$$

Now we multiply both parts of the above equality by the function $\sqrt{1-x^2}$ $U_{k-1}(x)$ and integrate from -1 to 1 taking into account the orthogonality property of Chebyshev's polynomials of second Kind:

$$\int\limits_{-1}^{1}U_{m-1}(x)U_{k-1}(x)\sqrt{1-x^2}dx=\begin{cases}0, & m\neq k,\\ \frac{\pi}{2}, & m=k,\quad k,m=1,2,\dots.\end{cases}$$

Performing simple operations for defining the unknown coefficients a_n, we obtain an infinite system of linear equations

$$a_k+\frac{1}{\lambda_0}\sum_{m=1}^{\infty}R_{km}a_m+\frac{1}{\lambda_0}\sum_{m=1}^{\infty}R'_{km}a_m=f_k,\quad k=1,2,\dots,\tag{4.21}$$

where

$$R_{mk}=\frac{2}{\pi m}\int\limits_{-1}^{1}U_{m-1}(x)U_{k-1}(x)\frac{\sqrt{1-x^2}}{d(x)}dx,$$

$$R'_{mk}=\frac{2}{\pi m}\int\limits_{-1}^{1}\sqrt{1-x^2}U_{k-1}(x)\Bigg(\int\limits_{-1}^{1}R(t,x)\sqrt{1-t^2}U_{m-1}(t)dt\Bigg)dx,$$

$$f_k=-\frac{2}{\pi\lambda_0}\int\limits_{-1}^{1}\sqrt{1-x^2}U_{k-1}(x)f(x)dx.\tag{4.22}$$

Thus integro-differential equation (4.1) under the boundary conditions $\lambda(\pm1)=0$ is equivalent to the infinite system (4.21).

It is known that the Chebyshev polynomials form a basis in the space $L_2(-1,1)$, the series (4.20) converges in the norm of the space $L_2(-1,1)$ and the corresponding sequences $\{a_m\}$ belong to l_2 by the Parseval equality $\|\lambda\|_{L_2(-1,1)}=\|a\|_{l_2}$, $a=\{a_m\}_{m=1}^{\infty}$.

Thus the following lemma is proved:

If $f(x)\in L_2(-1,1)$, then to any solution $\lambda(x)$ of the equation (4.1) *from the class $L_2(-1,1)$ there corresponds a sequence of numbers $\{a_m\}$ from the class l_2 satisfying the infinite system of linear algebraic equations* (4.21).

Conversely, if $f(x)\in L_2(1,1)$, then to any solution $\{a_m\}\in l_2$ of system (4.21) *there corresponds a solution $\lambda(x)\in L_2(-1,1)$ of the equation* (4.1).

Theorem 3. *If $f(x) \in L_2(-1,1)$, then the operators appearing in the equation* (4.21) *and acting in l_2 are continuous for all values of the parameter λ_0. The system* (4.21) *is uniquely solvable in l_2.*

Proof. Indeed, $x = \cos Q$, from (4.22) we have

$$R_{mk} = \frac{2}{\pi m} \int\limits_0^\pi \frac{\sin mQ \sin kQ}{d(\cos Q)} dQ =$$

$$= \frac{1}{\pi m} \int\limits_0^\pi \frac{\cos(m-k)Q}{d(\cos Q)} dQ - \frac{1}{\pi m} \int\limits_0^\pi \frac{\cos(m+k)Q}{d(\cos Q)} dQ,$$

which implies that $R_{mk} = O(m^{-1})$ for $m = k$, and R_{mk} tends to zero as $k \to \infty$, $m \to \infty$, with the rate, not less than k^{-1}, m^{-1}, respectively, i.e., $\sum_{m,k=1}^\infty R_{mk}^2 < \infty$, $S_k = O(k^{-1})$, $k \to \infty$, $s_k \equiv \sum_{m=1}^\infty |R_{mk}|$.

Let $R'_{mk} = \frac{1}{m} T'_{mk}$, where

$$T'_{mk} = \frac{2}{\pi} \int\limits_{-1}^1 \sqrt{1-x^2} U_{k-1}(x) \biggl(\int\limits_{-1}^1 R(t,x) \sqrt{1-t^2} U_{m-1}(t) dt \biggr) dx.$$

Note that $\{T'_{mk}\}_{m,k=1}^\infty$ are the Fourier coefficients of the function $R(x,t)$ quadratically summable in the square $-1 \le x$, $t \le 1$ with respect to the complete orthogonal system of functions $\{U_{k-1}(x) U_{m-1}(t)\}_{k,m=1}^\infty$. Therefore by the Bessel inequality, the double series $\sum_{k,m=1}^\infty |T'_{km}|^2$ converges, and hence the series $\sum_{k=1}^\infty T'_k$, where $T'_k = \sum_{m=1}^\infty |T'_{km}|^2$, converges as well. This implies that at least $T'_k = O\Bigl(k^{-(1+\varepsilon)}\Bigr)$ $(k \to \infty,\ \varepsilon > 0)$.

If we suppose that $S'_k = \sum_{m=1}^\infty |R'_{km}| = \sum_{m=1}^\infty \frac{1}{m} |T'_{km}|$, then, using the Cauchy–Buniakovsky inequality, we get

$$S'_k \le \biggl[\sum_{m=1}^\infty \frac{1}{m^2} \biggr]^{\frac{1}{2}} \biggl[\sum_{m=1}^\infty |T'_{km}|^2 \biggr]^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}} \sqrt{T'_k},$$

and hence $S'_k = O\Bigl(k^{-\frac{(1+\varepsilon)}{2}}\Bigr)$ as $k \to \infty$.

On the basis of the above-said it follows that the operators appearing in the equation (4.1) act in l_2 and are completely continuous for $\lambda_0 \in (0,\infty)$.

The free term of that system tends to zero as $k \to \infty$ with the rate not less than $k^{-\frac{(1+\varepsilon)}{2}}$. This circumstance allows one to assert that the infinite system (4.21) is quasi-completely regular.

Therefore the Hilbert alternative [19] on the solvability of infinite systems is quite applicable to the system (4.21). Since the solution of the integral equation corresponding to the system (4.21) does exist and is unique in the space $L_2(-1,1)$, by our lemma there exists a unique non-trivial solution of the infinite system (4.21) which belongs to the space l_2 and can be found with any degree of accuracy by the method of successive approximations. □

Note that in the particular case, when

$$d(x) = 1, \quad R_{km} = \begin{cases} \frac{1}{k}, & k = m, \\ 0, & k \neq m, \end{cases}$$

system (4.21) takes the form

$$a_k + \sum_{m=1}^{\infty} R^0_{km} a_m = f^\circ_k, \quad k = 1, 2, \ldots,$$

where $R^0_{km} = \frac{k}{\lambda_0 k + 1} R'_{km}$, $f^0_k = \frac{\lambda_0 k f_k}{\lambda_0 k + 1}$.

5. Problems for Anisotropic Plates

As is known [21], moments, cross-cutting forces (and hence stresses) in the theory of elasticity are expressed by deflections of a midsurface w which satisfies the differential equation of the fourth order

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + \\ + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = q, \tag{5.1}$$

where q is the load per unit of area distributed over the exterior surface. General expression for the function w depends on the roots μ_1, μ_2, $\overline{\mu}_1$, $\overline{\mu}_2$ of the characteristic equation

$$D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12} + 2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0; \tag{5.2}$$

the constants D_{ij}, by analogy with an isotropic plate, are called rigidities of an anisotropic plate.

For any elastic homogeneous material, the equation (5.2) has no real roots. Complex or purely imaginary numbers $\mu_1 = \alpha + i\beta$, $\mu_2 = \gamma + i\delta$ are called complex parameters of the bending.

General expression for deflections has the form

1) for different complex parameters ($\mu_1 \neq \mu_2$):

$$w = w_0 + 2\,\mathrm{Re}[w_1(z_1) + w_2(z_2)]; \tag{5.3}$$

2) for equal complex parameters ($\mu_1 = \mu_2$):

$$w = w_0 + 2\,\mathrm{Re}[w_1(z_1) + \overline{z}_1 w_2(z_1)]. \tag{5.4}$$

Here w_0 is a particular solution of the inhomogeneous equation (5.1) whose type depends on the distribution of the load q over the surface, w_1 and w_2 are arbitrary analytic functions of complex variables $z_1 = x + \mu_1 y$ and $z_2 = x + \mu_2 y$.

General expressions for the moments and cross-cutting forces (for the case $\mu_2 \neq \mu_1$) are of the form

$$\begin{aligned} M_x =& M_x^0 - 2\,\mathrm{Re}[p_1 w_1''(z_1) + p_2 w_2''(z_2)], \\ M_y =& M_y^0 - 2\,\mathrm{Re}[q_1 w_1''(z_1) + q_2 w_2''(z_2)], \\ H_{xy} =& H_{xy}^0 - 2\,\mathrm{Re}[r_1 w_1''(z_1) + r_2 w_2''(z_2)], \\ N_x =& N_x^0 - 2\,\mathrm{Re}[\mu_1 s_1 w_1'''(z_1) + \mu_2 s_2 w_2'''(z_2)], \\ N_x =& N_x^0 + 2\,\mathrm{Re}[s_1 w_1'''(z_1) + s_2 w_2'''(z_2)], \end{aligned} \tag{5.5}$$

where M_x^0, $M_y^0, \dots, N_y^0$ are the moments and cross-cutting forces corresponding to the function w_0:

$$\begin{aligned} p_i =& D_{11} + D_{12}\mu_i^2 + 2D_{16}\mu_i, \\ q_i =& D_{12} + D_{22}\mu_i^2 + 2D_{26}\mu_i, \\ r_i =& D_{16} + D_{26}\mu_i^2 + 2D_{66}\mu_i, \\ s_i =& \frac{D_{11}}{\mu_i} + 3D_{16} + (D_{12} + 2D_{66})\mu_i + D_{26}\mu_i^2, \\ s_i - r_i =& \frac{p_i}{\mu_i}, \quad s_i + r_i = -q_i\mu_i, \quad i = 1, 2. \end{aligned}$$

The functions $w_1'(z_1)$ and $w_2'(z)$ in the plate bent by forces distributed over its edge must satisfy a number of conditions, namely:

1) if the region of the plate is simply connected, then the functions $w_1'(z_1)$ and $w_2'(z_2)$ must be holomorphic and single-valued in their domains;

2) if the plate has a hole, but stresses distributed over its edges are balanced (the principal vector and the principal moment are equal to zero), then the functions $w_1'(z_1)$ and $w_2'(z_2)$ are holomorphic and single-valued in both regions;

3) if the plate region is bounded by several contours (plate with holes), and moreover, the principal vector and the principal moment of forces for at least one of the contour do not equal to zero, then the functions $w_1'(z_1)$ and $w_2'(z_2)$ will be many-valued.

If the plate with holes is infinite, then the case where M_x, $M_y, \dots, N_y$ remain bounded in the whole region of the plate is of particular interest. This requirement leads to some restrictions regarding the character of the load and the type of the functions $w_1'(z_1)$ and $w_1'(z_2)$. If stresses distributed over the edges of the hole are balanced, then for the moments and cross-cutting forces to be bounded, it is necessary and sufficient that the functions $w_1'(z_1)$ and $w_2'(z_2)$ outside of the hole to have the form $(B+iC)z_1 + \mathring{w}_1'(z_1)$, $(B'+iC')z_2 + \mathring{w}_2'(z_2)$, where B, B', C, C' are real numbers, $\mathring{w}_1(z_1)$, $\mathring{w}_2(z_2)$ are the functions, holomorphic outside of the hole, including the point at infinity. Then we can assume that $C = 0$, and $\mathring{w}_1'(\infty)$ and $\mathring{w}_2'(\infty)$ are arbitrary complex constants.

Thus outside of the hole, i.e., in the neighbourhood of the point at infinity

$$w_1'(z_1) = Bz_1 + \overset{\circ}{w}{}_1'(z_1), \quad w_2'(z_2) = (B' + iC')z_2 + \overset{\circ}{w}{}_2'(z_2). \tag{5.6}$$

The moments at infinity are expressed by B, B', C':

$$\begin{aligned} M_x^\infty &= -\operatorname{Re}[p_1 B + p_2(B' + iC')], \\ M_y^\infty &= -\operatorname{Re}[q_1 B + q_2(B' + iC')], \\ H_{xy}^\infty &= -\operatorname{Re}[r_1 B + r_2(B' + iC')], \quad N_x^\infty = 0, \quad N_y^\infty = 0. \end{aligned} \tag{5.7}$$

1. Consider a thin anisotropic unbounded plate reinforced over $y = 0$, $|x| < a$ by an elastic inclusion of variable bending rigidity $D_0(x)$. Bending moment $M_x^\infty = M$, $M_y^\infty = 0$ acts at infinity. It is required to find contact forces of interaction between the inclusion and the plate.

This problem is equivalent to finding a solution of the homogeneous equation corresponding to the equation (5.1), with boundary conditions (1.2)–(1.7) imposed on the inclusion.

By (5.4),

$$\begin{aligned} \frac{\partial w}{\partial x} &= 2\operatorname{Re}[w_1'(z_1) + w_2'(z_2)], \\ \frac{\partial w}{\partial y} &= 2\operatorname{Re}[\mu_1 w_1'(z_1) + \mu_2 w_2'(z_2)]. \end{aligned}$$

By virtue of (5.5), (1.3) yields

$$\begin{aligned} \langle w_1'(x) + w_2'(x) + \overline{w_1'(x)} + \overline{w_2'(x)} \rangle &= 0, \\ \langle \mu_1 w_1'(x) + \mu_2 w_2'(x) + \overline{\mu}_1 \overline{w_1'(x)} + \overline{\mu}_2 \overline{w_2'(x)} \rangle &= 0, \\ \langle q_1 w_1''(x) + q_2 w_2''(x) + \overline{q}_1 \overline{w_1''(x)} + \overline{q}_2 \overline{w_2''(x)} \rangle &= 0, \\ \langle s_1 w_1'''(x) + s_2 w_2'''(x) + \overline{s}_1 \overline{w_1'''(x)} + \overline{s}_2 \overline{w_2'''(x)} \rangle &= \mu(x), \quad |x| < a. \end{aligned} \tag{5.8}$$

Differentiating the first and the second equality twice, and the third equality only once, we obtain for the jumps $\langle w_1'''(x) \rangle$, $\langle w_2'''(x) \rangle$, $\langle \overline{w_1'''(x)} \rangle$, $\langle \overline{w_1'''(x)} \rangle$ a system of algebraic equations.

If

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & \overline{\mu}_1 & \overline{\mu}_2 \\ q_1 & q_2 & \overline{q}_1 & \overline{q}_2 \\ s_1 & s_2 & \overline{s}_1 & \overline{s}_2 \end{vmatrix} \neq 0,$$

then by solving this system we obtain

$$\begin{aligned} [w_1'''(x)]^- - [w_1'''(x)]^+ &= -\frac{\Delta_1}{\Delta}\mu(x), \\ [w_2'''(x)]^- - [w_2'''(x)]^+ &= \frac{\Delta_2}{\Delta}\mu(x), \end{aligned} \quad |x| < a, \tag{5.9}$$

where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ \mu_2 & \overline{\mu}_1 & \overline{\mu}_2 \\ q_2 & \overline{q}_1 & \overline{q}_2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ \mu_1 & \overline{\mu}_1 & \overline{\mu}_2 \\ q_1 & \overline{q}_1 & \overline{q}_2 \end{vmatrix}.$$

Taking into account that $w_1'''(\infty) = 0$ and $w_2'''(\infty) = 0$, solutions of boundary problems (5.9) can be represented by the formulas

$$w_1'''(z_1) = \frac{\Delta_1}{2\pi i\Delta} \int\limits_{-a}^{a} \frac{\mu(t)dt}{t - z_1}, \quad w_2'''(z_2) = -\frac{\Delta_2}{2\pi i\Delta} \int\limits_{-a}^{a} \frac{\mu(t)dt}{t - z_2},$$

where z_1 and z_2 are complex variable varying respectively in the domains S_1 and S_2 cut along the segment $(-a, a)$.

By the conditions (1.6), $w_1''(z_1)$ and $w_2''(z_2)$ are represented as

$$\begin{aligned} w_1''(z_1) &= -\frac{\Delta_1}{2\pi i\Delta} \int\limits_{-a}^{a} \ln(t - z_1)\mu(t)dt + B, \\ w_2''(z_2) &= \frac{\Delta_2}{2\pi i\Delta} \int\limits_{-a}^{a} \ln(t - z_2)\mu(t)dt + (B' + iC'). \end{aligned} \tag{5.10}$$

The constants B, B' and C' are defined from the system

$$\begin{aligned} &(p_1 + \overline{p}_1)B + p_2 B_1 + \overline{p}_2 \overline{B}_1 = -M, \\ &(q_1 + \overline{q}_1)B + q_2 B_1 + \overline{q}_2 \overline{B}_1 = 0, \\ &(r_1 + \overline{r}_1)B + r_2 B_1 + \overline{r}_2 \overline{B}_1 = 0, \quad B_1 = B' + iC'. \end{aligned}$$

Realizing the contact condition between the inclusion and the plate and taking into account (5.10) and the fact that $\frac{\partial^2 w(x,0)}{\partial x^2} = 2\,\mathrm{Re}[w_1''(x) + w_2''(x)]$, we see that the condition (1.4) takes the form

$$\frac{d^2}{dx^2} D_0(x) \left[\frac{\lambda_0}{\pi} \int\limits_{-a}^{a} \ln|t - x|\mu(t)dt + (B + B') \right] = -\mu(x), \quad |x| < 1,$$

where $\lambda_0 = \mathrm{Im}\,\frac{\Delta_1 - \Delta_2}{\Delta}$.

Integrating the last equation twice and introducing the notation $\lambda(x) = \int_{-a}^{x} dt \int_{-a}^{\tau} \mu(\tau)d\tau$, we arrive at the equation

$$\lambda(x) - \frac{D_0(x)\lambda_0}{\pi} \int\limits_{-a}^{a} \frac{\lambda'(t)dt}{t - x} = -(B + B')D_0(x), \quad |x| < a, \tag{5.11}$$

under the condition

$$\lambda(\pm a) = 0 \quad \text{and} \quad \lambda'(\pm a) = 0. \tag{5.12}$$

2. Consider now an infinite elastic plate with general anisotropy, having a circular hole of unit radius and strenghtened over: $y = 0$, $-c < x < -b$, $b < x < c$, $b > 1$ by an elastic inclusion of variable bending rigidity. The hole contour is rigidly clamped and the bending moment $M_x^\infty = M$, $M_y^\infty = 0$ acts at infinity. It is required to find contact stresses of interaction between the inclusion and the plate.

Just in the same way as in the previous problem, the above-posed problem is equivalent to finding a solution of the homogeneous problem (5.1) with the boundary conditions (1.2)–(1.7) imposed on the inclusion and the conditions

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \gamma \tag{5.13}$$

on the boundary of the hole, where γ is the boundary of the unit circle.

If S is the region of the plate, then the regions with varying functions $w_1(z_1)$ and $w_2(z_2)$ will be S_1 and S_2 which are obtained from S by the so-called affine transformation: $x_1 = x + \alpha y$, $y_1 = \beta y$ and $x_2 = x + \gamma y$, $y_2 = \delta y$ respectively, where $\alpha + i\beta = \mu_1$, $\gamma + i\delta = \mu_2$ ($\mu_1 \neq \mu_2$). Then we can represent complex variables z_1 and z_2 in the form

$$\begin{aligned} z_1 =& x + \mu_1 y = \frac{z+\overline{z}}{2} + \mu_1 \frac{z-\overline{z}}{2i} = \frac{1-i\mu_1}{2}\Big(z + \frac{1+\mu_1 i}{1-\mu_1 i}\overline{z}\Big), \\ z_2 =& x + \mu_2 y = \frac{1-i\mu_2}{2}\Big(z + \frac{1+\mu_2 i}{1-\mu_2 i}\overline{z}\Big). \end{aligned} \tag{5.14}$$

Satisfying boundary conditions (1.3) on the interval $(-c,-b)\cup(b,c)$, the solution of boundary problem (5.9) for $-c < x < -b$, $b < x < c$ is given by the formulas

$$\begin{aligned} w_1'(z_1) &= \frac{\Delta_1}{2\pi i \Delta}\int\limits_l (t-z_1)\ln(t-z_1)\mu(t)dt + Bz_1 + F_1(z_1), \\ w_2'(z_2) &= -\frac{\Delta_1}{2\pi i \Delta}\int\limits_l (t-z_2)\ln(t-z_2)\mu(t)dt + B_1 z_2 + F_2(z_2), \end{aligned} \tag{5.15}$$

$$l \equiv (-c,-b)\cup(b,c),$$

where $F_1(z_1)$ and $F_2(z_2)$ are analytic functions in the regions S_1 and S_2, respectively.

On the basis of (5.14), the conditions (5.13) on γ can be written as

$$2\,\mathrm{Re}\Big[F_1\Big(R_1\Big(\sigma + \frac{m_1}{\sigma}\Big)\Big) + F_2\Big(R_2\Big(\sigma + \frac{m_2}{\sigma}\Big)\Big)\Big] =$$

$$= -2\,\mathrm{Re}\Big[f_1\Big(R_1\Big(\sigma + \frac{m_1}{\sigma}\Big)\Big) + f_2\Big(R_2\Big(\sigma + \frac{m_2}{\sigma}\Big)\Big) +$$

$$+ BR_1\Big(\sigma + \frac{m_1}{\sigma}\Big) + B_1 R_2\Big(\sigma + \frac{m_2}{\sigma}\Big)\Big],$$

$$2\,\mathrm{Re}\Big[\mu_1 F_1\Big(R_1\Big(\sigma + \frac{m_1}{\sigma}\Big)\Big) + \mu_2 F_2\Big(R_2\Big(\sigma + \frac{m_2}{\sigma}\Big)\Big)\Big] =$$

$$-2\,\mathrm{Re}\Big[\mu_1 f_1\Big(R_1\Big(\sigma+\frac{m_1}{\sigma}\Big)\Big)+\mu_2 f_2\Big(R_2\Big(\sigma+\frac{m_2}{\sigma}\Big)\Big)+$$
$$+\mu_1 BR_1\Big(\sigma+\frac{m_1}{\sigma}\Big)+\mu_2 B_1R_2\Big(\sigma+\frac{m_2}{\sigma}\Big)\Big],$$

where we have introduced the following notation:

$$R_k=\frac{1-i\mu_k}{2},\quad m_k=\frac{1+i\mu_k}{1-i\mu_k},\quad k=1,2,\quad |\sigma|=1,$$
$$f_1(z_1)=\frac{\Delta_1}{2\pi i\Delta}\int\limits_l (t-z_1)\ln|t-z_1|\mu(t)dt,$$
$$f_2(z_2)=-\frac{\Delta_2}{2\pi i\Delta}\int\limits_l (t-z_2)\ln|t-z_2|\mu(t)dt.$$

Using the notation $\overset{\circ}{F}_k(\sigma)\equiv F_k\Big(R_k\Big(\sigma+\frac{m_k}{\sigma}\Big)\Big)$, $\overset{\circ}{f}_k(\sigma)\equiv f_k\Big(R_k\Big(\sigma+\frac{m_k}{\sigma}\Big)\Big)$, $k=1,2$, we rewrite the latter conditions as follows:

$$\begin{gathered}
\overset{\circ}{F}_1(\sigma)+\overset{\circ}{F}_2(\sigma)+\overline{\overset{\circ}{F}_1(\sigma)}+\overline{\overset{\circ}{F}_2(\sigma)}=\\
=-\overset{\circ}{f}_1(\sigma)-\overset{\circ}{f}_2(\sigma)-\overline{\overset{\circ}{f}_1(\sigma)}-\overline{\overset{\circ}{f}_2(\sigma)}-\\
-BR_1\Big(\sigma+\frac{m_1}{\sigma}\Big)-B_1R_2\Big(\sigma+\frac{m_2}{\sigma}\Big)-\\
-B\overline{R}_1\Big(\frac{1}{\sigma}+\overline{m}_1\sigma\Big)-\overline{B}_1\overline{R}_2\Big(\frac{1}{\sigma}+\overline{m}_2\sigma\Big),\\
\mu_1\overset{\circ}{F}_1(\sigma)+\mu_2\overset{\circ}{F}_2(\sigma)+\overline{\mu}_1\overline{\overset{\circ}{F}_1(\sigma)}+\overline{\mu}_2\overline{\overset{\circ}{F}_2(\sigma)}=\\
=\mu_1\overset{\circ}{f}_1(\sigma)-\mu_2\overset{\circ}{f}_2(\sigma)-\\
-\overline{\mu}_1\overline{\overset{\circ}{f}_1(\sigma)}-\overline{\mu}_2\overline{\overset{\circ}{f}_2(\sigma)}-\mu_1BR_1\Big(\sigma+\frac{m_1}{\sigma}\Big)-\mu_2B_1R_2\Big(\sigma+\frac{m_2}{\sigma}\Big)-\\
-\overline{\mu}_1B\overline{R}_1\Big(\frac{1}{\sigma}+\overline{m}_1\sigma\Big)-\overline{\mu}_2\overline{B}_1\overline{R}_2\Big(\frac{1}{\sigma}+\overline{m}_2\sigma\Big).
\end{gathered}\tag{5.16}$$

Bearing in mind the fact that the function $\overset{\circ}{F}_k(\sigma)$ of the point on the circumference γ must represent the values of a function $\overset{\circ}{F}_k(\zeta)$ holomorphic outside of γ $(k=1;2)$, on the basis of the Cauchy formula for an infinite domain we have

$$\frac{1}{2\pi i}\int\limits_\gamma\frac{\overset{\circ}{F}_k(\sigma)d\sigma}{\sigma-\zeta}=-\overset{\circ}{F}_k(\zeta),\quad \frac{1}{2\pi i}\int\limits_\gamma\frac{\overline{\overset{\circ}{F}_k(\sigma)}d\sigma}{\sigma-\zeta}=0,$$

where ζ is an arbitrary point outside of γ.

Moreover, the functions $\overset{\circ}{f}_k(\sigma)$ are the boundary values of the functions $\overset{\circ}{f}_k(\zeta)$, holomorphic in γ, and the functions $\overline{\overset{\circ}{f}_k(\sigma)}$ are the boundary values of the functions $\overline{\overset{\circ}{f}_k\Big(1/\overline{\zeta}\Big)}$ holomorphic outside of γ. Consequently, we have

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overset{\circ}{f}_k(\sigma)d\sigma}{\sigma-\zeta}=0,\quad \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\overset{\circ}{f}_k(\sigma)}d\sigma}{\sigma-\zeta}=-\overline{\overset{\circ}{f}}_k\Big(\frac{1}{\zeta}\Big);$$

the function $\zeta+\frac{m_k}{\zeta}$ is holomorphic in γ, except the point $\zeta=0$, where it has a pole with the principal part $\frac{m_k}{\zeta}$ and the function $\frac{1}{\zeta}+\overline{m}_k\zeta$ is holomorphic outside of γ, except the point $\zeta=\infty$, where it is of the type $\overline{m}_k\zeta+O\Big(\frac{1}{\zeta}\Big)$. Thus we have

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{(\sigma+\frac{m_k}{\sigma})d\sigma}{\sigma-\zeta}=-\frac{m_k}{\zeta},$$

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{(\frac{1}{\sigma}+\overline{m}_k\sigma)}{\sigma-\zeta}=-\frac{1}{\zeta}-\overline{m}_k\zeta+\overline{m}_k\zeta=-\frac{1}{\zeta}.$$

As a result, from (5.16) we get

$$\begin{aligned}-\overset{\circ}{F}_1(\zeta)-\overset{\circ}{F}_2(\zeta)&=\overline{\overset{\circ}{f}}_1\Big(\frac{1}{\zeta}\Big)+\overline{\overset{\circ}{f}}_2\Big(\frac{1}{\zeta}\Big)+\frac{\Gamma_1}{\zeta},\\ -\mu_1\overset{\circ}{F}_1(\zeta)-\mu_2\overset{\circ}{F}_2(\zeta)&=\overline{\mu}_1\overline{\overset{\circ}{f}}_1\Big(\frac{1}{\zeta}\Big)+\overline{\mu}_2\overline{\overset{\circ}{f}}_2\Big(\frac{1}{\zeta}\Big)+\frac{\Gamma_2}{\zeta},\end{aligned}\tag{5.17}$$

where

$$\Gamma_1=BR_1m_1+B_1R_2m_2+BR_1+\overline{B}_1\overline{R}_2,$$
$$\Gamma_2=\mu_1BR_1m_1+\mu_2B_1R_2m_2+\overline{\mu}_1BR_1+\overline{\mu}_2\overline{B}_1\overline{R}_2.$$

From (5.17) we have

$$\overset{\circ}{F}_1(\zeta)=-\overline{\overset{\circ}{f}}_1\Big(\frac{1}{\zeta}\Big)+\frac{\Gamma_2-\mu_2\Gamma_1}{\mu_2-\mu_1}\cdot\frac{1}{\zeta},$$
$$\overset{\circ}{F}_2(\zeta)=-\overline{\overset{\circ}{f}}_2\Big(\frac{1}{\zeta}\Big)+\frac{\mu_1\Gamma_1-\Gamma_2}{\mu_2-\mu_1}\cdot\frac{1}{\zeta},$$

and hence

$$
\begin{gathered}
F_1(z_1) = \\
= \frac{\overline{\Delta}_1}{2\pi i \overline{\Delta}} \int\limits_l \Big(t - \frac{2R_1}{z_1 + \sqrt{z_1^2 - 4R_1^2 m_1}}\Big) \ln \Big| t - \frac{2R_1}{z_1 + \sqrt{z_1^2 - 4R_1^2 m_1}} \Big| \mu(t)dt + \\
+ \frac{\Gamma_2 - \mu_2 \Gamma_1}{\mu_2 - \mu_1} \cdot \frac{2R_1}{z_1 + \sqrt{z_1^2 - 4R_1^2 m_1}}, \\
F_2(z_2) = \\
= - \frac{\overline{\Delta}_2}{2\pi i \overline{\Delta}} \int\limits_l \Big(t - \frac{2R_2}{z_2 + \sqrt{z_2^2 - 4R_2^2 m_2}}\Big) \ln \Big| t - \frac{2R_2}{z_2 + \sqrt{z_2^2 - 4R_2^2 m_2}} \Big| \mu(t)dt + \\
+ \frac{\mu_1 \Gamma_1 - \Gamma_2}{\mu_2 - \mu_1} \cdot \frac{2R_2}{z_2 + \sqrt{z_2^2 - 4R_2^2 m_2}}.
\end{gathered}
\tag{5.18}
$$

Taking into account (1.6), (5.15) and (5.18) for $\mu_1 = i\beta$, $\mu_2 = i\delta$, by (5.3) we have

$$
\begin{gathered}
\frac{\partial^2 w(x,0)}{\partial x^2} = w_1''(x) + w_2''(x) + \overline{w_1''(x)} +, \overline{w_2''(x)} = \\
= - \operatorname{Im} \frac{\Delta_1 - \Delta_2}{\pi \Delta} \int\limits_l \ln|t - x| \mu(t)dt - \frac{1+\beta}{\pi} \operatorname{Im} \frac{\Delta_1}{\Delta} \times \\
\times \frac{1}{\sqrt{x^2 - 1 + \beta^2}(x + \sqrt{x^2 - 1 + \beta^2})} \times \int\limits_l \ln \Big| t - \frac{1+\beta}{x + \sqrt{x^2 - 1 + \beta^2}} \Big| \mu(t)dt + \\
+ \frac{1+\delta}{\pi} \operatorname{Im} \frac{\Delta_2}{\Delta} \cdot \frac{1}{\sqrt{x^2 - 1 + \delta^2}(x + \sqrt{x^2 - 1 + \delta^2})} \times \\
\times \int\limits_l \ln \Big| t - \frac{1+\delta}{x + \sqrt{x^2 - 1 + \delta^2}} \Big| \mu(t)dt + \\
+ \frac{l_1}{\sqrt{x^2 - 1 + \beta^2}(x + \sqrt{x^2 - 1 + \beta^2})} + \\
+ \frac{l_2}{\sqrt{x^2 - 1 + \delta^2}(x + \sqrt{x^2 - 1 + \delta^2})} + 2B + 2B',
\end{gathered}
$$

where

$$
\begin{aligned}
l_1 &= (1+\beta)\Big[\frac{B(1+\beta)(\beta+\delta)}{\delta - \beta} + \frac{2B'(1+\delta)\delta}{\delta - \beta} - B(1-\beta)\Big], \\
l_2 &= (1+\delta)\Big[B'(\delta - 1) + \frac{2B\beta(1+\beta)}{\beta - \delta} + \frac{B'(1+\delta)(\beta+\delta)}{\beta - \delta}\Big].
\end{aligned}
$$

By the contact condition of between the inclusion and the plate, we obtain the following integral equation:

$$\frac{\varkappa_1}{\pi}\int\limits_l \ln|t-x|\lambda''(t)dt+\frac{\varkappa_2}{\pi}\beta(x)\int\limits_l \ln\left|t-\frac{1+\beta}{x+\sqrt{x^2-1+\beta^2}}\right|\lambda''(t)dt+$$
$$+\frac{\varkappa_3}{\pi}\delta(x)\int\limits_l \ln\left|t-\frac{1+\delta}{x+\sqrt{x^2-1+\delta^2}}\right|\lambda''(t)dt+$$
$$+\frac{\lambda(x)}{D_0(x)}=f(x),\quad x\in[-c,-b]\cup[b,c],\quad b>1,$$

where we have introduced the notation

$$\varkappa_1=-\operatorname{Im}\frac{\Delta_1-\Delta_2}{\Delta},\quad \varkappa_2=-(1+\beta)\operatorname{Im}\frac{\Delta_1}{\Delta},\quad \varkappa_2=(1+\delta)\operatorname{Im}\frac{\Delta_2}{\Delta},$$
$$\beta(x)=\frac{1}{\sqrt{x^2-1+\beta^2}(x+\sqrt{x^2-1+\beta^2})},$$
$$\delta(x)=\frac{1}{\sqrt{x^2-1+\delta^2}(x+\sqrt{x^2-1+\delta^2})}.$$
$$\lambda(x)\equiv\begin{cases}\int\limits_b^x dt\int\limits_b^t \mu(\tau)d\tau, & x\in[b,c],\\ \qquad f(x)=-l_1\beta(x)-l_2\delta(x)-2B-2B', & \\ \int\limits_x^{-b} dt\int\limits_t^{-b} \mu(\tau)d\tau, & x\in[-c,-b].\end{cases}$$

Introducing dimensionless coordinates $\zeta=\frac{x}{c}$, $\eta=\frac{t}{c}$ and dimensionless values, we can transform the above equation into the integro-differential equation

$$-\frac{\mathring{\varkappa}_1}{\pi}\left(\int\limits_{-1}^{-\rho}+\int\limits_{\rho}^{1}\right)\frac{\mathring{\lambda}'(\eta)d\eta}{\eta-\zeta}-\frac{\mathring{\varkappa}_2}{\pi}\mathring{\beta}(\zeta)\left(\int\limits_{-1}^{-\rho}+\int\limits_{\rho}^{1}\right)\frac{\mathring{\lambda}'(\eta)d\eta}{\eta-\frac{1+\beta}{c^2(\zeta+\sqrt{\zeta^2-\gamma_1})}}-$$
$$-\frac{\mathring{\varkappa}_3\mathring{\delta}(\zeta)}{\pi}\left(\int\limits_{-1}^{-\rho}+\int\limits_{\rho}^{1}\right)\frac{\mathring{\lambda}'(\eta)d\eta}{\eta-\frac{1+\delta}{c^2(\zeta+\sqrt{\zeta^2-\gamma_2})}}=$$
$$=\frac{\mathring{\lambda}(\zeta)}{\mathring{D}(\zeta)}+\mathring{f}(\zeta),\quad \zeta\in(-1,-\rho)\cup(\rho,1), \tag{5.19}$$

whose solution according to (1.6) must satisfy the boundary conditions

$$\lambda(\pm\rho)=0,\quad \lambda(\pm1)=0,\quad \lambda'(\pm\rho)=0,\quad \lambda'(\pm1)=0. \tag{5.20}$$

Here $\rho = \frac{b}{c}$,

$$\overset{\circ}{\varkappa}_i = \frac{\varkappa_i}{c} \quad (i = 1, 2, 3), \quad \gamma_1 = \frac{1-\beta^2}{c^2}, \quad \gamma_2 = \frac{1-\delta^2}{c^2},$$

$\overset{\circ}{\lambda}(\zeta) \equiv \lambda(c\zeta)$, $\overset{\circ}{D}(\zeta) \equiv D(c\zeta)$, $\overset{\circ}{f}(\zeta) = f(c\zeta)$, $\overset{\circ}{\beta}(\zeta) = \beta(c\zeta)$, $\overset{\circ}{\delta}(\zeta) \equiv \delta(c\zeta)$, $\overset{\circ}{\lambda}(-\zeta) = \overset{\circ}{\lambda}(\zeta)$, $\overset{\circ}{D}(-\zeta) = \overset{\circ}{D}(\zeta)$, $\overset{\circ}{f}(-\zeta) = \overset{\circ}{f}(\zeta)$.

Now the last equation, from two symmetrical segments we will transfer to the segment $[-1, 1]$. To this end, by means of the evenness of the functions appearing in that equation, we first transform them to the segment $[\rho, 1]$ and then pass to new variables using the formulas

$$s = \frac{2\zeta^2 - \rho^2 - 1}{1 - \rho^2}, \quad v = \frac{2\eta^2 - \rho^2 - 1}{1 - \rho^2}, \quad \rho \le \zeta, \eta \le 1, \quad -1 \le s, \quad v \le 1.$$

After simple calculations, (5.19) transforms into the equation

$$\frac{\widetilde{\varkappa}_1}{\pi} \int_{-1}^{1} \frac{\psi'(v)dv}{v - s} -$$

$$-\widetilde{\varkappa}_2 \widetilde{\beta}(s) \int_{-1}^{1} \frac{\psi'(v)dv}{c^4(d_1 v + d_2)(\sqrt{d_1 s + d_2} + \sqrt{d_1 s + d_2 - \gamma_1})^2 - (1+\beta)^2} -$$

$$-\widetilde{\varkappa}_3 \widetilde{\delta}(s) \int_{-1}^{1} \frac{\psi'(v)dv}{c^4(d_1 v + d_2)(\sqrt{d_1 s + d_2} + \sqrt{d_1 s + d_2 - \gamma_2})^2 - (1+\delta)^2} =$$

$$= \frac{\psi(s)}{\widetilde{D}(s)} + \widetilde{f}(s), \quad |s| < 1, \tag{5.21}$$

where

$$\widetilde{\varkappa}_1 = -\frac{\overset{\circ}{\varkappa}_1}{d_1}, \ \widetilde{\varkappa}_2 = \frac{2\overset{\circ}{\varkappa}_2 c^2(1+\beta)}{\pi}, \ \widetilde{\varkappa}_3 = \frac{2\overset{\circ}{\varkappa}_3 c^2(1+\delta)}{\pi},$$

$$\widetilde{\beta}(s) = \overset{\circ}{\beta}\Big(\sqrt{d_1 s + d_2}\Big)\Big(1 + \sqrt{1 - \frac{\gamma_1}{d_1 s + d_2}}\Big),$$

$$\widetilde{\delta}(s) = \overset{\circ}{\delta}\Big(\sqrt{d_1 s + d_2}\Big)\Big(1 + \sqrt{1 - \frac{\gamma_2}{d_1 s + d_2}}\Big),$$

$$\widetilde{D}(s) = \overset{\circ}{D}\big(\sqrt{d_1 s + d_2}\big)\sqrt{d_1 s + d_2}, \ \widetilde{f}(s) = \frac{\overset{\circ}{f}(s)}{\sqrt{d_1 s + d_2}},$$

$$\psi(s) = \overset{\circ}{\lambda}\big(\sqrt{d_1 s + d_2}\big), \ d_1 = \frac{1-\rho^2}{2}, \ d_2 = \frac{1+\rho^2}{2}.$$

Thus we have obtained the equation (5.21) with the same structure as that of equation (1.22), and now boundary conditions (5.20) take the form

$$\psi(\pm 1) = 0, \quad \psi'(\pm 1) = 0. \tag{5.22}$$

The techniques analogous to that presented in the foregoing section allows one to reduce integro-differential equations (5.21)-(5.22) to the Fredholm integral equation of second kind. On the other hand, this equation is equivalent to an infinite system of algebraic equations. Thus we have

$$\psi'(s) = \frac{1}{\sqrt{1-s^2}} \sum_{k=0}^{\infty} b_k T_k(s), \quad |s| < 1,$$

and for the determination of unknown coefficients b_k we get an infinite system of linear equations

$$b_m + \sum_{k=1}^{\infty} Q_{mk} b_k + \sum_{k=1}^{\infty} Q'_{mk} b_k = g_m, \quad m = 1, 2, \ldots, \tag{5.23}$$

where

$$Q_{mk} = \frac{1}{k\widetilde{\varkappa}_1} \int_{-1}^{1} \frac{(1-s^2) U_{m-1}(s) U_{k-1}(s)}{\overset{\circ}{D}(\sqrt{d_1 s + d_2})\sqrt{d_1 s + d_2}} ds,$$

$$Q'_{mk} = \frac{1}{\widetilde{\varkappa}_1 k} \int_{-1}^{1} \sqrt{1-s^2} U_{m-1}(s) \left(\int_{-1}^{1} Q(s,v) \sqrt{1-v^2} U_{k-1}(v) dv \right) ds,$$

$$g_m = \frac{1}{\widetilde{\varkappa}_1} \int_{-1}^{1} \sqrt{1-s^2} U_{m-1}(s) \widetilde{f}(s) ds,$$

$$Q(s,v) = -\frac{\widetilde{\varkappa}_2 c^4 d_1 \widetilde{\beta}(s)(\sqrt{d_1 s + d_2} + \sqrt{d_1 s + d_2 - \gamma_1})}{[c^4(d_1 v + d_2)(\sqrt{d_1 s + d_2} + \sqrt{d_1 s + d_2 - \gamma_1})^2 - (1+\beta)^2]^2} - $$
$$- \frac{\widetilde{\varkappa}_3 c^4 d_1 \widetilde{\delta}(s)(\sqrt{d_1 s + d_2} + \sqrt{d_1 s + d_2 - \gamma_2})}{[c^4(d_1 v + d_2)(\sqrt{d_1 s + d_2} + \sqrt{d_1 s + d_2 - \gamma_2})^2 - (1+\delta)^2]^2}.$$

We now pass to the investigation of regularity of the infinite system (5.23). Towards this end, in the integral expression for the kernel Q_{mk} we put $s = \cos Q$:

$$Q_{mk} = \frac{1}{k\widetilde{\varkappa}_1} \int_{0}^{\pi} \frac{\sin mQ \sin kQ \sin Q dQ}{\overset{\circ}{D}(\sqrt{d_1 \cos Q + d_2})\sqrt{d_1 \cos Q + d_2}} =$$

$$= \frac{1}{4\widetilde{\varkappa}_1 k} (N_{m+k-1} + N_{m-k+1} - N_{m-k-1} - N_{m+k+1}), \tag{5.24}$$

where $N_p = \int_0^\pi \frac{\sin pQdQ}{\overset{\circ}{D}(\sqrt{d_1\cos Q + d_2})\sqrt{d_1\cos Q+d_2}}$, p is any integer ($N_0 = 0$).

Integrating by parts, we easily get $|N_p| < \frac{A}{|p|}$, $A = \text{const}$. Compose now the sums $s_m = \sum_{k=1}^\infty |Q_{mk}|$. On the basis of (5.24) we find that

$$s_m \le \frac{A}{\widetilde{\varkappa}_1}\Bigg[\sum_{k=1}^\infty \frac{1}{k(m+k-1)} + \sum_{k=1}^\infty \frac{1}{k(m+k+1)} + \sum_{k=1}^{m-2} \frac{1}{k(m-k-1)} + \\ + \sum_{k=m}^\infty \frac{1}{k(k-m+1)} + \sum_{k=1}^{m} \frac{1}{k(m-k+1)} + \sum_{k=m+2}^\infty \frac{1}{k(k-m-1)}\Bigg]. \qquad (5.25)$$

Since

$$\sum_{k=1}^\infty \frac{1}{k(k+m)} = \frac{1}{m}\Big[c + \ln m + \frac{1}{2m} - l(m)\Big],$$

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} = \frac{2}{m}\Big[c + \ln(m-1) + \frac{1}{2(m-1)} - l(m-1)\Big],$$

$$\sum_{k=m+1}^\infty \frac{1}{k(k-m)} = \frac{1}{m}\Big[c + \ln m + \frac{1}{2m} - l(m)\Big],$$

$$l(m) = \frac{1}{m}\sum_{q=2}^\infty \frac{A_q}{(m+1)\dots(m+q-1)}, \quad m = 1,2,\dots, \qquad (5.26)$$

and all the sums appearing in (5.25) are of the type (5.26), we have $s_m = O\Big(\frac{1}{m^{1-\varepsilon}}\Big)$ as $m \to \infty$ (c is the well-known Euler constant, the coefficients $A_q = \frac{1}{q}\int_0^1 x(1-x)(2-x)\dots(q-1-x)dx$, $q = 2,3,\dots$). Hence we can assert that $\sum_{m,k=1}^\infty |Q_{mk}|^2 < \infty$.

Let $Q'_{mk} = \frac{1}{k}T'_{mk}$, where

$$T'_{mk} = \frac{1}{\widetilde{\varkappa}_1}\int_{-1}^{1}\sqrt{1-s^2}U_{m-1}(s)\Big(\int_{-1}^{1}Q(s,v)\sqrt{1-v^2}U_{k-1}(v)dv\Big)ds.$$

As is easily seen, $\{T'_{mk}\}_{k,m=1}^\infty$ are the Fourier coefficients of quadratically summable in the square $-1 \le s$, $v \le 1$ function over a whole orthogonal system of functions $\{U_{m-1}(s)U_{k-1}(v)\}_{m,k=1}^\infty$, and hence the series $\sum_{m,k=1}^\infty |T'_{mk}|^2 < \infty$-converges.

On the other hand, $s'_m = \sum_{k=1}^\infty |Q'_{mk}| \le \Big[\sum_{k=1}^\infty \frac{1}{k^2}\Big]^{\frac{1}{2}}\Big[\sum_{k=1}^\infty |T'_{mk}|^2\Big]^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}\sqrt{T_m}$, where $T_m = \sum_{k=1}^\infty |T'_{mk}|^2$ and the series $\sum_{m=1}^\infty T_m$ converges. This at least yields

$$s'_m = O\Big(\frac{1}{m^{\frac{1+\varepsilon}{2}}}\Big), \quad m \to \infty.$$

The free term of the system (5.23) tends to zero with the same rate. This fact allows one to state that the infinite system (5.23) is quasi-completely regular.

When the inclusion rigidity varies by the law $D^0(\zeta)=\sqrt{(1-\zeta^2)(\zeta^2-\rho^2)}$, the coefficient Q_{mk} of the system (5.23) is expressed by

$$Q_{mk}=\frac{1}{\widetilde{\varkappa}_1 k}\int\limits_0^\pi \frac{\sin mQ\sin kQdQ}{\sqrt{d_1\cos Q+d_2}}=$$

$$=\frac{1}{2\widetilde{\varkappa}_1 k}\int\limits_0^\pi \frac{\cos(m-k)QdQ}{\sqrt{d_1\cos Q+d_2}}-\frac{1}{2\widetilde{\varkappa}_1 k}\int\limits_0^\pi \frac{\cos(m+k)QdQ}{\sqrt{d_1\cos Q+d_2}}.$$

Thus we conclude that for $m=k$, $Q_{mk}=O(k^{-1})$ $(k\to\infty)$, for $m\neq k$ it tends to zero with the rate, not less than k^{-1}, m^{-1}, $(k\to\infty, m\to\infty)$ and hence the series $\sum_{m,k}^{\infty}|Q_{mk}|^2$ converges and the system remains quasi-completely regular.

The above-obtained results make it possible to draw the following conclusions:

1. Problems of a contact between finite or infinite, isotropic or anisotropic plates and an elastic inclusion are reduced to the integral differential equations with Prandtl type characteristic part.

2. The solution of the characteristic equation for the coefficient of singular operator $D_0(x)=d(a^2-x^2)^{n+\frac{1}{2}}$ ($d=\text{const}$, $n\geq 0$ is a natural number) is obtained effectively and the estimate $\lambda(x)=O\Big((a^2-x^2)^{n+\frac{1}{2}}\Big)$ as $x\to\pm a$ holds (for the right-hand side $f(x)=O\Big((a^2-x^2)^{n+\frac{1}{2}}\Big)$ $(x\to\pm a)$.

3. The above-obtained integral differential equations of the types (1.22), (2.9) and (5.21) are equivalent under certain boundary conditions to the Fredholm integral equation of second kind on the one hand, and to the infinite quasi-regular system of linear algebraic equations on the other hand.

4. On the basis of the uniqueness of contact problems the obtained integral equations have solutions in the corresponding spaces.

5. Under conditions of variable rigidity of the inclusion, or more precisely, if the rigidity function vanishes at the ends of the contact line, the question reduces to the integral equations whose characteristic part is the Prandtl equation with the "degenerated" (vanishing at the ends of the interval) coefficient. For higher degree of degeneration the equation goes out of the framework of the cases already studied. For such a coefficient, from a sufficiently wide class of functions we have managed to investigate the obtained equations, to get exact or approximate solutions and to estimate behaviour of unknown contact stresses at the ends of the contact line.

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