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TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. In the paper effective sufficient conditions are obtained for unique solvability and correctness of the mixed problem and of the Dirichlet problem for second order linear singular functional differential equations. Some of these conditions are nonimprovable and some of them generalize results which are well known for ardinary differential equations.

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Main Notation

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}^+ =]0, +\infty[$.

Let $\alpha \in \mathbb{R}$.

$[\alpha]$ is the integral part of the number α,

$$[\alpha]_+ = \frac{|\alpha| + \alpha}{2}, \quad [\alpha]_- = \frac{|\alpha| - \alpha}{2}.$$

$C(\,]a,b[)$ is the space of continuous and bounded functions $u :]a,b[\to \mathbb{R}$ with the norm

$$\|u\|_C = \sup\{|u(t)| : a < t < b\}.$$

$\widetilde{C}_{\text{loc}}(]a,b[)$ is the set of the functions $u :]a,b[\to \mathbb{R}$ absolutely continuous on each subsegment of $]a,b[$.

$\widetilde{C}'_{\text{loc}}(\,]a,b[)$ is the set of the functions $u :]a,b[\to \mathbb{R}$ absolutely continuous on each subsegment of $]a,b[$ along with their first order derivatives.

$L([a,b])$ is the space of summable functions $u : [a,b] \to \mathbb{R}$ with the norm

$$\|u\|_L = \int_a^b |u(s)|ds.$$

$L_\infty(]a,b])$ is the space of essentially bounded functions $u :]a,b[\to \mathbb{R}$ with the norm

$$\|u\| = \operatorname*{ess\,sup}_{t \in [a,b]} |u(t)|.$$

$L_{\text{loc}}(]a,b[)$ $(L_{\text{loc}}(]a,b[))$ is the set of the measurable functions $u :]a,b[\to \mathbb{R}$ $(u :]a,b] \to \mathbb{R})$, summable on each subsegment of $]a,b[$ $(]a,b])$.

Let $x, y :]a,b[\to]0,+\infty[$ be continuous functions.

$C_x(]a,b[)$ is the space of functions $u \in C(]a,b[)$ such that

$$\|u\|_{C,x} = \sup\left\{\frac{|u(t)|}{x(t)} : \quad a < t < b\right\} < +\infty.$$

$L_y([a,b])$ is the space of the functions $u \in L(]a,b[)$ such that

$$\|u\|_{L,y} = \int_a^b y(s)|u(s)|ds < +\infty.$$

$\mathcal{L}(C_x; L_y)$ is the set of the linear operators $h : C_x(]a,b[) \to L_y([a,b])$ such that

$$\sup\left\{|h(u)(\cdot)| : \|u\|_{C,x} \le 1\right\} \in L_y([a,b]).$$

$\sigma : L_{\text{loc}}(]a,b[) \to \widetilde{C}_{\text{loc}}(]a,b[)$ is the operator defined by

$$\sigma(p)(t) = \exp\left(\int_{\frac{a+b}{2}}^t p(s)ds\right) \quad \text{for} \quad a \le t \le b,$$

where $p \in L_{\mathrm{loc}}(]a,b[)$.

If $\sigma(p) \in L([a,b])$, then we define the operators σ_1 and σ_2 by

$$\sigma_1(p)(t) = \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(s)ds \int_t^b \sigma(p)(s)ds,$$

$$\sigma_2(p)(t) = \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(s)ds \quad \text{for} \quad a \le t \le b.$$

Let $f,\, g \in C(]a,b[)$ and $c \in [a,b]$. Then we write

$$f(t) = O(g(t)) \quad \big(f(t) = O^*(g(t))\big) \quad \text{as} \quad t \to c,$$

if

$$\limsup_{t\to c} \frac{|f(t)|}{|g(t)|} < +\infty \quad \Big(0 < \liminf_{t\to c} \frac{|f(t)|}{|g(t)|} \ \text{ and } \ \limsup_{t\to c} \frac{|f(t)|}{|g(t)|} < +\infty\Big).$$

Let A and B be normed spaces and let $\mathbb{U} : A \to \mathbb{B}$ be a linear operator. Then we denote the norm of the operator $\mathbb{U}$ as follows:

$$||\mathbb{U}||_{A\to\mathbb{B}}.$$

Introduction

During the last two decades the boundary value problems for functional differential equations attract the attention of many mathematicians and are intensively studied. At present the foundations of the general theory of such kind of problems are already laid and many of them are investigated in detail (see [1], [2], [19]–[23], [44] and references therein). Despite this fact, there remains a wide class of boundary value problems on the solvability of which not much is known. Among them are the two-point boundary value problems for linear singular functional differential equations of second order, and we devote our work to the investigation of these problems.

It should be noted that the present monograph is tightly connection with the works of I. T. Kiguradze [17], L. B. Shekhter [23] and A. G. Lomtatidze [27] in which for singular ordinary differential equations we developed the method of upper and lower Nagumo's functions in the case of boundary value problems and found the conditions under which Fredholm's alternative is valid in the case of linear equations. We introduced and described the set $\mathbb{V}_{0,i}$ (see Definition 1.1.2).

In the present work we consider the equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t) \tag{0.0.1}$$

under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{0.0.2$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{0.0.2$_2$}$$

and separately for the case of homogeneous conditions

$$u(a) = 0, \quad u(b) = 0,$$
$$u(a) = 0, \quad u'(b-) = 0,$$

where $c_1, c_2 \in \mathbb{R}$, $p_j \in L_{\rm loc}(]a,b[)$ $(j = 0,1,2)$ and $g : C(]a,b[) \to L_{\rm loc}(]a,b[)$ is a continuous linear operator. In studying these problems the use is made of the auxiliary equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) - h(u)(t),$$

where $h : C(]a,b[) \to L_{\rm loc}(]a,b[)$ is the nonnegative linear operator.

The question of the unique solvability of problems (0.0.1), (0.0.2$_i$) is studied in Chapter I. We introduced sets of two-dimensional vector functions $(p_0,p_1) :]a,b[\to \mathbb{R}^2$, $\mathbb{V}_{i,\beta}(]a,b[;h)$, $\beta \in [0,1]$ (see Definitions 1.1.3 and 1.1.4), which were found to be useful for our investigation. In Section 1.1, in terms of the sets $\mathbb{V}_{i,\beta}(]a,b[;h)$ we established theorems for the unique solvability of problems (0.0.1), (0.0.23$_i$). The question on the unique solvability of problems (0.0.1), (0.0.2$_{i0}$) in the space with weight $C_\lambda(]a,b[)$ is studied separately. In the same chapter we can find corollaries of basic theorems and and also the effective sufficient conditions for the unique solvability of the above-mentioned problems. Among them there occur unimprovable conditions and those which generalize the well-known results for ordinary differential equations.

In Chapter II we consider the question dealing with the correctness of problems (0.0.1), $(0.0.2_i)$ under the assumption that $(p_0, p_1) \in \mathbb{V}_{i,\beta}(]a, b[; h)$. The effective sufficient conditions guaranteeing the correctness of the above-mentioned problems are presented.

Everywhere in our work, special attention is given to the case, when the operator g in equation (0.0.1) is defined by the equality

$$g(u)(t) = \sum_{k=1}^{n} g_k(t) u(\tau_k(t)),$$

where $g_k \in L_{\rm loc}(]a, b[)$, $\tau_k : [a, b] \to [a, b]$ $(k = 1, \dots, n)$ are measurable functions.

CHAPTER I
UNIQUE SOLVABILITY OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR LINEAR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

§ 1.1. Statement of the Problem and Formulation of Basic Results

In this chapter we consider the linear equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t) \tag{1.1.1}$$

under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{1.1.2$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{1.1.2$_2$}$$

where p_0, $p_j \in L_{\rm loc}(]a,b[)$, $c_j \in \mathbb{R}$ $(j=1,2)$ and $g : C(]a,b[) \to L_{\rm loc}(]a,b[)$ is a continuous linear operator.

The equation (1.1.1) will also be studied separately in the weighted space $C_{x^\beta}(]a,b[)$ under the homogeneous boundary conditions

$$u(a) = 0, \quad u(b) = 0 \tag{1.1.2$_{10}$}$$

or

$$u(a) = 0, \quad u'(b-) = 0, \tag{1.1.2$_{20}$}$$

where $\beta \in]0,1]$ and

$$x(t) = \int_a^t \sigma(p_1)(s)\,ds \left(\int_t^b \sigma(p_1)(s)\,ds \right)^{2-i} \quad \text{for} \quad a \le t \le b.$$

When considering the problems (1.1.1), (1.1.2$_1$) and (1.1.1), (1.1.2$_{10}$), it will always be assumed that

$$\begin{gathered} p_j \in L_{\rm loc}(]a,b[) \;\; (j=0,1,2), \\ \sigma(p_1) \in L([a,b]), \quad p_0 \in L_{\sigma_1(p_1)}([a,b]), \end{gathered} \tag{1.1.3$_1$}$$

and when considering the problems (1.1.1), (1.1.2$_2$) and (1.1.1), (1.1.2$_{20}$) we will assume that

$$\begin{gathered} p_j \in L_{\rm loc}(]a,b]) \;\; (j=0,1,2), \\ \sigma(p_1) \in L([a,b]), \quad p_0 \in L_{\sigma_2(p_1)}([a,b]). \end{gathered} \tag{1.1.3$_2$}$$

Introduce the following definitions.

Definition 1.1.1. Let $i \in \{1,2\}$. We will say that $w \in C(]a,b[)$ is the lower (upper) function of the problem (1.1.1), (1.1.2$_i$) if:

(a) w' is of the form $w'(t) = w_0(t) + w_1(t)$, where $w_0 :]a,b[\to \mathbb{R}$ is absolutely continuous on each segment from $]a,b[$, the function $w_1 :]a,b[\to \mathbb{R}$ is nondecreasing (nonincreasing) and its derivative is almost everywhere equal to zero;

(b) almost everywhere on $]a,b[$ the inequality

$$w''(t) \geq p_0(t)w(t) + p_1(t)w'(t) + g(w)(t) + p_2(t)$$
$$\big(w''(t) \leq p_0(t)w(t) + p_1(t)w'(t) + g(w)(t) + p_2(t)\big)$$

is satisfied:

(c) there exists the limit $w'(b-)$ and

$$w(a) \leq c_1, \quad w^{(i-1)}(b-) \leq c_2 \quad \big(w(a) \geq c_1, \quad w^{(i-1)}(b-) \geq c_2\big).$$

Definition 1.1.2. Let $i \in \{1,2\}$. We will say that a two-dimensional vector function $(p_0,p_1) :]a,b[\to \mathbb{R}^2$ belongs to the set $\mathbb{V}_{i,0}(]a,b[)$ if the conditions (1.1.3$_i$) are fulfilled, the solution of the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t), \tag{1.1.4}$$
$$u(a) = 0, \quad \lim_{t \to a} \frac{u'(t)}{\sigma(p_1)(t)} = 1$$

has no zeros in the interval $]a,b[$ and $u^{(i-1)}(b-) > 0$.

Note that this definition is in a full agreement with that of the set $\mathbb{V}_{i,0}(]a,b[)$ given in [23] as the set of three-dimensional vector functions $(p_0,p_{11},p_{12}) :]a,b[\to \mathbb{R}^3$ if $p_{11}(t) = p_{12}(t) = p_1(t)$ almost everywhere on $]a,b[$.

Definition 1.1.3. Let $i \in \{1,2\}$ and $h : C(]a,b[) \to L_{\text{loc}}(]a,b[)$ be a continuous linear operator. We will say that a two-dimensional vector function $(p_0,p_1) :]a,b[\to \mathbb{R}^2$ belongs to the set $\mathbb{V}_{i,0}(]a,b[;h)$ if the conditions (1.1.3$_i$) are satisfied and the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) - h(u)(t)$$
$$u(a) = 0, \quad u^{(i-1)}(b-) = 0$$

has a positive upper function w on the segment $[a,b]$.

Definition 1.1.4. Let $i \in \{1,2\}$, $\beta \in]0,1]$ and $h : C(]a,b[) \to L_{\text{loc}}(]a,b[)$ be a continuous linear operator. We will say that a two-dimensional vector function $(p_0,p_1) :]a,b[\to \mathbb{R}^2$ belongs to the set $\mathbb{V}_{i,\beta}(]a,b[;h)$ if

$$(p_0,p_1) \in \mathbb{V}_{i,0}(]a,b[),$$

there exists a measurable function $q_\beta :]a,b[\to[0,+\infty[$ such that

$$\int_a^b |G(t,s)|q_\beta(s)\,ds = O^*(x^\beta(t))$$

as $t\to a$, $t\to b$ if $i=1$, and as $t\to b$ if $i=2$, where G is Green's function of the problem (1.1.4), $(1.1.2_{i0})$ and

$$x(t)=\int_a^t \sigma(p_1)(s)\,ds\bigg(\int_t^b \sigma(p_1)(s)\,ds\bigg)^{2-i} \quad \text{for} \quad a\le t\le b,$$

and the problem

$$u''(t)=p_0(t)u(t)+p_1(t)u'(t)-h(u)(t)-q_\beta(t),$$
$$u(a)=0, \quad u^{(i-1)}(b-)=0$$

on the interval $]a,b[$ has a positive upper function w such that

$$w(t)=O^*(x^\beta(t))$$

as $t\to a$, $t\to b$ if $i=1$ and as $t\to a$ if $i=2$.

1.1.1. Theorems on the Unique Solvability of the Problems (1.1.1), $(1.1.2_i)$ $(i=1,2)$.

Theorem 1.1.1$_i$. *Let* $i\in\{1,2\}$,

$$p_2\in L_{\sigma_i(p_1)}([a,b]) \tag{1.1.5$_i$}$$

and let the constants α, $\beta\in[0,1]$ *connected by the inequality*

$$\alpha+\beta\le 1 \tag{1.1.6}$$

be such that

$$(p_0,p_1)\in \mathbb{V}_{i,\beta}(]a,b[\,;h), \tag{1.1.7$_i$}$$

where

$$h\in \mathcal{L}\Big(C_{x^\beta};L_{\frac{x^\alpha}{\sigma(p_1)}}\Big)\cap\mathcal{L}\big(C;L_{\sigma_i(p_1)}\big) \tag{1.1.8$_i$}$$

is a nonnegative operator and

$$x(t)=\int_a^t \sigma(p_1)(s)\,ds\bigg(\int_t^b \sigma(p_1)(s)\,ds\bigg)^{2-i} \quad \textit{for} \quad a\le t\le b. \tag{1.1.9$_i$}$$

Let, moreover, a continuous linear operator $g : C(]a,b[) \to L_{\sigma_i(p_1)}([a,b])$ *be such that for any function* $u \in C(]a,b[)$ *almost everywhere in the interval* $]a,b[$ *the inequality*

$$|g(u)(t)| \le h(|u|)(t) \tag{1.1.10}$$

is satisfied. Then the problem (1.1.1), $(1.1.2_i)$ *has one and only one solution.*

Theorem 1.1.1$_{i0}$. *Let* $i \in \{1,2\}$ *and let the constants* $\alpha \in [0,1[$, $\beta \in]0,1]$ *connected by the inequality* (1.1.6) *be such that*

$$p_2 \in L_{\frac{x^{1-\beta}}{\sigma(p_1)}}([a,b]) \tag{1.1.11}$$

and the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ *satisfy the inclusion* $(1.1.7_i)$, *where*

$$h \in \mathcal{L}\big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\big) \tag{1.1.12}$$

is a nonnegative operator and the function $x :]a,b[\to \mathbb{R}^+$ *is defined by the equality* $(1.1.9_i)$. *Let, moreover, a continuous linear operator* $g : C_{x^\beta}(]a,b[) \to L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b])$ *be such that for any function* $u \in C_{x^\beta}(]a,b[)$ *almost everywhere in the interval* $]a,b[$ *the inequality* (1.1.10) *is satisfied. Then the problem* (1.1.1), $(1.1.2_{i0})$ *has one and only one solution in the space* $C_{x^\beta}(]a,b[)$.

Remark $1.1.1_i$. Let $i \in \{1,2\}$ and all the requirements of Theorem $1.1.1_i$ be satisfied. Then for any function $v_0 \in C(]a,b[)$ there exists a unique sequence $v_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, v_n is a solution of the problem

$$\begin{gathered} v''(t) = p_0(t)v_1(t) + p_1(t)v'(t) + g(v_{n-1})(t) + p_2(t), \\ v(a) = c_1, \quad v^{i-1}(b-) = c_2, \end{gathered} \tag{1.1.13$_i$}$$

and uniformly on $]a,b[$

$$\lim_{n\to\infty}(v_n(t) - u(t)) = 0, \quad \lim_{n\to\infty}\sigma_i(p_1)(t)(v_n'(t) - u'(t)) = 0, \tag{1.1.14}$$

where u is a solution of the problem (1.1.1), $(1.1.2_i)$.

Remark $1.1.1_{i0}$. Let $i \in \{1,2\}$ and all the requirements of Theorem $1.1.1_{i0}$ be satisfied. Then for any function $v_0 \in C_{x^\beta}(]a,b[)$ there exists a unique sequence $v_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, v_n is a solution of the problem

$$\begin{gathered} v''(t) = p_0(t)v(t) + p_1(t)v'(t) + g(v_{n-1})(t) + p_2(t), \\ v(a) = 0, \quad v^{i-1}(b-) = 0, \end{gathered} \tag{1.1.13$_{i0}$}$$

and uniformly on $]a,b[$

$$\lim_{n\to\infty}\frac{v_n(t) - u(t)}{x^\beta(t)} = 0, \quad \lim_{n\to\infty}\frac{x^\alpha(t)}{\sigma(p_1)(t)}(v_n'(t) - u'(t)) = 0, \tag{1.1.15}$$

where u is a solution of the problem (1.1.1), ($1.1.2_{i0}$).

We can easily give examples of the operator h and the function p_1 such that $h \in \mathcal{L}(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}})$ and $h \notin \mathcal{L}(C; L_{\sigma_i(p_1)})$.

Example 1.1.1. Let $\varepsilon > 0$, $p_1(t) \equiv 0$, $h(u)(t) = [(b-t)(t-a)]^{-2-\varepsilon}$ for $a \le t \le b$ and let $\tau : [a,b] \to \{a,b\}$ be a measurable function.

Example 1.1.2. Let $a = -1$, $b = 1$, $\alpha = \beta = \frac{1}{5}$, $p_1(t) \equiv 0$ and $h(u)(t) = (1-t^2)^{-3} u(\tau(t))$, $\tau(t) = \sqrt{1-(1-t^2)^{10}}$ for $-1 \le t \le 1$. Then it is clear that

$$\sigma(p_1)(t) = 1, \quad x(t) = 1 - t^2, \quad x^{1/5}(\tau(t)) = (1-t^2)^2 \quad \text{for} \quad -1 \le t \le 1$$

and

$$\alpha + \beta < \frac{1}{2}.$$

In such a case if $u_1 \in C_{x^{\frac{1}{5}}}([-1,1])$ it follows from the inequality

$$|u_1(\tau(t))| \le \delta x^{1/5}(\tau(t)) \quad \text{for} \quad -1 \le t \le 1,$$

where

$$\delta = \sup\left\{ \left| \frac{u_1(\tau(t))}{x^{1/5}(\tau(t))} \right| : \ -1 < t < 1 \right\},$$

that

$$\int_{-1}^{1} x^\alpha(s) h(u_1)(s)\, ds \le \delta \int_{-1}^{1} (1-s^2)^{-4/5}\, ds < +\infty,$$

i.e., the condition ($1.1.11_i$) is satisfied.

Let now $u_2(t) \equiv 1$. Then $u_2 \in C(\,]-1,1[\,)$ and

$$\int_{-1}^{1} x(s) h(u_2)(s)\, ds = \int_{-1}^{1} (1-s^2)^{-2}\, ds,$$

i.e., owing to the fact that the last integral does not exist, the condition ($1.1.8_1$) is violated.

Consider the case where $p_0(t) \equiv 0$, $p_1(t) \equiv 0$, i.e., when the equation (1.1.1) has the form

$$u''(t) = g(u)(t) + p_2(t). \tag{1.1.16}$$

Then the following theorem is valid.

Theorem 1.1.2$_1$. *Let* $\gamma \in [0,1[$,

$$p_2 \in L_x([a,b]) \tag{1.1.17}$$

and

$$g \in \mathcal{L}(C; L_{x^\gamma}) \tag{1.1.18}$$

be a nonnegative operator, where

$$x(t) = (t-a)(t-b) \quad \textit{for} \quad a \le t \le b. \tag{1.1.19$_1$}$$

Let, moreover, there exist constants $\alpha,\ \beta \in [0, \frac{1}{2}]$ *such that*

$$0 \le \beta < 1 - \gamma, \tag{1.1.20}$$

$$\alpha + \beta \le \frac{1}{2} \tag{1.1.21}$$

and

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds < 2^\beta \, \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.1.22}$$

Then the problem (1.1.16), (1.1.2$_1$) *has one and only one solution.*

Remark 1.1.2. Theorem 1.2.2$_1$ will remain valid if we replace the conditions (1.1.20) and (1.1.22) respectively by

$$0 < \beta < 1 - \gamma, \tag{1.1.23}$$

and

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds \le 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.1.24$_1$}$$

Theorem 1.1.2$_2$. *Let* $\gamma \in [0, 1[$ *and let a function* p_2 *and a nonnegative operator* g *satisfy respectively the inclusions* (1.1.17) *and* (1.1.18), *where*

$$x(t) = t - a \quad \textit{for} \quad a \le t \le b. \tag{1.1.19$_2$}$$

Let, moreover, there exist constants $\alpha,\ \beta \in [0, \frac{1}{2}]$ *such that the conditions* (1.1.20), (1.1.21) *are fulfilled and*

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds \le \frac{8}{b-a} \Big(\frac{b-a}{4}\Big)^{\alpha+\beta}. \tag{1.1.24$_2$}$$

Then the problem (1.1.16), (1.1.2$_2$) *has one and only one solution.*

Theorem 1.1.2$_{i0}$. *Let* $i \in \{1, 2\}$, $\gamma \in [0, 1[$, $\delta \in]0, 1-\gamma[$,

$$p_2 \in L_{x^\gamma}([a, b]) \tag{1.1.25}$$

and let

$$g \in \mathcal{L}(C_{x^\delta}; L_{x^\gamma}) \tag{1.1.26}$$

be a nonnegative operator, where the function x is defined by the equality $(1.1.19_i)$. *Let, moreover, there exist constants* $\alpha \in [0, \frac{1}{2}]$, $\beta \in]0, \frac{1}{2}]$, *such that*

$$\delta \le \beta < 1 - \gamma \tag{1.1.27}$$

and the conditions (1.1.21), $(1.1.24_i)$ *are satisfied. Then the problem* (1.1.16), $(1.1.2_{i0})$ *has in the space* $C_{x^\delta}(]a,b[)$ *one and only one solution.*

Remark 1.1.3. The condition (1.1.22) is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^b x^\alpha(s) g(x^\beta)(s)\, ds < 2^\beta \frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)} + \varepsilon \tag{1.1.28}$$

with no matter how small $\varepsilon > 0$.

Indeed, let

$$\alpha = 0, \quad \beta = 0, \quad a = -\frac{1}{2}, \quad b = \frac{1}{2},$$

$$\lambda = \frac{\varepsilon}{4(16+\varepsilon)}, \quad \mu = 16\lambda\sqrt{1 + \frac{1}{(16+\varepsilon)^2}},$$

$$g_0(t) = \begin{cases} 64\mu^2(16\mu^2 - (1+4t)^2)^{-\frac{3}{2}} & \text{for } t \in \Big] -\frac{1}{4} - \lambda, -\frac{1}{4} + \lambda \Big[\\ 64\mu^2(16\mu^2 - (1-4t)^2)^{-\frac{3}{2}} & \text{for } t \in \Big] \frac{1}{4} - \lambda, \frac{1}{4} + \lambda \Big[\\ 0 \ \text{ for } \Big[-\frac{1}{2}, -\frac{1}{4} - \lambda \Big] \cup \Big[-\frac{1}{4} + \lambda, \frac{1}{4} - \lambda \Big] \cup \Big[\frac{1}{4} + \lambda, \frac{1}{2} \Big] \end{cases},$$

$$p_2(t) = 0, \quad \tau(t) = -\frac{4}{16+\varepsilon} \operatorname{sign} t \quad \text{for} \quad -\frac{1}{2} \le t \le \frac{1}{2},$$

and

$$g(u)(t) = g_0(t) u(\tau(t)).$$

Then the problem (1.1.16), $(1.1.2_{10})$ can be rewritten as

$$u''(t) = g_0(t) u(\tau(t)), \tag{1.1.29}$$

$$u\Big(-\frac{1}{2}\Big) = 0, \quad u\Big(\frac{1}{2}\Big) = 0. \tag{1.1.30}$$

Note that for the operator g defined in such a way the condition (1.1.18) is satisfied for $\gamma = 0$ and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g(1)(s)\, ds = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_0(s)\, ds = 16 + \varepsilon,$$

i.e., instead of (1.1.22) the condition (1.1.28) is satisfied. In spite of this fact we can check directly that the function

$$u(t)=c\Big[\int\limits_{-\frac{1}{2}}^{t}\int\limits_{-\frac{1}{2}}^{s} g_0(\eta)\,\mathrm{sign}(-\eta)d\eta\,ds-\Big(4+\frac{\varepsilon}{4}\Big)\Big(t+\frac{1}{2}\Big)\Big]$$

is for any $c\in\mathbb{R}$ a solution of the problem (1.1.29), (1.1.30), i.e., the unique solvability is violated.

1.1.2. Effective Sufficient Conditions for the Unique Solvability of the Problem (1.1.1), (1.1.2$_i$) ($i=1,2$).

Corollary 1.1.1$_1$. *Let the function x be defined by* (1.1.9$_1$), *the constants α, $\beta\in[0,1]$ be connected by* (1.1.6), *the functions $p_j:]a,b[\to\mathbb{R}$ ($j=0,1,2$) satisfy* (1.1.3$_1$), (1.1.5$_1$),

$$[p_0]_-\in L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b]) \tag{1.1.31}$$

and for every function $u\in C(]a,b[)$ almost everywhere on interval $]a,b[$ the inequality (1.1.10) *is satisfied, where a nonnegative operator h satisfies the inclusion* (1.1.8$_1$). *Let, moreover,*

$$\begin{aligned}\Big[\Big(\int\limits_t^b\sigma(p_1)(\eta)d\eta\Big)^{\alpha}\int\limits_a^t\frac{([p_0(s)]_-x^\beta(s)+h(x^\beta)(s))}{\sigma(p_1)(s)}\Big(\int\limits_a^s\sigma(p_1)(\eta)d\eta\Big)^{\alpha}ds+\\+\Big(\int\limits_a^t\sigma(p_1)(\eta)d\eta\Big)^{\alpha}\int\limits_t^b\frac{([p_0(s)]_-x^\beta(s)+h(x^\beta)(s))}{\sigma(p_1)(s)}\Big(\int\limits_s^b\sigma(p_1)(\eta)d\eta\Big)^{\alpha}ds\Big]<\\<\frac{4}{\int\limits_a^b\sigma(p_1)(\eta)d\eta}\Bigg(\frac{\int\limits_a^b\sigma(p_1)(\eta)d\eta}{2}\Bigg)^{2(\alpha+\beta)}\quad \text{for}\quad a\le t\le b\end{aligned} \tag{1.1.32$_1$}$$

Then the problem (1.1.1), (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.1$_2$. *Let the function x be defined by* (1.1.9$_2$), *the constants α, $\beta\in[0,1]$ be connected by* (1.1.6), *the functions $p_j:]a,b[\to\mathbb{R}$ ($j=0,1,2$) satisfy* (1.1.3$_2$), (1.1.5$_2$), (1.1.31) *and for every function $u\in C(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where a* *nonnegative operator h satisfies* (1.1.8$_2$). *Let, moreover,*

$$\int_a^t \frac{([p_0(s)]_- x^\beta(s) + h(x^\beta)(s))}{\sigma(p_1)(s)} \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds +$$
$$+\Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha \int_t^b \frac{([p_0(s)]_- x^\beta(s) + h(x^\beta)(s))}{\sigma(p_1)(s)} ds <$$
$$< \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \quad \text{for} \quad a \le t \le b. \tag{1.1.32$_2$}$$

Then the problem (1.1.1), (1.1.2$_2$) *has one and only one solution.*

Corollary 1.1.1$_{i0}$. *Let $i \in \{1,2\}$, the function x be defined by* (1.1.9$_i$), *the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* (1.1.3$_i$), (1.1.11), (1.1.31) *and for any function $u \in C_{x^\beta}(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* (1.1.12). *Let, moreover,* (1.1.32$_i$) *be satisfied. Then the problem* (1.1.1), (1.1.2$_{i0}$) *has in the space $C_{x^\beta}(]a,b[)$ one and only one solution.*

Remark 1.1.4. Corollary 1.1.1$_i$ remains valid if we replace the conditions (1.1.8$_i$) and (1.1.32$_i$) respectively by the conditions

$$h \in \mathcal{L}(C; L_{\sigma_i(p_1)}), \tag{1.1.33}$$

and

$$\int_a^b \frac{([p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s)h(x^\beta)(s))}{\sigma(p_1)(s)} ds <$$
$$< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\right)^{2(\alpha+\beta)} \tag{1.1.34$_1$}$$

for $i = 1$ or by

$$\int_a^b \frac{([p_0(s)]_- x^{\alpha+\beta}(s)+x^\alpha(s)h(x^\beta)(s))}{\sigma(p_1)(s)} ds < \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \tag{1.1.34$_2$}$$

for $i = 2$, where the function x is defined by (1.1.9$_i$).

Remark 1.1.4$_0$. Corollary 1.1.1$_{i0}$ remains valid if we replace (1.1.32$_i$) by (1.1.34$_i$) and reject the condition (1.1.12) at all.

Consider the case where the equation (1.1.1) has the form

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + \sum_{k=1}^{n} g_k(t)u(\tau_k(t)) + p_2(t). \quad (1.1.35)$$

Corollary 1.1.2$_1$. *Let the function x be defined by* (1.1.9$_1$), *the constants α, $\beta \in [0,1]$ be defined by the inequality* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy the conditions* (1.1.3$_1$), (1.1.5$_1$), (1.1.31), *$\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ be measurable functions and*

$$g_k x^{\beta}(\tau_k) \in L_{\frac{x^{\alpha}}{\sigma(p_1)}}([a,b]), \quad g_k \in L_{\sigma_1(p_1)}([a,b]) \quad (k = 1,\dots,n). \quad (1.1.36_1)$$

Let, moreover,

$$\Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha} \int_a^t \frac{([p_0(s)]_- x^{\beta}(s) + \sum\limits_{k=1}^{n} |g_k(s)| x^{\beta}(\tau_k(s)))}{\sigma(p_1)(s)} \times$$

$$\times \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^{\alpha} ds + \Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^{\alpha} \times$$

$$\times \int_t^b \frac{([p_0(s)]_- x^{\beta}(s) + \sum\limits_{k=1}^{n} |g_k(s)| x^{\beta}(\tau_k(s)))}{\sigma(p_1)(s)} \Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha} ds <$$

$$< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \Bigg(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\Bigg)^{2(\alpha+\beta)} \quad \text{for} \quad a \le t \le b. \quad (1.1.37_1)$$

Then the problem (1.1.35), (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.2$_2$. *Let the function x be defined by* (1.1.9$_2$), *the constants α, $\beta \in [0,1]$ be connected by* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* (1.13$_2$), (1.1.5$_2$), (1.1.31), *$\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ be measurable functions and*

$$g_k x^{\beta}(\tau_k) \in L_{\frac{x^{\alpha}}{\sigma(p_1)}}([0,b]), \quad g_k \in L_{\sigma_2(p_1)}([a,b]) \quad (k = 1,\dots,n). \quad (1.1.36_2)$$

Let, moreover,

$$\int_0^t \frac{[p_0(s)]_- x^{\beta}(s) + \sum\limits_{k=1}^{n} |g_k(s)| x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)} \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^{\alpha} ds +$$

$$+\Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^{\alpha}\int_t^b \frac{[p_0(s)]_-x^{\beta}(s)+\sum\limits_{k=1}^{n}|g_k(s)|x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)}\,ds<$$

$$<\Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \quad \textit{for} \quad a\le t\le b. \qquad (1.1.37_2)$$

Then the problem (1.1.35), $(1.1.2_2)$ *has one and only one solution.*

Corollary 1.1.2$_{i0}$. *Let* $i\in\{1,2\}$, *the function* x *be defined by* $(1.1.9_i)$, *the constants* $\alpha\in[0,1[$, $\beta\in]0,1]$ *be connected by the inequality* (1.1.6), *the functions* $p_j:]a,b[\to\mathbb{R}$ $(j=0,1,2)$ *satisfy the conditions* $(1.1.3_i)$, (1.1.11), (1.1.31), $\tau_k:[a,b]\to[a,b]$ $(k=1,\dots,n)$ *be measurable functions and*

$$g_kx^{\beta}(\tau_k)\in L_{\frac{x^{\alpha}}{\sigma(p_1)}}([a,b]) \quad (k=1,\dots,n). \qquad (1.1.38)$$

Let, moreover, the conditions $(1.1.37_i)$ *be satisfied. Then the problem* (1.1.35), $(1.1.2_{i0})$ *has in the space* $C_{x^{\beta}}(]a,b[)$ *one and only one solution.*

Remark 1.1.5. Corollary $1.1.2_i$ remains valid if we replace the conditions $(1.1.36_i)$ and $(1.1.37_i)$ respectively by the conditions

$$g_k\in L_{\sigma_i(p_1)}([a,b]) \quad (k=1,\dots,n) \qquad (1.1.39)$$

and

$$\int_a^b \frac{[p_0(s)]_-x^{\alpha+\beta}(s)+x^{\alpha}\sum\limits_{k=1}^{n}|g_k(s)|x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)}\,ds<$$

$$<\frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta}\left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\right)^{2(\alpha+\beta)} \qquad (1.1.40_1)$$

for $i=1$ or by

$$\int_a^b \frac{[p_0(s)]_-x^{\alpha+\beta}(s)+x^{\alpha}(s)\sum\limits_{k=1}^{n}|g_k(s)|x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)}\,ds<$$

$$<\Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \qquad (1.1.40_2)$$

for $i=2$, where the function x is defined by $(1.1.9_i)$.

Remark 1.1.5_0. Corollary 1.1.2_{i0} remains valid if we replace (1.1.37_i) by (1.1.40_i) and reject the condition (1.1.38) at all.

Corollary 1.1.3_1. *Let the function x be defined by* (1.1.9_1)*, the constants α, $\beta \in [0,1]$ be connected by* (1.1.6)*, the functions g_k, $p_j :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n;\ j = 0,1,2)$ satisfy* (1.1.3_1)*,* (1.1.5_1)*,* (1.1.36_1)*, where $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ are measurable functions and*

$$p_0(t) \geq 0 \quad for \quad a < t < b. \tag{1.1.41}$$

Let, moreover, for any $m \in \{1,\dots,n\}$ the condition

$$\sum_{k=1}^{n} \int_a^{\tau_m(t)} \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^{b} \sigma(p_1)(\eta)d\eta \Big)^{\beta} \times$$
$$\times \Big(\int_a^{s} \sigma(p_1)(\eta)d\eta \Big)^{\alpha} ds \Big(\int_{\tau_m(t)}^{b} \sigma(p_1)(\eta)d\eta \Big)^{\alpha} +$$
$$+ \sum_{k=1}^{n} \int_{\tau_m(t)}^{b} \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^{b} \sigma(p_1)(\eta)d\eta \Big)^{\beta} \times$$
$$\times \Big(\int_s^{b} \sigma(p_1)(\eta)d\eta \Big)^{\alpha} ds \Big(\int_a^{\tau_m(t)} \sigma(p_1)(\eta)d\eta \Big)^{\alpha} <$$
$$< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2} \right)^{2(\alpha+\beta)}, \quad a \leq t \leq b, \tag{1.1.42_1}$$

be valid. Then the problem (1.1.35)*,* (1.1.2_1) *has one and only one solution.*

Corollary 1.1.3_2. *Let the function x be defined by the equality* (1.1.9_2)*, the constants α, $\beta \in [0,1]$ be connected by* (1.1.6)*, the functions $g_k, p_j :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n;\ j = 0,1,2)$ satisfy the conditions* (1.1.3_2)*,* (1.1.5_2)*,* (1.1.36_2)*,* (1.1.41)*, where $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ are measurable functions. Let, moreover, for any $m \in \{1,\dots,n\}$ the condition*

$$\sum_{k=1}^{n} \int_a^{\tau_m(t)} \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \Big)^{\beta} \Big(\int_a^{s} \sigma(p_1)(\eta)d\eta \Big)^{\alpha} ds +$$
$$+ \sum_{k=1}^{n} \int_{\tau_m(t)}^{b} \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \Big)^{b} ds \Big(\int_a^{\tau_m(t)} \sigma(p_1)(s)\, ds \Big)^{\alpha} <$$

$$< \left(\int_a^b \sigma(p_1)(\eta)d\eta \right)^{\alpha+\beta-1}, \quad a \le t \le b, \tag{1.1.42$_2$}$$

be valid. Then the problem (1.1.35), (1.1.2$_2$) *has one and only one solution.*

Corollary 1.1.3$_{i0}$. *Let* $i \in \{1,2\}$, *the function* x *be defined by* (1.1.9$_i$), *the constants* $\alpha \in [0,1[$, $\beta \in]0,1]$ *be connected by* (1.1.6), *the functions* g_k, $p_j :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n;\ j = 0,1,2)$ *satisfy* (1.1.3$_i$), (1.1.11), (1.1.38), (1.1.41), *where* $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ *are measurable functions. Let, moreover, for any* $m \in \{1,\dots,n\}$ *the condition* (1.1.42$_i$) *be valid. Then the problem* (1.1.35), (1.1.2$_{i0}$) *has in the space* $C_{x^\beta}(]a,b[)$ *one and only one solution.*

Remark 1.1.6. The condition (1.1.42$_i$) consisting of n separate inequalities can be replaced by one inequality

$$\sum_{k=1}^n \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} \left(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^b \sigma(p_1)(\eta)d\eta \right)^\beta \times$$
$$\times \left(\int_a^s \sigma(p_1)(\eta)d\eta \right)^\alpha ds \left(\int_t^b \sigma(p_1)(\eta)d\eta \right)^\alpha +$$
$$+ \sum_{k=1}^n \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} \left(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^b \sigma(p_1)(\eta)d\eta \right)^\beta \times$$
$$\times \left(\int_s^b \sigma(p_1)(\eta)d\eta \right)^\alpha ds \left(\int_a^t \sigma(p_1)(\eta)d\eta \right)^\alpha < \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \times$$
$$\times \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2} \right)^{2(\alpha+\beta)} \quad \text{for} \quad t \in \Theta_{\tau_1,\dots,\tau_n} \tag{1.1.43$_1$}$$

if $i = 1$ and

$$\sum_{k=1}^n \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} \left(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \right)^\beta \left(\int_a^s \sigma(p_1)(\eta)d\eta \right)^\alpha ds +$$
$$+ \sum_{k=1}^n \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} \left(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \right)^\beta ds \left(\int_a^t \sigma(p_1)(\eta)d\eta \right)^\alpha <$$

$$< \Big(\int_a^b \sigma(p_1)(\eta) \Big)^{\alpha+\beta-1} \quad \text{for} \quad t \in \Theta_{\tau_1,\dots,\tau_n} \tag{1.1.43$_2$}$$

if $i = 2$, where

$$\Theta_{\tau_1,\dots,\tau_n} = \bigcup_{k=1}^{n} \big\{ \tau_k(t) | \;\; a \le t \le b \big\}.$$

For clearness we will give one corollary for the equation

$$u''(t) = g_0(t)u(\tau(t)) + p_2(t). \tag{1.1.44}$$

Corollary 1.1.4$_i$. *Let $i \in \{1,2\}$, the constants α, $\beta \in [0,1]$ be connected by the inequality* (1.1.6)*, $\tau : [a,b] \to [a,b]$ be a measurable function and*

$$p_2, g_0 \in L_x([a,b]), \tag{1.1.45}$$

where

$$x(t) = (a-t)(b-t)^{2-i} \quad \textit{for} \quad a \le t \le b. \tag{1.1.46}$$

Let, moreover,

$$\int_a^b |g(s)| \big[(\tau(s)-a)(b-\tau(s))^{2-i}\big]^{\beta} \big[(s-a)(b-s)^{2-i}\big]^{\alpha} ds <$$

$$< \Big(\frac{2}{i}\Big)^{2(1-\alpha-\beta)} (b-a)^{\frac{2}{i}(\alpha+\beta)-1}. \tag{1.1.47$_i$}$$

Then the problem (1.1.44), (1.1.2$_i$) *has one and only one solution.*

Corollary 1.1.4$_{i0}$. *Let $i \in \{1,2\}$, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6)*, $\tau : [a,b] \to [a,b]$ a be measurable function,*

$$p_2 \in L_{x^{1-\beta}}([a,b]), \tag{1.1.48}$$

where the function x is defined by (1.1.46)*. Let, moreover, the condition* (1.1.47$_i$) *be satisfied. Then the problem* (1.1.44), (1.1.2$_{i0}$) *has one and only one solution in the space $C_{x^\beta}(]a,b[)$.*

Remark 1.1.7. In the case of the equation

$$u''(t) = g_0(t)u(t) + p_2(t) \tag{1.1.49}$$

the conditions (1.32$_1$), (1.1.34$_1$), (1.1.40$_1$), (1.1.42$_1$), (1.1.47$_1$) will take for $\alpha = \beta = 0$ the form

$$\int_a^b |g_0(s)| \, ds < \frac{4}{b-a}.$$

As is known, this condition is unimprovable in the sense that no matter how small $\varepsilon > 0$ is, the inequality

$$\int_a^b |g_0(s)|\,ds \le \frac{4}{b-a} + \varepsilon$$

does not guarantee the unique solvability of the problem (1.1.49), $(1.1.2_1)$. This implies that the corollaries corresponding to the above conditions are unimprovable in the above-mentioned sense.

Corollary $1.1.5_1$. *Let the function x be defined by $(1.1.9_1)$, the constants $\alpha, \beta \in [0,1]$ be connected by the inequality (1.1.6), the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy the conditions $(1.1.3_1)$, $(1.1.5_1)$ and for any function $u \in C(]a,b[)$ almost everywhere in the interval $]a,b[$ (1.1.10) be satisfied, where the nonnegative operator h satisfies the inclusion $(1.1.8_1)$. Let, moreover, in case $\beta < 1$,*

$$\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x^\beta)(t)}{x^\beta(t)} - p_0(t)\Big) \le 2\beta^2 \quad \textit{for} \quad a < t < b, \qquad (1.1.50_1)$$

and in case $\beta = 1$,

$$\operatorname*{ess\,sup}_{t\in]a,b[} \Big[\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x)(t)}{x(t)} - p_0(t)\Big)\Big] < 2 \qquad (1.1.51_1)$$

be satisfied. Then the problem (1.1.1), $(1.1.2_1)$ has one and only one solution.

Remark 1.1.8. The condition (1.1.51) is unimprovable in the sense that the validity of Corollary $1.1.5_1$ is violated if we replace it by the condition

$$\operatorname*{ess\,sup}_{t\in]a,b[} \Big[\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x)(t)}{x(t)} - p_0(t)\Big)\Big] \le 2\beta^2. \qquad (1.1.52)$$

Indeed, let $h(u) \equiv 0$, $p_1 \equiv 0$, $p_2 \equiv 0$. Then

$$\sigma(p_1)(t) = 1 \ \text{ and } \ x(t) = (b-t)(t-a) \ \text{ for } \ a \le t \le b$$

and the condition (1.1.52) will take the form

$$\operatorname*{ess\,sup}_{t\in]a,b[} \big(-(b-t)(t-a)p_0(t)\big) \le 2. \qquad (1.1.53)$$

If

$$p_0(t) = -\frac{2}{(b-t)(t-a)},$$

then the condition (1.1.53) is satisfied in the form of the equality, and at the same time, for any $c \in \mathbb{R}$ the function $c(b-t)(t-a)$ is a solution of the equation

$$u''(t) = -\frac{2}{(b-t)(t-a)}u(t), \qquad (1.1.54)$$

that is, the uniqueness of solution of the problem (1.1.54), ($1.1.2_{i0}$) is violated although the condition (1.1.52) along with the other requirements of Corollary $1.1.5_1$ is satisfied.

Corollary $1.1.5_2$. *Let the function x be defined by* ($1.1.9_2$)*, the constants α, $\beta \in [0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* ($1.1.3_2$), ($1.1.5_2$) *and for any function $u \in C(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where a nonnegative operator h satisfies the inclusion* ($1.1.8_2$)*. Let, moreover,*

$$\operatorname*{ess\,sup}_{t\in]a,b[} \left[\frac{x^{2-[\beta]}(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x^\beta)(t)}{x^\beta(t)} - p_0(t)\Big)\right] < \beta(1-\beta), \qquad (1.1.50_2)$$

$$\frac{x^{2-\beta}}{\sigma^2(p_1)}[p_0]_- \in L_\infty([a,b]) \qquad (1.1.55)$$

if $0 < \beta \le 1$ and

$$0 \le p_0(t) - h(1)(t) \quad for \quad a < t < b \qquad (1.1.51_2)$$

if $\beta = 0$ be satisfied. Then the problem (1.1.1), ($1.1.2_2$) *has one and only one solution.*

Remark 1.1.9. In the case $\beta = 1$, the condition (1.1.55) follows automatically from the condition ($1.1.50_2$).

Corollary $1.1.5_{10}$. *Let the function x be defined by* ($1.1.9_1$)*, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* ($1.1.3_1$), (1.1.11) *and for any function $u \in C_{x^\beta}(]a,b[)$ almost everywhere on the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* (1.1.12)*. Let, moreover, in case $0 < \beta < 1$ the condition* ($1.1.50_1$) *and in case $\beta = 1$ the condition* ($1.1.51_1$) *be satisfied. Then the problem* (1.1.1), ($1.1.2_{10}$) *has in the space $C_{x^\beta}(]a,b[)$ one and only one solution.*

Corollary $1.1.5_{20}$. *Let the function x be defined by* ($1.1.9_2$)*, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* ($1.1.3_2$), (1.1.11) *and for any function $u \in C_{x^\beta}(]a,b[)$ almost everywhere on the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* (1.1.12)*. Let, moreover, the conditions* ($1.1.50_2$) *and* (1.1.55) *be satisfied. Then the problem* (1.1.1), ($1.1.2_{20}$) *has one and only one solution in the space $C_{x^\beta}(]a,b[)$.*

Corollary $1.1.6_1$. *Let the functions $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ be measurable and the functions p_j, $p_k \in L_{\text{loc}}(]a,b[)$ $(k = 1,\dots,n;\ j = 0,1,2)$ as well as the constants $\lambda_{l,m} \in]0,+\infty[$, $\beta_m \in [0,1]$ $(l,m = 1,2)$, $c \in]a,b[$ be such that the conditions* ($1.1.3_1$), ($1.1.5_1$) *are satisfied,*

$$g_k \in L_{\sigma_1(p_1)}([a,b]) \qquad (1.1.56_1)$$

and

$$\int\limits_0^{+\infty} \frac{ds}{\lambda_{11} + \lambda_{12}s + s^2} > \frac{(c-a)^{1-\beta_1}}{1-\beta_1},$$
$$\int\limits_0^{+\infty} \frac{ds}{\lambda_{21} + \lambda_{22}s + s^2} > \frac{(b-c)^{1-\beta_2}}{1-\beta_2}. \tag{1.1.57$_1$}$$

Let, moreover,

$$(t-a)^{2\beta_2}\Big[p_0(t) - \sum_{k=1}^n |g_k(t)|\Big] \geq -\lambda_{11},$$
$$(t-a)^{\beta_1}\Big[p_1(t) + \frac{\beta_1}{t-a} - \sum_{k=1}^n |g_k(t)|(\tau_k(t)-t)\Big] \geq -\lambda_{12}$$
$$\text{for} \quad a < t < c,$$
$$(b-t)^{2\beta_2}\Big[p_0(t) - \sum_{k=1}^n |g_k(t)|\Big] \geq -\lambda_{12},$$
$$(b-t)^{\beta_2}\Big[p_1(t) - \frac{\beta_2}{b-t} - \sum_{k=1}^n |g_k(t)|(\tau_k(t)-t)\Big] \leq \lambda_{22}$$
$$\text{for} \quad c \leq t < b. \tag{1.1.58$_1$}$$

Then the problem (1.1.35), (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.6$_2$. *Let the functions* $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ *be measurable and the functions* $\widetilde{p_1}$, p_j, $g_k \in L_{\text{loc}}(]a,b])$ $(k = 1,\dots,n;\ j = 0,1,2)$ *as well as the constants* $\lambda_{l,m} \in]0,+\infty[$, $(l,m = 1,2)$, $\beta_r \in [0,1]$ $(r = 1,2,3)$, $c \in]\max(a, b-1); b]$, $\varepsilon > 0$ *and the dependent on them constant* $\alpha \in [0,1[$ *be such that the conditions*

$$\sigma(\widetilde{p_1}) \in L([a,b]), \quad p_j\sigma_2(\widetilde{p_1}) \in L([a,b]) \quad (j = 0,2),$$
$$g_k\sigma_2(\widetilde{p_1}) \in L([a,b]) \quad (k = 1,\dots,n) \tag{1.1.56$_2$}$$

and

$$\int\limits_\varepsilon^{+\infty} \frac{ds}{\lambda_{11} + \lambda_{12}s + s^2} > \frac{(c-a)^{1-\beta_1}}{1-\beta_1},$$
$$\int\limits_0^{+\infty} \frac{ds}{\lambda_{21} + \lambda_{22}s + s^2} > \frac{(b-c)^{1-\beta_2}}{1-\beta_2} \tag{1.1.57$_2$}$$

are satisfied. Let, moreover,

$$
\begin{gathered}
(t-a)^{2\beta_2}\Big[p_0(t)-\sum_{k=1}^{n}|g_k(t)|\Big]\geq -\lambda_{11},\\
(t-a)^{\beta_1}\Big[\widetilde{p_1}(t)+\frac{\beta_1}{t-a}-\sum_{k=1}^{n}|g_k(t)|(\tau_k(t)-t)\Big]\geq -\lambda_{12}\\
\text{for}\quad a<t<c,\\
(b-t)^{\beta_2-\beta_3}\Big[p_0(t)-\sum_{k=1}^{n}|g_k(t)|\Big]\geq -\alpha\lambda_{21},\\
(b-t)^{\beta_2}\Big[\widetilde{p_1}(t)+\frac{\beta_3}{b-t}-\sum_{k=1}^{n}|g_k(t)|(\tau_k(t)-t)\Big]\geq \lambda_{22}\\
\text{for}\quad c\leq t<b.
\end{gathered}
\tag{1.1.58$_2$}
$$

Then for any function $p_1\in L_{\mathrm{loc}}(]a,b])$ *such that*

$$p_1(t)\geq \widetilde{p_1}(t)\quad \text{for}\quad a<t<b, \tag{1.1.59}$$

the problem (1.1.35), (1.1.2$_2$) *has one and only one solution.*

Consider now corollaries of Theorems 1.1.2$_i$ and 1.1.2$_{i0}$ for the equation

$$u''(t)=\sum_{k=1}^{n}g_k(t)u(\tau_k(t))+p_2(t). \tag{1.1.60}$$

Corollary 1.1.7$_1$. *Let* $\gamma\in[0,1[$, *the function* $p_2:]a,b[\to\mathbb{R}$ *satisfy the inclusion* (1.1.17),

$$g_k\in L_{x^\gamma}([a,b])\quad (k=1,\dots,n) \tag{1.1.61}$$

and

$$g_k(t)\geq 0\quad (k=1,\dots,n)\quad \text{for}\quad a<t<b, \tag{1.1.62}$$

where

$$x(t)=(b-t)(t-a)\quad a\leq t\leq b. \tag{1.1.63$_1$}$$

Let, moreover, there exist constants α, $\beta\in[0,\frac{1}{2}]$ *such that*

$$0\leq\beta<1-\gamma,\quad \alpha+\beta\leq\frac{1}{2} \tag{1.1.64}$$

and

$$
\begin{gathered}
\sum_{k=1}^{n}\int_a^b g_k(s)(b-\tau_k(s))^{\beta}(\tau_k(s)-a)^{\beta}(b-s)^{\alpha}(s-a)^{\alpha}\,ds<\\
<2^{\beta}\frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}.
\end{gathered}
\tag{1.1.65}
$$

Then the problem (1.1.60), $(1.1.2_1)$ *has one and only one solution.*

Remark 1.1.10. Corollary $1.1.7_1$ remains valid if for $\beta \in]0, 1-\gamma[$ we replace the condition (1.1.65) by the following one:

$$\sum_{k=1}^{n} \int_a^b g_k(s)(b-\tau_k(s))^\beta (\tau_k(s)-a)^\beta (b-s)^\alpha (s-a)^\alpha \, ds \le$$
$$\le 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.1.66$_1$}$$

Corollary 1.1.7$_2$. *Let* $\gamma \in [0,1[$, *the functions* p_2, $p_k :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n)$ *satisfy the conditions* (1.1.17), (1.1.61), *and* (1.1.62), *where*

$$x(t) = t-a \quad for \quad a \le t \le b. \tag{1.1.63$_2$}$$

Let, moreover, there exist constants α, $\beta \in [0, \frac{1}{2}]$ *such that the conditions* (1.1.64) *and*

$$\sum_{k=1}^{n} \int_a^b g_k(s)(\tau_k(s)-a)^\beta (s-a)^\beta \, ds \le \frac{8}{b-a}\Big(\frac{b-a}{4}\Big)^{\alpha+\beta} \tag{1.1.66$_2$}$$

are satisfied. Then the problem (1.1.60), $(1.1.2_2)$ *has one and only one solution.*

Corollary 1.1.7$_{i0}$. *Let* $i \in \{1,2\}$, $\gamma \in [0,1[$, $\delta \in]0, 1-\gamma[$,

$$p_2 \in L_{x^\gamma}([a,b]), \;\; g_k x^\delta(\tau_k) \in L_{x^\gamma}([a,b]) \;\; (k=1,\dots,n),$$

and the condition (1.1.62) *be satisfied, where the function* x *is defined by* $(1.1.63_i)$. *Let, moreover, there exist constants* $\alpha \in [0, \frac{1}{2}]$, $\beta \in]0, \frac{1}{2}]$ *such that the conditions*

$$\delta \le \beta < 1-\gamma, \quad \alpha+\beta \le \frac{1}{2}$$

and $(1.1.66_i)$ *are satisfied. Then the problem* (1.1.60), $(1.1.2_{i0})$ *has in the space* $C_{x^\delta}(]a,b[)$ *one and only one solution.*

§ 1.2. Auxiliary Propositions

1.2.1. Statement of Auxiliary Problems and Some of Their Properties. Let us consider the linear equations

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(v)(t) + p_2(t), \tag{1.2.1}$$
$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(v)(t) \tag{1.2.1$_0$}$$

under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2, \tag{1.2.2$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{1.2.2$_2$}$$

as well as under the conditions

$$v(a) = 0, \quad v(b) = 0, \tag{1.2.2$_{10}$}$$

$$v(a) = 0, \quad v'(b-) = 0, \tag{1.2.2$_{20}$}$$

where $c_1,\ c_2 \in \mathbb{R}$ and $h : C(\,]a,b[\,) \to L_{\mathrm{loc}}(]a,b[)$ is a continuous linear operator and

$$p_j \in L_{\mathrm{loc}}(]a,b[)\ (j = 0,1,2),\ \ \sigma(p_1) \in L([a,b]),\ \ p_0 \in L_{\sigma_1(p_1)}([a,b]) \tag{1.2.3$_1$}$$

or

$$p_j \in L_{\mathrm{loc}}(]a,b])\ (j = 0,1,2),\ \ \sigma(p_1) \in L([a,b]),\ \ p_0 \in L_{\sigma_2(p_1)}([a,b]). \tag{1.2.3$_2$}$$

For this purpose we will need the homogeneous equation

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) \tag{1.2.4}$$

under the initial conditions

$$v(a) = 0, \quad \lim_{t\to a} \frac{v'(t)}{\sigma(p_1)(t)} = 1, \tag{1.2.5}$$

$$v(b) = 0, \quad \lim_{t\to b} \frac{v'(t)}{\sigma(p_1)(t)} = -1, \tag{1.2.5$_1$}$$

or

$$v(b) = 1, \quad v'(b-) = 0. \tag{1.2.5$_2$}$$

The facts mentioned in the remarks below or their analogues have been proved in [23], pp. 110–158.

Remark 1.2.1. Let measurable functions $p_0,\ p_1\ :]a,b[\to \mathbb{R}$ satisfy the conditions (1.2.3$_1$) and the functions v_1 and v_2 be respectively solutions of the problems (1.2.4), (1.2.5) and (1.2.4), (1.2.5$_1$). Then any linearly independent with v_j, $(j = 1,2)$ solution $\widetilde{v}$ of the equation (1.2.4) satisfies the condition

$$\widetilde{v}(a) \neq 0 \quad \text{for} \quad j = 1$$

and

$$\widetilde{v}(b) \neq 0 \quad \text{for} \quad j = 2.$$

Remark 1.2.2. Let $i \in \{1,2\}$ and

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a,b[). \tag{1.2.6$_i$}$$

Then the problem (1.2.4), (1.2.2$_{i0}$) has only the trivial solution and the unique Green's function G can be represented as:

$$G(t,s) = \begin{cases} -\dfrac{v_2(t)v_1(s)}{v_2(a)\sigma(p_1)(s)} & \text{for } a \le s < t \le b, \\ -\dfrac{v_2(s)v_1(t)}{v_2(a)\sigma(p_1)(s)} & \text{for } a \le t < s \le b, \end{cases} \tag{1.2.7}$$

where v_1 and v_2 are respectively the solutions of the problems (1.2.4), (1.2.5) and (1.2.4), (1.2.5$_i$), and

$$G(t,s) < 0 \quad \text{for} \quad (t,s) \in]a,b[\times]a,b[\,, \tag{1.2.8}$$

$$G(a,s) = 0, \quad G(b,s) = i-1 \quad \text{for} \quad a \le s \le b. \tag{1.2.9$_i$}$$

Remark 1.2.3. Let $i \in \{1,2\}$ and the inclusion (1.2.6$_i$) be satisfied. Then there exist constants c_*, $d_* \in \mathbb{R}^+$ such that the estimates

$$d_* \le \frac{v_1(t)}{\int\limits_a^t \sigma(p_1)(s)\,ds} \le c_*, \quad d_* \le \frac{v_2(t)}{(\int\limits_t^b \sigma(p_1)(s)\,ds)^{2-i}} \le c_* \tag{1.2.10$_i$}$$
$$\text{for} \quad a < t < b,$$

$$\frac{|v_1'(t)|}{\sigma(p_1)(t)} \le 1 + c_* \int\limits_a^t |p_0(s)|\sigma_2(p_1)(s)\,ds,$$

$$\frac{|v_2'(t)|}{\sigma(p_1)(t)} \le 2 - i + c_* \int\limits_t^b \frac{|p_0(s)|}{\sigma(p_1)(s)} \bigg(\int\limits_s^b \sigma(p_1)(\eta)\,d\eta \bigg)^{2-i} ds \tag{1.2.11$_i$}$$
$$\text{for} \quad a \le t < b$$

are valid, where v_1 and v_2 are respectively the solutions of the problems (1.2.4), (1.2.5) and (1.2.4), (1.2.5$_i$), and

$$\left|\frac{\partial^{j-1}G(t,s)}{\partial t^{j-1}}\right| \le$$
$$\le c_* \frac{\sigma_i(p_1)(s)}{[\sigma_i(p_1)(t)]^{j-1}} \ (j=1,2) \ \text{for} \ (t,s) \in]a,b[\times]a,b[\ \ (t \ne s). \tag{1.2.12$_i$}$$

Remark 1.2.4. Let $i \in \{1,2\}$, the conditions (1.2.3$_i$) be satisfied and the problem (1.2.4), (1.2.2$_i$) have lower w_1 and upper w_2 functions such that

$$w_1(t) \le w_2(t) \quad \text{for} \quad a \le t \le b.$$

Then the problem (1.2.4), (1.2.2$_i$) has at least one solution v such that

$$w_1(t) \le v(t) \le w_2(t) \quad \text{for} \quad a \le t \le b.$$

Remark 1.2.5. Let $i \in \{1,2\}$ and the inclusion $(1.2.6_i)$ be satisfied. Then every upper function w of the problem (1.2.4), $(1.2.2_{i0})$ is nonnegative in the interval $]a,b[$; moreover, if

$$w(a) + w^{(i-1)}(b-) \neq 0,$$

then w is positive on the interval $]a,b[$.

Remark 1.2.6. Let $i \in \{1,2\}$, the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy the conditions $(1.2.3_i)$ and

$$p_0(t) \geq 0 \quad \text{for} \quad a < t < b.$$

Then the inclusion $(1.2.6_i)$ is valid.

Lemma 1.2.1. *Let* $i \in \{1,2\}$ *and*

$$h \in \mathcal{L}(C; L_{\sigma_i(p_1)}) \tag{1.2.13$_i$}$$

where h *is a nonnegative operator. Then*

$$\mathbb{V}_{i,0}(]a,b[;h) \subset \mathbb{V}_{i,0}(]a,b[).$$

Proof. Let $(p_0,p_1) \in \mathbb{V}_{i,0}(]a,b[;h)$. Then the problem $(1.2.1_0)$, $(1.2.2_{i0})$ has a positive upper function w which because of the nonnegativeness of the operator h will at the same time be an upper function of the problem (1.2.4), $(1.2.2_{i0})$.

Consider first the case $i = 1$. For the equation (1.2.4) we pose the problem

$$v(a) = 0, \quad v(b) = w(b), \tag{1.2.14}$$

for which $\beta(t) \equiv 0$ and w are respectively lower and upper functions. Then by virtue of Remark 1.2.4, the problem (1.2.4), (1.2.14) has a solution v_0 such that

$$0 \leq v_0(t) \leq w(t) \quad \text{for} \quad a \leq t \leq b.$$

If we assume that $v_0(t_0) = 0$ for some $t_0 \in]a,b[$, then we will get the contradiction with the unique solvability of the Cauchy problem, i.e.,

$$v_0(t) > 0 \quad \text{for} \quad a < t \leq b. \tag{1.2.15}$$

As is seen from Remark 1.2.1 and the conditions (1.2.14) that v_1 a solution of the problem (1.2.4), $(1.2.5_1)$, and v_0 are linearly dependent, hence by virtue of (1.2.15),

$$v_1(t) > 0 \quad \text{for} \quad a < t \leq b,$$

i.e., as is seen from Definition 1.1.2, $(p_0,p_1) \in \mathbb{V}_{1,0}(]a,b[)$.

Let now $i = 2$, and for the equation (1.2.4) we pose the initial problem

$$v(b) = 0, \quad v'(b-) = -1$$

which, with regard for the conditions ($1.2.3_2$), has a unique solution $\widetilde{v}$ defined on the whole interval $[a,b]$. Then we choose $\varepsilon > 0$ such that the inequality

$$\varepsilon v(t) < w(t) \quad \text{for} \quad a < t < b \tag{1.2.16}$$

is satisfied; this is possible because the function w is positive. It is clear from (1.2.16) that

$$w_1(t) = w(t) - \varepsilon v(t)$$

is an upper function of the problem (1.2.4), ($1.2.2_{20}$) and

$$w_1'(b-) > 0, \quad w_1(t) > 0 \quad \text{for} \quad a \le t \le b.$$

We consider now for the equation (1.2.4) the problem

$$v(a) = 0, \quad v'(b-) = w_1'(b-), \tag{1.2.17}$$

for which $\beta(t) \equiv 0$ and w_1 are respectively lower and upper functions. Hence by virtue of Remark 1.2.4, the problem (1.2.4), (1.2.17) has a solution v_0 such that

$$0 \le v_0(t) \le w_1(t) \quad \text{for} \quad a < t < b$$

and

$$v_0(a) = 0, \quad v_0(b) > 0, \quad v_0'(b-) > 0.$$

Reasoning in the same way as for $i = 1$, we see that $(p_0, p_1) \in \mathbb{V}_{2,0}(]a,b[)$. ☐

Along with Lemma 1.2.1 we have proved the following

Lemma 1.2.2. *Let $i \in \{1,2\}$, the functions p_0, p_1 $:]a,b[\to \mathbb{R}$ satisfy the conditions ($1.2.3_i$) and, moreover, let the problem (1.2.4), ($1.2.2_{i0}$) have a positive upper function. Then the inclusion ($1.2.6_i$) is satisfied.*

Lemma 1.2.3. *Let $i \in \{1,2\}$, the functions p_0, p_1 $:]a,b[\to \mathbb{R}$ satisfy the inclusion ($1.2.6_i$) and the nonnegative operator h satisfy the inclusion ($1.2.13_i$). Let, moreover, $\rho_0 \in C(]a,b[)$ such that*

$$\rho_0(t) > 0 \quad \textit{for} \quad a < t < b \tag{1.2.18}$$

and

$$\sup\left\{\frac{1}{\rho_0(t)}\int_a^b |G(t,s)|h(\rho_0)(s)ds: \ a < t < b\right\} < 1, \tag{1.2.19}$$

where G is Green's function of the problem (1.2.4), ($1.2.2_{i0}$). Then there exists a continuous function $\rho : [a,b] \to \mathbb{R}^+$ such that

$$\max\left\{\frac{1}{\rho(t)}\int_a^b |G(t,s)|h(\rho)(s)ds: \ a \le t \le b\right\} < 1. \tag{1.2.20}$$

Proof. First of all we note that the existence of Green's function G of the problem (1.2.4), ($1.2.2_{i0}$) follows from Remark 1.2.2, and the boundedness of the integrals in the inequalities (1.2.19) and (1.2.20) for any continuous function ρ follows from the estimates ($1.2.12_i$) and the inclusion ($1.2.13_i$).

Consider now separately the case $i = 2$. By virtue of the equalities ($1.2.9_2$), the inequality (1.2.19) can be satisfied only under the conditions

$$\rho_0(a) \geq 0, \quad \rho_0(b) > 0. \tag{1.2.21}$$

Then (1.2.19) can be rewritten as

$$\int_a^b |G(t,s)|h(\rho_0)(s)ds < \rho_0(t) \quad \text{for} \quad a < t \leq b. \tag{1.2.22}$$

As is seen from the equalities ($1.2.9_2$), there exist positive constants r_1 and δ such that

$$\int_a^b |G(t,s)|h(1)(s)ds - 1 < 0 \quad \text{for} \quad a \leq t \leq a + \delta \tag{1.2.23}$$

and

$$\int_a^b |G(t,s)|h(1)(s)ds - 1 < r_1 \quad \text{for} \quad a \leq t \leq b. \tag{1.2.24}$$

On the other hand, from (1.2.22) it follows the existence of a constant $r_2 > 0$ such that

$$r_2 < \rho_0(t) - \int_a^b |G(t,s)|h(\rho_0)(s)\,ds \quad \text{for} \quad a + \delta \leq t \leq b. \tag{1.2.25}$$

Then from (1.2.22)–(1.2.25) we obtain

$$\frac{r_2}{r_1}\left(\int_a^b |G(t,s)|h(1)(s)\,ds - 1\right) \leq \rho_0(t) - \int_a^b |G(t,s)|h(\rho_0)(s)\,ds \quad \text{for} \quad a \leq t \leq b,$$

which implies the validity of the inequality (1.2.20) for the function $\rho(t) = \varepsilon + \rho_0(t)$, where $\varepsilon = \frac{r_2}{r_1}$.

To complete the proof of the lemma we note that for $i = 1$, unlike the case $i = 2$, the inequality (1.2.19) by virtue of ($1.2.9_i$) can be satisfied also for

$$\rho(a) > 0, \quad \rho(b) \geq 0$$

and for

$$\rho(a) \geq 0, \quad \rho(b) \geq 0$$

as well.

In these cases the above lemma can be proved similarly to the case of the conditions (1.2.21) with the only difference that the inequality (1.2.22) will be valid for $t \in [a, b[$ or $t \in]a, b[$, the inequality (1.2.23) for $t \in [b-\delta, b]$ or $t \in [a+\delta; b-\delta]$, and the inequality (1.2.25) will be considered for $t \in [a, b-\delta[$ or $t \in]a+\delta,\, b-\delta[$. $\square$

Lemma 1.2.4. *Let $i \in \{1, 2\}$,*

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a, b[; h), \qquad (1.2.26_i)$$

where the nonnegative operator h satisfies the inclusion $(1.2.13_i)$. *Then there exists a continuous function $\rho : [a, b] \to \mathbb{R}^+$ such that the inequality* (1.2.20) *holds, where G is Green's function of the problem* (1.2.4), $(1.2.2_{i0})$.

Proof. As is seen from the definition of the set $\mathbb{V}_{i,0}(\,]a, b[\,; h)$, the problem $(1.2.1_0)$, $(1.2.2_{i0})$ has on the interval $[a, b]$ a positive upper function w. Then we introduce a continuous operator $\chi : C(]a, b[) \to C(]a, b[)$ by the equality

$$\chi(y)(t) = \frac{1}{2}\Big[|y(x)| - |w(t) - y(t)| + w(t)\Big] \quad \text{for} \quad a \le t \le b \qquad (1.2.27)$$

which for any $v \in C(]a, b[)$ satisfies

$$0 \le \chi(v)(t) \le w(t) \quad \text{for} \quad a \le t \le b, \qquad (1.2.28)$$

and consider the problem

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(\chi(v))(t), \qquad (1.2.29)$$

$$v(a) = w(a), \quad v^{(i-1)}(b-) = w^{(i-1)}(b-). \qquad (1.2.30_i)$$

Note that from Lemma 1.2.1 and Remark 1.2.2 it follows the existence of Green's function of the problem (1.2.4), $(1.2.2_i)$. Introduce the operator $H : C(]a, b[) \to C(]a, b[)$ by the equality

$$H(g)(t) = v_0(t) + \int_a^b |G(t, s)| h(\chi(y))(s)\, ds,$$

where v_0 is a solution of the problem (1.2.4), $(1.2.30_i)$, and consider the equation

$$v(t) = H(v)(t) \qquad (1.2.31)$$

which is equivalent to the problem (1.2.29), $(1.2.30_i)$. Let us show that the operator H is compact. Let c_* be a constant mentioned in Remark 1.2.3,

$$r = c_* \int_a^b \sigma_i(p_1)(s) h(w)(s) ds,$$

$$\mathbb{B}_r = \big\{z \in C(]a, b[) : \ \|z - v_0\|_C \le r\big\},$$

and $(x_n)_{n=1}^{\infty}$ be any sequence from $\mathbb{B}_r$. Then from the estimate $(1.2.12_i)$ for the sequence $y_n(t) = H(x_n)(t)$, $n \in \mathbb{N}$, we have

$$\|v_0 - y_n\|_C \le r, \quad n \in \mathbb{N}. \tag{1.2.32}$$

Consider separately the case $i = 1$. By virtue of $(1.2.9_1)$, (1.2.28) and the fact that the function v_0 is continuous, for any constant $\varepsilon > 0$ there exist $a_1, b_1 \in]a, b[$, $a_1 < b_1$ such that

$$\max\big\{|v_0(t_1) - v_0(t_2)| : \ a \le t_1 \le t_2 \le a_1,\, b_1 \le t_1 \le t_2 \le b\big\} \le \frac{\varepsilon}{4}$$

and

$$\varepsilon^* \equiv \max\left\{\int_a^b |G(t,s)|h(\chi(x_n))(s)\,ds : \ a \le t \le a_1,\, b_1 \le t \le b\right\} \le \frac{\varepsilon}{8}.$$

Then for any $n \in \mathbb{N}$ the estimate

$$|y_n(t_1) - y_n(t_2)| \le \frac{\varepsilon}{4} + 2\varepsilon^* \le \frac{\varepsilon}{2},$$
$$\text{for} \quad a \le t_1 \le t_2 \le a_1, \ \ b_1 \le t_1 \le t_2 \le b,$$

is valid.

In the same way, by virtue of the estimates $(1.2.12_i)$, there exists a constant δ, $0 < \delta < \min(a_1 - a,\, b - b_1)$, such that for any $n \in \mathbb{N}$

$$|y_n(t_1) - y_n(t_2)| \le$$
$$\le (1 + r)\max\big\{|v_0'(t)| + \sigma_1^{-1}(p_1)(t) : \ a_1 - \delta < t < b + \delta\big\}|t_2 - t_1| \le \frac{\varepsilon}{2}$$
$$\text{for} \quad |t_1 - t_2| \le \delta, \quad a_1 - \delta \le t_j \le b_1 + \delta \ \ (j = 1, 2).$$

It follows from the last two estimates that if $t_j \in [a, b]$ $(j = 1, 2)$ and

$$|t_1 - t_2| \le \delta,$$

then

$$|y_n(t_1) - y_n(t_2)| \le \varepsilon, \quad n \in \mathbb{N}.$$

From this and from the inequality (1.2.32) we obtain that the sequence $(y_n)_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In case $i = 2$, the same follows from the possibility to choose for any $\varepsilon > 0$, owing to $(1.1.9_2)$, (1.2.28), $a_1 \in]a, b[$ and $0 < \delta < a_1 - a$ such that

$$\max\big\{|v_0(t_1) - v_0(t_2)| : \ a \le t_1 \le t_2 \le a_1\big\} \le \frac{\varepsilon}{4},$$
$$\max\left\{\int_a^b |G(t,s)|h(w)(s)\,ds : \ a \le t \le a_1\right\} \le \frac{\varepsilon}{4},$$

and

$$|y_n(t_1) - y_n(t_2)| \le$$
$$\le (1+r)\max\big\{|v_0'(t)| + \sigma_2^{-1}(p_1)(t): \ \ a_1 - \delta \le t \le b\big\}|t_1 - t_2| \le \frac{\varepsilon}{2}$$
$$\text{for} \quad |t_1 - t_2| \le \delta, \ \ a_1 - \delta \le t_j \le b \ \ (j = 1, 2).$$

Then according to the Arzella–Ascoli lemma, the operator H which is, as it is not difficult to show, continuous, transforms the ball $\mathbb{B}_r$ into its compact subset. In this case the equation (1.2.31), i.e., the problem (1.2.29), (1.2.30$_i$) has at least one solution, say v. Show that

$$0 < v(t) \le w(t) \quad \text{for} \quad a \le t \le b.$$

Let

$$v_1(t) = w(t) - v(t).$$

Then from the nonnegativeness of the operator h and also from the inequality (1.2.28) we have

$$v_1''(t) \le p_0(t)v_1(t) + p_1(t)v_1'(t) - h(w - \chi(v))(t) \le p_0(t)v_1(t) + p_1(t)v_1'(t)$$

and

$$v_1(a) = 0, \quad v_1^{(i-1)}(b-) = 0.$$

Hence v_1 is an upper function of the problem (1.2.4), (1.2.2$_{i0}$), and due to Remark 1.2.5,

$$v_1(t) \ge 0 \quad \text{for} \quad a < t < b,$$

i.e.,

$$v(t) \ge w(t) \quad \text{for} \quad a < t < b. \tag{1.2.33}$$

On the other hand, taking into account the inequality (1.2.28) and the fact that the operator h is nonnegative, from (1.2.29) and (1.2.30$_i$) we conclude that v is an upper function of the problem (1.2.4), (1.2.2$_{i0}$), i.e., by virtue of Remark 1.2.5,

$$v(t) > 0 \quad \text{for} \quad a \le t \le b. \tag{1.2.34}$$

It follows from (1.2.33) and (1.2.34) that the inequality $0 < v(t) \le w(t)$ is valid and hence

$$\chi(v)(t) = v(t) \quad \text{for} \quad a \le t \le b,$$

i.e., v as a solution of the equation (1.2.31) has the form

$$v(t) = v_0(t) + \int_a^b |G(t,s)|h(v)(s)\,ds \quad \text{for} \quad a \le t \le b, \tag{1.2.35}$$

where by Remark 1.2.5,

$$v_0(t) > 0 \quad \text{for} \quad a \le t \le b. \tag{1.2.36}$$

If we introduce the notation $\rho(t) = v(t)$ and take into consideration (1.2.36), then in view of (1.2.35) we can see that our lemma is valid. □

Lemma 1.2.5. *Let $i \in \{1,2\}$, the constants $\alpha \in [0,1[$ and $\beta \in]0,1]$ be connected by the inequality*

$$\alpha + \beta \le 1, \tag{1.2.37}$$
$$(p_0, p_1) \in \mathbb{V}_{i,\beta}(]a,b[;h), \tag{1.2.38$_i$}$$

where

$$h \in \mathcal{L}\Big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\Big) \tag{1.2.39$_i$}$$

is a nonnegative operator and

$$x(t) = \int_a^t \sigma(p_1)(s)\,ds \bigg(\int_t^b \sigma(p_1)(s)\,ds\bigg)^{2-i} \quad for \quad a \le t \le b. \tag{1.2.40$_i$}$$

Then there exists a positive function $\rho \in C(]a,b[)$ such that the inequality (1.2.20) is satisfied, where G is Green's function of the problem (1.2.4), (1.2.2$_i$) and

$$\rho(t) = O^*(x^\beta(t)) \tag{1.2.41}$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to a$ if $i = 2$.

Proof. As is seen from the definition of the set $\mathbb{V}_{i,\beta}(]a,b[;h)$, the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy the inclusion (1.2.6$_i$) from which by virtue of Remark 1.2.2 it follows the existence of Green's function of the problem (1.2.4), (1.2.2$_{i0}$), and there exists a measurable function $q_\beta :]a,b[\to [0,+\infty[$ such that the problem

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(v)(t) - q_\beta(t), \tag{1.2.42}$$
$$v(a) = 0, \quad v^{(i-1)}(b-) = 0 \tag{1.2.43$_i$}$$

has in the interval $]a,b[$ a positive upper function w, where

$$w(t) = O^*(x^\beta(t)) \quad \text{and} \quad \int_a^b |G(t,s)| q_\beta(s)\,ds = O^*(x^\beta(t)) \tag{1.2.44}$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to a$ if $i = 2$.

Introduce the operator χ as in the previous proof and let

$$H(y)(t) = \int_a^b |G(t,s)|\big(q_\beta(s) + h(\chi(y))(s)\big)\,ds.$$

As we can see from the conditions (1.2.39$_i$), (1.2.44), the operator χ transforms the space $C(]a,b[)$ into $C_{x^\beta}(]a,b[)$. Consider now the equations

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(\chi(v))(t) - q_\beta(t), \tag{1.2.45}$$

$$v(t) = H(v)(t) \tag{1.2.46}$$

and note that the problem (1.2.45), (1.2.43$_i$) is equivalent to the equation (1.2.46).

From the equality (1.2.7) by means of which Green's function is expressed, as well as from the estimates (1.2.10$_i$) and the conditions (1.2.44), for any $y \in C(]a,b[)$ we have

$$|H(y)(t)| \le r_0 x^{1-\alpha}(t) \int_a^t \frac{x^\alpha(s)}{\sigma(p_1)(s)} h(x^\beta)(s)\,ds +$$

$$+ \int_a^b |G(t,s)| q_\beta(s)\,ds < +\infty \quad \text{for} \quad a \le t \le b, \tag{1.2.47}$$

where

$$r_0 = \frac{c_*^2}{d_*} \sup\Big\{ \frac{w(t)}{x^\beta(t)} : \quad a < t < b \Big\}.$$

It follows from (1.2.37), (1.2.44) that the operator H transforms the space $C(]a,b[)$ into $C_{x^\beta}(]a,b[)$. Noticing that the right-hand side of the estimate (1.2.47) is independent of the function y, we make sure that a constant r exists such that for any $y \in C(]a,b[)$

$$\|H(y)\|_{C,x^\beta} \le r.$$

It is clear that this estimate is the more so valid if y belongs to the ball

$$\mathbb{B}_r = \big\{ z \in C_{x^\beta}(]a,b[) : \quad \|z\|_{C,x^\beta} \le r \big\}.$$

Repeating now the reasoning of the previous proof, we can see that the operator $H : C_{x^\beta}(]a,b[) \to C_{x^\beta}(]a,b[)$ is compact and hence there exists a solution v of the equation (1.2.46) such that

$$v \in C_{x^\beta}(]a,b[), \tag{1.2.48}$$

$$\chi(v)(t) = v(t) \quad \text{for} \quad a \le t \le b,$$

and

$$v(t) > 0 \quad \text{for} \quad a < t < b. \tag{1.2.49}$$

Then the following representation is valid:

$$v(t) = \int_a^b |G(t,s)| \big(h(v)(s) + q_\beta(s) \big)\,ds, \tag{1.2.50}$$

whence with regard for (1.2.49) we obtain the inequality

$$v(t) \geq \int\limits_a^b |G(t,s)| q_\beta(s)\, ds \quad \text{for} \quad a \leq t \leq b$$

which together with the conditions (1.2.44) and (1.2.48) implies that

$$v(t) = O^*(x^\beta(t)) \tag{1.2.51}$$

for $t \to a$, $t \to b$, if $i = 1$, and for $t \to a$ if $i = 2$. If we now take into consideration that owing to the conditions (1.2.44) and (1.2.51) we have

$$\inf\Big\{\frac{1}{v(t)} \int\limits_a^b |G(t,s)| q_\beta(s)\, ds : \ \ a < t < b\Big\} > 0,$$

then from (1.2.50) we obtain

$$\sup\Big\{\frac{1}{v(t)} \int\limits_a^b |G(t,s)| h(v)(s)\, ds : \ \ a < t < b\Big\} < 1. \tag{1.2.52}$$

Introducing the notation $\rho(t) = v(t)$, from (1.2.49), (1.2.51) and (1.2.52) we see that our lemma is valid. □

Lemma 1.2.6. *Let $i \in \{1,2\}$, the function x be defined by $(1.2.40_i)$, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by (1.2.37) and the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy $(1.2.38_i)$, where*

$$h \in \mathcal{L}\Big(C_{x^\beta};\, L_{\frac{x^\alpha}{\sigma(p_1)}}\Big) \cap \mathcal{L}\Big(C;\, L_{\sigma_i(p_1)}\Big) \tag{$1.2.53_i$}$$

is a nonnegative operator. Then there exists a continuous function $\rho : [a,b] \to \mathbb{R}^+$ such that the inequality (1.2.20) is satisfied, where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$.

Proof. By Lemma 1.2.5, from the fact that $h \in \mathcal{L}(C_{x^\beta};\, L_{\frac{x^\alpha}{\sigma(p_1)}})$ it follows the existence of the function $\rho_0 \in C(]a,b[)$ such that

$$\rho_0(t) > 0 \quad \text{for} \quad a < t < b$$

and

$$\sup\Big\{\frac{1}{\rho_0(t)} \int\limits_a^b |G(t,s)| h(\rho_0)(s)\, ds : \ \ a < t < b\Big\} < 1.$$

Then, taking into account that the operator h also belongs to $\mathcal{L}(C; L_{\sigma_i(p_1)})$, we can see by Lemma 1.2.3 that our lemma is valid. □

Lemma 1.2.7. *Let $i \in \{1,2\}$, the function $x :]a,b[\to \mathbb{R}^+$ be defined by* $(1.2.40_i)$ *and the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy the inclusion* $(1.2.6_i)$*. Then for any $\beta \in]0,1]$ we have*

$$\int_a^b |G(t,s)| \frac{\sigma^2(p_1)(s)}{x^{2-\beta-[\beta]}(s)}\, ds = O^*(x^\beta(s)) \tag{1.2.54}$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to a$ if $i = 2$, where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$.

Proof. By Remark 1.2.2 and the inclusion $(1.2.6_i)$ there exists Green's function G of the problem (1.2.4), $(1.2.2_{i0})$ which is expressed by the equality (1.2.7).

Consider the case $i = 1$ separately and note that

$$\int_t^b \sigma(p_1)(s)\, ds \geq \int_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds \quad \text{for} \quad a \leq t \leq \frac{a+b}{2}. \tag{1.2.55}$$

Then, taking into consideration (1.2.7), $(1.2.10_i)$ and (1.2.55), for any $\beta \in]0,1[$ we obtain for $t \in [a, \frac{a+b}{2}]$ the estimates

$$\int_a^b |G(t,s)| \frac{\sigma^2(p_1)(s)}{x^{2-\beta}(s)}\, ds \leq \frac{c_*^2}{v_2(a)} \Bigg[\frac{x^\beta(t)}{\beta \int\limits_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds} +$$

$$+ \frac{\big(\int_a^t \sigma(p_1)(s)\, ds\big)^\beta}{(1-\beta)\big(\int\limits_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds\big)^{1-\beta}} + \frac{\big(\int_a^b \sigma(p_1)(s)\, ds\big)^{1-\beta}}{\beta\big(\int\limits_a^{\frac{a+b}{2}} \sigma(p_1)(s)\, ds\big)^{2-\beta}} x^\beta(t) \Bigg] \leq$$

$$\leq \frac{c_*^2}{\beta v_2(a)} \bigg(\frac{1}{1-\beta} + \Big(\int_a^b \sigma(p_1)(s)\, ds \Big)^{1-\beta} \Big(\int_a^{\frac{a+b}{2}} \sigma(p_1)(s)\, ds \Big)^{\beta-2} \bigg) x^\beta(t)$$

and

$$\int_a^b |G(t,s)| \frac{\sigma^2(p_1)(s)}{x^{2-\beta}(s)}\, ds \geq \frac{d_*^2}{v_2(a)} \Big(\int_t^b \sigma(p_1)(s)\, ds \Big)^\beta \Big(\int_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds \Big)^{1-\beta} \times$$

$$\times \int_a^t \frac{\sigma(p_1)(s)\, ds}{\big(\int_a^s \sigma(p_1)(\eta)\, d\eta\big)^{1-\beta} \big(\int_s^b \sigma(p_1)(\eta)\, d\eta\big)^{2-\beta}} \geq$$

$$\geq \frac{d_*^2}{\beta v_2(a)}\,\frac{\Big(\int\limits_{\frac{a+b}{2}}^{b}\sigma(p_1)(s)\,ds\Big)^{1-\beta}}{\Big(\int\limits_a^b\sigma(p_1)(s)\,ds\Big)^{2-\beta}}\,x^\beta(t).$$

The last two estimates imply the validity of (1.2.54) as $t\to a$. Reasoning analogously for $t\in[\frac{a+b}{2},\,b]$, we can see that this equality is also valid as $t\to b$. Consider the case $\beta=1$. With regard for the equalities (1.2.7) and the estimates ($1.2.10_1$) we obtain

$$\frac{d_*^2}{2C_*}\leq\int\limits_a^b|G(t,s)|\sigma^2(p_1)(s)\,ds\,x^{-1}(t)\leq\frac{C_*^2}{2d_*}\ \text{ for }\ a<t<b. \tag{1.2.56}$$

It follows from (1.2.56) that our lemma is valid in the case $\beta=1$ as well.

Reasoning similarly, we can prove the lemma for $i=2$. $\square$

1.2.2. Auxiliary Propositions to Theorems ($1.1.2_i$), ($1.1.2_{i0}$) ($i=1,2$). Consider in the interval $]a,b[$ the equation

$$v''(t)=g(v)(t), \tag{1.2.57}$$

where $g: C(]a,b[)\to L_{\rm loc}(]a,b[)$ is a continuous linear operator. We will also need the equation

$$v''(t)=0\quad\text{for}\quad a\leq t\leq b. \tag{1.2.58}$$

Note that Green's function of the problem (1.2.58), ($1.2.2_{i0}$) has the form

$$G(t,s)=\begin{cases}-(s-a)\Big(\dfrac{b-t}{b-a}\Big)^{2-i} & \text{for}\quad a\leq s<t\leq b,\\[2mm] -(t-a)\Big(\dfrac{b-s}{b-a}\Big)^{2-i} & \text{for}\quad a\leq t<s\leq b.\end{cases} \tag{$1.2.59_i$}$$

Lemma $1.2.8_1$. *Let $\gamma\in[0,1[$, $\lambda\in[0;1-\gamma[$ and*

$$g\in\mathcal{L}(C_{x^\lambda};\,L_{x^\gamma}) \tag{1.2.60}$$

be a nonnegative operator, where

$$x(t)=(b-t)(t-a)\quad\textit{for}\quad a\leq t\leq b. \tag{$1.2.61_1$}$$

Let, moreover, there exist constants α, $\beta\in[0,\frac12]$ such that

$$\lambda\leq\beta<1-\gamma, \tag{1.2.62}$$

$$\alpha+\beta\leq\frac12, \tag{1.2.63}$$

and

$$\int\limits_a^b x^{\alpha}(s)g(x^{\beta})(s)\,ds < 2^{\beta}\frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \qquad (1.2.64_1)$$

Then the problem (1.2.57), $(1.2.2_{10})$ *has only the zero solution in the space* $C_{x^{\lambda}}(]a,b[)$.

Proof. Suppose to the contrary that the problem (1.2.57), $(1.2.2_{i0})$ has a nonzero solution $v_0 \in C_{x^{\lambda}}(]a,b[)$.

If v_0 is a function of constant signs, then from the nonnegativeness of the operator g we obtain

$$v_0''(t)\operatorname{sign} v_0(t) \geq 0 \quad \text{for} \quad a < t < b,$$

which together with the conditions $(1.2.2_{i0})$ contradicts the assumption $v_0(t_0) \not\equiv 0$, i.e., v_0 is a function of constant signs.

Using Green's function of the problem (1.2.58), $(1.2.2_{i0})$, v_0 can be represented as follows:

$$v_0(t) = -\frac{1}{b-a}\bigg((b-t)\int\limits_a^t (s-a)g(v_0)(s)\,ds + (t-a)\int\limits_t^b (b-s)g(v_0)(s)\,ds\bigg)$$
$$\text{for} \quad a \leq t \leq b$$

and hence for any β the estimate

$$\frac{v_0(t)}{[(b-t)(t-a)]^{\beta}} \leq$$
$$\leq \frac{[(b-t)(t-a)]^{1-(\gamma+\beta)}}{b-a}\int\limits_a^b [(b-s)(s-a)]^{\gamma} g(x^{\lambda})(s)\,ds \|v_0\|_{C,x^{\lambda}}$$
$$\text{for } a < t < b$$

is valid.

In the above estimate, taking into account the condition (1.2.60), if β satisfies the inequality (1.2.62), we get

$$\lim_{t\to a}\frac{v_0(t)}{[(b-t)(t-a)]^{\beta}} = 0, \quad \lim_{t\to b}\frac{v_0(t)}{[(b-t)(t-a)]^{\beta}} = 0.$$

These equalities imply the existence of points t_1, $t_2 \in]a,b[$ such that

$$\frac{v_0(t_1)}{(b-t_1)^{\beta}(t_1-a)^{\beta}} = \sup\Big\{\frac{v_0(t)}{(b-t)^{\beta}(t-a)^{\beta}} : \ a < t < b\Big\},$$
$$\frac{v_0(t_2)}{(b-t_2)^{\beta}(t_2-a)^{\beta}} = \inf\Big\{\frac{v_0(t)}{(b-t)^{\beta}(t-a)^{\beta}} : \ a < t < b\Big\}.$$

Without loss of generality we assume $t_1 < t_2$ and notice that by $(1.2.61_1)$ which defines the function x, we have

$$-g(x^\beta)(t)\frac{|v_0(t_2)|}{(b-t_2)^\beta(t_2-a)^\beta} \leq$$
$$\leq g(v_0)(t) \leq g(x^\beta)(t)\frac{|v_0(t_1)|}{(b-t_1)^\beta(t_1-a)^\beta} \quad \text{for} \quad a < t < b. \quad (1.2.65)$$

Recall also one simple numerical inequality

$$A \cdot B \leq \frac{(A+B)^2}{4}, \qquad (1.2.66)$$

where $A \geq 0$ and $B \geq 0$.

Suppose $c \in]t_1, t_2[$ and $v_0(c) = 0$. Then the following representations are valid:

$$v_0(t_1) = \frac{c-t_1}{c-a}\int_a^{t_1}(s-a)g(-v_0)(s)\,ds + \frac{t_1-a}{c-a}\int_{t_1}^{c}(c-s)g(-v_0)(s)\,ds$$

and

$$|v_0(t_2)| = \frac{b-t_2}{b-c}\int_c^{t_2}(s-c)g(v_0)(s)\,ds + \frac{t_2-a}{b-c}\int_{t_2}^{b}(b-s)g(v_0)(s)\,ds.$$

These representations with regard for the inequality (1.2.65), for any α, β satisfying the conditions of the lemma, result in

$$v_0(t_1) \leq \frac{[(c-t_1)(t_1-a)]^{1-\alpha}}{(c-a)[(b-t_2)(t_2-a)]^\beta}\int_a^c x^\alpha(s)g(x^\beta)(s)\,ds \cdot |v_0(t_2)| < +\infty$$

and

$$v_0(t_2) \leq \frac{[(b-t_2)(t_2-c)]^{1-\alpha}}{(b-c)[(b-t_1)(t_1-a)]^\beta}\int_c^b x^\alpha(s)g(x^\beta)(s)\,ds \cdot |v_0(t_1)| < +\infty.$$

Multiplying the above inequalities, by means of (1.2.66) we obtain

$$\lambda\int_a^b x^\alpha(s)g(x^\beta)(s)\,ds \geq 1, \qquad (1.2.67)$$

where

$$\lambda = \frac{1}{2}\sqrt{\frac{[(b-t_2)(t_2-c)(c-t_1)(t_1-a)]^{1-(\alpha+\beta)}[(t_2-c)(c-t_1)]^\beta}{(b-c)(c-a)(b-t_1)^\beta(t_2-a)^\beta}}.$$

Then by (1.2.66) we get the estimate

$$\lambda \le \frac{1}{2}\sqrt{\frac{[(b-c)(c-a)]^{1-2(\alpha+\beta)}(t_2-t_1)^{2\beta}}{4^{2-2(\alpha+\beta)+\beta}[(b-t_1)(t_2-a)]^{\beta}}},$$

whence using once more the inequality (1.2.66) and taking into consideration the fact that

$$(t_2-t_1)^{2\beta} \le [(b-t_1)(t_2-a)]^{\beta}, \tag{1.2.68}$$

we arrive at

$$\lambda \le \frac{b-a}{16\cdot 2^{\beta}}\Big(\frac{4}{b-a}\Big)^{2(\alpha+\beta)}. \tag{1.2.69}$$

Substituting the last inequality in (1.2.67), we obtain the contradiction with the condition ($1.2.64_1$), i.e., our assumption is invalid and $v_0(t)\equiv 0$. □

Lemma 1.2.8$_2$. *Let $\gamma\in[0,1[$, $\lambda\in[0,1-\gamma[$ and the nonnegative operator g satisfy the inclusion* (1.2.60), *where*

$$x(t)=t-a \quad for \quad a\le t\le b. \tag{1.2.61$_2$}$$

Let, moreover, there exist constants α, $\beta\in[0,\frac{1}{2}]$ such that the conditions (1.2.62), (1.2.63) *are satisfied and*

$$\int_a^b x^{\alpha}(s)g(x^{\beta})(s)\,ds \le \frac{8}{b-a}\Big(\frac{b-a}{4}\Big)^{\alpha+\beta}. \tag{1.2.64$_2$}$$

Then the problem (1.2.57), ($1.2.2_{20}$) *has only the zero solution in the space $C_{x^\lambda}(]a,b[)$.*

Proof. Suppose to the contrary that the problem (1.2.57), ($1.2.2_{20}$) has a nonzero solution $v_0\in C_{x^\lambda}(]a,b[)$. Similarly to the previous lemma we make sure that v_0 is of constant signs and the equality

$$\lim_{t\to a}\frac{v_0(t)}{(t-a)^{\beta}}=0$$

is valid for any $\beta\in[\lambda,1-\gamma[$. On the other hand, in any sufficiently small neighborhood of the point b, since $v_0'(b-)=0$, the equality

$$\operatorname{sign}\Big(\frac{v_0(t)}{(t-a)^{\beta}}\Big)'=-\operatorname{sign} v_0(t)$$

is satisfied. It follows from the last two equalities that the function $\frac{v_0(t)}{(t-a)^{\beta}}$ attains neither its minimum nor its maximum at the points a and b. Let

$$\max\Big\{\frac{v_0(t)}{(t-a)^{\beta}}:\ a\le t\le b\Big\}=\frac{v_0(t_1)}{(t_1-a)^{\beta}}$$

and

$$\min\Big\{\frac{v_0(t)}{(t-a)^\beta}:\quad a\le t\le b\Big\}=\frac{v_0(t_2)}{(t_2-a)^\beta}.$$

Then from the above-said it is clear that $t_1,\ t_2\ \in]a,b[$. Without loss of generality we assume $t_1 < t_2$ and let the point $c\ \in]t_1,t_2[$ be such that $v_0(c)=0$. Then from the inequality

$$-g(x^\beta)(t)\frac{|v_0(t_2)|}{(t_2-a)^\beta}\le g(v_0)(t)\le g(x^\beta)(t)\frac{|v_0(t_1)|}{(t_1-a)^\beta}\quad\text{for } a<t<b$$

and from the equalities

$$v_0(t_1)=\frac{c-t_1}{c-a}\int_a^{t_1}(s-a)g(-v_0)(s)\,ds+\frac{t_1-a}{c-a}\int_{t_1}^{c}(c-s)g(-v_0)(s)\,ds,$$

$$|v_0(t_2)|=\int_c^{t_2}(s-c)g(v_0)(s)\,ds+(t_2-c)\int_{t_2}^{b}g(v_0)(s)\,ds$$

we obtain

$$v_0(t_1)\le\frac{(c-t_1)(t_1-a)^{1-\alpha}}{(c-a)(t_2-a)^\beta}\int_a^c x^\alpha(s)g(x^\beta)(s)\,ds\cdot|v_0(t_2)|$$

$$|v_0(t_2)|\le\frac{(t_2-c)^{1-\alpha}}{(t_1-a)^\beta}\int_c^b x^\alpha(s)g(x^\beta)(s)\,ds\cdot v_0(t_1).$$

Multiplying these inequalities, with regard for (1.2.66) we get

$$\lambda\int_a^b x^\alpha(s)g(x^\alpha)(s)\,ds\ge 1,\tag{1.2.70}$$

where

$$\lambda=\frac{1}{2}\sqrt{\frac{[(t_1-a)(c-t_1)]^{1-(\alpha+\beta)}(t_2-c)^{1-\alpha}(c-t_1)^{\alpha+\beta}}{(c-a)(t_2-a)^\beta}}.$$

Then by (1.2.66) and $t_2-a>t_2-c$ we have

$$\lambda\le\frac{1}{2}\sqrt{\frac{(c-a)^{1-2(\alpha+\beta)}(t_2-c)^{1-2(\alpha+\beta)}[(c-t_1)(t_2-c)]^{\alpha+\beta}}{4^{1-(\alpha+\beta)}}}.$$

Applying once more (1.2.66), we can see that

$$\lambda\le\frac{(t_2-a)^{1-2(\alpha+\beta)}(t_2-t_1)^{\alpha+\beta}}{2\cdot 4^{1-(\alpha+\beta)}}.\tag{1.2.71}$$

Notice that from the conditions $t_1,\ t_2 \in]a,b[$ as well as from the fact that for none of $\alpha,\ \beta \in [0,\frac{1}{2}]$ the expressions $\alpha+\beta$ and $1-2(\alpha+\beta)$ vanish simultaneously, we obtain the estimate

$$(t_2-a)^{1-2(\alpha+\beta)}\cdot(t_2-t_1)^{\alpha+\beta} < (b-a)^{1-(\alpha+\beta)},$$

with regard for which in (1.2.71) we get

$$\lambda < \frac{(b-a)}{8}\Big(\frac{4}{b-a}\Big)^{\alpha+\beta}.$$

Substituting the latter inequality in (1.2.70), we obtain the contradiction with the condition ($1.2.64_2$), i.e., our assumption is invalid and $v_0(t)\equiv 0$. □

Remark 1.2.7. Lemma $1.2.8_1$ remains valid if for $\beta\neq 0$ we replace the condition ($1.2.64_1$) by

$$\int\limits_a^b x^{\alpha}(s)g(x^{\beta})(s)\,ds \le 2^{\beta}\frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.2.72}$$

Proof. If $\beta\neq 0$, then the inequality (1.2.68) will be strictly satisfied and hence the estimate (1.2.69) will take the form

$$\lambda < \frac{b-a}{16\cdot 2^{\beta}}\Big(\frac{4}{b-a}\Big)^{2(\alpha+\beta)}.$$

Taking into consideration the last inequality in (1.2.67), we obtain the contradiction with the condition (1.2.72) which indicates the possibility to replace in case $\beta\neq 0$ the condition ($1.2.64_1$) by (1.2.72). □

§ 1.3. Proof of Propositions on Existence and Uniqueness

1.3.1. Proof of Basic Theorems on Existence and Uniqueness of Solution of Two-Point Problems.

Proof of Theorem $1.1.1_i$. From the inclusions ($1.1.7_i$) and ($1.1.8_i$) and also from the fact that the operator h is nonnegative, for $\beta=0$ by virtue of Lemma 1.2.4 and for $\beta>0$ by virtue of Lemma 1.2.6 it follows that there exists a function $\rho\in C(]a,b[)$ such that

$$\rho(t)>0 \quad \text{for} \quad a\le t\le b \tag{1.3.1}$$

and

$$\sup\Big\{\frac{1}{\rho(t)}\int\limits_a^b |G(t,s)|h(\rho)(s)\,ds:\ \ a<t<b\Big\}<1, \tag{1.3.2}$$

where G is Green's function of the problem (1.2.4), ($1.2.2_{i0}$). Note that for any function $y\in C_{\rho}(]a,b[)$ the inequality

$$|y(t)|\le\rho(t)\|y\|_{C,\rho} \quad \text{for} \quad a\le t\le b \tag{1.3.3}$$

is valid and, owing to the estimates ($1.2.10_i$), the representation (1.2.7) of Green's function and the conditions (1.1.5)–($1.1.8_i$) and (1.1.10), we have

$$\Big|\int_a^b G(t,s)p_2(s)\,ds\Big| < +\infty, \quad \Big|\int_a^b G(t,s)g(y)(s)\,ds\Big| < +\infty,$$

$$\Big|\int_a^b G(t,s)h(y)(s)\,ds\Big| < +\infty.$$

Introduce the continuous operators $\mathbb{U}_0$, $\mathbb{U} : C_\rho(]a,b[) \to C_\rho(]a,b[)$ by the equalities

$$\begin{aligned} \mathbb{U}_0(y)(t) &= \int_a^b G(t,s)g(y)(s)\,ds, \\ \mathbb{U}(g)(t) &= u_0(t) + \mathbb{U}_0(y)(t) + \int_a^b G(t,s)p_2(s)\,ds, \end{aligned} \tag{1.3.4}$$

where u_0 is a solution of the problem (1.2.4), ($1.2.2_i$). Clearly every solution of the problem (1.1.1), ($1.1.2_i$) is a solution of the equation

$$u(t) = \mathbb{U}(u)(t) \tag{1.3.5}$$

and vice versa.

From the definition of the norm of the operator it follows that

$$\|\mathbb{U}_0\|_{C_\rho\to C_\rho} =$$
$$= \sup\left\{\Big\|\int_a^b G(t,s)g(y)(s)\,ds\Big\|_{C,\rho} : \ x \in C_\rho(]a,b[), \ \|y\|_{C,\rho} = 1\right\}$$

which with regard for (1.1.10), (1.3.1)–(1.3.3) implies

$$\|\mathbb{U}_0\|_{C_\rho\to C_\rho} < 1, \tag{1.3.6}$$

i.e., the operator $\mathbb{U}$ contracts the space $C_\rho(]a,b[)$ into itself for any $p_2 \in L_{\sigma_i(p_1)}([a,b])$ and any operator g satisfying (1.1.10). Then by virtue of the theorem on contracting map the equation (1.3.5) has in the space $C_\rho(]a,b[)$ and hence in $C(]a,b[)$ a unique solution because, by (1.3.1), any function from $C(]a,b[)$ belongs to the space $C_\rho(]a,b[)$ as well. It remains to notice that the unique solvability of the problem (1.1.1), ($1.1.2_i$) follows from the equivalence of that problem and the equation (1.3.5). □

Proof of Theorem $1.1.1_{i0}$. The inclusions ($1.1.7_i$), ($1.1.8_i$) and the nonnegativeness of the operator h imply by virtue of Lemma 1.2.5 the existence of a positive function $\rho \in C(]a,b[)$ such that

$$\rho(t) = O^*(x^\beta(t)) \tag{1.3.7}$$

as $t \to a$, $t \to b$, if $i = 1$, and as $t \to a$ if $i = 2$. Moreover, the condition (1.3.2) is satisfied, where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$. It is also clear that for any $y \in C_\rho(]a,b[)$ the inequality (1.3.3) is satisfied, and due to the estimates $(1.2.10_i)$ and the representation (1.2.7) of Green's functions we have

$$\begin{aligned} \left| \int_a^b G(t,s)h(y)(s)\,ds \right| &\le r_1 x^{1-\alpha}(t) \int_a^b \frac{x^\alpha(s)}{\sigma(p_1)(s)}\, h(x^\beta)(s)\,ds\, \|y\|_{C,x^\beta}, \\ \left| \int_a^b G(t,s)p_2(s)\,ds \right| &\le r_1 x^\beta(t) \int_a^b \frac{x^{1-\beta}(s)}{\sigma(p_1)(s)}\, |p_2(s)|\,ds \quad \text{for} \quad a \le t \le b, \end{aligned} \tag{1.3.8}$$

where

$$r_1 = \frac{c_*^2}{v_2(a)},$$

and the existence of integrals follows from the conditions (1.1.6), (1.1.11), (1.1.12). From (1.3.8) and (1.1.6), (1.1.10), (1.3.7) we also have that the operators

$$\mathbb{U}_0(y)(t) = \int_a^b G(t,s)g(y)(s)\,ds$$

and

$$\mathbb{U}(y)(t) = \mathbb{U}_0(y)(t) + \int_a^b G(t,s)p_2(s)\,ds$$

transform continuously the space $C_\rho(]a,b[)$ into itself. Repeating word by word the previous proof, we can see that the problem (1.1.1) $(1.1.2_{i0})$ has a unique solution u in the space $C_\rho(]a,b[)$. But as is seen from (1.3.7), u will be a unique solution in the space $C_{x^\beta}(]a,b[)$ as well. □

Proof of Remark $1.1.1_i$. Under the conditions of Theorem $1.1.1_i$, as is seen from its proof, the operator $\mathbb{U}$ contracts the space $C_\rho([a,b])$ into itself. Then from the theorem on contracting map it follows that for any function $v_0 \in C_\rho(]a,b[)$ the sequence $v_n : [a,b] \to \mathbb{R}$, where v_n is the unique solution of the equation

$$v_n(t) = \mathbb{U}(v_{n-1})(t) \tag{1.3.9}$$

tends to the unique solution u of the equation (1.3.5) with respect to the norm $\|\cdot\|_{C,\rho}$. We introduce the notation

$$\|\mathbb{U}_0\|_{C_\rho \to C_\rho} = \mu \quad \text{and} \quad \|u - v_1\|_{C,\rho} = \omega,$$

and notice that by virtue of (1.3.6), we have $\mu < 1$. Then, as is known, the estimate

$$\|u - v_n\|_{C,\rho} \le \omega\, \frac{\mu^n}{1-\mu}\,, \quad n \in \mathbb{N}, \tag{1.3.10}$$

is valid and for any $n \in \mathbb{N}$ with regard for (1.3.3) we obtain

$$|u(t) - v_n(t)| \le \omega\, \frac{\mu^n}{1-\mu}\, \|\rho\|_C \quad \text{for} \quad a \le t \le b. \tag{1.3.11}$$

Differentiating the difference of the equations (1.3.5) and (1.3.9) and taking into account the inequalities (1.1.10), (1.3.11) and the estimates ($1.2.12_i$) of Green's function, we obtain

$$\sup\big\{\sigma_i(p_1)(t)|v_n'(t) - u'(t)| : \ a < t < b\big\} \le \omega'\, \frac{\mu^n}{1-\mu}\,, \quad n \in \mathbb{N}, \tag{1.3.12}$$

where

$$\omega' = \omega c_* \|\rho\|_C \int\limits_a^b \sigma_i(p_1)(s) h(1)(s)\, ds.$$

The inequalities (1.3.11), (1.3.12) imply the validity of the estimates (1.1.14), and after differentiating twice the equality (1.3.9) we see that v_n is a solution of the problem ($1.1.13_i$). □

Proof of Remark $1.1.1_{i0}$. Let ρ be the function appearing in the proof of Theorem $1.1.1_{i0}$. Introduce the constants μ and ω and the functions $v_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, as in the previous proof. Reasoning as above, we make sure that the estimate (1.3.10) is valid, and by virtue of the condition (1.3.7) for any $n \in \mathbb{N}$ we have

$$\frac{|u(t) - v_n(t)|}{x^\beta(t)} \le \omega\, \frac{\mu^n}{1-\mu}\, \sup\Big\{\frac{\rho(t)}{x^\beta(t)} : \ a < t < b\Big\}. \tag{1.3.13}$$

On the other hand, differentiating the difference of the equations (1.3.5) and (1.3.9), with regard for the equality (1.2.7) and the estimates ($1.2.10_i$), ($1.2.11_i$), for any $n \in \mathbb{N}$ we obtain

$$\frac{x^\alpha(t)}{\sigma(p_1)(t)}\, |u'(t) - v_n'(t)| \le r\|u - v_n\|_{C,\rho} \quad \text{for} \quad a \le t \le b, \tag{1.3.14}$$

where

$$r = (1 + c^*)^2 \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\, x(s) + \sigma(p_1)(s)\, ds \int\limits_a^b \frac{x^\alpha(s)}{\sigma(p_1)(s)}\, h(x^\beta)(s)\, ds.$$

The inequalities (1.3.10), (1.3.13) and (1.3.14) imply the validity of the estimates (1.1.15), and having differentiated twice the equality (1.3.9) we see that v_0 is a solution of the problem ($1.1.13_{i0}$). □

Proof of Theorem 1.1.2$_i$. Let G be Green's function of the problem (1.2.58), (1.2.2$_{i0}$). Introduce the operator $\mathbb{U}_0$ and the function q by the equalities

$$\mathbb{U}_0(y)(t)=\int_a^b G(t,s)g(y)(s)\,ds,\quad q(t)=\int_a^b G(t,s)p_2(s)\,ds. \quad (1.3.15)$$

From the representation (1.2.59$_i$) of Green's function and from the conditions (1.1.17), (1.1.18) it follows that the operator $\mathbb{U}_0$ transforms continuously the space $C(]a,b[)$ into itself and $q\in C(]a,b[)$.

Consider now the equation

$$u(t)=\mathbb{U}_0(u)(t)+u_0(t)+q(t), \quad (1.3.16)$$

where $u_0(t)$ is a solution of the problem (1.2.58), (1.1.2$_i$). Every its solution is a solution of the problem (1.1.16), (1.1.2$_i$), and vice versa.

Let $r>0$, $\mathbb{B}_r=\{y\in C(]a,b[):\ \|y\|_C\le r\}$ and choose any sequence $(x_n)_{n=1}^{\infty}$ from $\mathbb{B}_r$. Let, moreover, $y_n(t)=\mathbb{U}_0(x_n)(t)$, $n\in\mathbb{N}$. Then

$$\|y_n\|_C\le r_1,\quad n\in\mathbb{N}, \quad (1.3.17)$$

where

$$r_1=r\int_a^b\Big(\frac{b-s}{b-a}\Big)^{2-i}(s-a)g(1)(s)\,ds.$$

Consider the case $i=1$ separately. From the definition of Green's function G, for any $\varepsilon>0$ it follows the existence of $a_1,\ b_1\in]a,b[$, where $a_1<b_1$, such that

$$\max\Big\{\int_a^b|G(t,s)|g(1)(s)\,ds:\ a\le t\le a_1,\ b_1\le t\le b\Big\}\le\frac{\varepsilon}{4}\,,$$

which implies the validity of the estimate

$$|y_n(t_1)-y_n(t_2)|\le\frac{\varepsilon}{2}\,,\ \ n\in\mathbb{N},\ \ \text{for}\ \ a\le t_1\le t_2\le a_1,\ \ b_1\le t_1\le t_2\le b.$$

It is also clear that there exists a constant δ, $0<\delta<\min(a_1-a,b-b_1)$ for which the following inequality is valid:

$$\begin{gathered}|y_n(t_1)-y_n(t_2)|\le\\ \le r_1\max\Big\{\frac{1}{(b-t)(t-a)}\ :\ a_1-\delta\le t\le b_1+\delta\Big\}|t_1-t_2|\le\frac{\varepsilon}{2}\\ \text{for}\ \ |t_1-t_2|\le\delta,\quad a_1-\delta\le t_j\le b_1+\delta\ \ (j=1,2).\end{gathered}$$

From the last two estimates we obtain that if $t_j\in[a,b]$ $(j=1,2)$ and

$$|t_1-t_2|\le\delta,$$

then

$$|y_n(t_1) - y_n(t_2)| \le \varepsilon, \quad n \in \mathbb{N}.$$

This and the inequality (1.3.17) imply that the sequence $(y_n)_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In case $i = 2$ the same follows from the possibility of choosing for any $\varepsilon > 0$, $a_1 \in]a, b[$ and $0 < \delta < a_1 - a$ such that

$$\max\left\{\int_a^b |G(t,s)|g(1)(s)\,ds : \ a \le t \le a_1\right\} < \frac{\varepsilon}{4},$$

$$|y_n(t_1) - y_n(t_2)| \le r_1 \max\left\{1 + \frac{1}{t-a} \ : \ a_1 - \delta \le t \le b\right\} |t_1 - t_2| \le \frac{\varepsilon}{2}$$
$$\text{for} \quad |t_1 - t_2| \le \delta, \quad a_1 - \delta \le t_j \le b \ \ (j = 1, 2).$$

Then by the Arzella–Ascoli lemma we obtain that $\mathbb{U}_0$ is a compact operator. Consequently, taking into account Fredholm's alternatives, the equation (1.3.16) is uniquely solvable if the homogeneous equation

$$u(t) = \mathbb{U}_0(u)(t) \tag{1.3.16$_0$}$$

has only the trivial solution in the space $C(]a, b[)$.

It remains to note that by virtue of the conditions (1.1.18)–(1.1.21) and (1.1.22) if $i = 1$ and ($1.1.24_2$) if $i = 2$, all the requirement of Lemma $1.2.8_i$ are satisfied for $\lambda = 0$, whence it follows that the problem (1.2.57), ($1.2.2_{i0}$), i.e., the equation ($1.3.16_0$) has only the trivial solution in the space $C(]a, b[)$. □

Proof of Remark 1.1.2 follows directly from Remark 1.2.7.

Proof of Theorem $1.1.2_{i0}$. Let x be a function defined by ($1.1.19_i$) and let G be Green's function of the problem (1.1.58), ($1.1.2_{i0}$) which is expressed by ($1.2.59_i$). Introduce the operator $\mathbb{U}_0$ and the function q by the equality (1.3.15). Then for any $y \in C_{x^\lambda}(]a, b[)$ the estimates

$$|\mathbb{U}_0(y)(t)| \le \frac{x^{1-\gamma}(t)}{(b-a)^{2-i}} \int_a^b x^{\gamma}(s) g(x^{\lambda})(s)\,ds\, \|y\|_{C,x^\lambda},$$

$$|q(t)| \le x^{1-\gamma}(t) \int_a^b x^{\gamma}(s)|p_2(s)|\,ds \quad \text{for} \quad a \le t \le b$$

are valid, from which by the conditions $\lambda \in]0, 1-\gamma[$ and (1.1.25), (1.1.26) it follows that $\mathbb{U}_0$ transforms continuously the space $C_{x^\lambda}(]a, b[)$ into itself and $q \in C_{x^\lambda}(]a, b[)$.

Consider now the equation

$$u(t) = \mathbb{U}_0(u)(t) + q(t) \tag{1.3.18}$$

which is equivalent to the problem (1.1.16), ($1.1.2_{i0}$), and the corresponding homogeneous equation ($1.3.16_0$).

As is seen from Lemma $1.2.8_i$ and Remark 1.2.7, by virtue of the conditions $\lambda \in]0, 1-\gamma[$, (1.1.21), ($1.1.24_i$) and (1.1.25)–(1.1.27) the problem (1.2.57), ($1.1.2_{i0}$), i.e., the equation ($1.3.16_0$), has in the space $C_{x^\lambda}(]a,b[)$ only the trivial solution. Then according to Fredholm's alternatives, to prove the validity of our theorem it remains to show that the operator $\mathbb{U}_0$ is compact. Let $r > 0$,

$$\mathbb{B}_r = \big\{ z \in C_{x^\lambda}(]a,b[) : \ \ \|z\|_{C,x^\lambda} \le r \big\}$$

$(x_n)_{n=1}^\infty$ be a sequence from $\mathbb{B}_r$ and $y_n(t) = \mathbb{U}_0(x_n)(t)$ for $n \in \mathbb{N}$.

Then as is seen from the definition of G, for any $n \in \mathbb{N}$ the estimate

$$|y_n^{(j)}(t)| \le r \, \frac{x^{1-j-\gamma}(t)}{(b-a)^{(1-j)(2-i)}} \int_a^b x^\gamma(s) g(x^\lambda)(s)\, ds \ \ (j = 0,1) \qquad (1.3.19_i)$$
$$\text{for} \quad a < t < b$$

is valid, which by virtue of the condition $\lambda \in]0, 1-\gamma[$ yields

$$\|y_n(t)\|_{C,x^\lambda} \le r_1, \qquad (1.3.20)$$

where

$$r_1 = \frac{r}{(b-a)^{2-i}} \int_a^b x^\gamma(s) g(x^\lambda)(s)\, ds \ \max\big\{ x^{1-(\lambda+\gamma)}(t) : \ \ a \le t \le b \big\}.$$

Consider now the case $i = 1$ separately. From ($1.3.19_1$) for $j = 0$ and for any $\varepsilon > 0$ follows the existence of a_1, $b_1 \in]a,b[$, where $a_1 < b_1$, such that

$$|y_n(t)| \le \frac{\varepsilon}{4}, \ \ n \in \mathbb{N}, \quad \text{for} \quad a \le t \le a_1, \ \ b_1 \le t \le b,$$

which implies the estimate

$$|y_n(t_1) - y_n(t_2)| \le \frac{\varepsilon}{2}, \ \ n \in \mathbb{N},$$
$$\text{for} \quad a \le t_1 < t_2 \le a_1, \ \ b_1 \le t_1 < t_2 \le b.$$

Moreover, from ($1.3.19_1$) for $j = 1$ it follows the existence of a constant δ such that

$$|y_n(t_1) - y_n(t_2)| \le r_2|t_1 - t_2| \le \frac{\varepsilon}{2}, \ \ n \in \mathbb{N},$$
$$\text{for} \quad a_1 - \delta \le t_l \le b_1 + \delta \ \ (l = 1,2),$$

where

$$r_2 = r \int_a^b x^\gamma(s) g(x^\lambda)(s)\, ds \ \max\big\{ x^{-\gamma}(t) : \ \ a_1 - \delta \le t \le b_1 + \delta \big\}.$$

It is clear from the last two estimates that if $t_l \in [a,b]$ $(l=1,2)$ and

$$|t_1 - t_2| \le \delta,$$

then for any $n \in \mathbb{N}$

$$|y_n(t_1) - y_n(t_2)| \le \varepsilon.$$

This and the estimate (1.3.20) imply that the sequence $(y_n)_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In case $i = 2$, by virtue of the estimates ($1.3.19_2$) the same follows from the possibility of choosing, for any $\varepsilon > 0$, of $a_1 \in]a,b[$ and $0 < \delta < a_1 - a$ such that

$$|y_n(t)| \le \frac{\varepsilon}{4}, \quad n \in \mathbb{N} \quad \text{for} \quad a \le t \le b,$$

and

$$|y_n(t_1) - y_n(t_2)| \le r_2 |t_1 - t_2| \le \frac{\varepsilon}{2}, \quad n \in \mathbb{N},$$
$$\text{for} \quad a_1 - \delta \le t_j \le b \;\; (j = 1,2),$$

where

$$r_2 = r \int\limits_a^b x^{\gamma}(s) g(\rho)(s)\, ds \, \max\big\{x^{-\gamma}(t): \;\; a_1 - \delta \le t \le b\big\}.$$

Then by the Arzella–Ascoli lemma we have that $\mathbb{U}_0$ is a compact operator. □

1.3.2. Proof of Effective Sufficient Conditions for Solvability of the Problems (1.1.1), ($1.1.2_i$) and (1.1.1), ($1.1.2_{i0}$) ($i = 1,2$). Before we proceed to proving the corollaries, we note that Green's function of the problem

$$v''(t) = p_1(t) v'(t), \tag{1.3.21}$$

$$v(a) = 0, \quad v^{(i-1)}(b-) = 0 \tag{$1.3.22_i$}$$

has the form

$$G_0(t,s) =$$
$$= \begin{cases} -\dfrac{1}{\sigma(p_1)(s)} \displaystyle\int\limits_a^s \sigma(p_1)(\eta)\, d\eta \Bigg(\dfrac{1}{\int_a^b \sigma(p_1)(\eta) d\eta} \displaystyle\int\limits_t^b \sigma(p_1)(\eta) d\eta \Bigg)^{2-i} \\ \qquad\qquad \text{for} \quad a \le s < t \le b, \\ -\dfrac{1}{\sigma(p_1)(s)} \displaystyle\int\limits_a^t \sigma(p_1)(\eta)\, d\eta \Bigg(\dfrac{1}{\int_a^b \sigma(p_1)(\eta) d\eta} \displaystyle\int\limits_s^b \sigma(p_1)(\eta)\, d\eta \Bigg)^{2-i} \\ \qquad\qquad \text{for} \quad a \le t < s \le b. \end{cases} \tag{$1.3.23_i$}$$

Proof of Corollary 1.1.1$_1$. It is clear that all the requirements of Theorem 1.1.1$_1$, except (1.1.7$_1$), follow directly from the conditions of our corollary. It remains only to show that the conditions (1.1.31), (1.1.32$_1$) imply the inclusion (1.1.7$_1$) as well.

Indeed, let $\beta > 0$ and

$$z_\lambda(t) = \bigg[\bigg(\int\limits_t^b \sigma(p_1)(\eta)d\eta\bigg)^\alpha \times$$

$$\times \int\limits_a^t \frac{[p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta(s))}{\sigma(p_1)(s)} \bigg(\int\limits_a^s \sigma(p_1)(\eta)d\eta\bigg)^\alpha ds +$$

$$+\bigg(\int\limits_a^t \sigma(p_1)(\eta)d\eta\bigg)^\alpha \int\limits_t^b \frac{[p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta(s))}{\sigma(p_1)(s)} \bigg(\int\limits_s^b \sigma(p_1)(\eta)d\eta\bigg) ds \times$$

$$\times \frac{\big(\int_a^b \sigma(p_1)(s)\,ds\big)^{1-2(\alpha+\beta)}}{2^{2-2(\alpha+\beta)}}. \qquad (1.3.24)$$

Then, as is seen from the conditions (1.1.31), (1.1.32$_1$), we can choose $\lambda > 0$ such that

$$z_\lambda(t) < 1 \quad \text{for} \quad a \le t \le b \qquad (1.3.25)$$

be satisfied.

Introduce also the notation

$$q_\beta(t) = \frac{\sigma^2(p_1)(t)}{x^{2-\beta-[\beta]}(t)}, \quad w_\varepsilon(t) = \varepsilon \int\limits_a^b |G_0(t,s)| q_\beta(s)\,ds,$$

$$w(t) = \int\limits_a^b |G_0(t,s)| \big([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s)\big)\,ds + w_\varepsilon(t),$$

where $\varepsilon \in \mathbb{R}^+$, G_0 is Green's function of the problem (1.3.21), (1.3.22$_1$) which is defined by the equality (1.3.23$_1$), and by Lemma 1.2.7,

$$w_\varepsilon(t) = O^*(x^\beta(t)) \;\; \text{as} \;\; t \to a, \;\; t \to b \qquad (1.3.26)$$

for any $\varepsilon > 0$. From the conditions (1.3.25), (1.3.26) we have the possibility of choosing the constant $\varepsilon > 0$ such that

$$z_\lambda(t) + \sup\Big\{\frac{w_\varepsilon(t)}{x^\beta(t)} \;:\; a < t < b\Big\} < 1 \quad \text{for} \quad a \le t \le b. \qquad (1.3.27)$$

By virtue of (1.3.23$_1$) we easily get the estimate

$$0 < w(t) \le z_\lambda(t)x^\beta(t) + w_\varepsilon(t) \quad \text{for} \quad a < t < b$$

which with regard for (1.3.27) results in

$$0 < w(t) \le x^\beta(t) \quad \text{for} \quad a < t < b. \tag{1.3.28}$$

The last inequality together with (1.3.26) means that

$$w(t) = O^*(x^\beta(t)) \quad \text{as} \quad t \to a, \quad t \to b. \tag{1.3.29}$$

On the other hand, it is clear that

$$w''(t) = -[p_0(t)]_-(\lambda + x^\beta(t)) + p_1(t)w'(t) - h(x^\beta)(s) - q_\beta(t).$$

Taking into account the inequality (1.3.28) and the fact that the operator h and the constant λ are nonnegative, the above equality results in

$$w(t)'' \le p_0(t)w(t) + p_1(t)w'(t) - h(w)(t) - q_\beta(t). \tag{1.3.30}$$

If we introduce the notation $\widetilde{w}(t) = \lambda + w(t)$, then

$$\widetilde{w}''(t) \le p_0(t)\widetilde{w}(t) + p_1(t)\widetilde{w}'(t), \tag{1.3.31}$$

where

$$\widetilde{w}(t) > 0 \quad \text{for} \quad a \le t \le b. \tag{1.3.32}$$

From the inequalities (1.3.31) and (1.3.32), by Lemma 1.2.2 we obtain the inclusion

$$(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[). \tag{1.3.33$_1$}$$

Then, as is seen from Remark 1.2.2, the problem (1.2.4), ($1.2.2_{i0}$) has Green's function G which is expressed by the equality (1.2.7). Using now the inequalities ($1.2.10_1$), we arrive at

$$\frac{d_*^2}{c_*} \le \varepsilon w_\varepsilon^{-1}(t) \int_a^b |G(t,s)| q_\beta(s)\, ds \le \frac{c_*^2}{d_*} \quad \text{for} \quad a \le t \le b$$

which with regard for the equality (1.3.26) yields

$$\int_a^b |G(t,s)| q_\beta(s)\, ds = O^*(x^\beta(s)) \quad \text{as} \quad t \to a, \ \ t \to b. \tag{1.3.34}$$

It remains to note that the conditions (1.2.28), (1.3.29), ($1.3.33_1$), (1.3.34) and the inequality (1.3.30), owing to Definition 1.1.4, ensure the inclusion ($1.2.7_1$) for $\beta > 0$.

Assume now that $\beta = 0$ and

$$w(t) = \int_a^b |G_0(t,s)| \big([p_0(s)]_- + h(1)(s)\big)\, ds + \varepsilon v(t), \tag{1.3.35}$$

where v is a solution of the equation (1.3.21) under the boundary conditions

$$v(a) = 1, \quad v(b) = 1,$$

and

$$z_0(t) = \Bigg[\bigg(\int\limits_t^b \sigma(p_1)(\eta)d\eta\bigg)^\alpha \int\limits_a^t \frac{([p_0(s)]_- + h(1)(s))}{\sigma(p_1)(s)} \bigg(\int\limits_a^s \sigma(p_1)(\eta)d\eta\bigg)^\alpha ds +$$
$$+\bigg(\int\limits_a^t \sigma(p_1)(\eta)d\eta\bigg)^\alpha \int\limits_t^b \frac{([p_0(s)]_- + h(1)(s))}{\sigma(p_1)(s)} \bigg(\int\limits_s^b \sigma(p_1)(\eta)d\eta\bigg)^\alpha ds\Bigg] \times$$
$$\times \frac{(\int\limits_a^b \sigma(p_1)(s)\,ds)^{1-2\alpha}}{4^{1-\alpha}}.$$

Then, as is seen from the condition $(1.1.32_1)$,

$$z_0(t) < 1 \quad \text{for} \quad a \le t \le b,$$

and hence we can choose $\varepsilon > 0$ small enough for the inequality

$$z_0(t) + \varepsilon v(t) < 1 \tag{1.3.36}$$

to be fulfilled for $a \le t \le b$. Notice that by virtue of the equalities $(1.3.23_1)$, we obtain the estimate

$$0 < w(t) \le z_0(t) + \varepsilon v(t) \quad \text{for} \quad a \le t \le b$$

which with regard for (1.3.36) implies

$$0 < w(t) \le 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.37}$$

On the other hand,

$$w''(t) = -[p_0(t)] + p_1(t)w'(t) - h(1)(t),$$

whence, taking into account (1.3.37) and the fact that the operator h is nonnegative, we obtain

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - h(w)(t).$$

Consequently, owing to Definition 1.1.3, the inclusion $(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[\,; h)$ is valid. □

Proof of Corollary $1.1.1_2$. It is clear that all the requirements of Theorem $1.1.1_2$, except $(1.1.7_2)$ follow directly from the conditions of our corollary. It remains to show that the conditions (1.1.31), $(1.1.32_1)$ imply the inclusion $(1.1.7_2)$ as well.

To this end, we introduce for $\beta > 0$ the functions z_λ and w by the equalities

$$z_\lambda(t) = \Big[\int_a^t \frac{([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s))}{\sigma(p_1)(s)}\Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds +$$

$$+\Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha \int_t^b \frac{([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s)}{\sigma(p_1)(s)} ds\Big] \times$$

$$\times \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{1-(\alpha+\beta)}$$

and

$$w(t) = \int_a^b |G_0(t,s)|([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s))\, ds + w_\varepsilon(t),$$

where G_0 is Green's function of the problem (1.3.21), ($1.3.22_2$), and w_ε is defined just as in the previous proof. Then reasoning in the same manner as when proving Corollary $1.1.1_1$, we make sure that the inclusion ($1.1.7_2$) is valid for $\beta > 0$.

In the case $\beta = 0$, we consider the function z_λ for $\lambda = 0$ and the function w defined by (1.3.35), where v is a solution of the equation (1.3.21) under the boundary conditions

$$v(a) = 1, \qquad v'(b-) = 1.$$

Then reasoning just in the same way as in proving Corollary $1.1.1_1$ for $\beta = 0$, we can see that the inclusion $(p_0, p_1) \in \mathbb{V}_{2,0}(]a,b[\,;h)$ is valid. □

Proof of Corollary $1.1.1_{i0}$. Coincides completely with that of Corollary $1.1.1_i$ for $\beta > 0$. □

Proof of Remark 1.1.4. Denote the left-hand side of ($1.1.32_i$) by w. Then it is obvious that

$$w(t) \le \int_a^b \frac{[p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s)h(x^\beta)(s)}{\sigma(p_1)(s)} ds \quad \text{for} \quad a \le t \le b,$$

i.e., it follows from ($1.1.34_i$) that the condition ($1.1.32_i$) is valid. On the other hand, ($1.1.34_i$) implies the inclusion

$$h \in \mathcal{L}\Big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\Big)$$

which together with (1.1.33) means that ($1.1.8_i$) is satisfied. □

Proof of Remark $1.1.4_0$. As is seen from the proof of Remark 1.1.4, the conditions $(1.1.32_i)$ and (1.1.12) follow simultaneously from $(1.1.34_i)$. □

Proof of Corollary $1.1.2_i$. Introduce the notation

$$g(u)(t)=\sum_{k=1}^{n} g_k(t)u(\tau_k(t)) \tag{1.3.38}$$

and

$$h(u)(t)=\sum_{k=1}^{n} |g_k(t)|u(\tau_k(t)). \tag{1.3.39}$$

Then for any $u \in C(]a,b[)$ almost everywhere on the interval $]a,b[$ the inequality (1.1.10) is satisfied, and as is seen from $(1.1.36_i)$, the inclusion $(1.1.8_i)$ is valid. It is also clear that the condition $(1.1.37_i)$ in our notation can be rewritten as $(1.1.32_i)$. Hence all the requirements of Corollary $1.1.1_i$ are fulfilled and our corollary is valid. □

Proof of Corollary $1.1.2_{i0}$. Define the operators g and h by the equalities (1.3.38) and (1.3.39) and note that from the condition (1.3.38) it follows the inclusion (1.1.12). Reasoning similarly as when proving the above corollary, we can see that our corollary is valid. □

Proof of Remark 1.1.5. Denote the left-hand side of $(1.1.37_i)$ by w. Then it is evident that

$$w(t) \le \int_a^b \frac{[p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s)\sum\limits_{k=1}^{n} |g_k(s)| x^\beta(\tau_k(s))}{\sigma(p_1)(s)}\, ds \quad \text{for} \quad a \le t \le b,$$

i.e., $(1.1.40_i)$ implies the validity of the condition $(1.1.37_i)$. On the other hand, $(1.1.40_i)$ implies the inclusion

$$g_k x^\beta(\tau_k) \in L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b])$$

which together with (1.1.39) means that $(1.1.36_i)$ is satisfied. □

Proof of Remark $1.1.5_0$. As is seen from the proof of Remark 1.1.5, the conditions $(1.1.37_i)$ and (1.1.38) follow simultaneously from $(1.1.40_i)$. □

Proof of Corollary $1.1.3_1$. It is clear that all the requirements of Theorem $1.1.1_i$, except $(1.1.7_i)$, follow directly from the conditions of our corollary. It remains to show that the conditions (1.1.41), $(1.1.42_1)$ imply the inclusion $(1.1.7_1)$ as well, where $h(u)(t)=\sum\limits_{k=1}^{n} |g_k(t)|u(\tau_k(t))$.

Indeed, let $\beta > 0$ and

$$z(t) = \Big[\sum_{k=1}^{n} \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} x^\beta(\tau_k(s)) \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds \Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^\alpha +$$

$$+ \sum_{k=1}^{n} \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} x^\beta(\tau_k(s)) \Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^\alpha ds \Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha\Big] \times$$

$$\times \frac{\big(\int_a^b \sigma(p_1)(\eta)d\eta\big)^{1-2(\alpha+\beta)}}{2^{2-2(\alpha+\beta)}}.$$

Then as is seen from $(1.1.42_1)$, for every $m \in \{1, \dots, n\}$

$$z(\tau_m(t)) < 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.40}$$

Moreover, let

$$w(t) = \sum_{k=1}^{n} \int_a^b |G_0(t,s)| g_k(s) x^\beta(\tau_k(s))\, ds + w_\varepsilon(t),$$

where the function w_ε is defined in the same way as in proving Corollary $1.1.1_1$, $\varepsilon > 0$, G_0 is Green's function of the problem (1.3.21), $(1.3.22_1)$ defined by the equality $(1.3.23_1)$ and by Lemma 1.2.7,

$$w_\varepsilon(t) = O^*(x^\beta(t)) \quad \text{as} \quad t \to a, \quad t \to b, \tag{1.3.41}$$

for any $\varepsilon > 0$. From the conditions (1.3.40), (1.3.41) it follows that we can choose a constant $\varepsilon > 0$ such that for every $m \in \{1, \dots, n\}$

$$z(\tau_m(t)) + \sup\Big\{\frac{w_\varepsilon(\tau_m(t))}{x^\beta(\tau_m(t))} : \ a < t < b\Big\} < 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.42}$$

Using the equality $(1.3.23_1)$ we can easily obtain the estimate

$$0 \le w(t) \le z(t)x^\beta(t) + w_\varepsilon(t) \quad \text{for} \quad a \le t \le b, \tag{1.3.43}$$

whence by virtue of (1.3.42) for every $m \in \{1, \dots, n\}$ the inequality

$$0 \le w(\tau_m(t)) \le x^\beta(\tau_m(t)) \quad \text{for} \quad a < t < b \tag{1.3.44}$$

is valid. Analogously, from (1.3.41) and (1.3.43) it follows the estimate

$$0 < w(t) \le r_0 x^\beta(t) \quad \text{for} \quad a < t < b, \tag{1.3.45}$$

where

$$r_0 = \sup\Big\{z(t) + \frac{w_\varepsilon(t)}{x^\beta(t)} : \ a < t < b\Big\} < +\infty,$$

and according to (1.3.41) we get

$$w(t) = O^*(x^\beta(t)) \quad \text{as} \quad t \to a, \quad t \to b. \tag{1.3.46}$$

On the other hand, it is clear that

$$w''(t) = p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)| x^{\beta}(\tau_k(t)) - q_{\beta}(t),$$

which with regard for the conditions (1.1.41) and (1.3.44) results in

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)| w(\tau_k(t)) - q_{\beta}(t), \quad (1.3.47)$$

where, as is seen from Remark 1.2.6,

$$(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[). \quad (1.3.48)$$

Then, as we have shown in proving Corollary $1.1.1_1$,

$$\int_a^b |G(t,s)| q_{\beta}(s)\, ds = O^*(x^{\beta}(t)) \ \text{ as } \ t \to a, \ \ t \to b, \quad (1.3.49)$$

where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$. It remains to notice that the conditions (1.3.45), (1.3.46), (1.3.48), (1.3.49) and the inequality (1.3.47) by virtue of Definition 1.1.4 imply the inclusion $(1.1.7_1)$ for $\beta > 1$.

Suppose now that $\beta = 0$ and

$$w(t) = \sum_{k=1}^{n} \int_a^b |G_0(t,s)||g_k(s)|\, ds + \varepsilon v(t), \quad (1.3.50)$$

where v is a solution of the equation (1.3.21) under the boundary conditions

$$v(a) = 1 \quad \text{and} \quad v(b) = 1.$$

Then, as is seen from the condition $(1.1.42_1)$, for every $m \in \{1, \dots, n\}$

$$z(\tau_m(t)) < 1 \quad \text{for} \quad a \le t \le b$$

and hence for every $m \in \{1, \dots, n\}$ we can choose $\varepsilon > 0$ small enough for the inequality

$$z(\tau_m(t)) + \varepsilon v(\tau_m(t)) \le 1 \quad \text{for} \quad a \le t \le b. \quad (1.3.51)$$

to be fulfilled. Note that from the positiveness of v and also from $(1.3.23_1)$ we have the estimate

$$0 < w(t) \le z(t) + \varepsilon v(t) \quad \text{for} \quad a \le t \le b$$

which by virtue of (1.3.51) for every $m \in \{1, \dots, n\}$ yields

$$0 < w(\tau_m(t)) \le 1 \quad \text{for} \quad a \le t \le b. \quad (1.3.52)$$

On the other hand,

$$w''(t) = p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|$$

which with regard for (1.1.41) and (1.3.52) gives

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|w(\tau_k(t)).$$

Hence, owing to Definition 1.1.3, the inclusion $(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[\,; h)$, is valid, where $h(u)(t) = \sum\limits_{k=1}^{n} |g_k(t)|u(\tau_k(t))$. □

Proof of Corollary 1.1.3$_2$. It is clear that all the requirements of Theorem 1.1.1$_2$, except (1.1.7$_2$), follow directly from the conditions of our corollary. It remains to show that the inclusion (1.1.7$_2$) follows from the condition (1.1.41), (1.1.42$_1$) as well.

To this end, we introduce for $\beta > 0$ the functions z and w by the equalities

$$z(t) = \Bigg[\sum_{k=1}^{n} \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} x^\beta(\tau_k(s)) \bigg(\int_a^t \sigma(p_1)(\eta)d\eta \bigg)^\alpha ds +$$
$$+ \sum_{k=1}^{n} \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} x^\beta(\tau_k(s))\, ds \bigg(\int_a^t \sigma(p_1)(\eta)d\eta \bigg)^\alpha \Bigg] \bigg(\int_a^b \sigma(p_1)(\eta)d\eta \bigg)^{1-(\alpha+\beta)}$$

and

$$w(t) = \sum_{k=1}^{n} \int_a^b |G_0(t,s)|\, |g_k(s)| x^\beta(\tau_k(s))\, ds + w_\varepsilon(t),$$

where G_0 is Green's function of the problem (1.3.21), (1.3.22$_2$) and w_ε is defined in the same way as in proving Corollary 1.1.1$_1$. Reasoning just as in proving Corollary 1.1.3$_1$, we make sure that the inclusion (1.1.7$_2$) is valid for $\beta > 0$.

In the case $\beta = 0$ we consider the function w defined by the equality (1.3.50), where v is a solution of the equation (1.3.21) for the boundary conditions

$$v(a) = 1, \qquad v'(b-) = 1.$$

Then, reasoning analogously as in proving Corollary 1.1.3$_1$ for $\beta = 0$, we can see that the inclusion $(p_0, p_1) \in \mathbb{V}_{2,0}(]a, b[\,; h)$ is valid. □

Proof of Corollary 1.1.3$_{i0}$. Coincides completely with that of Corollary 1.1.3$_i$ for $\beta > 0$. □

Proof of Remark 1.1.6. If the inequality (1.1.43$_i$) is satisfied for $t \in \theta_{\tau_1,\dots,\tau_n}$, then it will especially be satisfied on each of the sets θ_{τ_m}, where $m \in \{1, \dots, n\}$, i.e., each of the n inequalities of (1.1.42$_i$) will be satisfied. □

Proof of Corollary $1.1.4_i$ ($1.1.4_{i0}$). It is sufficient to substitute $p_0 \equiv 0$, $p_1 \equiv 0$, $k = 1$ in Remark $1.1.5_i$ ($1.1.5_{i0}$). □

Proof of Corollary $1.1.5_1$. It is clear that all the requirements of Theorem $1.1.1_1$, except ($1.1.7_1$), follow directly from the conditions of our corollary. It remains to show that the inclusion ($1.1.7_1$) follows from the conditions ($1.1.50_1$) for $0 \le \beta < 1$ and ($1.1.51_1$) for $\beta = 1$ as well.

Consider first the case $0 < \beta < 1$. Let x be a function defined by the equality ($1.1.9_1$). Then

$$(x^\beta(t))'' = p_1(t)(x^\beta(t))' - 2\beta^2 \frac{\sigma^2(p_1)(t)}{x^{1-\beta}(t)} -$$

$$-\beta(1-\beta)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\bigg(\Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^2 + \Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^2\bigg). \qquad (1.3.53)$$

From the condition ($1.1.50_1$) and the fact that the operator h is nonnegative it follows that

$$-\frac{x^{2-\beta}(t)}{\sigma^2(p_1)(t)} p_0(t) \le 2\beta^2 \bigg(\int_a^b \sigma(p_1)(\eta)d\eta\bigg)^{2(1-\beta)} \quad \text{for } a < t < b.$$

Moreover,

$$0 \le \lambda p_0(t) + \beta(1-\beta)\min\bigg\{\Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^2 +$$

$$+\Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^2 : \ a \le s \le b\bigg\}\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}, \qquad (1.3.54)$$

where

$$\lambda = \frac{1-\beta}{2\beta}\bigg(\int_a^b \sigma(p_1)(\eta)d\eta\bigg)^{-2(1-\beta)} \times$$

$$\times \min\bigg\{\Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^2 + \Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^2 : \ a \le s \le b\bigg\}.$$

Let $w(t) = x^\beta(t) + \lambda$, and rewrite the identity (1.3.53) as

$$w''(t) = p_0(t)w(t) + p_1(t)w'(t) - \Big(p_0(t)x^\beta(t) + 2\beta^2\frac{\sigma^2(p_1)(t)}{x^{1-\beta}(t)}\Big) -$$

$$-\bigg[\lambda p_0(t) + \beta(1-\beta)\bigg(\Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^2 + \Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^2\bigg)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\bigg].$$

Then, taking into account the fact that the operator h is nonnegative, from the condition $(1.1.50_1)$ and the inequality (1.3.54) we obtain

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t), \tag{1.3.55}$$

i.e., owing to Lemma 1.2.2 the inclusion

$$(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[) \tag{1.3.56}$$

is satisfied. Then, as is seen from Remark 1.2.2, there exists Green's function G of the problem (1.2.4), $(1.2.2_{i0})$, and by Lemma 1.2.6,

$$\int_a^b |G(t,s)| q_\beta(s)\, ds = O^*(x^\beta(t)) \quad \text{for} \quad t \to a, \quad t \to b, \tag{1.3.57}$$

where

$$q_\beta(t) = \frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}.$$

Let now

$$\varepsilon = \beta(1-\beta) \min \Big\{ \Big(\int_a^s \sigma(p_1)(\eta) d\eta \Big)^2 + \\ + \Big(\int_s^b \sigma(p_1)(\eta) d\eta \Big)^2 : \ a \le t \le b \Big\} \tag{1.3.58}$$

and rewrite (1.3.53) in the form

$$(x^\beta(t))'' = p_0(t)x^\beta(t) + p_1(t)(x^\beta(t))' - h(x^\beta)(t) - \varepsilon q_\beta(t) - \\ -\Big(p_0(t)x^\beta(t) - h(x^\beta)(t) + 2\beta^2 \frac{\sigma^2(p_1)(t)}{x^{1-\beta}(t)} \Big) - \Big[\beta(1-\beta) \Big(\Big(\int_a^t \sigma(p_1)(\eta) d\eta \Big)^2 + \\ + \Big(\int_t^b \sigma(p_1)(\eta) d\eta \Big)^2 \Big) - \varepsilon \Big] \frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}. \tag{1.3.59}$$

Taking into account $(1.1.50_1)$ and (1.3.58), we obtain

$$(x^\beta(t))'' \le p_0(t)x^\beta(t) + p_1(t)(x^\beta(t))' - h(x^\beta)(t) - \varepsilon q_\beta(t) \\ \text{for} \quad a < t < b. \tag{1.3.60}$$

From (1.3.56), (1.3.57), and (1.3.60), by virtue of Definition 1.1.4 we conclude that the inclusion $(1.1.7_1)$ is satisfied for $0 < \beta < 1$.

Assume now that $\beta = 0$. Then the condition $(1.1.50_1)$ takes the form

$$0 \le p_0(t) - h(1)(t) \quad \text{for} \quad a < t < b,$$

from which we can see that the function $w(t) \equiv 1$ satisfies the inequality

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - h(w)(t),$$

i.e., owing to Definition 1.1.3 we can conclude that the inclusion $(1.1.7_1)$ is satisfied for $\beta = 0$.

Finally we consider the case $\beta = 1$ and note that

$$x''(t) = p_1(t)x'(t) - 2\sigma^2(p_1)(t). \tag{1.3.61}$$

It follows from $(1.1.51_1)$ that there exist constants ε, $\mu \in]0,1[$ such that

$$\operatorname*{ess\,sup}_{t\in\,]a,b[} \left(\frac{x(t)}{\sigma^2(p_1)(t)} \Big(\frac{h(x)(t)}{x(t)} - p_0(t) \Big) \right) < 2\mu^2 \tag{1.3.62}$$

and

$$\operatorname*{ess\,sup}_{t\in\,]a,b[} \left(\frac{x(t)}{\sigma^2(p_1)(t)} \Big(\frac{h(x)(t)}{x(t)} - p_0(t) \Big) \right) < 2 - \varepsilon. \tag{1.3.63}$$

Taking into account the fact that the operator h is nonnegative, from the condition (1.3.62) we get

$$-\frac{x^{2-\mu}(t)}{\sigma^2(p_1)(t)} p_0(t) \le 2\mu^2 \Big(\int_a^b \sigma(p_1)(\eta)d\eta \Big)^{2(1-\mu)} \quad \text{for} \quad a < t < b.$$

Reasoning in the same way as for $0 < \beta < 1$, from the last inequality as well as from (1.3.62) we can see that the function $w(t) = x^\mu(t) + \lambda$, where

$$\lambda = \frac{1-\mu}{2\mu} \Big(\int_a^b \sigma(p_1)(\eta)d\eta \Big)^{-2(1-\mu)} \times$$

$$\times \min\left\{ \Big(\int_a^s \sigma(p_1)(\eta)d\eta \Big)^2 + \Big(\int_s^b \sigma(p_1)(\eta)d\eta \Big)^2 : \ a \le s \le b \right\},$$

satisfies (1.3.55), i.e., the inclusion (1.3.56) is satisfied and there exists Green's function G of the problem (1.2.4), $(1.2.2_{i0})$. As is seen from Lemma 1.2.7, if $q_1(t) = \sigma^2(p_1)(t)$, then

$$\int_a^b |G(t,s)| q_1(s)\, ds = O^*(x(s)) \quad \text{as} \quad t \to a, \ \ t \to b. \tag{1.3.64}$$

We rewrite now the identity (1.3.61) as follows:

$$\begin{aligned} x''(t) = p_0(t)x(t) + p_1(t)x'(t) - h(x)(t) - \varepsilon q_1(t) + \\ + \big(h(x)(t) - p_0(t)x(t) - (2-\varepsilon)\sigma^2(p_1)(t) \big). \end{aligned}$$

The latter with regard for (1.3.63) yields

$$x''(t) \le p_0(t)x(t) + p_1(t)x'(t) - h(x)(t) - \varepsilon q_1(t) \quad \text{for} \quad a < t < b. \quad (1.3.65)$$

From (1.3.56), (1.3.64), and (1.3.65), according to Definition 1.1.4 we conclude that the inclusion $(1.1.7_1)$ is satisfied for $\beta = 1$. □

Proof of Corollary $1.1.5_2$. It is clear that all the requirements of Theorem $1.1.1_2$, except $(1.1.7_2)$, follow directly from the conditions of our corollary. It remains to show that the inclusion $(1.1.7_2)$ follows from the conditions $(1.1.50_2)$, (1.1.56) for $0 < \beta \le 1$ and from $(1.1.51_2)$ for $\beta = 1$.

First we consider the case $0 < \beta < 1$. Let x be the function defined by $(1.1.9_2)$. Then

$$(x^\beta(t))'' = p_1(t)(x^\beta(t))' - \beta(1-\beta)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}. \quad (1.3.66)$$

From $(1.1.50_2)$ it follows the existence of a constant $\varepsilon > 0$ such that

$$\operatorname*{ess\,sup}_{t\in\,]a,b[} \Big[\frac{x^2(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x^\beta)(t)}{x^\beta(t)} - p_0(t)\Big)\Big] < \beta(1-\beta) - \varepsilon \quad (1.3.67)$$

and likewise from the inclusion (1.1.55) it follows the existence of a constant λ such that

$$-\lambda\frac{x^{2-\beta}(t)}{\sigma^2(p_1)(t)}p_0(t) < \varepsilon \quad \text{for} \quad a < t < b. \quad (1.3.68)$$

Let $w(t) = x^\beta(t) + \lambda$, and rewrite the identity (1.3.66) in the form

$$w''(t) = p_0(t)w(t) + p_1(t)w'(t) - \Big(p_0(t)x^\beta(t) + \lambda p_0(t) + \beta(1-\beta)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\Big),$$

whence with regard for (1.3.67), (1.3.68) and the fact that the operator h is nonnegative we can see that the inequality (1.3.55) is valid, i.e., by virtue of Lemma 1.2.2 the inclusion

$$(p_0, p_1) \in \mathbb{V}_{2,0}(]a,b[) \quad (1.3.69)$$

is satisfied. Then, as is seen from Remark 1.2.2, there exists Green's function G of the problem (1.2.4), $(1.2.2_{20})$, and by Lemma 1.2.7,

$$\int_a^b |G(t,s)|q_\beta(s)\,ds = O^*(x^\beta(s)) \quad \text{as} \quad t \to a, \quad (1.3.70)$$

where

$$q_\beta(t) = \frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}.$$

Rewrite now (1.3.66) as

$$(x^\beta(t))'' = p_0(t)x^\beta(t) + p_1(t)(x^\beta(t))' - h(x^\beta) - \varepsilon q_\beta(t) +$$

$$+\Big(h(x^\beta)(t)-p_0(t)x^\beta(t)-(\beta(1-\beta)-\varepsilon)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\Big).$$

This equality by virtue of the condition (1.3.67) enables us to see that (1.3.60) is satisfied. From the conditions (1.3.60), (1.3.69), (1.3.70) and according to Definition 1.1.4, we can conclude that the inclusion $(1.1.7_2)$ is satisfied for $0<\beta<1$.

Assume now that $\beta=1$. From the condition $(1.1.50_2)$ for $\beta=1$ it follows the existence of a constant $\varepsilon>0$ such that

$$\operatorname*{ess\,sup}_{t\in\,]a,b[}\left[\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x)(t)}{x(t)}-p_0(t)\Big)\right]<-\varepsilon. \tag{1.3.71}$$

Then it is clear from the negativeness of the operator h that

$$p_0(t)\geq 0 \quad \text{for} \quad a<t<b,$$

i.e., by virtue of Remark 1.2.6, the inclusion (1.3.69) is satisfied and hence there exists Green's function G of the problem (1.2.4), $(1.2.2_{20})$. As is seen from lemma 1.2.7, if $q_1(t)=\sigma^2(p_1)(t)$, then

$$\int_a^b |G(t,s)|q_1(s)\,ds=O^*(x(t)) \quad \text{as} \quad t\to a. \tag{1.3.72}$$

Note that

$$x''(t)=p_0(t)x(t)+p_1(t)x'(t)-h(x)(t)-\varepsilon q_1(t)+$$
$$+\big(h(x)(t)-p_0(t)x(t)+\varepsilon\sigma^2(p_1)(t)\big),$$

whence with regard for (1.3.71) we see that (1.3.65) is satisfied.

From the conditions (1.3.65), (1.3.69), (1.3.72), owing to Definition 1.1.4 we conclude that the inclusion $(1.1.7_2)$ is satisfied for $\beta=1$ as well.

The proof of the given and of the previous corollary is identical for the case $\beta=0$. □

Proof of Corollary $1.1.5_{i0}$. Coincides completely with that of Corollary $1.1.5_i$ for $0<\beta\leq 1$. □

Proof of Corollary $1.1.6_1$. Let

$$h(u)(t)=\sum_{k=1}^n |g_k(t)|u(\tau_k(t)). \tag{1.3.73}$$

Then we can see from $(1.1.56_1)$ that the inclusion $(1.1.8_1)$ is satisfied for $\beta=0$. It is also clear that all the requirements of Theorem $1.1.1_1$ for $\alpha=1$, $\beta=0$, except $(1.1.7_1)$, follow directly from the conditions of our corollary. It remains to show that the conditions $(1.1.57_1)$, $(1.1.58_1)$ imply the inclusion $(1.1.7_1)$ as well.

Without restriction of generality we assume that $c \in]a,b[$. Then by $(1.1.57_1)$ there exist γ_m, η_m $(m=1,2)$ such that

$$0 \le \gamma_m < \eta_m < +\infty \quad (m=1,2)$$

and

$$\begin{aligned} \int\limits_{\gamma_1}^{\eta_1} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} &= \frac{(c-a)^{1-\beta_1}}{1-\beta_1}, \\ \int\limits_{\gamma_2}^{\eta_2} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} &= \frac{(b-c)^{1-\beta_2}}{1-\beta_2}. \end{aligned} \tag{1.3.74}$$

Introduce the functions φ_1 and φ_2 by

$$\int\limits_{\varphi_1(t)}^{\eta_1} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} = \frac{(t-a)^{1-\beta_1}}{1-\beta_1} \quad \text{for} \quad a \le t \le c$$

and

$$\int\limits_{\varphi_2(t)}^{\eta_2} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} = \frac{(b-t)^{1-\beta_2}}{1-\beta_2} \quad \text{for} \quad c \le t \le b.$$

From (1.3.74) we have

$$\gamma_1 < \varphi_1(t) < \eta_1 \ \text{ for } \ a < t < c, \ \ \gamma_2 < \varphi_2(t) < \eta_2 \ \text{ for } \ c < t < b \ \text{ and} \\ \varphi_m(c) = \gamma_m \quad (m=1,2).$$

Introduce also the function w by

$$w(t) = \exp\left(\int\limits_c^t (s-a)^{-\beta_1}\varphi_1(s)\,ds\right) \quad \text{for} \quad a \le t < c,$$

$$w(t) = \exp\left(\int\limits_t^c (b-s)^{-\beta_2}\varphi_2(s)\,ds\right) \quad \text{for} \quad c \le t \le b.$$

Then

$$\begin{gathered} w'(t) > 0 \quad \text{for} \quad a < t < c, \quad w'(t) < 0 \quad \text{for} \quad c \le t < b, \\ w(t) > 0 \quad \text{for} \quad a \le t \le b, \end{gathered} \tag{1.3.75}$$

$$w \in \widetilde{C}'_{\text{loc}}(\,]a,c[) \cap \widetilde{C}'_{\text{loc}}(\,]c;b[), \quad w(c-) \ge w(c+), \tag{1.3.76}$$

and the equalities

$$
\begin{aligned}
w''(t) &= -\frac{\lambda_{11}}{(t-a)^{2\beta_1}}\,w(t) - \Big[\frac{\lambda_{12}}{(t-a)^{\beta_1}} + \frac{\beta_1}{t-a}\Big]\,w'(t) \\
&\qquad\qquad\qquad\qquad \text{for} \quad a < t < c, \\
w''(t) &= -\frac{\lambda_{21}}{(b-t)^{2\beta_2}}\,w(t) + \Big[\frac{\lambda_{22}}{(b-t)^{\beta_2}} + \frac{\beta_2}{b-t}\Big]\,w'(t) \\
&\qquad\qquad\qquad\qquad \text{for} \quad c \le t < b
\end{aligned} \tag{1.3.77}
$$

are valid.

From the above equalities, by virtue of (1.3.75) it follows that

$$
w''(t) \le 0 \quad \text{for} \quad a < t < b. \tag{1.3.78}
$$

On the other hand, taking into account the conditions ($1.1.58_1$) in the equalities (1.3.77), we obtain

$$
\begin{aligned}
w''(t) \le \Big(p_0(t) - \sum_{k=1}^{n} |g_k(t)|\Big)\,w(t) + p_1(t)w'(t) - \\
-w'(t)\sum_{k=1}^{n} |g_k(t)|\big(\tau_k(t) - t\big) \quad \text{for} \quad a < t < b.
\end{aligned} \tag{1.3.79}
$$

Analogously, from (1.3.78) it follows

$$
\int\limits_{t}^{\tau_k(t)} w'(s)\,ds \le w'(t)\big(\tau_k(t) - t\big) \quad (k = 1, \dots, n) \ \text{for} \quad a < t < b.
$$

Taking this inequality into consideration, from (1.3.79) we can see that

$$
w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|\,w\big(\tau_k(t)\big) \ \text{for} \ a < t < b.
$$

The latter inequality together with (1.3.75), (1.3.76) and by virtue of Definition 1.1.3 shows that the inclusion $(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[; h)$ is satisfied. □

Proof of Corollary $1.1.6_2$. We define the operator h by the equality (1.3.73). Note also that if $p_1 \in L_{\mathrm{loc}}(]a, b])$, then from the conditions (1.1.56) and (1.1.59) we obtain

$$
\begin{gathered}
\sigma(p_1) \in L([a, b]), \ \ p_j\sigma_2(p_1) \in L([a, b]) \ \ (j = 0, 2), \\
g_k\sigma_2(p_1) \in L([a, b]) \ \ (k = 1, \dots, n),
\end{gathered}
$$

i.e., the conditions ($1.1.3_2$), ($1.1.5_2$), and ($1.1.8_2$), are satisfied where $\beta = 0$, $\alpha = 1$. Then just as in the previous proof it remains to show that from the conditions ($1.1.57_2$)–(1.1.59) it follows the inclusion ($1.1.7_2$) for $\beta = 0$.

Without restriction of generality we assume that $c \in]a,b[$. Then by virtue of $(1.1.57_2)$ there exist constants γ_m, η_m $(m=1,2)$ such that

$$\varepsilon \le \gamma_1 < \eta_1 < +\infty, \quad 0 < \gamma_2 < \eta_2 < +\infty$$

and (1.3.74) is satisfied. Introduce the functions φ_1 and φ_2 by

$$\int\limits_{\varphi_1(t)}^{\eta} \frac{ds}{\lambda_{11} + \lambda_{12}s + s^2} = \frac{(t-a)^{1-\beta_1}}{1-\beta_1} \quad \text{for} \quad a \le t < c,$$

$$\int\limits_{\gamma_2}^{\varphi_2(t)} \frac{ds}{\lambda_{21} + \lambda_{22}s + s^2} = \frac{(b-t)^{1-\beta_2}}{1-\beta_2} \quad \text{for} \quad c \le t \le b.$$

From (1.3.74) we have

$$\gamma_1 < \varphi_1(t) < \eta_1 \ \text{ for } \ a < t < c, \ \ \gamma_2 < \varphi_2(t) < \eta_2 \ \text{ for } \ c < t < b,$$
$$\varphi_1(c) = \gamma_1 \quad \varphi_2(c) = \eta_2.$$

Introduce likewise the function w by the equalities

$$w(t) = \exp\Big(\int\limits_a^t (s-a)^{-\beta_1}\varphi_1(s)\,ds\Big) \quad \text{for} \quad a \le t < c,$$

$$w(t) = \exp\Big(\alpha\int\limits_c^t (b-s)^{-\beta_3}\varphi_2(s)\,ds\Big) \quad \text{for} \quad c \le t \le b,$$

where $0 < \alpha < \min\Big(1; \frac{\gamma_1}{\eta_2}(b-c)^{-\beta_3}(c-a)^{-\beta_1}\Big)$, i.e.,

$$\alpha \in]0,1[. \tag{1.3.80}$$

Then

$$w'(t) > 0 \ \text{ for } \ t \in]a,c[\cup]c,b[, \ \ w(t) > 0 \ \text{ for } \ a \le t \le b, \tag{1.3.81}$$

$$w \in \widetilde{C}'_{\text{loc}}(]a,c[) \cap \widetilde{C}'_{\text{loc}}(]c;b[), \ \ w(c-) \ge w(c+), \ \ w'(b-) \ge 0, \tag{1.3.82}$$

and the equalities

$$w''(t) = -\frac{\lambda_{11}}{(t-a)^{2\beta_1}}w(t) - \Big[\frac{\lambda_{12}}{(t-a)^{\beta_1}} + \frac{\beta_1}{t-a}\Big]w'(t) \quad \text{for} \quad a < t < c \tag{1.3.83}$$

and

$$w''(t) = -\frac{\alpha\lambda_{21}}{(b-t)^{\beta_2-\beta_3}}w(t) - \Big[\frac{\lambda_{22}}{(b-t)^{\beta_2}} + \frac{\beta_3}{b-t}\Big]w'(t) -$$
$$-\alpha\big[1-\alpha(b-t)^{\beta_2+\beta_3}\big](b-t)^{\beta_3-\beta_2}w(t)\varphi_2^2(t), \ \ \text{for} \ \ c < t < b \tag{1.3.84}$$

are valid. Note also that the condition $c \in [\max(a, b-1); b]$ and (1.3.80) imply

$$1 - \alpha(b-t)^{\beta_2+\beta_3} \geq 0 \quad \text{for} \quad c \leq t \leq b.$$

Taking this into account in the equality (1.3.84), we obtain

$$w''(t) \leq -\frac{\alpha\lambda_{21}}{(b-t)^{\beta_2-\beta_3}}\,w(t) - \Big[\frac{\lambda_{22}}{(b-t)^{\beta_2}} + \frac{\beta_3}{b-t}\Big]\,w'(t). \qquad (1.3.85)$$
$$\text{for} \quad a \leq t < b.$$

From (1.3.83) and (1.3.85), according to the condition (1.3.81), it is clear that the inequality (1.3.78) is satisfied.

On the other hand, taking into account in (1.3.83) and (1.3.85) the conditions $(1.1.58_2)$, we get

$$w''(t) \leq \Big(p_0(t) - \sum_{k=1}^{n} |g_k(t)|\Big)\,w(t) + \widetilde{p}_1(t)w'(t) -$$
$$- w'(t)\sum_{k=1}^{n} |g_k(t)|\big(\tau_k(t) - t\big) \quad \text{for} \quad a < t < b,$$

which with regard for (1.3.81) and (1.1.59) imply that (1.3.79) is satisfied. Reasoning in the same way as in the previous proof, we see that the inclusion $(p_0, p_1) \in \mathbb{V}_{2,0}(]a, b[\,; h)$ is valid. □

Proof of Corollary $1.1.7_1$. It is not difficult to notice that if we introduce the notation

$$g(u)(t) = \sum_{k=1}^{n} g_k(t)u\big(\tau_k(t)\big),$$

then the inequality (1.1.22) will be satisfied, and from (1.1.61), (1.1.62) it follows that the conditions (1.1.17) and (1.1.18) are valid. That is, all the requirements of Theorem $1.1.2_1$ are fulfilled and this implies that our corollary is valid. □

Proof of Remark 1.1.10. Follows directly from that of Remark 1.1.2. □

Corollaries $1.1.7_2$ and $1.1.7_{i0}$ are proved analogously to Corollary $1.1.7_1$.

CHAPTER II

CORRECTNESS OF TWO-POINT PROBLEMS FOR LINEAR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

§ 2.1. Statement of the Problem and Formulation of Main Results

2.1.1. Statement of the Problem.

Let us Consider the functional differential equations

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t), \tag{2.1.1}$$

$$u''(t) = p_{0k}(t)u(t) + p_{1k}(t)u'(t) + g_k(u)(t) + p_{2k}(t), \quad k \in \mathbb{N}, \tag{2.1.1$_k$}$$

under one of the following the boundary conditions

$$u(a) = 0, \quad u(-b) = 0, \tag{2.1.2$_{10}$}$$

$$u(a) = 0, \quad u'(b-) = 0; \tag{2.1.2$_{20}$}$$

$$u(a) = c_1, \quad u(b) = c_2, \tag{2.1.2$_1$}$$

$$u(a) = c_1, \quad u'(b-) = c_2; \tag{2.1.2$_2$}$$

$$u(a) = c_{1k}, \quad u(b) = c_{2k}, \tag{2.1.2$_{1k}$}$$

$$u(a) = c_{1k}, \quad u'(b-) = c_{2k}, \tag{2.1.2$_{2k}$}$$

where c_l, $c_{l_k} \in \mathbb{R}$, $(l = 1, 2; k \in \mathbb{N})$, g, $g_k : C(]a, b[) \to L_{\text{loc}}(]a, b[)$, $k \in \mathbb{N}$, are continuous operators,

$$\begin{gathered} p_1, p_j \in L_{\text{loc}}(]a, b[) \quad \sigma(p_1) \in L([a, b]), \\ p_j \in L_{\sigma_1(p_1)}([a, b]) \ (j = 0, 2) \end{gathered} \tag{2.1.3$_1$}$$

if $i = 1$,

$$\begin{gathered} p_1, p_j \in L_{\text{loc}}(]a, b]) \quad \sigma(p_1) \in L([a, b]), \\ p_j \in L_{\sigma_2(p_1)}([a, b]) \quad (j = 0, 2) \end{gathered} \tag{2.1.3$_2$}$$

if $i = 2$, and $p_{jk} :]a, b[\to \mathbb{R}$ $(j = 0, 1, 2;\ k \in \mathbb{N})$ are measurable functions.

The correctness of the problem (2.1.1), (2.1.2$_i$) will be studied under the assumption that the inclusion

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a, b[\,; h)$$

is satisfied. (Effective sufficient conditions for the above inclusion to be fulfilled are given in §1.1, where

$$|g(x)(t)| \le h(|x|)(t)$$

almost everywhere in the interval $]a, b[$ for every $x \in C(]a, b[)$.)

Consider also the following linear equation

$$u''(t) = p_{0k}(t)u(t) + p_{1k}(t)u'(t) + p_{2k}(t). \tag{2.1.4$_k$}$$

Let G_k be Green's function of the problem $(2.1.4_k)$, $(2.1.2_{i0})$ and $r \in \mathbb{R}^+$. Then we denote the set

$$\left\{y(t): \ \ y(t) = \alpha_1 \widetilde{v}_k(t) + \int_a^b G_k(t,s) g_k(x)(s)\,ds, \ \ \alpha_1 \in [0,r], \ \ \|x\|_C \le r\right\}$$

by $\mathbb{B}_{r,k}$ if $\widetilde{v}_k$ is a solution of the problem $(2.1.4_k)$, $(2.1.2_{i0})$, and by $\mathbb{B}'_{r,k}$ if $\widetilde{v}_k$ is a solution of the problem $(2.1.4_k)$, $(2.1.2_{ik})$.

Throughout this chapter the use will also be made of the notation

$$I_i(x)(t) = \int_a^t x(s)\,ds \biggl(\int_t^b x(s)\,ds\biggr)^{2-i} \quad \text{for} \quad a \le t \le b,$$

where $x \in L([a,b])$.

2.1.2. Formulation of Main Results.

Theorem 2.1.1$_i$. *Let* $i \in \{1,2\}$, *the continuous linear operators* g, $g_k, h : C(]a,b[) \to L_{\text{loc}}(]a,b[)$ $(k \in \mathbb{N})$, *the measurable functions* p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1,2;\ k \in \mathbb{N})$ *and the constants* $\alpha \in [a,b]$, $\gamma \in]1,+\infty[$, $\beta, \mu \in \mathbb{R}$ *be such that*

$$0 \le \beta < \mu < \frac{\gamma-1}{\gamma-\alpha}\,, \tag{2.1.5}$$

$$\begin{gathered} \sigma^\gamma(p_1) \in L([a,b]), \ \ \int_a^b \frac{|p_j(s)|}{\sigma(p_1)(s)}\, I_i^\mu(\sigma^\alpha(p_1))(s)\,ds < +\infty \ \ (j=0,2), \\ \int_a^b \frac{h(1)(s)}{\sigma(p_1)(s)}\, I_i^\mu(\sigma^\alpha(p_1))(s)\,ds < +\infty, \end{gathered} \tag{2.1.6}$$

where h is a non-negative operator and uniformly on the segment $[a,b]$

$$\begin{gathered} \lim_{k\to\infty} \int_a^t |p_1(s) - p_{1k}(s)|\,ds = 0, \\ \lim_{k\to\infty} \int_a^t \frac{p_j(s) - p_{jk}(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds = 0 \ \ \ (j=0,2), \end{gathered} \tag{2.1.7}$$

$$\lim_{k\to\infty} \biggl(\sup\biggl\{\biggl|\int_a^t \frac{g(y)(s) - g_k(y(s))}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\biggr| : \ a \le t \le b, \ \ y \in \mathbb{B}_{1k}\biggr\}\biggr) = 0. \tag{2.1.8}$$

Moreover, let

$$(p_0, p_1) \in \mathbb{V}_{i,0}(\,]a,b[, h), \tag{2.1.9}$$

where for every $x \in C(]a,b[)$ *almost everywhere in the interval* $]a,b[$ *the inequality*

$$|g(x)(t)| \le h(|x|)(t) \tag{2.1.10}$$

is satisfied. Then there exists a number k_0 *such that if* $k > k_0$*, then the problem* $(2.1.1_k)$, $(2.1.2_{i0})$ *has a unique solution* u_k *and uniformly in the interval* $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)(u(t)-u_k(t)) = 0, \tag{2.1.11}$$

$$\lim_{k\to\infty} \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)}(u'(t)-u_k'(t)) = 0, \tag{2.1.12}$$

where u *is the solution of the problem* (2.1.1), $(2.1.2_{i0})$.

Theorem 2.1.2$_i$. *Let* $i \in \{1,2\}$*, the continuous linear operators* g, $g_k, h : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ $(k \in \mathbb{N})$*, the measurable functions* p_j, $p_{jk} : (]a,b[) \to \mathbb{R}$ $(j = 0,1,2;\, k \in \mathbb{N})$ *and the constans* $\alpha \in [a,b]$, $\gamma \in]1,+\infty[$, c_l, c_{lk}, β, $\mu \in \mathbb{R}$ $(l = 1,2;\; k \in \mathbb{N})$ *be such that conditions* (2.1.5)–(2.1.7), (2.1.9), (2.1.10) *and also*

$$\lim_{k\to\infty}\left(\sup\left\{\Big|\int_a^t \frac{g(y)(s)-g_k(y(s))}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds\Big| : a \le t \le b,\ x \in \mathbb{B}'_{1k}\right\}\right) = 0 \tag{2.1.13}$$

and

$$\lim_{k\to\infty} c_{lk} = c_l \quad (l = 1,2) \tag{2.1.14}$$

are satisfied. Then there exists a number k_0 *such that if* $k > k_0$*, the problem* $(2.1.1_k)$, $(2.1.2_{i0})$ *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12) *and*

$$\lim_{k\to\infty}\big(u(t)-u_k(t)\big) = 0 \tag{2.1.15}$$

are satisfied, where u *is the solution of the problem* (2.1.1), $(2.1.2_{i0})$.

2.1.3. Corollaries of Theorems $(2.1.1_i)$ $(2.1.2_i)$ $(i = 1,2)$.

Corollary 2.1.1$_i$. *Let* $i \in \{1,2\}$*, the continuous linear operators* g, $g_k, h : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ $(k \in \mathbb{N})$*, the measurable functions* η, p_j, p_{jk} $:]a,b[\to \mathbb{R}$ $(j = 0,1,2;\, k \in \mathbb{N})$ *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}^+$

be such that the conditions (2.1.5)–(2.1.7), (2.1.9), (2.1.10) *are satisfied and for every* $y \in \widetilde{C}(]a,b[)$ *almost everywhere on the interval* $]a,b[$

$$\big|g_k(y)(t) - g(y)(t)\big| \le \eta(t)\|y\|_C \quad (k \in \mathbb{N}) \tag{2.1.16}$$

and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \int_a^t \frac{g_k(y)(s) - g(y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds = 0, \tag{2.1.17}$$

where

$$\int_a^b \frac{\eta(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds < +\infty. \tag{2.1.18}$$

Then there exists a number k_0*, such that for* $k > k_0$ *the problem* $(2.1.1_k)$, $(2.1.2_{i0})$ *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the equalities* (2.1.11), (2.1.12) *are satisfied, where* u *is the solution of the problem* (2.1.1), $(2.1.2_{i0})$.

Corollary 2.1.2$_i$. *Let* $i \in \{1,2\}$*, the continuous linear operators* g, g_k, h : $C(]a,b[) \to L_{\rm loc}(]a,b[)$ $(k \in \mathbb{N})$*, the measurable functions* η, p_j, p_{jk} $:]a,b[\to \mathbb{R}$, $(j = 0,1,2;\ k \in \mathbb{N})$ *and constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}^+$ *be such that the conditions* (2.1.5)–(2.1.7), (2.1.9), (2.1.10), (2.1.14), *and* (2.1.16)–(2.1.18) *are satisfied. Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* $(2.1.1_k)$, $(2.1.2_{ik})$ *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12), (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.1), $(2.1.2_i)$.

Consider now the case where the equations (2.1.1) and $(2.1.1_k)$ are of the form

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + \sum_{m=1}^{n} g_{0m}(t)u(\tau_{0m}(t)) + p_2(t) \tag{2.1.19}$$

and

$$u''(t) = p_{0k}(t)u(t) + p_{1k}(t)u'(t) + \sum_{m=1}^{n} g_{km}(t)u(\tau_{km}(t)) + p_{2k}(t), \tag{$2.1.19_k$}$$

where g_{0m}, g_{km} $:]a,b[\to \mathbb{R}$ and τ_{0m}, τ_{km} : $[a,b] \to [a,b]$ $(m = 1,\dots,n,$ $k \in \mathbb{N})$ are measurable functions.

Corollary 2.1.3$_i$. *Let* $i \in \{1,2\}$*, the measurable functions* η, g_{0m}, g_{km}, p_j, p_{jk} $:]a,b[\to \mathbb{R}$, τ_{0m}, τ_{km} : $[a,b] \to [a,b]$, $(m = 1,\dots,n;\ j = 0,1,2;\ k \in$

$\mathbb{N}$) *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ *be such that conditions* (2.1.5), (2.1.7), (2.1.18) *as well as*

$$\sigma^{\gamma}(p_1) \in L([a,b]),$$

$$\int_a^b \Big[|p_j(s)| + \sum_{m=1}^n |g_{0m}(s)|\Big] \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}{\sigma(p_1)(s)}\, ds < +\infty \quad (j = 0, 2), \tag{2.1.20}$$

$$\Big|\sum_{m=1}^n \big(g_{0m}(t) - g_{km}(t)\big)\Big| \le \eta(t) \quad (k \in \mathbb{N}) \tag{2.1.21}$$

are satisfied, and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \sum_{m=1}^n \Big| \int_a^t \frac{g_{km}(s) - g_{0m}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\, ds\Big| = 0, \tag{2.1.22}$$

$$\operatorname{ess\,sup}\left\{ I_i^{\beta-\mu}(\sigma^{\alpha}(p_1))(t) \sum_{m=1}^n \Big| \int_{\tau_{0m}(t)}^{\tau_{km}(t)} \frac{\sigma(p_1)(s)}{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}\, ds\Big| : a<t<b\right\} \to 0 \quad \text{as } k \to +\infty. \tag{2.1.23}$$

Let also the condition (2.1.9) *be satisfied, where*

$$h(x)(t) = \sum_{m=1}^n |g_{0m}(t)| x(\tau_{0m}(t)).$$

Then there exists a number k_0 *such that for* $k > k_0$ *the problem* ($2.1.19_k$), ($2.1.2_{i0}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.11), (2.1.12) *are satisfied, where* u *is the solution of the problem* (2.1.19), ($2.1.2_{i0}$).

Corollary 2.1.4$_i$. *Let* $i \in \{1,2\}$, *the measurable functions* η, g_{0m}, g_{km}, p_j, p_{jk} $:]a,b[\to \mathbb{R}$, τ_{0m}, $\tau_{km} : [a,b] \to [a,b]$, ($m = 1,\dots,n$; $j = 0,1,2$; $k \in \mathbb{N}$) *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, c_l, c_{lk}, β, $\mu \in \mathbb{R}$ ($l = 1,2$; $k \in \mathbb{N}$) *be such that the conditions* (2.1.5), (2.1.7), (2.1.9), (2.1.14), (2.1.18), (2.1.20)–(2.1.23) *are satisfied, where* $h(x)(t) = \sum_{m=1}^n |g_{0m}(t)| x(\tau_{0m}(t))$. *Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* ($2.1.19_k$), ($2.1.2_{ik}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12), (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.19), ($2.1.2_i$).

Corollary 2.1.5$_i$. *Let* $i \in \{1,2\}$, *the measurable functions* η, g_{0m}, g_{km}, p_j, p_{jk} $:]a,b[\to \mathbb{R}$, τ_{0m}, $\tau_{km} : [a,b] \to [a,b]$, ($m = 1,\dots,n$; $j = 0,1,2$; $k \in \mathbb{N}$) *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ *be such that the* *conditions* (2.1.5), (2.1.7), (2.1.18), (2.1.22) *as well as*

$$\sigma^\gamma(p_1)\in L([a,b]),\quad \int\limits_a^b \frac{|p_j(s)|}{\sigma(p_1)(s)}\, I_i^\mu(\sigma^\alpha(p_1))(s)\,ds<+\infty\ \ (j=0,2), \tag{2.1.24}$$

$$\sum_{m=1}^n \big(|g_{km}(t)|+|g_{0m}(t)|\big)\le \eta(t)\ \ (k\in\mathbb{N})\ \ for\ \ a<t<b \tag{2.1.25}$$

and

$$\operatorname{ess\,sup}\Big\{\sum_{m=1}^n |\tau_{0m}(t)-\tau_{km}(t)|:\ \ a\le t\le b\Big\}\to 0\ \ for\ \ k\to+\infty \tag{2.1.26}$$

are satisfied. Let also the condition (2.1.9) *be satisfied, where* $h(x)(t)=\sum\limits_{m=1}^n |g_{0m}(t)|x(\tau_{0m}(t))$. *Then there exists a number* k_0 *such that for* $k>k_0$ *the problem* (2.1.19$_k$), (2.1.2$_{i0}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.11), (2.1.12) *are satisfied, where* u *is the solution of the problem* (2.1.19), (2.1.2$_{i0}$).

Corollary 2.1.6$_i$. *Let* $i\in\{1,2\}$, *the measurable functions* η, g_{0m}, g_{km}, p_j, p_{jm} $]a,b[\to\mathbb{R}$ τ_{0m}, $\tau_{km}:[a,b]\to[a,b]$, $(m=1,\dots,n;\ j=0,1,2;\ k\in\mathbb{N})$ *and the constants* $\alpha\in[0,1]$, $\gamma\in]1,+\infty[$, c_l, c_{lk}, β, $\mu\in\mathbb{R}$ $(l=1,2;\ k\in\mathbb{N})$ *be such that the conditions* (2.1.5), (2.1.7), (2.1.9), (2.1.14), (2.1.18), (2.1.22) *and* (2.1.24)–(2.1.26) *are satisfied, where* $h(x)(t)=\sum\limits_{m=1}^n |g_{0m}(t)|x(\tau_{0m}(t))$. *Then there exists a number* k_0 *such that for* $k>k_0$ *the problem* (2.1.19$_k$), (2.1.2$_{ik}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12), (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.19), (2.1.2$_{i0}$).

For more clearness, let us consider the equations

$$u''(t)=g_0(t)u(\tau_0(t))+p_2(t), \tag{2.1.27}$$

$$u''(t)=g_{0k}(t)u(\tau_k(t))+p_{2k}(t), \tag{2.1.27$_k$}$$

where g_0, g_{0k}, p_2, p_{2k}; $]a,b[\to\mathbb{R}$, and τ_0, τ_{0k}; $[a,b]\to[a,b]$ $(k\in\mathbb{N})$ are measurable functions.

Corollary 2.1.7$_i$. *Let* $i\in\{1,2\}$, *the measurable functions* η, g_0, g_{0k}, p_2, p_{2k} $:]a,b[\to\mathbb{R}$, τ_0, $\tau_k:[a,b]\to[a,b]$, $(k\in\mathbb{N})$ *and the constants* β, $\mu\in\mathbb{R}$ *be such that the conditions*

$$\beta<\mu<1, \tag{2.1.28}$$

$$|g_0(t)|+|g_{0k}(t)|\le\eta(t)\quad for\quad a<t<b, \tag{2.1.29}$$

$$\int_a^b |p_2(s)|(s-a)^{\mu}(b-s)^{\mu(2-i)}\,ds < +\infty,$$
$$\int_a^b \eta(s)(s-a)^{\beta}(b-s)^{\beta(2-i)}\,ds < +\infty \tag{2.1.30}$$

are satisfied, and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty}\int_a^t \big(p_2(s)-p_{2k}(s)\big)(s-a)^{\beta}(b-s)^{\beta(2-i)}\,ds = 0,$$
$$\lim_{k\to\infty}\int_a^t \big(g_0(s)-g_{0k}(s)\big)(s-a)^{\beta}(b-s)^{\beta(2-i)}\,ds = 0 \tag{2.1.31}$$

and

$$\operatorname{ess\,sup}\big\{|\tau_0(t)-\tau_k(t)| : \ a\le t\le b\big\} \to 0 \ \text{ as } \ k\to+\infty. \tag{2.1.32}$$

Let, moreover, the inclusion

$$(0,0)\in \mathbb{V}_{i,0}(]a,b[;h) \tag{2.1.33}$$

be satisfied, where $h(x)(t) = |g_0(t)|x(\tau_0(t))$. *Then there exists a number* k_0, *such that for* $k > k_0$, *the problem* (2.1.27$_k$), (2.1.2$_{i0}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the conditions* (2.1.11), (2.1.12) *are satisfied, where* u *is a solution of the problem* (2.1.27), (2.1.2$_{i0}$).

Corollary 2.1.8$_i$. *Let* $i\in\{1,2\}$, *the measurable functions* η, g_{0m}, g_{0k}, p_2, p_{2k} $:]a,b[\to\mathbb{R}$, τ_0, $\tau_k : [a,b]\to[a,b]$, $(k\in\mathbb{N})$ *and the constants* c_l, c_{lk}, β, $\mu\in\mathbb{R}$ $(l=1,2;\ k\in\mathbb{N})$ *be such that the conditions* (2.1.14) *and* (2.1.28)–(2.1.33) *are satisfied, where* $h(x)(t) = |g_0(t)|x(\tau_0(t))$. *Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* (2.1.27$_k$), (2.1.2$_{ik}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12), (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.27), (2.1.2$_i$).

§ 2.2. Auxiliary Propositions

2.2.1. Correctness of the Initial Problem for Linear Second Order Ordinary Differential Equations. Consider on the interval $]a,b[$ the equations

$$v''(t) = p_0(t)v(t) + p_1(t)u'(t) \tag{2.2.1}$$

and

$$v''(t) = p_{0k}(t)v(t) + p_{1k}(t)v'(t), \quad k\in\mathbb{N}, \tag{2.2.1$_k$}$$

where

$$p_0,\, p_1 \in L_{\rm loc}(]a,b[), \quad \sigma(p_1) \in L([a,b]), \quad p_0 \in L_{\sigma_1(p_1)},([a,b]) \qquad (2.2.2_1)$$

$$p_{0k}, p_{1k} \in L_{\rm loc}(]a,b[), \quad k \in \mathbb{N}, \qquad (2.2.3_1)$$

or

$$p_0,\, p_1 \in L_{\rm loc}(]a,b]), \quad \sigma(p_1) \in L(\,[a,b]\,), \quad p_0 \in L_{\sigma_2(p_1)}([a,b]), \qquad (2.2.2_2)$$

$$p_{0k},\, p_{1k} \in L_{\rm loc}(\,]a,b]), \quad k \in \mathbb{N}, \qquad (2.2.3_2)$$

and the following initial conditions:

$$v(a) = 0, \quad \lim_{t\to a} \frac{v'(t)}{\sigma(p_1)(t)} = 1, \qquad (2.2.4_1)$$

$$v(a) = 0, \quad \lim_{t\to a} \frac{v'(t)}{\sigma(p_{1k})(t)} = 1, \qquad (2.2.4_k)$$

$$v(b) = 0, \quad \lim_{t\to b} \frac{v'(t)}{\sigma(p_1)(t)} = -1, \qquad (2.2.5_1)$$

$$v(b) = 0, \quad \lim_{t\to b} \frac{v'(t)}{\sigma(p_{1k})(t)} = -1, \qquad (2.2.5_{1k})$$

$$v(b) = 1, \quad v'(b) = 0. \qquad (2.2.5_2)$$

Remark 2.2.1. It has been shown in [23] that for the conditions $(2.2.2_i)$ the problems (2.2.1), (2.2.4) and (2.2.1), $(2.2.5_i)$ are uniquely solvable. Analogously, if

$$p_{0k},\, p_{1k} \in L_{\rm loc}(]a,b[), \;\; \sigma(p_{1k}) \in L([a,b]), \;\; p_{0k} \in L_{\sigma_1(p_{1k})}([a,b]),$$

then the problems $(2.2.1_k)$, $(2.2.4_k)$ and $(2.2.1_k)$, $(2.2.5_{1k})$ are uniquely solvable, and if

$$p_{0k},\, p_{1k} \in L_{\rm loc}(]a,b]), \;\; \sigma(p_{1k}) \in L([a,b]), \;\; p_{0k} \in L_{\sigma_2(p_{1k})}([a,b]),$$

then the problems $(2.2.1_k)$, $(2.2.4_k)$ and $(2.2.1_k)$, $(2.2.5_2)$ are uniquely solvable as well.

For brevity we introduce the notation

$$\Delta p_{jk}(t) = p_j(t) - p_{jk}(t) \;\; (j = 0,1,2;\; k \in \mathbb{N}) \;\text{ for }\; a < t < b.$$

Lemma 2.2.1$_1$. *Let the measurable functions $p_j,\ p_{jk} :]a,b[\to \mathbb{R}$ ($j = 0,1;\ k \in \mathbb{N}$) and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, $\beta,\ \mu \in \mathbb{R}$ such that*

$$0 \le \beta < \mu \le \frac{\gamma - 1}{\gamma - \alpha}\,, \qquad (2.2.6)$$

$$\sigma^\gamma(p_1) \in L([a,b]), \quad \int_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\, I_1^\mu(\sigma^\alpha(p_1))(s)\, ds < +\infty \qquad (2.2.7_1)$$

and uniformly on the segment $[a,b]$ the conditions

$$\lim_{k\to\infty}\int_a^t \frac{\Delta p_{0k}(s)}{\sigma(p_1)(s)}\, I_1^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds=0,\quad \lim_{k\to\infty}\int_a^t |\Delta p_{1k}(s)|\,ds=0 \qquad (2.2.8_1)$$

be satisfied. Then there exists a number k_0 such that for $k>k_0$ the problem $(2.2.1_k)$, $(2.2.4_{1k})$ has a unique solution v_{1k} and the problem $(2.2.1_k)$, $(2.2.5_{1k})$ has a unique solution v_{2k}, and uniformly on the interval $]a,b[$

$$\lim_{k\to\infty}\big(v_{1k}(t)-v_1(t)\big)\bigg(\int_a^t \sigma(p_1)(s)\,ds\bigg)^{-1}=0, \qquad (2.2.9_{11})$$

$$\lim_{k\to\infty}\big(v_{2k}(t)-v_2(t)\big)\bigg(\int_t^b \sigma(p_1)(s)\,ds\bigg)^{-1}=0 \qquad (2.2.9_{12})$$

and

$$\lim_{k\to\infty}\frac{v_{1k}'(t)-v_1'(t)}{\sigma(p_1)(t)}\bigg(\int_t^b \sigma^{\alpha}(p_1)(s)\,ds\bigg)^{\mu}=0, \qquad (2.2.10_{11})$$

$$\lim_{k\to\infty}\frac{v_{2k}'(t)-v_2'(t)}{\sigma(p_1)(t)}\bigg(\int_a^t \sigma^{\alpha}(p_1)(s)\,ds\bigg)^{\mu}=0, \qquad (2.2.10_{12})$$

where v_1 and v_2 are the solutions of the problems (2.2.1), $(2.2.4_1)$ *and* (2.2.1), $(2.2.5_1)$, *respectively.*

Proof. It is clear from the definition of the constants α, β, γ, μ that

$$\beta-\mu<0,\quad 0<\frac{1-\alpha\beta}{1-\beta}<\frac{1-\alpha\mu}{1-\mu}\le\gamma. \qquad (2.2.11)$$

Hence

$$\sigma^{\alpha}(p_1),\quad \sigma^{\frac{1-\alpha\beta}{1-\beta}}(p_1),\quad \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)\in L([a,b]). \qquad (2.2.12)$$

Using the Hölder inequality, we obtain

$$\int_{t_1}^{t_2}\sigma(p_1)(s)\,ds\le\bigg(\int_{t_1}^{t_2}\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\bigg)^{1-\mu}\times$$

$$\times\bigg(\int_{t_1}^{t_2}\sigma^{\alpha}(p_1)(s)\,ds\bigg)^{\mu}\quad\text{for}\quad a\le t_1\le t_2\le b, \qquad (2.2.13)$$

$$\int_a^b \frac{\sigma(p_1)(s)}{\big(\int_a^s \sigma^\alpha(p_1)(\eta)\,d\eta\big)^\beta}\,ds \le$$

$$\le \Big(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int_a^b \frac{\sigma^\alpha(p_1)(s)}{\big(\int_a^s \sigma^\alpha(p_1)(\eta)\,d\eta\big)^{\frac{\beta}{\mu}}}\,ds\Big)^\mu =$$

$$= \Big(\frac{\mu}{\mu-\beta}\Big)^\mu \Big(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu-\beta}, \quad (2.2.14)$$

$$\int_a^b \frac{\sigma(p_1)(s)}{\big(\int_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\big)^\beta}\,ds \le$$

$$\le \Big(\frac{\mu}{\mu-\beta}\Big)^\mu \Big(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu-\beta}, \quad (2.2.15)$$

where the existence of the integrals follows from (2.2.12). By means of (2.2.14), (2.2.15) we easily get

$$\int_a^b \frac{\sigma(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds \le 2\Big(\frac{\mu}{\mu-\beta}\Big)^\mu I_1^{-\beta}(\sigma^\alpha(p_1))\Big(\frac{a+b}{2}\Big) \times$$
$$\times\Big(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu-\beta} < +\infty. \quad (2.2.16)$$

It is also evident that for every $\delta \in [0,1[$

$$\int_a^b \frac{\sigma^\alpha(p_1)(s)}{I_i^\delta(\sigma^\alpha(p_1))(s)}\,ds < +\infty. \quad (2.2.17)$$

By virtue of condition $(2.2.8_1)$, for every $\varepsilon > 1$ there exists a number k_0 such that for $k > k_0$

$$\varepsilon^{-1} \le \sigma(\Delta p_{1k})(t) \le \varepsilon \quad \text{for} \quad a \le t \le b. \quad (2.2.18)$$

We now proceed to the proof of the lemma. Taking into account the conditions $(2.2.7_1)$, (2.2.12) and the inequality (2.2.13), the inequality

$$\int_a^b |p_0(s)|\sigma_1(p_1)(s)\,ds \le \int_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\, I_1^\mu(\sigma^\alpha(p_1))(s)\,ds \times$$

$$\times\Big(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{2(1-\mu)} < +\infty \tag{2.2.19}$$

is valid, i.e. the conditions $(2.2.2_1)$ are satisfied. In this case, owing to Remark 2.2.1, the problems (2.2.1), (2.2.4) and (2.2.1), $(2.2.5_1)$ are uniquely solvable. Integrating by parts and using (2.2.18), we arrive at

$$\Big|\int_a^b \frac{p_{0k}(s)}{\sigma(p_{1k})(s)} I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds\Big| \leq$$

$$\leq \Big|\int_a^b \frac{\Delta p_{0k}(s)}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds\Big| + \int_a^b \frac{|p_0(s)|}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds \leq$$

$$\leq A_k \int_a^b \Big|\Big(\sigma(\Delta p_{1k})(s)\frac{I_1^\mu(\sigma^\alpha(p_{1k}))(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\Big)'\Big|\,ds +$$

$$+\varepsilon^3 \int_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\, I_1^\mu(\sigma^\alpha(p_1))(s)\,ds \quad \text{for} \quad k > k_0, \tag{2.2.20}$$

where

$$A_k = \sup\Big\{\Big|\int_{t_1}^{t_2} \frac{\Delta p_{0k}(s)}{\sigma(p_1)(s)}\, I_1^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ a \leq t_1 < t_2 \leq b\Big\}.$$

In view of $(2.2.8_1)$

$$\lim_{k\to\infty} A_k = 0, \tag{2.2.21}$$

and by virtue of (2.2.18) the estimate

$$\Big|\Big(\sigma(\Delta p_{1k})(t)\frac{I_1^\mu(\sigma^\alpha(p_{1k}))(t)}{I_1^\beta(\sigma^\alpha(p_1))(t)}\Big)'\Big| \leq \varepsilon^3|\Delta p_{1k}(t)|I_1^{\mu-\beta}(\sigma^\alpha(p_1))(t) +$$

$$+(\mu+\beta)\varepsilon^3 \int_a^b \sigma^\alpha(p_1)(s)\,ds \frac{\sigma^\alpha(p_1)(t)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(t)} \quad \text{for} \quad a < t < b$$

is valid. Substituting the latter in (2.2.20) and taking into account $(2.2.7_1)$, $(2.2.8_1)$, (2.2.17) and (2.2.21), we can see that a constant $r_0 \in \mathbb{R}^+$ exist, such that

$$\sup\Big\{\int_a^b \frac{|p_{0k}(s)|}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds : \ k > k_0\Big\} < r_0. \tag{2.2.22}$$

In the same way we get

$$p_{0k} \in L_{\sigma_1(p_{1k})}([a,b]) \quad \text{for} \quad k > k_0,$$

where in view of (2.2.18)

$$\sigma(p_{1k}) \in L([a,b]) \quad \text{for} \quad k > k_0,$$

which together with the conditions ($2.2.3_i$) and Remark 2.2.1 imply that the problems ($2.2.1_k$), ($2.2.4_k$) and ($2.2.1_k$), ($2.2.5_{1k}$) are uniquely solvable for $k > k_0$.

Note that the function $w_{jk}(t) = v_j(t) - v_{jk}(t)$ $(j = 1,2;\ k > k_0)$ is a solution of the equation

$$\begin{aligned} v''(t) = p_{0k}(t)v(t) + p_{1k}(t)v'(t) + \\ +\Delta p_{0k}(t)v_j(t) + \Delta p_{1k}(t)v_j'(t) \ \ (j = 1,2) \end{aligned} \tag{2.2.23}$$

and

$$w_{1k}(a) = 0, \ \lim_{t \to a} \frac{w_{1k}'(t)}{\sigma(p_{1k})(t)} = \sigma(\Delta p_{1k})(a) - 1, \tag{$2.2.24_1$}$$

$$w_{2k}(b) = 0, \ \lim_{t \to b} \frac{w_{2k}'(t)}{\sigma(p_{1k})(t)} = 1 - \sigma(\Delta p_{1k})(b), \tag{$2.2.24_2$}$$

where in view of ($2.2.8_1$),

$$\lim_{k \to \infty} \big\| 1 - \sigma(\Delta p_{1k}) \big\|_C = 0. \tag{2.2.25}$$

Consider first the case $j = 1$. From (2.2.23), ($2.2.24_1$) we have

$$\begin{aligned} \frac{w_{1k}'(t)}{\sigma(p_{1k})(t)} = \sigma(\Delta p_{1k})(t) - 1 + \int_a^t \Delta p_{0k}(s) \, \frac{v_1(s) - w_{1k}(s)}{\sigma(p_{1k})(s)} \, ds + \\ + \int_a^t \frac{p_0(s) w_{1k}(s) + \Delta p_{1k}(s) v_1'(s)}{\sigma(p_{1k})(s)} \, ds \quad \text{for} \quad a < t < b, \end{aligned} \tag{2.2.26}$$

where the existence of integrals follows from the estimate ($1.2.10_1$), ($1.2.11_1$) and the conditions ($2.2.7_1$), ($2.2.8_1$). From (2.2.26), integration by parts results in

$$\begin{aligned} \frac{|w_{1k}'(t)|}{\sigma(p_{1k})(t)} \leq \big|1 - \sigma(\Delta p_{1k})(a)\big| + A_k \int_a^t \Big| \Big(\frac{v_1(s) - w_{1k}(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)} \, \sigma(\Delta p_{1k})(s) \Big)' \Big| \, ds + \\ + \int_a^t \frac{|p_0(s) w_{1k}(s) + \Delta p_{1k}(s) v_1'(s)|}{\sigma(p_{1k})(s)} \, ds \quad \text{for} \quad a < t < b, \end{aligned} \tag{2.2.27}$$

where in view of (2.2.18),

$$\int_a^t \left| \Big(\frac{v_1(s) - w_{1k}(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\, \sigma(\Delta p_{1k})(s) \Big)' \right| ds \leq$$

$$\leq \varepsilon \int_a^t \frac{|w_{1k}'(s)| + |v_1'(s)|}{I_1^\beta(\sigma^\alpha(p_1))(s)} + \big(|w_{1k}(s)| + |v_1(s)|\big) h_k(s)\, ds$$

with

$$h_k(t) = \frac{|\Delta p_{1k}(t)|}{I_1^\beta(\sigma^\alpha(p_1))(t)} + \beta \int_a^b \sigma^\alpha(p_1)(s)\, ds \frac{\sigma^\alpha(p_1)(t)}{I_1^{1+\beta}(\sigma^\alpha(p_1))(t)} \quad \text{for } a < t < b.$$

Substituting the latter inequality in (2.2.27), with regard for (2.2.18) we get

$$\frac{|w_{1k}'(t)|}{\sigma(p_1)(t)} \leq \varepsilon^2 A_k \int_a^t \frac{|w_{1k}'(s)|}{I_1^\beta(\sigma^\alpha(p_1))(s)}\, ds +$$

$$+\varepsilon^2 \left[\big\| 1 - \sigma(\Delta p_{1k}) \big\|_C + \int_a^t f_k(s) |w_{1k}(s)| + q_k(s)\, ds \right], \qquad (2.2.28)$$

where

$$f_k(t) = \frac{|p_{0k}(t)|}{\sigma(p_1)(t)} + A_k h_k(t),$$

$$q_k(t) = \frac{|v_1'(t)|}{\sigma(p_1)(t)} \Big(|\Delta p_{1k}(t)| + A_k \frac{\sigma(p_1)(t)}{I_1^\beta(\sigma^\alpha(p_1))(t)} \Big) + A_k h_k(t) |v_1(t)|$$

$$\text{for} \quad a < t < b.$$

From (2.2.28), using Gronwall-Bellman's lemma, it follows that

$$|w_{1k}'(t)| \leq r_k \sigma(p_1)(t) \bigg(\big\| 1 - \sigma(\Delta p_{1k}) \big\|_C +$$

$$+ \int_a^t f_k(s) |w_{1k}(s)| + q_k(s)\, ds \bigg) \quad \text{for} \quad a < t < b, \qquad (2.2.29)$$

where

$$r_k = \varepsilon^2 \left[1 + \exp\Big(\varepsilon^2 A_k \int_a^b \frac{\sigma(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\, ds \Big) \right] \quad \text{for} \quad k > k_0$$

and by virtue of (2.2.16), (2.2.21),

$$\sup\{r_k : \; k > k_0\} < +\infty. \qquad (2.2.30)$$

Let us now introduce the notation

$$z_k = |w_{1k}(t)|\Big(\int\limits_a^t \sigma(p_1)(s)\,ds\Big)^{-1} \quad \text{for} \quad a < t < b.$$

Integrating (2.2.29) from a to t, dividing by $\int\limits_a^t \sigma(p_1)(s)\,ds$ and using integration by parts, by virtue of the inequalities (2.2.13) and

$$\int\limits_s^t \sigma(p_1)(s)\,ds\Big(\int\limits_a^t \sigma(p_1)(s)\,ds\Big)^{-1} \le$$

$$\le \int\limits_s^b \sigma(p_1)(s)\,ds\Big(\int\limits_a^b \sigma(p_1)(s)\,ds\Big)^{-1} \quad \text{for} \quad a < s \le t < b$$

we obtain

$$z_k(t) \le r\int\limits_a^t f_k(s) I_1^\mu(\sigma^\alpha(p_1))(s) z_k(s)\,ds + \widetilde{r}_k \ \text{ for } \quad a < t < b,$$

where

$$r = \sup\big\{r_k : \quad k > k_0\big\}\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{2(1-\mu)}\Big(\int\limits_a^b \sigma(p_1)(s)\,ds\Big)^{-1},$$

$$\widetilde{r}_k = r\Bigg[\frac{\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}}{\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds}\int\limits_a^b q_k(s)\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\mu ds + $$

$$+\big\|1 + \sigma(\Delta p_{1k})\big\|_C\Bigg].$$

Applying Gronwall–Bellman's lemma, from the latter inequality we get

$$z_k(t) \le \widetilde{r}_k \exp\Bigg(r\int\limits_a^b f_k(s) I_1^\mu(\sigma^\alpha(p_1))(s)\,ds\Bigg) \ \text{ for } \ a < t < b. \qquad (2.2.31)$$

By virtue of (2.2.18) we note that the estimate

$$\int\limits_a^b f_k(s) I_1^\mu(\sigma^\alpha(p_1))(s)\,ds \le \varepsilon^3\int\limits_a^b \frac{|p_{0k}(s)|}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds +$$

$$+A_k\Big[\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\int\limits_a^b |\Delta p_{1k}(s)|\,ds+$$

$$+\beta\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\,ds\Big] \quad \text{for} \quad k>k_0$$

is valid, which with regard for the conditions $(2.2.8_1)$, (2.2.17) with $\delta = 1+\beta-\mu$ and the condition (2.2.22) results in

$$\sup\Big\{\int\limits_a^b f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)\,ds: \quad k>k_0\Big\}<+\infty. \tag{2.2.32}$$

Just in the same way, taking into account the estimates $(1.2.10_1)$, $(1.2.11_1)$ and the inequality (2.2.13), we obtain

$$\int\limits_a^b q_k(s)\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\mu ds \le$$

$$\le\int\limits_a^b |\Delta p_{1k}(s)|+A_k\,\frac{\sigma(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds\Big[\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^\mu+$$

$$+c^*\int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\,I_1^\mu(\sigma^\alpha(p_1))(s)\,ds\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big]+$$

$$+c^*A_k\left(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)ds\right)^{1-\mu}\Big[\beta\int\limits_a^b \sigma^\alpha(p_1)(s)ds\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\,ds+$$

$$+\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\int\limits_a^b |\Delta p_{1k}(s)|\,ds\Big] \quad \text{for} \quad k>k_0,$$

By virtue of the inequalities (2.2.16), (2.2.17) with $\delta = 1+\beta-\mu$ and the conditions $(2.2.7_1)$, $(2.2.8_1)$ and (2.2.21)

$$\lim_{k\to\infty}\int\limits_a^b q_k(s)\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\mu ds=0 \tag{2.2.33}$$

which together with (2.2.25) implies

$$\lim_{k\to\infty}\widetilde{r}_k=0. \tag{2.2.34}$$

Substituting (2.2.32) and (2.2.34) in (2.2.31) we get

$$\lim_{k\to\infty}\|z_k\|_C = 0, \tag{2.2.35}$$

i.e., the condition $(2.2.9_{11})$ is satisfied.

Applying (2.2.13), we see from (2.2.29) that

$$\frac{|w'_{1k}(t)|}{\sigma(p_1)(t)}\bigg(\int_t^b \sigma^\alpha(p_1)(s)\,ds\bigg)^\mu \le$$
$$\le \widetilde{r}\bigg(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\bigg)^{1-\mu}\bigg[\|z_k\|_C\int_a^b f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)\,ds +$$
$$+\int_a^b q_k(s)\bigg(\int_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\bigg)^\mu ds\bigg] +$$
$$+\widetilde{r}\big\|1-\sigma(\Delta p_{1k})\big\|_C\bigg(\int_a^b \sigma^\alpha(p_1)(s)ds\bigg)^\mu \quad \text{for } a<t<b,$$

where $\widetilde{r} = \sup\{r_k : k > k_0\}$. The above inequality with regard for (2.2.25), (2.2.32), (2.2.33) and (2.2.35) implies that the condition $(2.2.10_{11})$ is valid.

Consider now the case $j = 2$. Let $k > k_0$. Then for w_{2k}, i.e., for a solution of the problem (2.2.23), $(2.2.24_2)$ the representation

$$-\frac{w'_{2k}(t)}{\sigma(p_{1k})(t)} = \sigma(\Delta p_{1k})(t) - 1 + \int_t^b \Delta p_{0k}(s)\frac{v_2(s)-w_{2k}(s)}{\sigma(p_{1k})(s)}\,ds +$$
$$+\int_t^b \frac{p_{0k}(s)w_{2k}(s)+\Delta p_{1k}v'_2(s)}{\sigma(p_{1k})(s)}\,ds \quad \text{for} \quad a<t<b$$

is valid. Repeating the arguments presented for $j = 1$, where f_k, h_k are defined as before,

$$q_k(t) = \Big(|\Delta p_{1k}(t)| + A_k\,\frac{\sigma(p_1)(t)}{I_1^\beta(\sigma^\alpha(p_1))(t)}\Big)\,\frac{|v'_2(t)|}{\sigma(p_1)(t)} + A_k h_k(t)|v_2(t)|,$$
$$z_k(t) = |w_{2k}(t)|\bigg(\int_t^b \sigma(p_1)(s)ds\bigg)^{-1}$$

and

$$\widetilde{r}_k = r\Bigg[\frac{\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}}{\int\limits_a^b \sigma^{\alpha}(p_1)(s)\,ds}\int\limits_a^b q_k(s)\Big(\int\limits_a^s \sigma^{\alpha}(p_1)(\eta)\,d\eta\Big)^{\mu} ds +$$

$$+\big\|1+\sigma(\Delta p_{1k})\big\|_C\Bigg],$$

we see that the conditions $(2.2.9_{12})$, $(2.2.10_{12})$ are valid. □

Lemma 2.2.1$_2$. *Let the measurable functions p_j, $p_{jk} :]a,b[\,\to \mathbb{R}$ $(j=0,1;$ $k\in\mathbb{N})$ and the constants $\alpha\in[0,1]$, $\gamma\in\,]1,+\infty[$, β, $\mu\in\mathbb{R}$ be such that the conditions* (2.2.6) *are satisfied,*

$$\sigma^{\gamma}(p_1)\in L([a,b]),\quad \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\,I_2^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds<+\infty \qquad (2.2.7_2)$$

and uniformly on the segment $[a,b]$ the conditions

$$\lim_{k\to\infty}\int\limits_a^t \frac{\Delta p_{0k}(s)}{\sigma(p_1)(s)}\,I_2^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds=0,$$
$$\lim_{k\to\infty}\int\limits_a^t |\Delta p_{1k}(s)|\,ds=0 \qquad (2.2.8_2)$$

are satisfied. Then there exists a number k_0 such that for $k>k_0$ the problem $(2.2.1_k)$, $(2.2.4_k)$ *has a unique solution v_{1k} and the problem* $(2.2.1_k)$, $(2.2.5_2)$ *has a unique solution v_{2k}, and uniformly on the interval $]a,b[$*

$$\lim_{k\to\infty}\big(v_{1k}(t)-v_1(t)\big)\bigg(\int\limits_a^t \sigma(p_1)(s)\,ds\bigg)^{-1}=0, \qquad (2.2.9_{21})$$

$$\lim_{k\to\infty}\big(v_{2k}(t)-v_2(t)\big)=0 \qquad (2.2.9_{22})$$

and

$$\lim_{k\to\infty}\frac{v_{1k}'(t)-v_1'(t)}{\sigma(p_1)(t)}=0, \qquad (2.2.10_{21})$$

$$\lim_{k\to\infty}\frac{v_{2k}'(t)-v_2'(t)}{\sigma(p_1)(t)}\bigg(\int\limits_a^t \sigma^{\alpha}(p_1)(s)\,ds\bigg)^{\mu}=0, \qquad (2.2.10_{22})$$

where v_1 and v_2 are the solutions of the problems (2.2.1), (2.2.4) *and* (2.2.1), $(2.2.5_2)$, *respectively.*

Proof. Repeating word by word the previous proof for the case $j = 1$ and replacing everywhere I_1 by I_2, we can see that the problems $(2.2.1_k)$, $(2.2.4_k)$ and $(2.2.1_k)$, $(2.2.5_2)$ are uniquely solvable, the condition $(2.2.9_{21})$ is satisfied and for the function $w_{1k}(t) = v_1(t) - v_{1k}(t)$ the representation

$$\frac{|w'_{1k}(t)|}{\sigma(p_1)(t)} \leq r_k \Big(\|z_k\|_C \int_a^t f_k(s) \Big(\int_a^s \sigma^\alpha(p_1)(\eta)\, d\eta \Big)^\mu ds + \\ + \int_a^t q_k(s)\, ds + \|1 - \sigma(\Delta p_{1k})\|_C \Big) \quad \text{for} \quad a < t \leq b \tag{2.2.36}$$

is valid, where the functions f_k, q_k and z_k are defined in the previous proof. Using the same technique as when proving the relations (2.2.25), (2.2.32), (2.2.33), we obtain

$$\sup \Big\{ \int_a^b f_k(s) I_2^\mu(\sigma^\alpha(p_1))(s)\, ds : \quad k > k_0 \Big\} < +\infty,$$

$$\lim_{k \to \infty} \int_a^b q_k(s)\, ds = 0, \quad \lim_{k \to \infty} \|1 - \sigma(\Delta p_{1k})\|_C = 0$$

and

$$\lim_{k \to \infty} \|z_k\|_C = 0,$$

from which it follows with regard for (2.2.36) that the condition $(2.2.10_{21})$ is valid.

Note that the function $w_{2k}(t) = v_2(t) - v_{2k}(t)$ satisfies the conditions

$$w_{2k}(b) = 0, \quad w'_{2k}(b) = 0,$$

i.e., the representation

$$\frac{|w'_{2k}(t)|}{\sigma(p_{1k})(t)} = - \int_t^b \Delta p_{0k}(s)\, \frac{w_{2k}(s)}{\sigma(p_{1k})(s)}\, ds - \int_t^b \Delta p_{0k}(s)\, \frac{v_2(s)}{\sigma(p_{1k})(s)}\, ds - \\ - \int_t^b \frac{p_0(s) w_{1k}(s) + \Delta p_{1k}(s) v'_2(s)}{\sigma(p_{1k})(s)}\, ds \quad \text{for} \quad a < t \leq b$$

is valid. Repeating the arguments taking place in the proof of Lemma 2.2.1 for $j = 2$, we come to the conclusion that the conditions $(2.2.9_{12})$ and $(2.2.10_{22})$ are valid. But owing to the condition $p_1 \in L_{loc}(]a, b])$, it follows from $(2.2.9_{12})$ that $(2.2.9_{22})$ is valid. □

Lemma 2.2.2. *Let* $i \in \{1,2\}$, *the measurable functions* p_j, $p_{jk} :]a,b[\to \mathbb{R}$ *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ *be such that the conditions* (2.2.6), $(2.2.7_i)$, $(2.2.8_i)$ *and*

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a,b[) \tag{$2.2.37_i$}$$

are satisfied. Then there exists a number k_0 *such that for* $k > k_0$

$$(p_{0k}, p_{1k}) \in \mathbb{V}_{i,0}(]a,b[). \tag{$2.2.38_i$}$$

Proof. Let $i = 1$ and v_1, v_2, v_{1k}, v_{2k} be solutions of the problems (2.2.1), (2.2.4), (2.2.1), $(2.2.5_1)$, $(2.2.1_k)$, $(2.2.4_k)$, $(2.2.1_k)$, $(2.2.5_{1k})$ respectively, whose existence and uniqueness follow from Remark 2.2.1.

As is seen from Definition 1.1.2 of the set $\mathbb{V}_{1,0}(]a,b[)$ and Remark 1.2.1, $v_1(b) > 0$ and $v_1(a) > 0$. Then by virtue of Remark 1.2.5 and the inclusion $(2.2.37_i)$,

$$v_1(t) + v_2(t) > 0 \quad \text{for} \quad a \le t \le b,$$

hence if

$$c = \min\big\{v_1(t) + v_2(t): \quad a \le t \le b\big\},$$

then

$$c > 0. \tag{2.2.39}$$

On the other hand, by Lemma $2.2.1_i$, there exists a number k_0 such that for any $k > k_0$

$$-\frac{c}{2} < v_{jk}(t) - v_j(t) \ \ (j = 1,2) \quad \text{for} \quad a \le t \le b. \tag{2.2.40}$$

Thus for the solution v_k of the equation $(2.2.1_k)$, where

$$v_k(t) = v_{1k}(t) + v_{2k}(t),$$

the estimate

$$v_k(t) = \big(v_{1k}(t) - v_1(t)\big) + \big(v_{2k}(t) - v_2(t)\big) + \big(v_1(t) + v_2(t)\big)$$

is valid from which with regard for (2.2.39) and (2.2.40) we obtain

$$v_k(t) > 0 \quad \text{for} \quad a \le t \le b.$$

This inequality by virtue of Lemma 1.2.2 means that the inclusion $(2.2.38_i)$ is true. ☐

Consider now the boundary conditions

$$u(a) = 0, \quad u(b) = 0 \tag{$2.2.41_1$}$$

and

$$u(a) = 0, \quad u'(b-) = 0. \tag{$2.2.41_2$}$$

The following Lemma is valid.

Lemma 2.2.3. *Let $i \in \{1,2\}$, the measurable functions f, p_j, $p_{jk} :]a,b[\to \mathbb{R}$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$) and*

$$\int_a^b \frac{|f(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds < +\infty. \tag{2.2.42}$$

Then there exists a number k_0 such that for $k > k_0$ the problem (2.2.1$_k$), (2.2.41$_i$) has a unique Green's function G_k, and uniformly in the interval $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \int_a^b |G(t,s) - G_k(t,s)|\, |f(s)|\, ds = 0, \tag{2.2.43}$$

$$\lim_{k\to\infty} \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \int_a^b \Big|\frac{\partial(G(t,s) - G_k(t,s))}{\partial t}\Big|\, |f(s)|\, ds = 0, \tag{2.2.44}$$

where G is Green's function of the problem (2.2.1), (2.2.41$_i$).

Proof. By Lemma 2.2.2$_i$, for $k > k_0$ the inclusion (2.2.38$_i$) is satisfied. Then as is seen from Remark 1.2.2, the inclusions (2.2.37$_i$) and (2.2.38$_i$) imply the existence of the functions G and G_k, respectively, where G is defined by the equality (1.2.7), and

$$G_k(t,s) = \begin{cases} -\dfrac{v_{2k}(t)v_{1k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} & \text{for} \quad a \le s < t \le b, \\ -\dfrac{v_{1k}(t)v_{2k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} & \text{for} \quad a \le t < s \le b, \end{cases} \tag{2.2.45}$$

where v_{1k} is the solution of the problem (2.2.1$_k$), (2.2.4$_{ik}$) and v_{2k} is that of the problem (2.2.1$_k$), (2.2.5$_{1k}$) for $i = 1$ and of the problem (2.2.1$_k$), (2.2.5$_2$) for $i = 2$.

From the estimates (1.2.10$_i$), (1.2.11$_i$) and the equalities (2.2.9$_{i1}$), (2.2.9$_{i2}$), (2.2.10$_{i1}$), (2.2.10$_{i2}$) it follows the existence of constants d_1 and d_2, such that on the interval $]a,b[$ the estimates

$$\begin{gathered} v_{1k}(t)\Big(\int_a^t \sigma(p_1)(s)\, ds\Big)^{-1} \le d_1, \quad v_{2k}(t)\Big(\int_t^b \sigma(p_1)(s)\, ds\Big)^{i-2} \le d_1 \\ \text{for} \quad k > k_0, \\ v_1(t)\Big(\int_a^t \sigma(p_1)(s)\, ds\Big)^{-1} \le d_1, \quad v_2(t)\Big(\int_t^b \sigma(p_1)(s)\, ds\Big)^{i-2} \le d_1 \end{gathered} \tag{2.2.46}$$

and

$$\frac{|v_{1k}'(t)|}{\sigma(p_1)(t)}\Big(\int_t^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu(2-i)} \le d_1, \quad \frac{|v_{2k}'(t)|}{\sigma(p_1)(t)}\Big(\int_a^t \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu} \le d_1$$

$$\text{for} \quad k > k_0, \tag{2.2.47}$$

$$\frac{|v_1'(t)|}{\sigma(p_1)(t)}\Big(\int_t^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu(2-i)} \le d_1, \quad \frac{|v_2'(t)|}{\sigma(p_1)(t)}\Big(\int_a^t \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu} \le d_1,$$

as well as

$$v_{2k}(a) \ge d_2 \ \text{ for } \ k > k_0, \quad v_2(a) \ge d_2 \tag{2.2.48}$$

are valid.

Introduce now the notation $w_{lk}^{(j)}(t) = v_l^{(j)}(t) - v_{lk}^{(j)}(t)$ $(l = 1, 2;\ j = 0, 1;$ $k \in \mathbb{N})$ and

$$\omega_{1k} = \sup\Big\{|w_{1k}(t)|\Big(\int_a^t \sigma(p_1)(s)\,ds\Big)^{-1} : \ a < t \le b\Big\},$$

$$\omega_{2k} = \sup\Big\{|w_{2k}(t)|\Big(\int_t^b \sigma(p_1)(s)\,ds\Big)^{i-2} : \ a \le t < b\Big\},$$

$$\omega_{1k}' = \sup\Big\{\frac{|w_{1k}'(t)|}{\sigma(p_1)(t)}\Big(\int_t^b \sigma^\alpha(p_1)(s)\,ds\Big)^{(2-i)\mu} : \ a < t < b\Big\},$$

$$\omega_{2k}' = \sup\Big\{\frac{|w_{2k}'(t)|}{\sigma(p_1)(t)}\Big(\int_a^t \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu} : \ a < t < b\Big\}.$$

Then as is seen from Lemma $2.2.1_i$,

$$\lim_{k\to\infty} \omega_{jk} = 0, \quad \lim_{k\to\infty} \omega_{jk}' = 0 \ \ (j = 1, 2). \tag{2.2.49}$$

It is also clear that the equality

$$\Big(\frac{I_i^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\Big)^j \int_a^b \Big|\frac{\partial^j}{\partial t^j}\,(G_k(t,s) - G(t,s))\Big|\,|f(s)|\,ds =$$

$$= \Big(\frac{I_i^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\Big)^j \int_a^t \Big|\frac{v_{2k}^{(j)}(t)v_{1k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} - \frac{v_2^{(j)}(t)v_1(s)}{v_2(a)\sigma(p_1)(s)}\Big|\,|f(s)|\,ds +$$

$$+ \int_t^b \Big|\frac{v_{1k}^{(j)}(t)v_{2k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} - \frac{v_1^{(j)}(t)v_2(s)}{v_2(a)\sigma(p_1)(s)}\Big|\,|f(s)|\,ds \ \ (j = 0, 1) \tag{2.2.50}$$

$$\text{for} \quad a < t < b$$

is valid.

Let $j = 0$. With regard for the inequalities (2.2.18) and (2.2.46) we obtain the estimate

$$\int_a^t \Big| \frac{v_{2k}(t)v_{1k}(s)}{v_{2k}(a)\sigma(\Delta p_{1k})(s)} - \frac{v_2(t)v_1(s)}{v_2(a)\sigma(\Delta p_{1k})(s)} \Big| \, |f(s)| \, ds \le$$

$$\le \frac{\varepsilon}{v_{2k}(a)} \bigg[|w_2(t)| \int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |v_{1k}(s)| \, ds +$$

$$+ |v_2(t)| \bigg(\int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |w_{1k}(s)| \, ds + \frac{|w_{2k}(a)|}{v_2(a)} \int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |v_1(s)| \, ds \bigg) \bigg] +$$

$$+ \frac{\|1 - \sigma(\Delta p_{1k})\|_C}{v_2(a)} \, v_2(t) \int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |v_1(s)| \, ds \le$$

$$\le r_k I_i^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \le t \le b,$$

where

$$r_k = \varepsilon \, \frac{d_1}{d_2} \bigg[\omega_{1k} + \omega_{2k} \Big(1 + \frac{d_1}{d_2} \int_a^b \sigma(p_1)(s) \, ds \Big) + \frac{d_1}{\varepsilon} \, \big\| 1 - \sigma(\Delta p_{1k}) \big\|_C \bigg] \times$$

$$\times \int_a^b \frac{|f(s)|}{\sigma(p_1)(s)} \, I_i^{\mu}(\sigma^{\alpha}(p_1))(s) \, ds$$

and in view of the conditions (2.2.8$_i$), (2.2.42), and (2.2.49),

$$\lim_{k \to \infty} r_k = 0. \tag{2.2.51}$$

Having analogously estimated the second integral in (2.2.50) for $j = 0$, we obtain for any $k > k_0$

$$I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \int_a^b |G(t,s) - G_k(t,s)| \, |f(s)| \, ds \le 2r_k \quad \text{for} \quad a < t < b$$

which in view of (2.2.51) implies the validity of the condition (2.2.43).

Similarly, from the equality (2.2.50) for $j = 1$, with regard for (2.2.18), (2.2.46) and (2.2.47), for any $k > k_0$ we get

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \int_a^b \Big| \frac{\partial (G(t,s) - G_k(t,s))}{\partial t} \Big| \, |f(s)| \, ds \le \widetilde{r}_k \quad \text{for} \quad a < t < b,$$

where

$$\widetilde{r}_k = 2\varepsilon \frac{d_1}{d_2} \Bigl(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds \Bigr) \int_a^b \frac{|f(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds \times$$

$$\times \Bigl[\omega'_{1k} + \omega'_{2k} + \omega_{1k} + \omega_{2k} \Bigl(1 + \frac{d_1}{d_2} \int_a^b \sigma(p_1)(s)\,ds \Bigr) + \frac{d_1}{\varepsilon}\, \|1 - \sigma(\Delta p_{1k})\|_C \Bigr].$$

By the conditions (2.2.8$_i$), (2.2.42), and (2.2.49),

$$\lim_{k\to\infty} \widetilde{r}_k = 0$$

which guarantees the validity of the condition (2.2.44). □

Lemma 2.2.4. *Let $i \in \{1,2\}$, the measurable functions f, p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1;\ k \in \mathbb{N})$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, $\beta, \mu \in \mathbb{R}$ satisfy conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$) *and*

$$\int_a^b \frac{|f(s)|}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds < +\infty. \tag{2.2.52}$$

Then there exist a constant $r_1 \in \mathbb{R}^+$ and a number k_0 such that for $k > k_0$ the problem (2.2.1$_k$), (2.2.42$_i$) *has a unique Green's function G_k, and*

$$\Bigl| \int_a^b G_k(t,s) f(s)\,ds \Bigr| \le r_1 \max \Bigl\{ \Bigl| \int_a^t \frac{f(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds \Bigr| :\ a \le t \le b \Bigr\} \times$$
$$\times I_i^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \le t \le b \tag{2.2.53}$$

and

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \Bigl| \int_a^b \frac{\partial G_k(t,s)}{\partial t} f(s)\,ds \Bigr| \le$$
$$\le r_1 \max \Bigl\{ \Bigl| \int_a^t \frac{f(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds :\ a \le t \le b \Bigr\} \tag{2.2.54}$$
$$\text{for} \quad a < t < b.$$

Proof. In the proof of the previous lemma it has been shown that under the conditions of that lemma the problem (2.2.1$_k$), (2.2.42$_i$) has a unique Green's function G_k which is represented by the equality (2.2.45).

Consider separately the case $i = 1$. First we note that in view of (2.2.12) and (2.2.17) the inequality

$$\int\limits_{t_1}^{t_2} \frac{\sigma(p_1)(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\,ds \le \Big(\int\limits_{t_1}^{t_2} \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu} \times$$
$$\times\Big(\int\limits_a^b \frac{\sigma^{\alpha}(p_1)(s)}{I_1^{\frac{\beta}{\mu}}(\sigma^{\alpha}(p_1))(s)}\,ds\Big)^{\mu} < +\infty \quad \text{for} \quad a \le t_1 < t_2 \le b \qquad (2.2.55)$$

is valid. Integrating by parts and applying (2.2.48), we get

$$\Big|\int\limits_a^b \frac{\partial^j G(t,s)}{\partial t^j}\,f(s)\,ds\Big| \le$$
$$\le \frac{2}{d_2}\max\Big\{\Big|\int\limits_a^t \frac{f(s)}{\sigma(p_1)(s)}\,I_1^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds\Big| : \quad a \le t \le b\Big\} \times$$
$$\times\Big[|v_{2k}^{(j)}(t)|\int\limits_a^t \Big|\Big(\frac{v_{1k}(s)\sigma(\Delta p_{1k})(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\Big)'\Big|\,ds +$$
$$+|v_{1k}^{(j)}(t)|\int\limits_t^b \Big|\Big(\frac{v_{2k}(s)\sigma(\Delta p_{1k})(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\Big)'\Big|\,ds \quad (j=0,1) \quad \text{for} \quad a < t < b. \qquad (2.2.56)$$

Using now the estimates (2.2.46), (2.2.55), we obtain

$$|v_{2k}(t)|\int\limits_a^t \Big|\Big(\frac{v_{1k}(s)\sigma(\Delta p_{1k})(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\Big)'\Big|\,ds \le \varepsilon d_1\Big(\int\limits_t^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu} \times$$
$$\times\int\limits_a^t \frac{\sigma(p_1)(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\,\frac{v_{1k}'(s)}{\sigma(p_1)(s)}\Big(\int\limits_s^b \sigma^{\alpha}(p_1)(\eta)\,d\eta\Big)^{\mu} ds +$$
$$+\varepsilon d_1^2 I_1^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\Big[\Big(\int\limits_a^b \sigma^{\alpha}(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\int\limits_a^b |\Delta p_{1k}(s)|\,ds +$$
$$+\int\limits_a^b \sigma^{\alpha}(p_1)(s)\,ds\int\limits_a^b \frac{\sigma^{\alpha}(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^{\alpha}(p_1))(s)}\,ds\Big] \le$$
$$\le \widetilde{r}_1 I_1^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \le t \le b, \qquad (2.2.57)$$

where

$$\widetilde{r}_1 = \varepsilon d_1^2 \Bigg(\Big(\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{\frac{\beta}{\mu}}(\sigma^\alpha(p_1))(s)}\, ds \Big)^\mu + $$
$$ + \Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\, ds \Big)^{2(\mu-\beta)} \sup \Big\{ \int\limits_a^b |\Delta p_{1k}(s)|\, ds : \quad k > k_0 \Big\} + $$
$$ + \int\limits_a^b \sigma^\alpha(p_1)(s)\, ds \int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\, ds \Bigg).$$

Analogously we have

$$|v_{1k}(t)| \int\limits_t^b \Big| \Big(\frac{v_{2k}(s)\sigma(\Delta p_{1k})(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)} \Big)' \Big|\, ds \le$$
$$\le \widetilde{r}_1 I_1^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \le t \le b, \tag{2.2.58}$$
$$\frac{I_1^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\, |v_{2k}'(t)| \int\limits_a^t \Big| \Big(\frac{v_{1k}(s)\sigma(\Delta p_{1k})(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)} \Big)' \Big|\, ds \le$$
$$\le \widetilde{r}_2 \quad \text{for} \quad a < t < b \tag{2.2.59}$$

and

$$\frac{I_1^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\, |v_{1k}'(t)| \int\limits_t^b \Big| \Big(\frac{v_{2k}(s)\sigma(\Delta p_{1k})(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)} \Big)' \Big|\, ds \le$$
$$\le \widetilde{r}_2 \quad \text{for} \quad a < t < b, \tag{2.2.60}$$

where

$$\widetilde{r}_2 = \varepsilon d_1^2 \Bigg[\int\limits_a^b \frac{\sigma_1(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\, ds + \Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\, ds \Big)^{1-\mu} \times$$
$$\times \Big(\sup \Big\{ \int\limits_a^b |\Delta p_{1k}|\, ds : k > k_0 \Big\} \Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\, ds \Big)^{2(\mu-\beta)} + $$
$$ + \int\limits_a^b \sigma^\alpha(p_1)(s)\, ds \int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\, ds \Big) \Bigg].$$

Let us now introduce the notation

$$r_1 = \frac{4}{d_2} \max(\widetilde{r}_1; \widetilde{r}_2).$$

Substituting the estimates (2.2.57), (2.2.58) in (2.2.56) for $j = 0$, we see that the condition (2.2.53) is valid. Taking then into account (2.2.59), (2.2.60) in (2.2.56) for $j = 1$, we are convinced of the validity of (2.2.54).

For $i = 2$ the lemma is proved analogously. □

Lemma 2.2.5. *Let $i \in \{1, 2\}$, the measurable functions p_j, $p_{jk} :]a, b[\to \mathbb{R}$ ($j = 0, 1$; $k \in \mathbb{N}$) and the constants $\alpha \in [0, 1]$, $\gamma \in]1, +\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$). Then there exists a number k_0 such that for $k > k_0$ the problem (2.2.1$_k$), (2.2.41$_k$) has a unique Green's function G_k for which the estimate*

$$\left|\frac{d^j G_k(t,s)}{dt^j}\right| \le c' \, \frac{\sigma_i(p_1)(s)}{[\sigma_i(p_1)(t)]^j} \quad (j = 0, 1) \quad \textit{for} \quad a < t, s < b, \ \ t \ne s, \tag{2.2.61}$$

is valid, where c' is a constant.

Proof. The existence of Green's function under the given conditions has been shown in Lemma 2.2.3. Similarly, by virtue of the estimate (1.2.12$_i$) from Remark 1.2.3,

$$\left|\frac{d^j G_k(t,s)}{dt^j}\right| \le c^* \, \frac{\sigma_i(p_{1k})(s)}{[\sigma_i(p_{1k})(t)]^j} \quad (j = 0, 1) \quad \text{for} \quad a < t, s < b, \ \ t \ne s,$$

whence with regard for the inequalities (2.2.18) and (2.2.48) follows the validity of our lemma. □

Consider now the equations

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) + p_2(t), \tag{2.2.62}$$

$$v''(t) = p_{0k}(t)v(t) + p_{1k}(t)v'(t) + p_{2k}(t), \tag{2.2.62$_k$}$$

where p_2, $p_{2k} \in L_{loc}(]a, b[)$ ($k \in \mathbb{N}$) and the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{2.2.63$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{2.2.63$_2$}$$

and

$$u(a) = c_{1k}, \quad u(b) = c_{2k} \tag{2.2.63$_{1k}$}$$

or

$$u(a) = c_{1k}, \quad u'(b-) = c_{2k}, \tag{2.2.63$_{2k}$}$$

where c_l, $c_{lk} \in \mathbb{R}$ ($l = 1, 2$; $k \in \mathbb{N}$). Then the following lemma is valid.

Lemma 2.2.6. *Let $i \in \{1,2\}$, the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ ($j = 0,1,2$; $k \in \mathbb{N}$) and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$),*

$$\int_a^b \frac{|p_2(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds < +\infty \tag{2.2.64}$$

and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \int_a^t \frac{p_2(s) - p_{2k}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\, ds = 0. \tag{2.2.65}$$

Then there exists a number k_0 such that for $k > k_0$:

(a) *the problem (2.2.62$_k$), (2.2.41$_i$) has a unique solution $\widetilde{v}_k$, and uniformly on the interval $]a,b[$*

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\big(\widetilde{v}(t) - \widetilde{v}_k(t)\big) = 0, \tag{2.2.66}$$

$$\lim_{k\to\infty} \frac{\widetilde{v}'(t) - \widetilde{v}_k'(t)}{\sigma(p_1)(t)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(t) = 0, \tag{2.2.67}$$

where $\widetilde{v}$ is a solution of the problem (2.2.61), (2.2.41$_i$);

(b) *the problem (2.2.62$_k$), (2.2.63$_{ik}$) has a unique solution $\widetilde{v}_k$, and if*

$$\lim_{k\to\infty} c_{lk} = c_l \quad (l = 1,2), \tag{2.2.68}$$

then uniformly on the interval $]a,b[$ the conditions (2.2.67) and

$$\lim_{k\to\infty} \big(\widetilde{v}(t) - \widetilde{v}_k(t)\big) = 0 \tag{2.2.69}$$

are satisfied, where $\widetilde{v}$ is a solution of the problem (2.2.62), (2.2.63$_i$);

(c) *the sequence $(\widetilde{v}_k)_{k=1}^{\infty}$, where $\widetilde{v}_k$ is a solution of the problem (2.2.62$_k$), (2.2.41$_i$), ((2.2.62$_k$), (2.2.63$_{ik}$)), is uniformly bounded and equicontinuous.*

Proof. First we prove the validity of proposition (a). It has been mentioned in the proof of Lemma 2.2.3 that under the above-mentioned conditions the problems (2.2.1), (2.2.41$_i$), and (2.2.1$_k$), (2.2.41$_i$) for $k > k_0$ have a unique Green's function G and G_k, respectively.

Let

$$\widetilde{v}(t) = \int_a^b G(t,s)p_2(s)\, ds \quad \text{and} \quad \widetilde{v}_k(t) = \int_a^b G_k(t,s)p_{2k}(s)\, ds.$$

Then

$$\widetilde{v}^{(j)}(t) - \widetilde{v}_k^{(j)}(t) = \int_a^b \frac{\partial^j G_k(t,s)}{\partial t^j}\big(p_2(s) - p_{2k}(s)\big)\, ds +$$

$$+\int_a^b \frac{\partial^j \Delta G_k(t,s)}{\partial t^j}\, p_2(s)\, ds \ \ (j=0,1) \quad \text{for} \quad a<t<b.$$

Taking into account the equalities (2.2.43), (2.2.44) of Lemma 2.2.3 and the equalities (2.2.53), (2.2.54) of Lemma 2.2.4, by means of the conditions (2.2.64), (2.2.65) we make sure that the equalities (2.2.66) and (2.2.67) are valid.

Now we proceed to proving proposition (b). Let v_0 and v_{0k} be solutions of the problems (2.2.1), (2.2.63$_i$) and (2.2.1$_k$), (2.2.63$_{ik}$), respectively. Then

$$\widetilde{v}(t) = v_0(t) + \int_a^b G(t,s)p_2(s)\, ds \quad \widetilde{v}_k(t) = v_{0k}(t) + \int_a^b G(t,s)p_{2k}(s)\, ds$$

and

$$\widetilde{v}^{(j)}(t) - \widetilde{v}_k^{(j)}(t) = v_0^{(j)}(t) - v_{0k}^{(j)}(t) + \int_a^b \frac{\partial^j G_k(t,s)}{\partial t^j}\big(p_2(s) - p_{2k}(s)\big)\, ds +$$

$$+\int_a^b \frac{\partial^j \Delta G_k(t,s)}{\partial t^j}\, p_2(s)\, ds \ \ (j=0,1) \quad \text{for} \quad a<t<b,$$

where

$$v_0(t) - v_{0k}(t) =$$

$$= c_1\, \frac{v_2(t)}{v_2(a)} - c_{1k}\, \frac{v_{2k}(t)}{v_{2k}(a)} + c_2\, \frac{v_1(t)}{v_1(b)} - c_{2k}\, \frac{v_{1k}(t)}{v_{1k}(b)} \quad \text{for} \quad a \le t < b$$

and v_j, v_{jk} $(j=1,2;\ k \ge k_0)$ are the solutions mentioned in Lemma 2.2.1$_i$. It follows from the given representation, Lemma 2.2.1$_i$ and the condition (2.2.68) that uniformly in the interval $]a,b[$

$$\lim_{k\to\infty} \big(v_0(t) - v_{0k}(t)\big) = 0$$

and

$$\lim_{k\to\infty} \frac{v_0'(t) - v_{0k}'(t)}{\sigma(p_1)(t)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(t) = 0.$$

Next, reasoning analogously as in proving proposition (a), we can see that the conditions (2.2.67), (2.2.69) are valid.

The validity of proposition (c) follows immediately from (2.2.66) ((2.2.69)) and also from

$$\big|\widetilde{v}_k(t_1) - \widetilde{v}_k(t_2)\big| \le \big|\widetilde{v}_k(t_1) - \widetilde{v}(t_1)\big| + \big|\widetilde{v}_k(t_2) - \widetilde{v}(t_2)\big| + \big|\widetilde{v}(t_1) - \widetilde{v}(t_2)\big| \le$$

$$\le 2\|\widetilde{v}_k - v\|_C + \big|\widetilde{v}(t_1) - \widetilde{v}(t_2)\big|,$$

where t_1, $t_2 \in [a,b]$. $\square$

Remark 2.2.2. It is not difficult to notice that if the condition (2.1.8) is satisfied, then for any fixed $r \in \mathbb{R}^+$ the equality

$$\lim_{k\to\infty}\left(\sup\left\{\Big|\int_a^t \frac{g_k(x)(s)-g(x)(s)}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds\Big| :\right.\right.$$
$$\left.\left. a\le t\le b,\ \ x\in \mathbb{B}_{r,k}\right\}\right)=0 \tag{2.2.70}$$

is valid. The same is true for the set $\mathbb{B}'_{r,k}$.

Lemma 2.2.7. *Let $i\in\{1,2\}$, the measurable functions $p_j,\ p_{jk} :]a,b[\to\mathbb{R}$ ($j=0,1,2$; $k\in\mathbb{N}$) and the constants $\alpha\in[0,1]$, $\gamma\in]1,+\infty[$, $\beta,\ \mu\in\mathbb{R}$ satisfy the conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$), (2.2.64) *and* (2.2.65). *Moreover, let continuous linear operators $g,\ g_k : C(]a,b[)\to L_{loc}(]a,b[)$, be such that the condition* (2.1.8) *is satisfied. Then for every fixed $r\in\mathbb{R}^+$ the sequence $(z_k)_{k=1}^{\infty}$*

$$z_k(t)=\alpha_k\widetilde{v}_k(t)+\int_a^b G_k(t,s)g_k(x_k)(s)\,ds,$$

is uniformly bounded and equicontinuous, where $\widetilde{v}_k$ is a solution of the problem (2.2.62$_k$), (2.2.41$_i$), *G_k is the Green's function of that problem, and for every $\alpha_k\in[0,r]$, $x_k\in\mathbb{B}_{r,k}$ ($k\in\mathbb{N}$).*

Proof. Introduce the notation

$$\widetilde{z}_k(t)=\int_a^b G_k(t,s)g_k(x_k)(s)\,ds,\quad w_k(t)=\int_a^b G(t,s)g(x_k)(s)\,ds,$$

where G is Green's function of the problem (2.2.62), (2.2.41$_i$).

Similarly to the proof of Lemma 1.2.4 we see that

$$\sup\big\{\|w_k\|_C : k\in\mathbb{N}\big\}<+\infty$$

and for any $\varepsilon>0$ there exists a constant $\delta>0$ such that for every $k\in\mathbb{N}$

$$\big|w_k(t_1)-w_k(t_2)\big|<\varepsilon \quad\text{for}\quad |t_1-t_2|<\delta. \tag{2.2.71}$$

On the other hand, from the inequality

$$\big|\widetilde{z}_k(t)-w_k(t)\big|\le\Big|\int_a^b \big(G_k(t,s)-G(t,s)\big)g(x_k)(s)\,ds\Big|+$$
$$+\Big|\int_a^b G_k(t,s)\big(g_k(x_k)(s)-g(x_k)(s)\big)\,ds\Big|$$

by virtue of Lemmas 2.2.3–2.2.4 and Remark 2.2.2 with all conditions satisfied, we obtain

$$\lim_{k\to\infty} \|\widetilde{z}_k - w_k\|_C = 0 \tag{2.2.72}$$

which, owing to the inequality

$$\begin{aligned}\big|\widetilde{z}_k(t_1) - \widetilde{z}_k(t_2)\big| \le \big|\widetilde{z}_k(t_1) - w_k(t_1)\big| + \big|\widetilde{z}_k(t_2) - w_k(t_2)\big| + \\ + |w_k(t_2) - w_k(t_1)| \le 2\|\widetilde{z}_k - w_k\|_C + |w_k(t_2) - w_k(t_1)|\end{aligned}$$

with regard for (2.2.71) and (2.2.72), implies the uniform boundedness and equicontinuity of the sequence $(\widetilde{z}_k)_{k=1}^{\infty}$. This together with proposition (c) of Lemma 2.2.5 proves our lemma. □

Remark 2.2.3. Lemma 2.2.7 remains valid if $\widetilde{v}_k$ is a solution of the problem $(2.2.62_k)$, $(2.2.63_{ik})$, $x_k \in \mathbb{B}'_{r,k}$ $(k \in \mathbb{N})$ and

$$\lim_{k\to\infty} c_{lk} = c_l \quad (l = 1, 2).$$

Lemma 2.2.8. *Let functions $\mathbb{V}_k \in L_\infty(]a, b[)$ and $H_k \in L([a, b])$ $(k \in \mathbb{N})$ be such that uniformly on $[a, b]$*

$$\lim_{k\to\infty} \int_a^t H_k(s)\,ds = 0, \tag{2.2.73}$$

$$\operatorname{ess\,sup}\big\{|\mathbb{V}_k(t) - \mathbb{V}(t)| : \ a \le t \le b\big\} \to 0 \quad \textit{as} \quad k \to +\infty, \tag{2.2.74}$$

and let there exist a function $\eta \in L([a, b])$ such that everywhere on the interval $]a, b[$

$$|H_k(t)| \le \eta(t) \quad (k \in \mathbb{N}). \tag{2.2.75}$$

Then uniformly on the segment $[a, b]$

$$\lim_{k\to\infty} \int_a^t H_k(s)\mathbb{V}_k(s)\,ds = 0.$$

This lemma is a particular case of Lemma 2.1 from [19].

§ 2.3. Proof of Main Results

2.3.1. Proof of Theorems $2.1.1_i$, $2.1.2_i$ $(i = 1, 2)$.

Proof of Theorem $2.1.1_i$. From the inclusion (2.1.9), by Lemma 1.2.1 we obtain $(p_0, p_1) \in \mathbb{V}_{i,0}(]a, b[)$, which, owing to Lemma 2.2.2 for $k > k_0$, implies $(p_{0k}, p_{1k}) \in \mathbb{V}_{i,0}(]a, b[)$. From Remark 1.2.2 follows the unique solvability of the problems (2.2.61), $(2.1.2_{i0})$ and $(2.2.61_k)$, $(2.1.2_{i0})$. Denote by $\widetilde{v}$, $\widetilde{v}_k$ and G, G_k, respectively, solutions and Green's functions of these problems.

Then the problems (2.1.1), $(2.1.2_{i0})$ and $(2.1.1_k)$, $(2.1.2_{i0})$ are equivalent, respectively, to the equations

$$u(t) = \mathbb{U}_0(u)(t) + \widetilde{v}(t) \tag{2.3.1}$$

and

$$u(t) = \mathbb{U}_k(u)(t) + \widetilde{v}_k(t), \tag{$2.3.1_k$}$$

where the continuous linear operators $\mathbb{U}_k$, $\mathbb{U}_0 : C(]a,b[) \to C(]a,b[)$ are defined by the equalities

$$\mathbb{U}_0(x)(t) = \int_a^b G(t,s)g(x)(s)\,ds \quad \text{and} \quad \mathbb{U}_k(x)(t) = \int_a^b G_k(t,s)g_k(x)(s)\,ds.$$

If $\rho : [a,b] \to \mathbb{R}^+$ is the function mentioned in the proof of Theorem $1.1.1_i$, then as is seen from that proof, there exists a constant $\lambda_0 \in [0,1[$ such that

$$||\mathbb{U}_0||_{C_\rho \to C_\rho} < \lambda_0. \tag{2.3.2}$$

Suppose that the equation

$$u(t) = \mathbb{U}_k(u)(t) \tag{$2.3.1_{0k}$}$$

has a non-zero solution u_{0k}. Not restricting the generality, we assume that

$$||u_{0k}||_{C,\rho} = 1 \quad \text{for} \quad k > k_0, \tag{2.3.3}$$

in which case $||u_{0k}||_C \leq ||\rho||_C$, i.e., if we introduce the notation $r = ||\rho||_C$, then

$$u_{0k} \in \mathbb{B}_{rk} \quad \text{for} \quad k > k_0. \tag{2.3.4}$$

Also, from $(2.3.1_{0k})$, (2.3.3), by Lemma 2.2.7 it follows that the sequence $(u_{0k})_{k=1}^{\infty}$ is uniformly bounded and equicontinuous. Hence by the Arzella–Ascoli lemma, not restricting the generality we can assume that there exists a function $u_0 \in C(]a,b[)$ such that uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} u_{0k}(t) = u_0(t). \tag{2.3.5}$$

It is clear from the equations (2.3.3), (2.3.5) that

$$||u_0||_{C,\rho} = 1. \tag{2.3.6}$$

Let us now introduce the notation

$$\Delta p_{jk}(t) = p_j(t) - p_{jk}(t) \ \ (j = 0,1,2), \quad \Delta G_k(t,s) = G(t,s) - G_k(t,s),$$
$$\Delta g_k(x)(t) = g(x)(t) - g_k(x)(t) \ \ (k \in \mathbb{N}).$$

For u_{0k}, when $k > k_0$, the representation

$$u_{0k}(t) = \mathbb{U}_0(u_{0k})(t) + \int_a^b \Delta G_k(t,s) g(u_{0k})(s)\,ds +$$
$$+ \int_a^b G_k(t,s) \Delta g_k(u_{0k})(s)\,ds \quad (k \in \mathbb{N}) \quad \text{for} \quad a \le t \le b \qquad (2.3.7)$$

is valid. Taking into account (2.3.4), (2.3.5), Remark 2.2.2, equality the (2.2.43) of Lemma 2.2.3 and also the equality (2.2.53) of Lemma 2.2.4 with all conditions satisfied, and then passing in (2.3.7) to limit as $k \to +\infty$, we get

$$u_0(t) = \mathbb{U}_0(u_0)(t)$$

which, with regard for (2.3.2), (2.3.6), results in the estimate

$$\|u_0\|_{C,\rho} < 1.$$

But this contradicts (2.3.6). Hence our assumption is invalid and the equation ($2.3.1_{0k}$) has only the zero solution, and because of its Fredholm property the equation ($2.3.1_k$) is uniquely solvable. The unique solvability of the equation (2.3.1) follows from Theorem $1.1.1_i$.

Let u and u_k be respectively solutions of the equations (2.3.1) and ($2.3.1_k$),

$$\begin{gathered} w_k(t) = u(t) - u_k(t) \quad \text{for} \quad k > k_0, \\ \lambda_k = \begin{cases} \|u_k\|_{C,\rho} & \text{for} \quad \|u_k\|_{C,\rho} > 1, \\ 1 & \text{for} \quad \|u_k\|_{C,\rho} \le 1, \end{cases} \\ \widetilde{u}_k(t) = \lambda_k^{-1} u_k(t) \end{gathered} \qquad (2.3.8)$$

and

$$\rho_k(t) = \frac{\widetilde{v}(t) - \widetilde{v}_k(t)}{\lambda_k} + \int_a^b \Delta G_k(t,s) g(\widetilde{u}_k)(s)\,ds + \int_a^b G_k(t,s) \Delta g_k(\widetilde{u}_k)(s)\,ds.$$

Then for w_k the representation

$$w_k(t) = \mathbb{U}_0(w_k)(t) + \lambda_k \rho_k(t) \quad \text{for} \quad a \le t \le b \qquad (2.3.9)$$

is valid, and if $r = \|\rho\|_C$, then

$$\widetilde{u}_k \in \mathbb{B}_{r,k}. \qquad (2.3.10)$$

In such a case, taking into account proposition (a) of Lemma 1.2.6, Remark 2.2.2, the equation (2.2.43) of Lemma 2.2.3 and also the equation (2.2.53) of Lemma 2.2.4, we obtain

$$\lim_{k\to\infty} \|\rho_k\|_{C,\rho} = 0. \tag{2.3.11}$$

On the other hand, from (2.3.9), with regard for (2.3.2), we get the estimate

$$\|w_k\|_{C,\rho} \le \alpha_k \lambda_k \quad \text{for} \quad k > k_0, \tag{2.3.12}$$

where

$$\alpha_k = \frac{\|\rho_k\|_{C,\rho}}{1-\lambda_0}$$

and by virtue of (2.3.11),

$$\lim_{k\to\infty} \alpha_k = 0. \tag{2.3.13}$$

Suppose now that we can extract from the sequence $(\lambda_k)_{k=1}^{\infty}$ a sequence $(\lambda_{k_m})_{m=1}^{\infty}$ such that $\lambda_{k_m} \ge 1$ for $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \lambda_{k_m} = +\infty, \tag{2.3.14}$$

and note that by our definition of the function w_k the inequality

$$\lambda_{k_m} - \|u\|_{C,\rho} \le \|w_{k_m}\|_{C,\rho} \tag{2.3.15}$$

is valid. Substituting now the inequality (2.3.12) in (2.3.15) and taking into account (2.3.13), we can see that this contradicts (2.3.14), i.e., our assumption is invalid, and there exists a constant $\lambda \in \mathbb{R}^+$ such that

$$\lambda_k \le \lambda \quad \text{for} \quad k > k_0 \tag{2.3.16}$$

which, with regard for (2.3.12), yields

$$\lim_{k\to\infty} \|w_k\|_{C,\rho} = 0. \tag{2.3.17}$$

Now we notice that (2.3.9) and (2.3.16) imply

$$|w_k^{(j)}(t)| \le \frac{d^j}{dt^j}\,\mathbb{U}_0(w_k)(t) + \lambda|\rho_k^{(j)}(t)| \;\; (j=0,1) \quad \text{for} \quad a<t<b. \tag{2.3.18$_j$}$$

Applying the estimates (2.2.46)–(2.2.48) and the inequalities (2.2.13), (2.2.10), we arrive at

$$\big|\mathbb{U}_0(w_k)(t)\big| \le r'\|w_k\|_{C,\rho} I_i^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \le t \le b, \tag{2.3.19}$$

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)}\left|\frac{d}{dt}\,\mathbb{U}_0(w_k)(t)\right| \le r'\|w_k\|_{C,\rho} \quad \text{for} \quad a<t<b, \tag{2.3.20}$$

where

$$r' = \frac{d_1^2}{d_2} \int\limits_a^b \frac{h(\rho)(s)}{\sigma(p_1)(s)}\, I_i^\mu(\sigma^\alpha(p_1))(s)\, ds.$$

By definition of the function $\widetilde{u}_k$, in view of the inequality (2.1.10) and the equalities (2.2.43), (2.2.44) of Lemma 2.2.3, we make sure that uniformly on the interval $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\left|\int\limits_a^b \Delta G_k(t,s) g(\widetilde{u}_k)(s)\, ds\right| = 0 \tag{2.3.21}$$

and

$$\lim_{k\to\infty} \frac{I_i^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\left|\int\limits_a^b \frac{d\Delta G_k(t,s)}{dt}\, g(\widetilde{u}_k)(s)\, ds\right| = 0. \tag{2.3.22}$$

Just in the same way, taking into account the inclusion (2.3.10) and the equalities (2.2.53), (2.2.54) of Lemma 2.2.4, we can see that

$$\left|\int\limits_a^b G_k(t,s)\Delta g_k(\widetilde{u}_k)(s)\, ds\right| \leq$$

$$\leq r_1 \sup\left\{\left|\int\limits_a^t \frac{\Delta g_k(x)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\, ds\right| : \ a \leq t \leq b, \ \ x \in \mathbb{B}_{r,k}\right\} \times$$

$$\times I_i^{1-\mu}(\sigma^\alpha(p_1))(t) \quad \text{for} \quad a \leq t \leq b, \tag{2.3.23}$$

$$\frac{I_i^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\left|\int\limits_a^b \frac{d}{dt}\, G_k(t,s)\Delta g_k(\widetilde{u}_k)(s)\, ds\right| \leq$$

$$\leq r_1 \sup\left\{\left|\int\limits_a^t \frac{\Delta g_k(x)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\, ds\right| :\right.$$

$$\left. a \leq t \leq b, \ \ x \in \mathbb{B}_{r,k}\right\} \quad \text{for} \quad a < t < b. \tag{2.3.24}$$

It is clear from the equalities (2.3.21)–(2.3.24), proposition (a) of Lemma 2.2.5 and also from the condition (2.1.8) and Remark 2.2.2 that uniformly on the interval $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^\alpha(p_1))(t)\rho_k(t) = 0 \tag{2.3.25}$$

and

$$\lim_{k\to\infty} \frac{\rho_k(t)}{\sigma(p_1)(t)}\, I_i^\mu(\sigma^\alpha(p_1))(t) = 0. \tag{2.3.26}$$

Multiplying ($2.3.18_0$) by $I_i^{\mu-1}(\sigma^\alpha(p_1))(t)$ and taking into consideration (2.3.17), (2.3.19) and (2.3.25) we see that the condition (2.1.11) is valid. Analogously, multiplying ($2.3.18_1$) by $\sigma^{-1}(p_1)(t)I_i^\mu\sigma^\alpha(p_1)(t)$ and taking into account (2.3.17), (2.3.20) and (2.3.26), we make sure that the condition (2.1.12) is valid. □

Proof of Theorem $2.1.2_i$. Reasoning in the same way as in the previous proof for the function $w_k(t) = u(t) - u_k(t)$, where u_k is a solution of the problem ($2.1.1_k$), ($2.1.2_{ik}$), using Remark 2.2.3 and proposition (b) of Lemma 2.2.6, we get the equality (2.3.17) which is the same as the condition (2.1.15). The proof of the condition (2.1.12) coincides completely with its proof in Theorem $2.1.1_i$. □

2.3.2. Proof of Corollaries.

Proof of Corollary $2.1.1_i$. It is sufficient to show that (2.1.8) follows from (2.1.16)–(2.1.18). Suppose to the contrary that the condition (2.1.18) is violated. Then there exist $\varepsilon > 0$, a sequence of positive numbers $(k_m)_{m=1}^{\infty}$ and a sequence of functions

$$y_m \in \mathbb{B}_{k_m} \tag{2.3.27}$$

such that

$$\max\left\{\Big|\int_a^t \frac{\Delta g_{k_m}(y_m)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ \ a \le t \le b\right\} > \varepsilon. \tag{2.3.28}$$

From (2.3.27) it follows

$$y_m(t) = \alpha_{1m}\widetilde{v}_{k_m}(t) + \int_a^b G_{k_m}(t,s)g_{k_m}(x_m)(s)\,ds \ \ (m \in \mathbb{N}), \tag{2.3.29}$$

where $x_m \in C(]a,b[)$ $(m \in \mathbb{N})$ and

$$0 \le \alpha_{1m} \le 1 \quad (m \in \mathbb{N}), \tag{2.3.30}$$

$$\|x_m\|_C \le 1 \quad (m \in \mathbb{N}). \tag{2.3.31}$$

Introduce the notation

$$z_m(t) = \int_a^b G_{k_m}(t,s)g_{k_m}(x_m)(s)\,ds \quad (m \in \mathbb{N})$$

and rewrite z_m as follows:

$$z_m(t) = \int_a^b G_{k_m}(t,s)\Delta g_{k_m}(x_m)(s)\,ds + \int_a^b G_{k_m}(t,s)g(x_m)(s)\,ds.$$

Then according to (2.1.10), (2.1.16), and (2.1.31) the inequality

$$|z_m^{(j)}(t)| \le \int_a^b \Big|\frac{\partial^j}{\partial t^j}\,\Delta G_{k_m}(t,s)\Big|\big(\eta(s)+h(1)(s)\big)\,ds +$$

$$+\int_a^b \Big|\frac{\partial^j}{\partial t^j}\,G(t,s)\Big|\big(\eta(s)+h(1)(s)\big)\,ds \;\; (j=0,1) \qquad (2.3.32_j)$$

is valid. By the conditions (2.1.6) and (2.1.18),

$$\int_a^b \frac{\eta(s)+h(1)(s)}{\sigma(p_1)(s)}\,I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds < +\infty$$

owing to which from $(2.3.32_0)$, in view of the equality (2.2.43) of Lemma 2.2.3 and by Lemma 2.2.5 we obtain the existence of a constant λ_1 such that

$$\|z_m\|_C < \lambda_1 \quad (m \in \mathbb{N}). \qquad (2.3.33)$$

Consider now the case $i = 1$ separately. From $(2.3.32_j)$ $(j = 0,1)$, by Lemmas 2.2.3 and 2.2.5 and the fact that

$$G(a,s) = G(b,s) = 0 \quad \text{for} \quad a < s < b$$

we can choose for any $\varepsilon_0 > 0$ constants m_0, a_1, b_1, δ, where

$$a < a_1 < b_1 < b, \quad \delta < \min(a_1 - a, b - b_1),$$

such that

$$|z_m(t)| \le \frac{\varepsilon_0}{4}, \quad m > m_0 \quad \text{for} \quad a \le t \le a_1, \quad b_1 \le t \le b,$$

i.e.,

$$\big|z_m(t_1) - z_m(t_2)\big| \le \frac{\varepsilon_0}{2}, \quad m > m_0, \quad \text{for} \quad a \le t_1, t_2 \le a_1, \quad b_1 \le t_1, t_2 \le b, \qquad (2.3.34)$$

and $A\delta < \frac{\varepsilon_0}{2}$, where

$$A = \sup\big\{|z_m'(t)| : \;\; a_1 - \delta < t < b_1 + \delta, \;\; m > m_0\big\} < +\infty,$$

i.e.,

$$\begin{gathered} \big|z_m(t_1) - z_m(t_2)| \le A|t_1 - t_2| < \frac{\varepsilon_0}{2}, \quad m > m_0 \\ \text{for} \quad a_1 - \delta < t_1, t_2 < b_1 + \delta, \quad |t_1 - t_2| < \delta. \end{gathered} \qquad (2.3.35)$$

The uniform boundedness and equicontinuity of the sequence $(z_m)_{m=1}^{\infty}$ follows from (2.3.33)–(2.3.35). Then by the Arzella–Ascoli lemma, not restricting the generality, we assume that uniformly on the segment $[a,b]$

$$\lim_{m\to\infty} z_m(t) = z(t). \qquad (2.3.36)$$

Notice now that however close may be a_1 from a and b_1 from b, the inequality (2.3.35) remains valid if we choose δ sufficiently small. Therefore, passing in (2.3.35) to limit, we can see that z is absolutely continuous on any segment contained in $]a,b[$, i.e.,

$$z \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b]). \tag{2.3.37}$$

On the other hand, in view of (2.3.30), not restricting the generality, we can assume that

$$\lim_{m\to\infty} \alpha_{1m} = \alpha_0,$$

which together with proposition (a) of Lemma 2.2.6 implies

$$\lim_{m\to\infty} \alpha_{1m}\widetilde{v}_{k_m}(t) = \alpha_0 \widetilde{v}(t) \quad \text{uniformly on} \quad [a,b], \tag{2.3.38}$$

where $\widetilde{v}$ is a solution of the problem (2.2.62), $(2.1.2_{i0})$.

Further, taking into account (2.3.36)–(2.3.38) in (2.3.29), we conclude that uniformly on the segment $[a,b]$

$$\lim_{m\to\infty} y_m(t) = y(t), \tag{2.3.39}$$

where

$$y \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b]). \tag{2.3.40}$$

The same takes place in the case $i=2$ owing to the fact that the relations

$$G(a,s) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} G(t,s)\Big|_{t=b} = 1 \quad \text{for} \quad a < s < b$$

follow from the inequalities

$$\big|z_m(t_1) - z_m(t_2)\big| \le \frac{\varepsilon_0}{2}, \quad m > m_0 \quad \text{for} \quad a \le t_1, t_2 \le a_1$$

and

$$\big|z_m(t_1) - z_m(t_2)\big| \le A_1|t_1 - t_2| \le \frac{\varepsilon_0}{2}, \quad m > m_0$$
$$\text{for} \quad a_1 - \delta < t_1, t_2 \le b, \quad |t_1 - t_2| < \delta$$

with

$$A_1 = \sup\big\{|z_m'(t)| : \quad a_1 - \delta < t < b, \quad m > m_0\big\} < +\infty,$$

and from the condition (2.3.38).

Finally, the conditions (2.1.16)–(2.1.18) and (2.3.39) imply

$$\max\bigg\{\Big|\int_a^t \frac{\Delta g_{k_m}(y_m)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\, ds\Big| : \quad a \le t \le b\bigg\} \le$$

$$\le \max\bigg\{\Big|\int_a^t \frac{\Delta g_{k_m}(y_m - y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\, ds\Big| : \quad a \le t \le b\bigg\} +$$

$$+\max\Big\{\Big|\int_a^t \frac{\Delta g_{k_m}(y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ a \le t \le b\Big\} \le$$

$$\le \int_a^b \frac{\eta(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\,\|y_m - y\|_C +$$

$$+\max\Big\{\Big|\int_a^t \frac{\Delta g_{k_m}(y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ a \le t \le b\Big\} \to 0$$

$$\text{as } \ m \to +\infty.$$

But this contradicts (2.3.28) and proves the validity of our corollary. □

Proof of Corollary $2.1.2_i$. Coincides completely with that of the previous corollary with the only difference that the functions $\widetilde{v}_k$ and $\widetilde{v}$ in (2.3.38) are solutions of the problems $(2.2.62_k)$, $(2.2.2_{ik})$ and (2.2.62), $(2.1.2_i)$, respectively, where the validity of the equality (2.3.38) follows from proposition (b) of Lemma 2.2.6. □

Proof of Corollary $2.1.3_i$. It can be easily verified that under the notation

$$\begin{aligned} g(x)(t) &= \sum_{m=1}^{n} g_{0m}(s)x(\tau_{0m}(t)), \\ g_k(x)(t) &= \sum_{m=1}^{n} g_{km}(t)x(\tau_{km}(t)) \end{aligned} \tag{2.3.41}$$

all the requirements of Theorem $2.1.1_i$, except for (2.1.8), are satisfied.

First we show the existence of a constant λ_1 such that

$$\sup\Big\{\Big\|\frac{y'}{\sigma(p_1)}\, I_i^\mu(\sigma^\alpha(p_1))\Big\|_C : \ y \in \mathbb{B}_{1k}, \ \ k > k_0\Big\} \le \lambda_1. \tag{2.3.42}$$

To this end we choose arbitrarily $k_1 > k_0$ and $y_1 \in \mathbb{B}_{k_1}$. Then there exist $\alpha_1 < 1$, $x_1 \in C(]a,b[)$, $\|x_1\|_C \le 1$ such that

$$y_1(t) = \alpha_1 \widetilde{v}_{k_1}(t) + \int_a^b G_{k_1}(t,s) g_{k_1}(x_1)(s)\,ds,$$

where $\widetilde{v}_{k_1}$ is a solution of the problem $(2.2.62_k)$, $(2.1.2_{i0})$. Next,

$$|y_1'(t)| \le |\widetilde{v}_{k_1}'(t)| + \int_a^b \Big|\frac{\partial G_{k_1}(t,s)}{\partial t}\Big|\eta(s)\,ds +$$

$$+ \int_a^b \Big|\frac{\partial G(t,s)}{\partial t}\Big| h(1)(s)\,ds \quad \text{for } \ a < t < b.$$

By virtue of the equality (2.2.67) of Lemma 2.2.6, there exists a constant λ_2 such that for any $k \geq k_0$

$$\Big\| \frac{\widetilde{v}_k'}{\sigma(p_1)} I_i^{\mu}(\sigma^{\alpha}(p_1)) \Big\|_C < \lambda_2. \tag{2.3.43}$$

Taking into account (2.3.43), the representation (2.2.45) of Green's function the estimates (2.2.46)–(2.2.48), the inequality (2.2.13) and the conditions (2.1.18), (2.2.20) and (2.2.21), we make sure that the estimate (2.3.42) is valid, where

$$\lambda_1 = \lambda_2 + \frac{d_1^2}{d_2^2} \Big(\int_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s) ds \Big)^{1-\mu} \Big(\int_a^b \frac{\eta(s) + h(1)(s)}{\sigma(p_1)(s)} I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds \Big).$$

We now notice that if

$$\lim_{k\to\infty} \Big(\sup \Big\{ \sum_{m=1}^n \Big| \int_a^t \frac{g_{0m}(s) - g_{km}(s)}{\sigma(p_1)(s)} I_i^{\beta}(\sigma^{\alpha}(p_1))(s) y(\tau_{km}(s))\, ds \Big| : \\ a \leq t \leq b, \ \ y \in \mathbb{B}_{1k} \Big\} \Big) = 0 \tag{2.3.44}$$

and

$$\lim_{k\to\infty} \Big(\sup \Big\{ \sum_{m=1}^n \Big| \int_a^t \frac{g_{0m}(s)}{\sigma(p_1)(s)} I_i^{\beta}(\sigma^{\alpha}(p_1))(s) \int_{\tau_{km}(s)}^{\tau_{0m}(s)} y'(\eta)\, d\eta\, ds \Big| : \\ a \leq t \leq b, \ \ y \in \mathbb{B}_{1k} \Big\} \Big) = 0, \tag{2.3.45}$$

then the condition (2.1.8) is satisfied.

Reasoning analogously to the proof of Corollary 2.1.1$_i$, we obtain that (2.3.44) is satisfied if for any $y \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b])$

$$\lim_{k\to\infty} \Big(\sum_{m=1}^n \Big| \int_a^t \frac{g_{0m}(s) - g_{km}(s)}{\sigma(p_1)(s)} I_i^{\beta}(\sigma^{\alpha}(p_1))(s) y(\tau_{km}(s))\, ds \Big| \Big) = 0. \tag{2.3.46}$$

On the other hand, from (2.1.23) it follows that

$$\operatorname{ess\,sup} \Big\{ \sum_{m=1}^n \big| \tau_{0m}(t) - \tau_{km}(t) \big| : \ \ a \leq t \leq b \Big\} \to 0 \quad \text{as} \quad k \to +\infty,$$

and hence for every $y \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b])$

$$\operatorname{ess\,sup} \Big\{ \sum_{m=1}^n \big| y(\tau_{km}(t)) - y(\tau_{0m}(t)) \big| : a \leq t \leq b \Big\} \to 0 \\ \text{as} \ \ k \to +\infty. \tag{2.3.47}$$

Then (2.1.21), (2.1.22), and (2.3.47) and lemma 2.2.8 imply the validity of the equality (2.3.46).

The validity of the equality (2.3.45) follows from the estimate (2.3.42), the condition (2.1.23) and the inequalities

$$\Big| \int\limits_a^t \frac{g_{0m}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s) \int\limits_{\tau_{km}(s)}^{\tau_{0m}(s)} y'(\eta)\, d\eta\, ds \Big| \le$$

$$\le \int\limits_a^b \frac{|g_{0m}(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds \times$$

$$\times \operatorname{ess\,sup} \Big\{ I_i^{\beta-\mu}(\sigma^{\alpha}(p_1))(t) \Big| \int\limits_{\tau_{km}(t)}^{\tau_{0m}(t)} \frac{\sigma(p_1)(s)ds}{I_i^{\mu}(\sigma(p_1))(s)} \Big| : \ \ a \le t \le b \Big\} \times$$

$$\times \Big\| \frac{y'}{\sigma(p_1)}\, I_i^{\mu}(\sigma^{\alpha}(p_1)) \Big\|_C \ \ (m = 1, \dots, n;\ \ k \in \mathbb{N}) \ \ \text{for} \ \ a \le t \le b. \qquad \square$$

Proof of Corollary 2.1.4$_i$. Coincides with the previous proof with the only difference that in the inequality (2.3.42) we will assume that $y \in \mathbb{B}'_{1k}$, i.e., the validity of (2.3.43) with $\widetilde{v}_k$ as a solution of the problem (2.1.4$_k$), (2.1.2$_{ik}$) will be shown by means of proposition (b) of Lemma 2.2.6. $\square$

Proof of Corollary 2.1.5$_i$. It is not difficult to notice that the conditions (2.1.18), (2.1.25) yield

$$\int\limits_a^b \frac{|g_{0m}(s)|}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)ds < +\infty \quad (m = 1, \dots, n), \qquad (2.3.48)$$

whence, owing to the fact that $\beta < \mu$, together with (2.1.24), we obtain the validity of the conditions (2.1.20), (2.1.21). That is, as it has been shown in the proof of Lemma 2.1.3$_i$, all the requirements of Theorem 2.1.1$_i$, except for (2.1.8), are satisfied.

On the other hand, the condition (2.1.8) under the notation (2.3.41) follows from the conditions (2.3.44), (2.3.45). Repeating now word by word the proof of Corollary 2.1.3$_i$, by the condition (2.1.26) we can see that (2.3.42) and (2.3.44) are valid.

Choosing $\mu_1 > \mu$ so as to satisfy

$$\mu_1 < 1, \quad \frac{1 - \alpha\mu_1}{1 - \mu_1} \le \delta,$$

analogously to the inequalities (2.2.15),(2.2.16) we obtain

$$\int\limits_a^b \frac{\sigma(p_1)(s)}{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}\,ds \le \Big(2I_1^{\mu}(\sigma^{\alpha}(p_1))\Big(\frac{a+b}{2}\Big)\Big)^{2-i}\Big(\frac{\mu_1}{\mu_1-\mu}\Big)\times$$
$$\times\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu_1}{1-\mu_1}}(p_1)(s)\,ds\Big)^{1-\mu_1}\Big(\int\limits_a^b \sigma^{\alpha}(p_1)(s)\,ds\Big)^{\mu_1-\mu} < +\infty.$$

From this and also from the condition (2.1.26), owing to the absolute continuity of the Lebesgue integral it follows that

$$\operatorname{ess\,sup}\Big\{\sum_{m=1}^{n}\Big|\int\limits_{\tau_{km}(t)}^{\tau_{0m}(t)} \frac{\sigma(p_1)(s)}{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}\,ds\Big| :\ a\le t\le b\Big\}\to 0 \qquad (2.3.49)$$
$$\text{for}\quad k\to+\infty.$$

Then the validity of the equality (2.3.45) follows from the conditions (2.3.48), (2.3.49) and also from the estimate (2.3.42) and the inequality

$$\Big|\int\limits_a^t \frac{g_{0m}(s)}{\sigma(p_1)(s)}\,I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\int\limits_{\tau_{km}(s)}^{\tau_{0m}(s)} y'(\eta)\,d\eta\,ds\Big| \le$$
$$\le\Big|\int\limits_a^b \frac{|g_{0m}(s)|}{\sigma(p_1)(s)}\,I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds\times$$
$$\times\operatorname{ess\,sup}\Big\{\Big|\int\limits_{\tau_{km}(t)}^{\tau_{0m}(t)} \frac{\sigma(p_1)(s)}{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}\,ds\Big| :\ a\le t\le b\Big\}\times$$
$$\times\Big\|\frac{y'}{\sigma(p_1)}\,I_i^{\mu}(\sigma^{\alpha}(p_1))\Big\|_C \quad (m=1,\dots,n;\ \ k\in\mathbb{N}). \qquad \square$$

Proof of Corollary 2.1.6_i. Coincides with the previous proof with the only difference that in the inequality (2.3.42) it will be assumed that $y\in\mathbb{B}'_{1k}$, i.e., the validity of the inequality (2.3.43) with $\widetilde{v}_k$ as a solution of the problem (2.1.4_k), (2.1.2_{ik}) will be shown by means of proposition (b) of Lemma 2.2.6. $\square$

Proof of Corollary 2.1.7_i (2.1.8_i). It is easily seen that for any $\alpha\in[0,1]$ and $\gamma>1$, by conditions (2.1.28)–(2.1.32) ((2.1.28)–(2.1.32), (2.1.14)), all the requirements of Corollary (2.1.5_i) ((2.1.6_i)) are satisfied for $p_j\equiv 0$, $p_{jk}\equiv 0$ ($j=0,1$; $k\in\mathbb{N}$), $n=1$, whence it follows that our corollary is valid. $\square$

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