Mem. Differential Equations Math. Phys. 12(1997), 13–21

Heinrich Begehr

ITERATIONS OF POMPEIU OPERATORS

Abstract. The Pompeiu operator T was extensively used by I. N. Vekua in his treatment of generalized Cauchy-Riemann systems. In the case of several complex variables when polydomains are considered, proper combinations of different T-operators for different components of the variable lead to a particular solution of the inhomogeneous Cauchy-Riemann system. This is applied to solve explicitely the Dirichlet problem in the unit polydisc for the inhomogeneous pluriharmonic system in the case of two complex variables.

რეზიუმე. პომპეიუს T ოპერატორი ინტენსიურად გამოიყენებოდა ი. ვეკუას მიერ განზოგადებული კოში-რიმანის სისტემების შესწავლისას. მრავალი კომპლექსური ცვლადის შემთხვევაში, როცა პოლიარეებია განხილული, სხვადასხვა ცვლადის მიმართ სხვადასხვა T-ოპერატორების შესაფერის კომბინაციებს მივყავართ არაერთგვაროვანი კოში-რიმანის სისტემის კერძო ამონახსნის აგებამდე. ეს ფაქტი გამოყენებულია არაერთგვაროვანი პლურიჰარმონიული სისტემისათვის ერთეულოვან პოლიდისკში დირიხლეს ამოცანის ცხადი სახის ამოსახსნელად ორი კომპლექსური ცვლადის შემთხვევაში.

In his theory of generalized analytic functions, I. N. Vekua has intensively studied the Pompeiu operator

$$Tf(z) := -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C},$$

and its complex conjugate

$$\overline{T}f(z) := -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}, \quad z \in \mathbb{C},$$

for different function spaces and different kinds of domains in the complex plane $\mathbb{C}$, see [11]. Because $\partial/\partial\overline{z}$ is left–inverse to T as is $\partial/\partial z$ to $\overline{T}$, the complex Laplace operator $\partial^2/(\partial z\partial\overline{z})$ is left–inverse to $T\overline{T}$. The operator $\partial^2/\partial\overline{z}^2$ is similary related to T^2. Hence, iterations of T and $\overline{T}$ lead to

1991 *Mathematics Subject Classification.* 47G10, 35J05, 35C15.

Key words and phrases. Pompeiu operator, Cauchy Pompeiu representation, higher order equations, one and several complex variables, boundary value problems, polydomains.

integral operators related to certain differential operators. This is true also in the case of several variables.

Three different situations will be considered. Arbitrary iterations of T and $\overline{T}$ are studied for general plane domains. For the unit disc, the iteration of Vekua's operator $\widetilde{T}$ is presented. At last, some particular cases in $\mathbb{C}^n$ are looked at.

1. Arbitrary Plane Domains

The Cauchy–Pompeiu representation formulas

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + (Tw_{\overline{\zeta}})(z) = \varphi(z) + (Tw_{\overline{\zeta}})(z), \quad z \in D,$$
$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} + (\overline{T}w_{\zeta})(z) = \overline{\psi(z)} + (\overline{T}w_{\zeta})(z), \quad z \in D,$$

for regular plane domains $D \subset \mathbb{C}$ and $w \in C^1(D;\mathbb{C}) \cap C^0(\overline{D};\mathbb{C})$ (see, e.g., [2,5,6,12]) are basic for the following. Here φ and ψ are analytic functions.

Theorem 1. *Let $D \subset \mathbb{C}$ be a regular domain. Then any $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ can be represented as*

$$w(z) = \varphi(z) + \overline{z}\psi(z) + \frac{1}{\pi}\int_D \frac{\overline{\zeta - z}}{\zeta - z} w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)d\xi d\eta, \quad z \in D,$$
$$w(z) = \varphi(z) + \overline{\psi(z)} - \frac{2}{\pi}\int_D \log|\zeta - z|\, w_{\overline{\zeta}\zeta}(\zeta)d\xi d\eta, \quad z \in D,$$

with some analytic functions φ and ψ.

Proof. From the Cauchy–Pompeiu formula we get $w(z) = \varphi_1(z) + Tw_{\overline{\zeta}}(z)$, $w_{\overline{z}}(z) = \varphi_2(z) + (Tw_{\overline{\zeta}\,\overline{\zeta}})(z)$ with some analytic functions φ_1 and φ_2. Hence $w(z) = \varphi_1(z) + T\varphi_2(z) + (T^2 w_{\overline{\zeta}\,\overline{\zeta}})(z)$. From the Cauchy–Pompeiu formula-the equality $\overline{z}\varphi_2(z) = \varphi_3(z) + T\varphi_2(z)$ follows with an analytic function φ_3. Moreover,

$$\begin{aligned} T^2 f(z) &= \frac{1}{\pi^2}\int_D f(\widetilde{\zeta})\int_D \frac{d\xi d\eta}{(\widetilde{\zeta} - \zeta)(\zeta - z)} d\widetilde{\xi} d\widetilde{\eta} = \\ &= \frac{1}{\pi}\int_D \frac{f(\widetilde{\zeta})}{\widetilde{\zeta} - z}\frac{1}{\pi}\int_D \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \widetilde{\zeta}}\right) d\xi d\eta d\xi d\widetilde{\eta} = \\ &= \frac{1}{\pi}\int_D \left[\frac{\overline{\widetilde{\zeta} - z}}{\zeta - z} - \frac{\varphi_4(\widetilde{\zeta}) - \varphi_4(z)}{\widetilde{\zeta} - z}\right] f(\widetilde{\zeta})\widetilde{\xi} d\widetilde{\eta} = \\ &= \frac{1}{\pi}\int_D \frac{\overline{\zeta - z}}{\zeta - z} f(\zeta) d\xi d\eta + \varphi_5(z) \end{aligned}$$

with analytic functions φ_4 and φ_5. This proves the first formula. Similarly, from $w(z) = \varphi_1(z) + (Tw_{\overline{\zeta}})(z)$ and $w_{\overline{z}}(z) = \overline{\varphi_2(z)} + (\overline{T}w_{\overline{\zeta}\zeta})(z)$ with analytic functions φ_1, φ_2 we see $w(z) = \varphi_1(z) + (T\overline{\varphi_2})(z) + (T\overline{T}w_{\overline{\zeta}\zeta})(z)$. The Cauchy–Pompeiu formula gives for a primitive ϕ_2 of φ_2 in D $\overline{\phi_2(z)} = \varphi_3(z) + (T\overline{\varphi_2})(z)$ with some analytic function φ_3. In order to reformulate

$$T\overline{T}f(z) = \frac{1}{\pi^2}\int_D f(\widetilde{\zeta})\int_D \frac{d\xi d\eta}{(\overline{\widetilde{\zeta} - \zeta})(\zeta - z)} d\widetilde{\xi}d\widetilde{\eta},$$

consider in $D_\varepsilon := D\backslash\{z : |z - \widetilde{\zeta}| \leq \varepsilon\}$

$$\log|\widetilde{\zeta} - z|^2 = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \log|\widetilde{\zeta} - \zeta|^2 \frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D_\varepsilon} \frac{1}{\overline{\zeta - \widetilde{\zeta}}}\frac{d\xi d\eta}{\zeta - z}.$$

Letting ε tend to zero, we get

$$\log|\widetilde{\zeta} - z|^2 = \frac{1}{2\pi i}\int_{\partial D} \log|\widetilde{\zeta} - \zeta|^2 \frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} \frac{1}{\overline{\zeta - \widetilde{\zeta}}}\frac{d\xi d\eta}{\zeta - z}.$$

Thus

$$\begin{aligned} T\overline{T}f(z) &= \frac{1}{\pi}\int_D f(\zeta)\log|\zeta - z|^2 d\xi d\eta - \\ &- \frac{1}{2\pi^2 i}\int_D\int_{\partial D} \log|\widetilde{\zeta} - \zeta|^2 \frac{d\zeta}{\zeta - z} f(\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta} = \\ &= \varphi_4(z) + \frac{1}{\pi}\int_D f(\zeta)\log|\zeta - z|^2 d\xi d\eta \end{aligned}$$

which proves the second formula. ■

The integral operators

$$T_{0,2}f(z) := \frac{1}{\pi}\int_D \frac{\overline{\zeta - z}}{\zeta - z} f(\zeta)d\xi d\eta \ , \ T_{1,1}f(z) := \frac{2}{\pi}\int_D \log|\zeta - z| f(\zeta)d\xi d\eta$$

were used in [9] (see [2]). Of course $T_{1,1}f$ differs from

$$Sf(z) := -\frac{2}{\pi}\int_D g(z,\zeta)f(\zeta)d\xi d\eta$$

only by a complex-valued harmonic function, i.e., by a sum $\varphi(z) + \overline{\psi(z)}$ with analytic functions φ and ψ. Here g is the Green function of D.

In the same manner, one can construct a hierarchy of integral operators providing integral representation formulas for functions w through their higher order derivatives $\partial^{m+n}w/(\partial z^m\partial\overline{z}^n)$ up to polyanalytic functions. These operators are (see [6,7]) for entire $m, n, 0 \le m+n, 0 < m^2+n^2$,

$$T_{m,n}f(z) := \int_D K_{m,n}(z-\zeta)f(\zeta)d\xi d\eta =$$

$$= \begin{cases} \dfrac{(-m)!(-1)^{-m}}{(n-1)!\pi}\displaystyle\int_D \frac{(\overline{z-\zeta})^{n-1}}{(z-\zeta)^{-m+1}}f(\zeta)d\xi d\eta \ \text{ for } \ m \le 0, \\ \dfrac{(-n)!(-1)^{-n}}{(m-1)!\pi}\displaystyle\int_D \frac{(z-\zeta)^{m-1}}{(\overline{z-\zeta})^{-n+1}}f(\zeta)d\xi d\eta \ \text{ for } \ n \le 0, \\ \dfrac{1}{(m-1)!(n-1)!\pi}\displaystyle\int_D (z-\zeta)^{m-1}(\overline{z-\zeta})^{n-1}\times \\ \times\left[\log|z-\zeta|^2 - \displaystyle\sum_{k=1}^{m-1}\frac{1}{k} - \sum_{l=1}^{n-1}\frac{1}{l}\right] f(\zeta)d\xi d\eta \ \text{ for } \ 0 < m, n. \end{cases}$$

Moreover,

$$T_{0,0}f := f, \quad T_{m,n}f := \frac{\partial^{m+n}f}{\partial z^m\partial\overline{z}^n} \ \text{ for } \ m, n < 0,$$

see [1]. These integral operators have the following properties for $f \in L_p(\overline{D})$, $1 < p$.

(1) $\overline{T_{m,n}f} = T_{n,m}\overline{f}$.

(2) There exists a constant M such that

$$|T_{m,n}f(z_1) - T_{m,n}f(z_2)| \le M\|f\|_{L_p(\overline{D})}|z_1-z_2|^\alpha$$

with $\alpha = 1$ for $2 < m+n$, $\alpha = (p-2)/p$ for $m+n = 1$, for $|z_1|, |z_2| \le R$, $0 < R$. M depends only on m, n, p in the case $2 \le m+n \le 3$ also on D and for $4 \le m+n$ also on D and R.

(3) For $m+n = 0 < m^2+n^2$, the operators $T_{m,n}$ are of Calderon–Zygmund type mapping $L_p(\mathbb{C})$, $1 < p$, into itself. They have to be understood as Cauchy principal value integrals and satisfy $\|T_{m,n}f\|_{L_p(\mathbb{C})} \le M(p)\|f\|_{L_p(\mathbb{C})}$.

(4) $T_{m,n}f$ has generalized derivatives for $1 \le m+n$

$$\frac{\partial}{\partial z}T_{m,n}f = T_{m-1,n}f, \quad \frac{\partial}{\partial\overline{z}}T_{m,n}f = T_{m,n-1}f,$$
$$\frac{\partial^{k+l}T_{m,n}f}{\partial z^k\partial\overline{z}^l} = T_{m-k,n-l}f \ \text{ for } \ k+l \le m+n.$$

(5) For $f \in L_2(\mathbb{C})$, we have

$$T_{m,-m}T_{k,-k}f = T_{m+k,-m-k}f, \quad T_{m,-m}T_{-m,m}f = f.$$

$T_{m,-m}$ is a unitary operator from $L_2(\mathbb{C})$ into itself, $\|T_{m,-m}f\|_{L_2(\mathbb{C})} = \|f\|_{L_2(\mathbb{C})}$. Its inverse and adjoint operator is $T_{-m,m}$.

As an example for a Pompeiu kind representation formula of higher order, the next result can be proven from the classical Pompeiu formula (see [2,11]) by induction.

Theorem 2. *If* $w \in C^n(\overline{D};\mathbb{C})$*, then*

$$w(z) = \sum_{k=0}^{n-1} \varphi_k(z)\overline{z}^k + \frac{1}{(n-1)!\pi}\int_D \frac{(\overline{z-\zeta})^{n-1}}{z-\zeta}\frac{\partial^n w(\zeta)}{\partial\overline{\zeta}^n}d\xi d\eta$$

with analytic functions φ_k*,* $0 \le k \le n-1$*.*

Proof. For $n = 1$ this is the classical Cauchy–Pompeiu formula $w(z) = \varphi_0(z) + (T_{0,1}w_{\overline{\zeta}})(z)$. Assume

$$\omega(z) = \sum_{k=0}^{n-2} \widetilde{\varphi}_k(z)\overline{z}^k + (T_{0,n-1}\partial^{n-1}\omega(\zeta)/\partial\overline{\zeta}^{n-1})(z)$$

holds for $\omega \in C^{n-1}(\overline{D};\mathbb{C})$. Then applying this formula to $\omega = w_{\overline{z}}$ and inserting the result in the Cauchy–Pompeiu formula for w, we obtain

$$\begin{aligned} w(z) &= \varphi_0(z) + T_{0,1}\left[\sum_{k=0}^{n-2} \widetilde{\varphi}_k(\zeta)\overline{\zeta}^k + T_{0,n-1}\partial^n w(\zeta)/\partial\overline{\zeta}^n\right](z) = \\ &= \varphi_0(z) + \sum_{k=0}^{n-2}\left[\frac{\widetilde{\varphi}_k(z)}{k+1}\overline{z}^{k+1} + \psi_k(z)\right] + (T_{0,n}\partial^n w(\zeta)/\partial\overline{\zeta}^n)(z) = \\ &= \sum_{k=0}^{n-1} \varphi_k(z) + (T_{0,n}\partial^n w/\partial\overline{\zeta}^n)(z)\ . \end{aligned}$$

Here ψ_k are analytic functions given by

$$\widetilde{\varphi}_k(z)\overline{z}^{k+1}/(k+1) = \psi_k(z) + (T_{0,1}\widetilde{\varphi}_k(\zeta)\overline{\zeta}^k)(z). \quad \blacksquare$$

For a general higher order Cauchy–Pompeiu formula see [2,6].

2. The Unit Disc

Besides the T–operator, Vekua [11] has introduced the $\widetilde{T}$–operator for the unit disc D. It has the same properties as the T–operator. Additionally it satisfies $\mathrm{Re}\,\widetilde{T}f(z) = 0$ for $|z| = 1$.

In fact it is uniquely given by the solution to the Schwarz problem $\mathrm{Re}\,w(z) = 0$ on $|z| = 1$ for the inhomogeneous Cauchy–Riemann equation $w_{\overline{z}} = f$ in $|z| < 1$. Its general solution is with arbitrary analytic φ given as $w = \varphi + Tf$. By the Schwarz condition, the analytic function φ satisfies $\mathrm{Re}\,\varphi = -\mathrm{Re}\,Tf$ on $|z| = 1$. The Schwarz formula (see [2,3,4,11]) defines φ as

$$\varphi(z) = -\frac{1}{\pi}\int_{|\zeta|<1} \overline{f(\zeta)}\frac{z}{1-z\overline{\zeta}}d\xi d\eta + ic$$

with arbitrary $c \in \mathbb{R}$. Hence,

$$w(z) = \widetilde{T}f(z) := -\frac{1}{\pi}\int_{|\zeta|<1}\left[\frac{f(\zeta)}{\zeta - z} + \frac{z\overline{f(\zeta)}}{1 - z\overline{\zeta}}\right] d\xi d\eta + ic.$$

From here $w(z) = (S_1 w_{\overline{\zeta}})(z) + i\,\mathrm{Im}\,w(0)$ follows with

$$S_1 f(z) := -\frac{1}{2\pi}\int_{|\zeta|<1}\left[\frac{\zeta + z}{\zeta - z}\frac{f(\zeta)}{\zeta} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right] d\xi d\eta.$$

This operator has the following properties (see [2,6]):

(1) For $f \in L_1(\overline{D})$, the function $S_1 f$ has generalized derivatives $(S_1 f)_{\overline{z}} = f$ and $(S_1 f)_z = \widetilde{\Pi} f$, where $\widetilde{\Pi}$ is the operator given by Vekua [11] as

$$\widetilde{\Pi} f(z) := -\frac{1}{\pi}\int_{|\zeta|<1}\left[\frac{f(\zeta)}{(\zeta - z)^2} + \frac{\overline{f(\zeta)}}{(1 - z\overline{\zeta})^2}\right] d\xi d\eta, \quad z \in D,$$

and satisfies $\|\widetilde{\Pi}\|_{L_2(\overline{D})} = 1$.

(2) $S_1 f$ satisfies homogeneous Schwarz conditions Re $S_1 f = 0$ on ∂D and the side condition $\mathrm{Im}\, S_1 f(0) = 0$.

(3) Iteration of S_1 leads for $|z| < 1$ to

$$S_1^k f(z) = S_k f(z) := \frac{(-1)^k}{2\pi(k-1)!}\int_{|\zeta|<1}\left(2\mathrm{Re}, (\zeta - z)\right)^{k-1} \times$$
$$\times\left[\frac{\zeta + z}{\zeta - z}\frac{f(\zeta)}{\zeta} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right] d\xi d\eta.$$

It satisfies $\partial S_k f/\partial \overline{z} = S_{k-1} f$ and is a particular solution to the Schwarz problem

$$\frac{\partial^k S_k f}{\partial \overline{z}^k} = f \text{ in } D, \quad \mathrm{Re}\frac{\partial^\kappa S_k f}{\partial \overline{z}^\kappa} = 0 \text{ on } \partial D,$$
$$\mathrm{Im}\,\frac{\partial^\kappa S_k f}{\partial \overline{z}^\kappa}\Big|_{z=0} = 0, \quad 0 \le \kappa \le k-1.$$

For $0 \le \kappa \le k-1$, the z–derivatives $\partial^\kappa S_k f/\partial z^\kappa$ are weakly singular integrals, while for $\kappa = k$,

$$\frac{\partial^k S_k f(z)}{\partial z^k} = \frac{(-1)^k k}{\pi}\int_{|\zeta|<1}\left[\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2} + \right.$$
$$\left. + \left(\frac{1 + \zeta(\zeta - z)}{1 - z\overline{\zeta}}\right)^{k-1}\frac{\overline{f(\zeta)}}{(1 - z\overline{\zeta})^2}\right] d\xi d\eta$$

is a singular integral operator. The L_2–norm of this operator is not yet known.

3. Polydomains in $\mathbb{C}^n$

A polydomain D^n in $\mathbb{C}^n$ is the Cartesian product of plane domains D_k, i.e., $D^n := \times_{k=1}^{n} D_k$. If one studies the overdetermined inhomogeneous Cauchy–Riemann system $w_{\overline{z_k}} = f_k$ $1 \le k \le n$, in D^n satisfying the compatibility conditions $f_{k\overline{z_l}} = f_{l\,\overline{z_k}}$, $1 \le k, l \le n$, a particular solution is given by a proper combination of T–operators. Denoting

$$T_k f(z_k) = -\frac{1}{\pi} \int_{D_k} f(\zeta_k) \frac{d\xi_k d\eta_k}{\zeta_k - z_k}$$

for $f \in L_1(\overline{D}_k)$, one can see that the general solution to the Cauchy–Riemann system is

$$w = \varphi + \sum_{\nu=1}^{n} (-1)^{\nu-1} \sum_{1 \le k_1 < \ldots < k_\nu \le n} T_{k_\nu} T_{k_{\nu-1}} \ldots T_{k_1} f_{k_1 \overline{\zeta_{k_2}} \ldots \overline{\zeta_{k_\nu}}}$$

with an arbitrary analytic function φ in D^n (see [3,4,8]). Here the last sum is taken over all ordered multiindices $\{k_1, k_2, \ldots, k_\nu\} \subset \{1, 2, \ldots, n\}$. For $n = 2$, e.g., $w = \varphi + T_1 f_1 + T_2 f_2 - T_2 T_1 f_{1\overline{\zeta_2}}$.

Integral representations of this form were given already in [10], see also [8].

As in the plane case, higher order systems can be treated similarly. The inhomogeneous pluriharmonic system $u_{z_k \overline{z_l}} = f_{kl}$, $1 \le k$, $l \le n$, in D^n, e.g., satisfying the compatibility conditions $f_{klz_i} = f_{ilz_k}$, $f_{kl\,\overline{z_j}} = f_{kj\overline{z_l}}$, $1 \le i, j, k, l \le n$, has a particular solution in the following form. For fixed $l, 1 \le l \le n$, the general solution to this anti–Cauchy–Riemann system is

$$u_{\overline{z_l}} = \overline{\psi_l} + \sum_{\mu=1}^{n} (-1)^{\mu-1} \sum_{1 \le k_1 < \ldots < k_\mu \le n} \overline{T_{k_\mu}} \ldots \overline{T_{k_1}} f_{k_1 l \zeta_{k_2} \ldots \zeta_{k_\mu}} =: F_l$$

with an analytic function ψ_l. Choosing ψ_l such that this inhomogeneous Cauchy–Riemann system, $1 \le l \le n$, satisfies the compatibility conditions $F_{l\overline{z_j}} = F_{j\overline{z_l}}$, $1 \le j$, $l \le n$, we have

$$u_0 = \sum_{\mu,\nu=1}^{n} (-1)^{\mu+\nu} \sum_{\substack{1 \le k_1 < \ldots < k_\mu \le n \\ 1 \le l_1 < \ldots < l_\nu \le n}} T_{l_\nu} \ldots T_{l_1} \overline{T_{k_\mu}} \ldots \overline{T_{k_1}} f_{k_1 l_1 \zeta_{k_2} \ldots \zeta_{k_\mu} \overline{\zeta_{l_2}} \ldots \overline{\zeta_{l_\nu}}} +$$
$$+ \sum_{\nu=1}^{n} (-1)^{\mu-1} \sum_{1 \le l_1 < \ldots < l_\nu \le n} T_{l_\nu} \ldots T_{l_1} \overline{\psi_{l_1 \zeta_{l_2}} \ldots \zeta_{l_\nu}}.$$

Because any pluriharmonic function, i.e., any solution to the homogeneous pluriharmonic system $f_{kl} = 0, 1 \le k$, $l \le n$, is the sum $\varphi + \overline{\psi}$ with two analytic functions φ, ψ, the general solution for the pluriharmonic system is then $u = \varphi + \overline{\psi} + u_0$. Of course u_0 can be simplified. Here only the case

$n = 2$ is studied. For the general case see [4], Chap. 5.3. For $n = 2$ the functions ψ_1, ψ_2 have to be chosen such that

$$\psi_{1z_2}(z) = \frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2} \overline{f_{21}(\zeta)}\frac{d\zeta_1}{\zeta_1 - z_1}\,\frac{d\overline{\zeta_2}}{\zeta_2 - z_2},$$
$$\psi_{2z_1}(z) = \frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2} \overline{f_{12}(\zeta)}\frac{d\overline{\zeta_1}}{\zeta_1 - z_1}\,\frac{d\zeta_2}{\zeta_2 - z_2}.$$

A particular solution is

$$\begin{aligned} u_0 = {} & T_1\overline{T_1}f_{11} + T_2\overline{T_2}f_{22} + T_1\overline{T_2}f_{21} + T_2\overline{T_1}f_{12} - T_1\overline{T_1}\,\overline{T_2}f_{11\zeta_2} - \\ & - T_2\overline{T_2}\,\overline{T_1}f_{22\zeta_1} - T_1\overline{T_1}T_2 f_{11\overline{\zeta_2}} - T_2\overline{T_2}T_1 f_{22\overline{\zeta_1}} + \\ & + T_1\overline{T_1}T_2\overline{T_2}f_{11\zeta_2\overline{\zeta_2}} + T_1\overline{\psi_1} + T_2\overline{\psi_2} - T_1T_2\overline{\psi_{1\zeta_2}}. \end{aligned}$$

In [3,4] the following result is proven.

Theorem 3. *Let f_{11}, f_{12}, f_{21}, f_{22} satisfy the above mentioned the compatibility conditions in $D^2 = \{(z_1z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ and γ be continuous on $\partial_0 D^2 = \{(z_1z_2) \in \mathbb{C}^2 : |z_1| = 1, |z_2| = 1\}$ satisfying*

$$\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma(\zeta)\left[\frac{z_1}{\zeta_1 - z_1}\,\frac{\overline{z_2}}{\overline{\zeta_2 - z_2}} + \frac{\overline{z_1}}{\overline{\zeta_1 - z_1}}\,\frac{z_2}{\zeta_2 - z_2}\right]\frac{d\zeta_1}{\zeta_1}\,\frac{d\zeta_2}{\zeta_2} +$$
$$+\frac{1}{\pi^2}\int_{D^2}\left[f_{12}\frac{z_1}{1 - z_1\overline{\zeta_1}}\,\frac{\overline{z_2}}{1 - \overline{z_2}\zeta_2} + f_{21}\frac{\overline{z_1}}{1 - \overline{z_1}\zeta_1}\,\frac{z_2}{1 - z_2\overline{\zeta_2}}\right]d\xi_1 d\eta_1 d\xi_2 d\eta_2 = 0.$$

Then the Dirichlet problem $u_{z_k\overline{z_l}} = f_{kl}$, $1 \le k, l \le 2$, in D^2 , $u = \gamma$ on $\partial_0 D^2$ is uniquely solvable by

$$\begin{aligned} u(z) = {} & \frac{1}{(2\pi i)^2}\int\limits_{\partial_0 D^2}\gamma(\zeta)\frac{1 - |z_1|^2}{|\zeta_1 - z_1|^2}\,\frac{1 - |z_2|^2}{|\zeta_2 - z_2|^2}\,\frac{d\zeta_1}{\zeta_1}\,\frac{d\zeta_2}{\zeta_2} + \\ & + \frac{1}{\pi}\int\limits_{|\zeta_1|<1} f_{11}(\zeta_1, z_2)\log\left|\frac{\zeta_1 - z_1}{1 - \overline{z_1}\zeta_1}\right|^2 d\xi_1 d\eta_1 + \\ & + \frac{1}{\pi}\int\limits_{|\zeta_2|<1} f_{22}(z_1, \zeta_2)\log\left|\frac{\zeta_2 - z_2}{1 - \overline{z_2}\zeta_2}\right|^2 d\xi_2 d\eta_2 + \\ & + \frac{1}{\pi^2}\int\limits_{D^2}\left\{f_{12}(\zeta)\frac{1}{\overline{\zeta_1 - z_1}}\left(\frac{1}{\zeta_2 - z_2} + \frac{\overline{z_2}}{1 - \overline{z_2}\zeta_2}\right) + \right. \\ & + f_{21}(\zeta)\left(\frac{1}{\zeta_1 - z_1} + \frac{\overline{z_1}}{1 - \overline{z_1}\zeta_1}\right)\frac{1}{\overline{\zeta_2 - z_2}} + \\ & + f_{11\overline{\zeta_2}}(\zeta)\log\left|\frac{\zeta_1 - z_1}{1 - \overline{z_1}\zeta_1}\right|^2\left(\frac{1}{\zeta_2 - z_2} + \frac{\overline{z_2}}{1 - \overline{z_2}\zeta_2}\right) + \end{aligned}$$

$$+ f_{11\zeta_2}(\zeta)\log\left|\frac{\zeta_1 - z_1}{1-\overline{z_1}\zeta_1}\right|^2 \frac{1}{\overline{\zeta_2 - z_2}} +$$
$$+ f_{22\overline{\zeta_1}}(\zeta)\left(\frac{1}{\zeta_1 - z_1} + \frac{\overline{z_1}}{1-\overline{z_1}\zeta_1}\right)\log\left|\frac{\zeta_2 - z_2}{1-\overline{z_2}\zeta_2}\right|^2 +$$
$$+ f_{22\zeta_1}(\zeta)\frac{1}{\overline{\zeta_1 - z_1}}\log\left|\frac{\zeta_2 - z_2}{1-\overline{z_2}\zeta_2}\right|^2 +$$
$$+ f_{11\zeta_2\overline{\zeta_2}}(\zeta)\log\left|\frac{\zeta_1 - z_1}{1-\overline{z_1}\zeta_1}\right|^2 \log\left|\frac{\zeta_2 - z_2}{1-\overline{z_2}\zeta_2}\right|\Bigg\} d\xi_1 d\eta_1 d\xi_2 d\eta_2 .$$

It is easily seen that u satisfies the Dirichlet condition. Also $u_{z_1\overline{z_1}} = f_{11}, u_{z_2\overline{z_2}} = f_{22}$ can be verified without difficulties. In order to calculate $u_{z_1\overline{z_2}}$ and $u_{z_2\overline{z_1}}$, the above condition has to be used.

Solvability conditions are characteristic for boundary value problems in several complex variables. In general, they fail to be unconditionally solvable, see [4,8].

References

1. M. Akal, Boundary value problems for complex elliptic partial differential equations of higher order. *PhD thesis, FU Berlin,* 1996; *Shaker Verlag, Aachen,* 1996.

2. H. Begehr, Complex analytic methods for partial differential equations. An introductory text. *World Scientific, Singapore,* 1994.

3. H. Begehr and A. Dzhuraev, The Schwarz problem for Cauchy–Riemann systems in several complex variables. *In: Analysis and Topology, Eds. C. Andreian Cazacu, O. Lehto, Th. M. Rassias. World Scientific, Singapore,* 1997 (*to appear*).

4. H. Begehr and A. Dzhuraev, An introduction to several complex variables and boundary value problems. *Addison Wesley Longman, Harlow,* 1997.

5. H. Begehr and R. P. Gilbert, Transformations, transmutations and kernel functions, I, II. *Longman, Harlow,* 1992, 1993.

6. H. Begehr and G. N. Hile, A hierarchy of integral operators. *Preprint Univ. Hawaii,* 1993; *Rocky Mountain J.* (*to appear*).

7. H. Begehr and G. N. Hile, Higher order Cauchy Pompeiu operators. *In: Operator Theory for Complex and Hypercomplex Analysis, Eds. E. Ramirez de Arellano, N. Salinas, M. Shapiro, N. Vasilevski, Contemporary Math.,* 1997 (*to appear*).

8. H. Begehr and G. C. Wen, Nonlinear elliptic boundary value problems and their applications. *Addison Wesley Longman, Harlow,* 1996.

9. A. Dzhuraev, Methods of singular integral equations. *Longman, Harlow,* 1992.

10. W. Tutschke, Partielle komplexe Differentialgleichungen in einer und mehreren komplexen Variablen. *Deutscher Verlag Wiss., Berlin,* 1977.

11. I. N. Vekua, Generalized analytic functions. *Pergamon Press, Oxford,* 1962.

12. G. C. Wen and H. Begehr, Boundary value problems for elliptic equations and systems. *Longman, Harlow,* 1990.

(Received 4.07.1997)

Author's address:
FU Berlin, I. Mathematisches Institut
Arnimallee 3, D–14195 Berlin
Germany