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ESTIMATES OF THE LOWER EXPONENT OF THE TWO-DIMENSIONAL LINEAR SYSTEM UNDER PERRON PERTURBATIONS

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Consider a linear system

$$\dot{x} = A(t)x, \quad x \in R^n, \quad t \geq 0, \tag{1_A}$$

with a piecewise continuous bounded matrix of coefficients $A(\cdot)$, the characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$, the incorrectness coefficients of Perron [1] $\sigma_P(A)$ and Grobman [1] $\sigma_G(A)$, and with a normal ordered system $X(t) = [X_1(t), \dots, X_n(t)]$ of its solutions $X_i(t)$. Along with the system (1_A) consider perturbed systems (1_{A+Q}) with piecewise continuous Perron perturbations $Q(\cdot)$ determined by the condition $\lambda[Q] \equiv \overline{\lim\limits_{t\to+\infty}} \frac{1}{t} \ln \|Q(t)\| < -\sigma_P(A)$.

For Perron perturbations only three following results are known: 1) the upper exponents $\lambda_2(A)$ and $\lambda_2(A+Q)$ of the two-dimensional systems (1_A) and (1_{A+Q}),respectively, coincide [1]; 2) generally speaking, the lower exponents $\lambda_1(A)$ and $\lambda_1(A+Q)$ of these systems have not this property [2]; 3) all characteristic exponents of three and higher-order systems (1_A) are, generally speaking, unstable (N. A. Izobov, S. N. Batan). Therefore, two-dimensional systems play a special role in the study of the behaviour of their characteristic exponents (lower and upper) under Perron perturbations.

For the two-dimensional system (1_A), introduce the angle $\gamma(t) \equiv \measuredangle\{X_1(t),\ X_2(t)\}$ between the solutions $X_1(t)$ and $X_2(t)$ forming its normal system of solutions.

Theorem 1. *For the lower exponent $\lambda_1(A+Q)$ of the two-dimensional system under any Perron perturbation $Q(\cdot)$ the following is true:* 1) $\lambda_1(A+Q) = \lambda_1(A)$ *if* $\sigma_P(A) = \sigma_G(A)$; 2) $\lambda_2(A) > \lambda_1(A+Q) > 2\lambda_1(A) - \lambda_2(A)$ *and nonstrict* $\lambda_1(A+Q) \geq \lambda[X_1 \sin\gamma]$ *otherwise.*

Scheme of proof. 1. The equality $\lambda_1(A+Q) = \lambda_1(A)$ if $\sigma_P(A) = \sigma_G(A)$ is a consequence of the Grobman theorem.

2. Supposing without loss of generality (1_A) to be a lower-triangular system, we transform (1_{A+Q}) by $y = X(t)z$ to

$$\dot{z} = \tilde{Q}(t)z, \quad z \in R^2, \quad t \geq 0, \tag{2}$$

a system of linear asymptotic balance ($\lambda[\tilde{Q}] < 0$). This allows us to prove the inequalities

$$\lambda[X_1 \sin\gamma] \leq \lambda_1(A+Q) < \lambda_2. \tag{3}$$

3. In the case due to (3) $\lambda[X_1 \sin\gamma] \leq 2\lambda_1(A) - \lambda_2(A)$ we establish that the Perron lower-triangular perturbation $Q_T(\cdot)$ preserves the characteristic exponents of the initial system (1_A), as well as of the conjugate one (1_{-A^T}), and their Perron incorrectness coefficient is invariant. This allows to include the lower-triangular part $Q_T(\cdot)$ of the

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Perron perturbation $Q(\cdot)$ into the very matrix $A(t)$ and to suppose that only the element $q_{12}(t)$ with the exponent $\lambda[q_{12}] < -\sigma_P(A)$ of the matrix $Q(t)$ is nonzero.

4. Assume that the inequality $\lambda(A+Q) \le 2\lambda_1(A) - \lambda_2(A)$, the opposite to one under investigation, is satisfied. Then the second component $y_2(t) = x_{12}(t)z_1(t) + x_{22}(t)z_2(t)$ of the solution $y(t) = X(t)z(t)$ of (1_{A+Q}), realizing the lower exponent $\lambda_1(A+Q)$ with the appropriate solution $z(t) = (z_1(t), z_2(t))$ of (2) (or $(1_{\widetilde{Q}})$) has the exponents of its terms $\lambda[x_{21}z_1] = \lambda_1(A)$, $\lambda[x_{22}z_2] < \lambda_1(A)$, and, as a result, it has the exponent $\lambda[y_2] = \lambda_1(A+Q) = \lambda_1(A)$, which contradicts to the assumption and to the inequality $\lambda_1(A) < \lambda_2(A)$ following from the condition $\sigma_P(A) < \sigma_G(A)$. ■

The exactness of the bounds of the lower exponent $\lambda(A+Q)$ of (1_{A+Q}) is established by Theorems 2 and 3 below.

Theorem 2. *For any numbers $\lambda_2 > \lambda_1$, $\sigma_0 \ge 2(\lambda_2 - \lambda_1)$, $\alpha \in (2\lambda_1 - \lambda_2, \lambda_1]$ and $\beta \in [\alpha, \alpha + \lambda_2 - \lambda_1)$ there exists a two-dimensional system (1_A) with an infinitely differentiable bounded matrix of coefficients $A(\cdot)$, the characteristic exponents $\lambda_i(A) = \lambda_i$, $i = 1, 2$, and the Perron incorrectness coefficient $\sigma_P(A) = \sigma_0$. Moreover, for any $\lambda \in [\alpha, \beta]$ there exists an analytical Perron perturbation $Q_\lambda(\cdot)$ such that the system (1_{A+Q_λ}) has a lower exponent $\lambda_1(A + Q_\lambda) = \lambda$*

Scheme of a proof. 1. Fix numbers $\theta > 1$ and $\theta_0 \in (1, \theta^{1/4})$. Using the points $\tau_k \equiv (\theta\theta_0^{-1})^k$ and $t_k \equiv (\theta\theta_0)^k$, $k \ge 0$, define the functions $f_1(t)$ and $f_2(t)$ on the half-axis $t \ge 0$ as follows. On the segments $[t_k, \tau_{k+1}]$ with zero and even k define them by $f_i(t) = (i-1)\lambda_i' + (i-2)\delta_i$, $i = 1, 2$, while with odd k by $f_i(t) = (2-i)\lambda_i' + (1-i)\delta_i$, $i = 1, 2$. Here $\lambda_1' = \alpha$, $\lambda_2' = \lambda_2$, and the numbers δ_1 and δ_2 satisfy $\lambda_1 + \delta_2 = \lambda_2 + \delta_1 = \sigma_0$. On the intervals (τ_k, t_k) put

$$f_i(t) = f[t; f_i(\tau_k), f_i(t_k)] \equiv$$
$$\equiv f_i(\tau_k) + [f_i(t_k) - f_i(\tau_k)] \exp\{-\ln^2(t/\tau_k) \exp[-\ln^{-2}(t/\tau_k)]\}, \tag{4}$$

using for this purpose an analogue of the well-known infinitely differentiable function (see B. Gelbaum and J. Olmsted "Counterexamples in Analysis"); on the interval $[0, 1)$ these functions are continued as the constants $f_i(1)$. It is easy to check that $a_{ii}(t) \equiv d[tf_i(t)]/dt$ are bounded and infinitely differentiable.

2. We will build the matrix $A(\cdot)$ of (1_A) as lower-triangular with already defined diagonal coefficients $a_{ii}(t)$ and the off-diagonal coefficient $a_{21}(t, \sigma) = -e^{-\sigma t}$, $t \ge 0$, with such $\sigma > \sigma_0$ that the second component $x_{21}(t, \sigma)$ of its solution $x(t) = (\exp tf_1(t), x_{21}(t, \sigma))$ has the exponent $\lambda[x_{21}] = \lambda[x] = \lambda_1$. For the proof of the existence of such $\sigma > 0$ we establish the Lipschitz condition $|\lambda[x_{21}(\cdot, \sigma_2)] - \lambda[x_{21}(\cdot, \sigma_1)]| \le \eta|\sigma_2 - \sigma_1|$ with a constant $\eta = \eta(\varepsilon_0) > 1$, whose exponent $\lambda[x_{21}(\cdot, \sigma)]$ satisfies $\sigma_1, \sigma_2 \ge \sigma_0 + \varepsilon_0$, as well as the estimates

$$\lambda_1 + \delta - \varepsilon \ge \lambda[x_{21}(\cdot, \sigma_0 + \varepsilon)] \ge$$
$$\ge \lambda_1 + (\delta - \varepsilon)\theta_0^2 - (\lambda_2 - \lambda_1)(\theta_0^2 - 1) \tag{5}$$

for all $\varepsilon > 0$ satisfying $\lambda_2 - \lambda_1 > \delta - \varepsilon$, where $\delta = \alpha - (2\lambda_1 - \lambda_2)$. Due to the proved continuity on $\sigma > 0$ of the exponent $\lambda[x_{21}(\cdot, \sigma)]$ and inequalities (5), we obtain now, taking $\theta_0 - 1 > 0$ small enough, the existence of the required $\sigma_1 > \sigma_0 : \lambda[x_{21}(\cdot, \sigma_1)] = \lambda_1$. The required equality $\sigma_P(A) = \sigma_0$ is established via the proof of the inequality $\lambda[x_{21}/(x_1x_2)] \le \delta_2$.

3. Let $B_\sigma(t)$ be an analytical Perron perturbation with the unique nonzero element $b_{21}(t, \sigma) = -a_{21}(t, \sigma) - \exp(-\sigma t)$, $\sigma > \sigma_0$, $t \ge 0$. Arguing as in the part 2, we establish the existence of such number $\sigma_2 = \sigma(\lambda) > \sigma_0$ that the second component of the solution $y(t) = (\exp tf_1(t), y_{21}(t, \sigma))$ of $(1_{A+B_{\sigma_2}})$ realizing its lower exponent $\lambda_1(A + B_{\sigma_2})$ has the exponent $\lambda[y_{21}(\cdot, \sigma_2)] = \lambda$. To complete the proof, it is sufficient to put $B_{\sigma(\lambda)} \equiv Q_\lambda(t)$, $t \ge 0$.

The condition $\alpha = \lambda[X_1 \sin\gamma] > 2\lambda_1(A) - \lambda_2(A)$ is carried out for the system (1_A) constructed in the proof of Theorem 2, with normal system of solutions $X = [X_1, X_2]$. There is the question: what will occur to a lower exponent $\lambda_1(A+Q)$ of perturbed system in case of Perron perturbation $Q(\cdot)$ and of the fulfilment of an opposite condition $\alpha = \lambda[X_1 \sin\measuredangle\{X_1, X_2\}] \le 2\lambda_1(A) - \lambda_2(A)$. The partial answer to it is given by the following

Theorem 3. *For any numbers $\alpha \le 2\lambda_1 - \lambda_2 < \lambda_1 < \beta < \lambda_2$ and $\sigma_0 > \lambda_2 - \alpha$ there exists a two-dimensional system (1_A) with a bounded infinitely differentiable matrix of coefficients $A(\cdot)$, a normal ordered system of solutions $[X_1(t), X_2(t)]$ satisfying $\lambda[X_1 \sin\measuredangle\{X_1, X_2\}] = \alpha$, the characteristic exponents $\lambda_i(A) = \lambda_i$, $i = 1, 2$, the Perron incorrectness coefficient $\sigma_P(A) = \sigma_0$ and such, that for any $\lambda \in [\lambda_1, \beta]$ there exists a system (1_{A+Q}) with infinitely differentiable Perron perturbation $Q(\cdot)$ and lower exponent $\lambda_1(A+Q) = \lambda$.*

Scheme of a proof. 1. Fix a parameter θ satisfying $1 < \sqrt{\theta} < \min\{[\sigma_0 + (i-1)(\alpha - \beta)]/(\sigma_0 + \alpha - \lambda_i)\}$, $i = 1, 2$, and introduce the points $t_k = \theta^k$, $t_{ik} = t_k\theta^{i/4}$, $t_{ik}^{\mp} = t_{ik}\theta^{\mp 1/16}$, $i = 0, 1, 2, 3$, and $k \ge 0$. Define the functions $f_1(t) = \alpha$ for all $t \ge 0$ and $f_2(t)$ as follows: 1) $f_2(t) = \lambda_1 - \sigma_0 = -\delta_2$ for $t \in [0, t_{00}^-]$; 2) on the segments $[t_{ik}^+, t_{i+1,k}^-]$ for $i = 0, 1, 2, 3$ and $t_{4k}^- = t_{0,k+1}^-$ put $f_2(t) = -\delta_2$ for $t \in [t_{1k}^+, t_{2k}^-] \bigcup [t_{3k}^+, t_{0,k+1}^-]$, $f_2(t) = \lambda_2$ for $t \in [t_{0k}^+, t_{1k}^-]$ and $f_2(t) = c_k$ for $t \in [t_{2k}^+, t_{3k}^-]$, $k \ge 0$, with a number c_k, defined below; 3) on the remaining intervals (t_{ik}^-, t_{ik}^+) put (see (4)) $f_2(t) = f[t; f_2(t_{ik}^-), f_2(t_{ik}^+)]$, $i = 0, 1, 2, 3$, $k \ge 0$. We define the diagonal coefficients of the required lower-triangular matrix $A(t)$ by $a_{ii}(t) = [tf_i(t)]'$, $t \ge 0$.

Let the coefficient $a_{21}(t)$ be equal to 1 on the segments $[t_{1k}^+, t_{2k}^-]$, $k \ge 0$, on the segments $[t_{3k}^+, t_{0,k+1}^-]$, let it be equal to a constant $b_k \in (-1, 0)$, which is equal to the ratio of the integrals of the function $\exp(\alpha + \delta_2)\tau$ over the segments $[t_{1k}^+, t_{2k}^-]$ and $[t_{3k}^+, t_{0,k+1}^-]$ and on all remaining intervals of the half-axis $t \ge 0$ let it be equal to 0.

2. Using the form of the constructed functions $a_{ii}(t)$ and $a_{21}(t)$, for the second component $x_{21}(t)$ of the solution $x(t)$ of the solution lower-triangular system (1_A) with initial value $x(0) = (1, 0)$ we establish the existence of a bounded sequence ($\{c_k\}$ the numbers c_k are used in the definition of $f_2(t)$ on the intervals $[t_k, t_{k+1})$), that the equalities $x_{21}(\tau_k)\exp(-\lambda_1\tau_k) = \max_{t_k \le t \le t_{k+1}} [x_{21}(t) \times \exp(-\lambda_1 t)] = (2\alpha + 2\delta_2)^{-1}$, $k \ge k_0$, are fulfilled. From them we have $\lambda[x] = \lambda[x_{21}] = \lambda_1$. Let $q_{12}(t)$ be equal to $\exp(-\sigma\tau_k)$ on the segments $[\tau_k, \tau_k + 1]$, $\forall k \ge k_1$, and be equal to 0 on the complement with respect to the half-axis $t \ge 0$. Put all other elements of the matrix $Q(\cdot)$ to be zero. By the transformation $y = X(t)z$ we transform (1_{A+Q}) to $(1_{\tilde{Q}})$ of the form (2) with $\lambda[\tilde{Q}] < 0$.

3. Let a solution $y(t) = (y_1(t), y_2(t))$ of (1_{A+Q}) realize the lower exponent $\lambda_1(A+Q)$. For an appropriate solution $z(t) = (z_1(t), z_2(t))$ of $(1_{\tilde{Q}})$ we have: $z_1(t) \to d_1 \ne 0$ as $t \to +\infty$, $\lambda[z_2] < 0$. We detect the exponent $\lambda[x_{22}z_2]$ from the parameter $\sigma > \sigma_0$ and the existence of such its value $\sigma = \sigma_1 > \sigma_0$, for which $\lambda[x_{22}z_2] = \lambda \in (\lambda_1, \beta]$. Then we receive the required relations $\lambda[y_2] = \lambda[x_{22}z_2] = \lambda > \alpha = \lambda[y_1]$ from the representation of $y_2(t)$.

4. We transform the constructed piecewise-constant functions $a_{21}(t)$ and $q_{12}(t)$ into infinitely-differentiable ones $a_{21}^*(t)$ and $q_{12}^*(t)$ by means of replacing of them by the functions $f[t; a_{21}(\tau_k), a_{21}(t_k)]$ and $f[t; q_{12}(\tau_k), q_{12}(t_k)]$ on all intervals (τ_k, t_k), one endpoint of which concides with a point of discontinuity of these functions, of such a small length, that the systems (1_A) and (1_{A^*}), (1_{A+Q}) and $(1_{A^*+Q^*})$ are pairwise asymptoticaly equivalent (this is possible according to the Yu. S. Bogdanov–S. A. Mazanik theorem). ■

Problem. Find out whether Theorem 3 is true in the case $\lambda \in (2\lambda_1 - \lambda_2, \lambda_1)$.

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