ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THIRD ORDER NONLINEAR DIFFERENCE EQUATIONS OF NEUTRAL TYPE

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Abstract. In the paper we consider the difference equation of neutral type

$$\Delta^3[x(n) - p(n)x(\sigma(n))] + q(n)f(x(\tau(n))) = 0, \quad n \in \mathbb{N}(n_0), \tag{E}$$

where $p, q\colon \mathbb{N}(n_0) \to \mathbb{R}_+$; $\sigma, \tau\colon \mathbb{N} \to \mathbb{Z}$, σ is strictly increasing and $\lim\limits_{n\to\infty} \sigma(n) = \infty$; τ is nondecreasing and $\lim\limits_{n\to\infty} \tau(n) = \infty$, $f\colon \mathbb{R} \to \mathbb{R}$, $xf(x) > 0$. We examine the following two cases:

$$0 < p(n) \leqslant \lambda^* < 1, \quad \sigma(n) = n - k, \quad \tau(n) = n - l,$$

and

$$1 < \lambda_* \leqslant p(n), \quad \sigma(n) = n + k, \quad \tau(n) = n + l,$$

where k, l are positive integers. We obtain sufficient conditions under which all nonoscillatory solutions of the above equation tend to zero as $n \to \infty$ with a weaker assumption on q than the usual assumption $\sum\limits_{i=n_0}^{\infty} q(i) = \infty$ that is used in literature.

Keywords: neutral type difference equation, third order difference equation, nonoscillatory solutions, asymptotic behavior

MSC 2000: 39A10

1. Introduction

Consider the third order neutral difference equation

$$\Delta^3[x(n) - p(n)x(\sigma(n))] + q(n)f(x(\tau(n))) = 0, \quad n \in \mathbb{N}(n_0), \tag{E}$$

where $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots\}$, n_0 is fixed in $\mathbb{N} = \{0, 1, 2, \ldots\}$ such that $\sigma(n_0) \geqslant 0$, $\tau(n_0) \geqslant 0$. Let Δ denote the forward difference operator defined by $\Delta x(n) =$

$x(n+1)-x(n)$, $\Delta^{i+1}x(n)=\Delta(\Delta^i x(n))$ for $i=1,2,\ldots$, $\Delta^0 x(n)=x(n)$. We examine the following two cases:

$$0<p(n)\leqslant\lambda^*<1,\quad \sigma(n)=n-k,\quad \tau(n)=n-l,$$

and

$$1<\lambda_*\leqslant p(n),\quad \sigma(n)=n+k,\quad \tau(n)=n+l,$$

where k,l are positive integers. Let $\mathbb{Z}$ denote the set of integers. We introduce the following hypotheses:

(H1) $p,q\colon\ \mathbb{N}(n_0)\longrightarrow\mathbb{R}_+$;

(H2) $\sigma\colon\ \mathbb{N}\longrightarrow\mathbb{Z}$, σ is strictly increasing and $\lim\limits_{n\longrightarrow\infty}\sigma(n)=\infty$;

(H3) $\tau\colon\ \mathbb{N}\longrightarrow\mathbb{Z}$, τ is nondecreasing and $\lim\limits_{n\longrightarrow\infty}\tau(n)=\infty$;

(H4) $f\colon\ \mathbb{R}\longrightarrow\mathbb{R}$, with $xf(x)>0$ for $x\neq 0$ and such that there exists a constant $M>0$ such that $|f(x)|\geqslant M\,|x|$ for all x.

For $k\in\mathbb{N}$ we use the usual factorial notation

$$n^{\underline{k}}=n(n-1)\ldots(n-k+1)\qquad\text{with}\qquad n^{\underline{0}}=1.$$

By a solution of equation (E) we mean a real sequence which is defined for all $n\in\mathbb{N}$ and satisfies equation (E) for n sufficiently large. We consider only such solutions which are nontrival for all large n. As usual a solution x of equation (E) is called oscillatory if for any $L\geqslant n_0$ there exists $n\geqslant L$ such that $x(n)x(n+1)\leqslant 0$. Otherwise it is called nonoscillatory.

In recent years there has been increasing interest in the study of the qualitative theory of neutral difference equations. For example, the first and second order difference equations of neutral type have been investigated in [5], [6], [8], [9], [12], [14]. For higher order difference equations we refer to [4], [10], [11], [13], [15]. In most of the papers [5], [6], [7], [8], [10], [11] it is assumed that the coefficient q satisfies the divergent condition of the series

$$\sum_{i=n_0}^{\infty}q(i)=\infty. \tag{1}$$

Our aim in this paper is to study the asymptotic behavior of solutions of equation (E) when (1) does not neccesarily hold.

2. Some basic lemmas

To prove our results we need the following lemmas which can be found in [9].

Lemma 1. *Suppose conditions* (H1), (H2) *and*

$$0 < p(n) \leqslant 1 \quad \text{for } n \geqslant n_0$$

hold. Let x be a nonoscillatory solution of the inequality

$$x(n)[x(n) - p(n)x(\sigma(n))] < 0.$$

(i) *Suppose that $\sigma(n) < n$ for $n \geqslant n_0$. Then x is bounded. If, moreover,*

$$0 < p(n) \leqslant \lambda^* < 1 \qquad \text{for } n \geqslant n_0 \tag{2}$$

for some positive constant λ^*, then $\lim\limits_{n \longrightarrow \infty} x(n) = 0$.

(ii) *Suppose that $\sigma(n) > n$ for $n \geqslant n_0$. Then x is bounded away from zero. If, moreover,* (2) *holds, then $\lim\limits_{n \longrightarrow \infty} |x(n)| = \infty$.*

Lemma 2. *Suppose conditions* (H1), (H2) *and*

$$p(n) \geqslant 1 \qquad \text{for } n \geqslant n_0$$

hold. Let x be a nonoscillatory solution of the inequality

$$x(n)[x(n) - p(n)x(\sigma(n))] > 0.$$

(i) *Suppose that $\sigma(n) > n$ for $n \geqslant n_0$. Then x is bounded. If, moreover,*

$$1 < \lambda_* \leqslant p(n) \quad \text{for } n \geqslant n_0 \tag{3}$$

for some positive constant λ_*, then $\lim\limits_{n \longrightarrow \infty} x(n) = 0$.

(ii) *Suppose that $\sigma(n) < n$ for $n \geqslant n_0$. Then x is bounded away from zero. If, moreover,* (3) *holds, then $\lim\limits_{n \longrightarrow \infty} |x(n)| = \infty$.*

The next lemma can be found in [1], [12].

Lemma 3. *Assume g is a positive real sequence and m is a positive integer. If*

$$\liminf_{n\to\infty} \sum_{i=n}^{n+m-1} g(i) > \Big(\frac{m}{m+1}\Big)^{m+1},$$

then

(i) *the difference inequality*

$$\Delta u(n) - g(n)u(n+m) \geqslant 0$$

has no eventually positive solution,

(ii) *the difference inequality*

$$\Delta u(n) - g(n)u(n+m) \leqslant 0$$

has no eventually negative solution.

3. Main results

We begin by classifying all possible nonoscillatory solutions of equations (E) on the basis of the well known Kiguradze's Lemma [15] (also see [1, Theorem 1.8.11]).

Lemma 4. *Let y be a sequence of real numbers and let $y(n)$ and $\Delta^m y(n)$ be of constant sign with $\Delta^m y(n)$ not eventually identically zero. If*

$$\delta y(n)\Delta^m y(n) < 0, \tag{4}$$

then there exist integers $\ell \in \{0, 1, \dots, m\}$ and $\widetilde{N} > 0$ such that $(-1)^{m+\ell-1}\delta = 1$ and

$$\begin{aligned} y(n)\Delta^j y(n) > 0 \quad &\text{for } j = 0, 1, \dots, \ell, \\ (-1)^{j-\ell} y(n)\Delta^j y(n) > 0 \quad &\text{for } j = \ell+1, \dots, m, \end{aligned}$$

for $n \geqslant \widetilde{N}$.

A sequence y satisfying (5) is called Kiguradze's sequence of degree ℓ.

Let x be a nonoscillatory solution of equation (E) and let

$$u(n) = x(n) - p(n)x(\sigma(n)), \quad n \in \mathbb{N}(n_0). \tag{6}$$

It is clear that u is eventually of one sign, so that either

$$x(n)[x(n) - p(n)x(\sigma(n))] > 0 \tag{7}$$

or

$$x(n)[x(n) - p(n)x(\sigma(n))] < 0 \tag{8}$$

for all sufficiently large n.

Let $\mathcal{N}_\ell^+$ [or $\mathcal{N}_\ell^-$] denote the set of solutions x of equation (E) satisfying (7) [or (8)] and for which $u(n) = x(n) - p(n)x(\sigma(n))$ is of degree ℓ. One can observe that if (7) holds that the condition (4) is fulfilled with $\delta = 1$. Since $m = 3$, so $(-1)^{m+\ell-1}\delta = 1$ if ℓ is even. But $\ell \in \{0, 1, \ldots, m\}$. Therefore $\ell = 0$ or $\ell = 2$. Similarly, if (8) holds, then $\ell = 1$ or $\ell = 3$. Hence we have the following classification of the set $\mathcal{N}$ of all nonoscillatory solutions of equation (E):

$$\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \mathcal{N}_1^- \cup \mathcal{N}_3^-. \tag{9}$$

First we will consider the case when $\sigma(n) = n - k, \quad \tau(n) = n - l$.

Theorem 1. *Assume* (H1)–(H4) *hold. Let* $0 < p(n) \leqslant \lambda^* < 1$, *where* λ^* *is a positive constant,* $\sigma(n) = n - k$, $\tau(n) = n - l$, *where* k, l *are positive integers and* $k > l$. *If*

$$\sum_{i=n_0}^{\infty} i^2 q(i) = \infty, \tag{10}$$

$$\limsup_{n\to\infty} (n-1)^{\underline{2}} \sum_{i=n+1+l}^{\infty} q(i) > \frac{8}{M}, \tag{11}$$

then every nonoscillatory solution of equation (E) *tends to zero as* $n \to \infty$.

P r o o f. By our assumptions, equation (E) takes on the form

$$\Delta^3(x(n) - p(n)x(n-k)) + q(n)f(x(n-l)) = 0, \quad n \in \mathbb{N}(n_0). \tag{E1}$$

Let x denote a nonoscillatory solution of (E1). Without loss of generality we may assume that x is an eventually positive solution of equation (E1). So, there exists an integer $n_1 \geqslant n_0$ such that $x(n-l) > 0$ for all $n \geqslant n_1$. One can observe that if $u(n) < 0$ then Lemma 1 implies that $\lim_{n\longrightarrow\infty} x(n) = 0$. Then $\lim_{n\longrightarrow\infty} u(n) = 0$, too. It means that the sequence u is increasing. Therefore $\mathcal{N}_1^- = \emptyset$ and $\mathcal{N}_3^- = \emptyset$. By (9), there are two cases to consider:

(A) $\quad u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0$,

(B) $\quad u(n) > 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0$,

eventually.

Case (A). Let

$$u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_1.$$

From (6) we have $u(n) < x(n)$. Summing equation (E1) from n to ∞ we get

$$\Delta^2 u(n) \geqslant \sum_{i=n}^{\infty} q(i) f(x(i-l)) \geqslant M \sum_{i=n}^{\infty} q(i) x(i-l).$$

Since $x(n-l) > u(n-l)$ we get

$$\Delta^2 u(n) > M \sum_{i=n}^{\infty} q(i) u(i-l). \tag{12}$$

Summing by parts we obtain the identity

$$\begin{aligned} \sum_{i=N}^{n-1} i^2 \Delta^3 u(i) &= n^2 \Delta^2 u(n) - 2n \Delta u(n+1) + 2u(n+2) \\ &\quad - N^2 \Delta^2 u(N) + 2N \Delta u(N+1) - 2u(N+2). \end{aligned} \tag{13}$$

Hence, using (E1) we arrive at

$$\sum_{i=N}^{n-1} i^2 q(i) f(x(i-l)) \leqslant -n^2 \Delta^2 u(n) + 2n \Delta u(n+1) + N^2 \Delta^2 u(N) + 2u(N+2).$$

By (H4)

$$M \sum_{i=N}^{n-1} i^2 q(i) u(i-l) \leqslant -n^2 \Delta^2 u(n) + 2n \Delta u(n+1) + N^2 \Delta^2 u(N) + 2u(N+2)$$

and

$$M u(N-l) \sum_{i=N}^{n-1} i^2 q(i) \leqslant -n^2 \Delta^2 u(n) + 2n \Delta u(n+1) + N^2 \Delta^2 u(N) + 2u(N+2).$$

In view of (10) this implies that

$$\lim_{n \to \infty} [2n \Delta u(n+1) - n^2 \Delta^2 u(n)] = \infty. \tag{14}$$

Thus

$$\Delta u(n+1) \geqslant \frac{n-1}{2} \Delta^2 u(n) \tag{15}$$

for $n \geqslant n_2$ where n_2 is sufficiently large. One can calculate:

$$\begin{aligned}\sum_{i=n_2}^{n-1} 2i\Delta u(i+1) &= [2iu(i+1)]_{n_2}^{n} - \sum_{i=n_2}^{n-1} \Delta 2iu(i+2) \\ &= 2nu(n+1) - 2n_2u(n_2+1) - 2\sum_{i=n_2}^{n-1} u(i+2)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=n_2}^{n-1} i^{\underline{2}}\Delta^2 u(i) &= \left[i^{\underline{2}}\Delta u(i)\right]_{n_2}^{n} - \sum_{i=n_2}^{n-1} 2i\Delta u(i+1) \\ &= n^{\underline{2}}\Delta u(n) - n_2^{\underline{2}}\Delta u(n_2) - \sum_{i=n_2}^{n-1} 2i\Delta u(i+1).\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{i=n_2}^{n-1} [2i\Delta u(i+1) - i^{\underline{2}}\Delta^2 u(i)] \\ &= 4nu(n+1) - 4n_2u(n_2+1) - 4\sum_{i=n_2}^{n-1} u(i+2) - u^2\Delta u(n) + u_2^{\underline{2}}\Delta u(n_2) \\ &\leqslant -n^{\underline{2}}\Delta u(n) + 4nu(n+1) + n_2^{\underline{2}}\Delta u(n_2).\end{aligned}$$

It means that $4nu(n+1) \geqslant n^{\underline{2}}\Delta u(n)$ and by (15) we get

$$4nu(n+1) \geqslant \frac{n(n-1)(n-2)}{2}\Delta^2 u(n-1),$$

which implies

$$\lim_{n\to\infty}[-n^{\underline{2}}\Delta u(n) + 4nu(n+1) + n_2^{\underline{2}}\Delta u(n_2)] = \infty.$$

Hence $u(n+1) \geqslant \frac{1}{8}(n-1)^{\underline{2}}\Delta^2 u(n-1)$ for sufficiently large n. From the above inequality and (12) we get

$$u(n+1) \geqslant \frac{(n-1)^{\underline{2}}}{8}M \sum_{i=n+1+l}^{\infty} q(i)u(i-l) \geqslant \frac{(n-1)^{\underline{2}}}{8}Mu(n+1) \sum_{i=n+1+l}^{\infty} q(i).$$

Therefore $8 \geqslant (n-1)^{\underline{2}}M \sum_{i=n+1+l}^{\infty} q(i)$, which contradicts (11).

Case (B). Let

$$u(n) > 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_3 \geqslant n_1.$$

Then there exists $\lim_{n\to\infty} u(n) = L \geqslant 0$. We claim that $L = 0$. Otherwise $L > 0$, then $L \leqslant u(n-l) \leqslant x(n-l)$. From (12) we have $\Delta^2 u(n) \geqslant ML \sum_{i=n}^{\infty} q(i)$.

Summing the above inequality from n to ∞ we get

$$-\Delta u(n) \geqslant ML \sum_{j=n}^{\infty} \sum_{i=j}^{\infty} q(i).$$

Summing once again from n_4 to ∞ we obtain

$$u(n_4) \geqslant ML \sum_{s=n_4}^{\infty} \sum_{j=s}^{\infty} \sum_{i=j}^{\infty} q(i) = ML \sum_{s=n_4}^{\infty} \frac{(i-n+2)^2}{2!} q(i),$$

which contradicts (10). Therefore $\lim_{n\to\infty} u(n) = 0$. Then $u(n) \leqslant 1$ for $n \geqslant n_5 \geqslant n_3$, where n_5 is large enough.Then

$$x(n) = p(n)x(n-k) + u(n) \leqslant p(n)x(n-k) + 1 \leqslant \lambda^* x(n-k) + 1 \tag{16}$$

for $n \geqslant n_6$.

We claim that x is bounded and $\lim_{n\to\infty} x(n) = 0$.

First suppose that x is unbounded. Then there exists a sequence $(n_s)_{s=1}^{\infty}$ such that $\lim_{s\to\infty} n_s = \infty$, $\lim_{s\to\infty} x(n_s) = \infty$ and $x(n_s) = \max_{n_0 \leqslant s \leqslant n_s} x(s)$.

Using (16) we get $x(n_s) \leqslant \lambda^* x(n_s - k) + 1 \leqslant \lambda^* x(n_s) + 1$, then $x(n_s) \leqslant (1-\lambda^*)^{-1}$, which contradicts the unboundedness of x.

Now, suppose that $\limsup_{n\to\infty} x(n) = c > 0$. Then there exists a sequence $(n_t)_{t=1}^{\infty}$ such that $\lim_{t\to\infty} n_t = \infty$, $\lim_{t\to\infty} x(n_t) = c$. This implies that for sufficient large t we have

$$x(n_t - k) \geqslant \frac{x(n_t) - u(n_t)}{\lambda^*} \geqslant \frac{x(n_t)}{\lambda^*}.$$

Choose $\varepsilon > 0$ such that $\varepsilon < (1-\lambda^*)c/\lambda^*$. Then $c/\lambda^* \leqslant \limsup_{t\to\infty} x(n_t - k) \leqslant c + \varepsilon$, hence $\varepsilon \geqslant c(1-\lambda^*)/\lambda^*$, which is a contradiction. This completes the proof.

Now we will consider the case when $\sigma(n) = n + k$, $\tau(n) = n + l$.

Theorem 2. *Let $1 < \lambda_* \leqslant p(n)$, where λ_* is a positive constant, $\sigma(n) = n + k$, $\tau(n) = n + l$, where k, l are positive integers and $l \geqslant k + 3$. Assume that there exists a sequence $\alpha\colon \mathbb{N} \to \mathbb{R}$ such that $n < \alpha(n)$. If*

$$\liminf_{n\to\infty} M \sum_{i=n}^{n+l-k-1} \sum_{j=i}^{\alpha(i)} \frac{(j-i+1)}{p(j+l-k)} q(j) > \Big(\frac{l-k}{l-k+1}\Big)^{l-k+1} \tag{17}$$

and

$$\limsup_{n\to\infty} M \sum_{i=n-l+k}^{n-3} (n-i-1)^{\underline{2}} \frac{q(i)}{p(i+l-k)} > 2, \tag{18}$$

then every nonoscillatory solution of equation (E) *tends to zero as $n \to \infty$.*

P r o o f. Assume that conditions (17) and (18) hold. Equation (E) takes on the form

$$\Delta^3(x(n) - p(n)x(n+k)) + q(n)f(x(n+l)) = 0, \quad n \in \mathbb{N}(n_0). \tag{E2}$$

Assume that x is an eventually positive solution of equation (E2). Then there exists an integer $n_1 \geqslant n_0$ such that $x(n) > 0$ for all $n \geqslant n_1$. By (9), there are four cases to consider:

(A) $u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0,$
(B) $u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0,$
(C) $u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) < 0, \quad \Delta^3 u(n) < 0,$
(D) $u(n) > 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0,$
eventually, for $\mathcal{N}_2^+, \mathcal{N}_1^-, \mathcal{N}_3^-, \mathcal{N}_0^+$, respectively.

C a s e (A). Let

$$u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_2 \geqslant n_1.$$

From (6) for $\sigma(n) = n + k$ we have

$$x(n) = u(n) + p(n)x(n+k) > u(n) \tag{19}$$

and

$$x(n) > p(n)x(n+k) > x(n+k),$$

which implies that x is bounded. But $u(n) < x(n)$ eventually which is a contradiction with the unboundedness of u.

C a s e (B). Let

$$u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_3 \geqslant n_1.$$

In [1], problem 1.9.35 p. 43 one can find the following formula:

$$\Delta^r u(n) = \sum_{i=r}^{m-1} (-1)^{\underline{i-r}} \frac{(t-n+i-r-1)^{\underline{i-r}}}{(i-r)!} \Delta^i u(t) + (-1)^{m-r} \frac{1}{(m-r-1)!} \sum_{j=n}^{t-1} (j-n+m-r-1)^{\underline{m-r-1}} \Delta^m u(j),$$

where $m, r, t \in \mathbb{N}$ and $t > m \geqslant n_0$, $0 \leqslant r < m$.

Applying the above equality to equation (E2) for $r = 1$ we get

$$\Delta u(n) = \sum_{i=1}^{2} (-1)^{(i-1)} \frac{(s-n+i-2)^{\underline{i-1}}}{(i-1)!} \Delta^i u(s) - \sum_{j=n}^{s-1} (j-n+1) q(j) f[x(j+l)]$$

for $s \geqslant n \geqslant n_3$.

Therefore we have

$$\Delta u(n) \leqslant - \sum_{j=n}^{s-1} (j-n+1) q(j) f(x(j+l)) \quad \text{for } s \geqslant n \geqslant n_3.$$

By (H4)

$$\Delta u(n) \leqslant -M \sum_{j=n}^{s-1} (j-n+1) q(j) x(j+l) \quad \text{for } n \geqslant n_3. \tag{20}$$

From (19) we get

$$-x(n+l) \leqslant \frac{u(n+l-k)}{p(n+l-k)}. \tag{21}$$

Putting (21) into (20) we obtain

$$\Delta u(n) \leqslant M \sum_{j=n}^{s-1} (j-n+1) \frac{q(j) u(j+l-k)}{p(j+l-k)}.$$

Let $s = \alpha(n) + 1$. Then we have

$$\Delta u(n) \leqslant M \sum_{j=n}^{\alpha(n)} (j-n+1) \frac{q(j) u(j+l-k)}{p(j+l-k)},$$

hence

$$\Delta u(n) - Mu(n+l-k) \sum_{j=n}^{\alpha(n)} (j-n+1) \frac{q(j)}{p(j+l-k)} \leqslant 0.$$

By Lemma 3 with regard to (17) for $m = l-k$ we obtain that the above inequality has no eventually negative solution, which is a contradiction.

Case (C). Let

$$u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) < 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_4 \geqslant n_1.$$

From discrete Taylor's formula we have

$$u(n) = \sum_{i=0}^{2} \frac{(n-n_4)^{\underline{i}}}{i!} \Delta^i[u(n_4)] + \frac{1}{2} \sum_{j=n_4}^{n-3} (n-j-1)^{\underline{2}} \Delta^3 u(j), \quad n > n_4,$$

where $n^{\underline{i}} = n(n-1)(n-2)\dots(n-i+1)$ and $n^{\underline{0}} = 1$. Therefore we obtain

$$u(n) \leqslant \frac{1}{2} \sum_{j=n_4}^{n-3} (n-j-1)^{\underline{2}} \Delta^3 u(j).$$

By (E2) and (H4) we have

$$-u(n) \geqslant \frac{1}{2} \sum_{j=n_4}^{n-3} (n-j-1)^{\underline{2}} q(j) f(x(j+l)) \geqslant \frac{M}{2} \sum_{j=n_4}^{n-3} (n-j-1)^{\underline{2}} [q(j)x(j+l)].$$

Using (21) in the above inequality we get

$$-u(n) \geqslant -\frac{M}{2} \sum_{j=n_4}^{n-3} (n-j-1)^{\underline{2}} \ \frac{q(j)u(j+l-k)}{p(j+l-k)}.$$

Let $n_4 = n-l+k$. Then

$$-u(n) \geqslant -\frac{M}{2} u(n) \sum_{j=n-l+k}^{n-3} (n-j-1)^{\underline{2}} \ \frac{q(j)}{p(j+l-k)}.$$

Therefore

$$\frac{2}{M} \geqslant \sum_{j=n-l+k}^{n-3} (n-j-1)^{\underline{2}} \ \frac{q(j)}{p(j+l-k)},$$

which contradicts (18).

Case (D). Let

$$u(n) > 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_5 \geqslant n_1.$$

By Lemma 2 it follows that $\lim\limits_{n \longrightarrow \infty} x(n) = 0$. This completes the proof. □

R e m a r k 1. One can observe that condition (H4) is fulfilled, for instance, with functions of the form $f(x) = (|x^\alpha| + c)\operatorname{sgn} x$ where $\alpha \geqslant 1$, $c > 0$. Particularly, for $\alpha = 2$ and $c = 1$, condition (H4) holds for each constant $M \in (0, 2)$.

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