DIRECTOIDS WITH SECTIONALLY ANTITONE INVOLUTIONS AND SKEW MV-ALGEBRAS

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Abstract. It is well-known that every MV-algebra is a distributive lattice with respect to the induced order. Replacing this lattice by the so-called directoid (introduced by J. Ježek and R. Quackenbush) we obtain a weaker structure, the so-called skew MV-algebra. The paper is devoted to the axiomatization of skew MV-algebras, their properties and a description of the induced implication algebras.

Keywords: directoid, antitone involution, sectionally switching mapping, MV-algebra, NMV-algebra, WMV-algebra, skew MV-algebra, implication algebra

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It was shown in [3] that every MV-algebra can be considered as a distributive lattice with sectionally antitone involutions satisfying a certain compatibility condition which can be expressed in the form of Exchange Identity where the term operation $x \to y = \neg x \oplus y$ is considered. Analogously, when a lattice is substituted by a commutative directoid, the resulting MV-like algebra called a non-associative MV-algebra in [5] is obtained. This approach was generalized in [10] where the axiomatic system of non-associative MV-algebras was slightly modified. Then the resulting algebra, called the weak MV-algebra, is neither associative nor commutative but it still satisfies the Łukasiewicz axiom

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

On the other hand, weak MV-algebras have the property that every section $[p, 1]$ can be equipped with polynomial operations $\oplus_p$ and $\neg_p$ such that $([p, 1]; \oplus_p, \neg_p, 1)$ is a weak MV-algebra again.

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Our aim is to replace the commutative directoid (alias λ-semilattice) by a general one which need not be commutative. The resulting algebra, called here the skew MV-algebra will be surely weaker than that of [10]. However, the new axiomatic system is still very simple and fully readable. In fact, this shows the power of directoids equipped with sectional involutions.

1. Basic concepts

The concept of the directoid was introduced by J. Ježek and R. Quackenbush [11] in order to axiomatize algebraic structures defined on upward directed ordered sets. In a certain sense, directoids generalize semilattices. For the reader's convenience, we repeat definitions and basic properties of these concepts.

An ordered set $(A;\leqslant)$ is *upward directed* if $U(x,y)\neq\emptyset$ for every $x,y\in A$, where $U(x,y)=\{a\in A;\ x\leqslant a \text{ and } y\leqslant a\}$. Elements of $U(x,y)$ are referred to as common upper bounds of x,y. Of course, if $(A;\leqslant)$ has a greatest element then it is upward directed.

Let $(A;\leqslant)$ be an upward directed set and let $\vee$ denote a binary operation on A. The pair $\mathcal{A}=(A;\vee)$ is called a *directoid* if

(i) $x\vee y\in U(x,y)$ for all $x,y\in A$;

(ii) if $x\leqslant y$ then $x\vee y=y$ and $y\vee x=y$.

The following axiomatization of directoids was given in [11]:

Proposition. *A groupoid* $\mathcal{A}=(A;\vee)$ *is a directoid if and only if it satisfies the identities*

(D1) $x\vee x=x$;

(D2) $(x\vee y)\vee x=x\vee y$;

(D3) $y\vee(x\vee y)=x\vee y$;

(D4) $x\vee((x\vee y)\vee z)=(x\vee y)\vee z$ *(skew associativity).*

Then the binary relation $\leqslant$ *defined on* A *by the rule*

$$x\leqslant y \quad \text{if and only if} \quad x\vee y=y$$

is an order and $x\vee y\in U(x,y)$ *for each* $x,y\in A$.

A directoid $\mathcal{A}=(A;\vee)$ is called *commutative* if it satisfies the identity

(D5) $x\vee y=y\vee x$.

It was shown in [11] that commutative directoids are axiomatized by the identities (D1), (D4) and (D5).

Let us denote the greatest element of an ordered set by 1 and the least by 0. We call a directoid *bounded* if it has both 0 and 1.

Let $(A; \leqslant, 1)$ be an ordered set with the greatest element 1. For $p \in A$, the interval $[p, 1]$ will be called a *section*. A mapping f of $[p, 1]$ into itself will be called a *sectional mapping*. To distinguish sectional mappings on different sections, we introduce the following notation: if f is a sectional mapping on $[p, 1]$ and $x \in [p, 1]$ then $f(x)$ will be denoted by x^p. A sectional mapping on $[p, 1]$ is called a *switching mapping* if $p^p = 1$ and $1^p = p$ and it is called an *involution* if $x^{pp} = x$ for each $x \in [p, 1]$. Of course, any involution is a bijection and if a sectional mapping on $[p, 1]$ is a switching involution then

$$x^p = 1 \text{ iff } x = p \quad \text{and} \quad x^p = p \text{ iff } x = 1.$$

$(A; \leqslant, 1)$ will be called an ordered set *with sectionally switching involutions* if there is a sectional switching involution on the section $[p, 1]$ for each $p \in A$.

As is well-known, MV-algebras were introduced in the late fifties of the 20th century by C. C. Chang [6] as an algebraic semantics of the Łukasiewicz many-valued sentential logic. More precisely, an *MV-algebra* is any algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV2) $x \oplus y = y \oplus x$;
(MV3) $x \oplus 0 = x$;
(MV4) $\neg\neg x = x$;
(MV5) $x \oplus 1 = 1$ (where $1 := \neg 0$);
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The prototypical example of an MV-algebra is the algebra $\Gamma(G, u) = ([0, u], \oplus, \neg, 0)$, where $(G, +, -, 0, \vee, \wedge)$ is an Abelian lattice-ordered group, $0 < u \in G$ and $[0, u] = \{x \in G \colon 0 \leqslant x \leqslant u\}$, and the operations $\oplus$ and $\neg$ are defined via $x \oplus y := (x + y) \wedge u$ and $\neg x := u - x$, respectively. D. Mundici proved (see e.g. [7]) that every MV-algebra $\mathcal{A}$ is isomorphic to an MV-algebra $\Gamma(G, u)$.

Another well-known fact is that for any MV-algebra $\mathcal{A}$, the relation $\leqslant$ given by

(A) $x \leqslant y \quad \Leftrightarrow \quad \neg x \oplus y = 1$

is a lattice order on A where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ are the lattice operations, and the top and the bottom element is 1 and 0, respectively.

Moreover, for any MV-algebra $\mathcal{A}$ and $p \in A$, one can define a structure of an MV-algebra on the section $[p, 1]$ in a natural way as follows:

(B) $x \oplus_p y = \neg(\neg x \oplus p) \oplus y \quad$ and $\quad \neg_p x = \neg x \oplus p$.

In the recent years a non-commutative generalization of MV-algebras has been introduced and studied by G. Georgescu and A. Iorgulescu [8] and independently by J. Rachůnek [12] under the name pseudo MV-algebras.

Another approach to generalize MV-algebras by omitting associativity (MV1) but keeping commutativity (MV2) was done by the first author and J. Kühr [5]. More precisely, they considered algebras $(A; \oplus, \neg, 0)$ of type (2,1,0) satisfying the axioms (MV2)–(MV6), where the axiom (MV1) is substituted by two axioms

(C) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$;

(D) $\neg x \oplus (x \oplus y) = 1$.

These algebras are called NMV-algebras (non-associative MV-algebras) [5]. Clearly, every MV-algebra satisfies the axioms (C) and (D) as well.

To clarify the role of the axiom (C), let us note that its validity enables us to prove that the relation $\leqslant$ defined by (A) remains transitive (hence being an order relation). From the logical point of view, such a property is quite natural since in all reasonable logics the set of truth values should be partially ordered.

We have seen that the sections in an MV-algebra form MV-algebras as given by (B). However, this is not true for NMV-algebras: it turns out that for an NMV-algebra A, the sections $[p, 1]$ have the structure of an NMV-algebra as defined by (B) if and only if $\oplus$ is associative. In other words, an NMV-algebra shares the above property if and only if it is an MV-algebra.

This fact motivated R. Halaš and L. Plojhar [10] to find a new class of generalized MV-algebras admitting the same structure on sections. They defined and investigated the so-called WMV-algebras.

An algebra $(A; \oplus, \neg, 0)$ is called a *weak MV-algebra* (or *WMV-algebra* for short) if it satisfies the axioms

(W1) $\neg\neg x = x$;

(W2) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$;

(W3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(W4) $x \oplus 0 = 0 \oplus x = x$;

(W5) $x \oplus 1 = 1 \oplus x = 1 \quad (1 := \neg 0)$;

(W6) $\neg y \oplus (\neg x \oplus y) = 1$;

(W7) $p \leqslant x \leqslant y \Rightarrow \neg y \oplus p \leqslant \neg x \oplus p$.

These algebras can be viewed as commutative directoids, (alias λ-semilattices) with respect to the induced order.

In what follows we replace the commutative directoid by a general one which need not be commutative. Thus the resulting algebra will be surely weaker than the WMV-algebra.

Definition 1. Let $\mathcal{D} = (D; \vee)$ be a bounded directoid with sectionally switching involutions. Define

$$x \oplus y = (x^0 \vee y)^y, \quad \neg x = x^0.$$

Then $\mathcal{A}(D) = (D; \oplus, \neg, 0)$ will be called a *skew MV-algebra*.

Theorem 1. *Let $\mathcal{D} = (D; \vee)$ be a bounded directoid with sectionally switching involutions and $\mathcal{A}(D)$ its skew MV-algebra. Then the following identities are satisfied:*

(1) $\neg\neg x = x$ *(double negation);*
(2) $x \oplus 0 = 0 \oplus x = x$;
(3) $\neg x \oplus (y \oplus x) = 1$;
(4) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$;
(5) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y$;
(6) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$.

P r o o f. By definition, we have

(1) $\neg\neg x = x^{00} = x$;
(2) $x \oplus 0 = (x^0 \vee 0)^0 = (x^0)^0 = x^{00} = x$, $0 \oplus x = (0^0 \vee x)^x = (1 \vee x)^x = 1^x = x$;
(3) $\neg x \oplus (y \oplus x) = (x \vee (y^0 \vee x)^x)^{(y^0 \vee x)^x} = ((y^0 \vee x)^x)^{(y^0 \vee x)^x} = 1$.

Clearly, $\neg(\neg x \oplus y) \oplus y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$, since $(x \vee y)^y \geqslant y$. We use this fact in the sequel:

(4) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = \neg x \oplus ((x \vee y) \vee z) = (x \vee ((x \vee y) \vee z))^{(x \vee y) \vee z} = ((x \vee y) \vee z)^{(x \vee y) \vee z} = 1$;
(5) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus x) \oplus x = (x \vee y) \vee x = x \vee y = \neg(\neg x \oplus y) \oplus y$;
(6) $\neg(\neg(x \oplus y) \oplus y) \oplus y = (x \oplus y) \vee y = (x^0 \vee y)^y \vee y = (x^0 \vee y)^y = x \oplus y$. □

Axiom (5) of Theorem 1 is a weak form of the Łukasiewicz axiom. Moreover, (2) is (MV3), (3) is a modification of (D) and (4) is (C) mentioned in the introduction.

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be an algebra satisfying* (1), (2) *and* (3). *Then* $\neg 1 = 0$, $\neg 0 = 1$ *and the following identities are satisfied:*

(C1) $x \oplus \neg x = 1 = \neg x \oplus x$;
(C2) $x \oplus 1 = 1 = 1 \oplus x$;
(C3) $\neg y \oplus (\neg(\neg x \oplus y) \oplus y) = 1$.

P r o o f. Obviously, $\neg 1 = \neg\neg 0 = 0$ by (1). If we put $x = 0 = y$ in (3) and apply (2), we get $1 = \neg 0 \oplus (0 \oplus 0) = \neg 0$.

(C1) Putting $y = 0$ in (3), we get by (2): $1 = \neg x \oplus (0 \oplus x) = \neg x \oplus x$. Putting $y = 0$ and $x = \neg x$ in (3), we obtain by (2) and (1): $1 = \neg\neg x \oplus (0 \oplus \neg x) = x \oplus \neg x$.

(C2) Applying (3), (1) and (C1), we obtain: $1 = \neg\neg x \oplus (x \oplus \neg x) = x \oplus 1$. By (3) and (2) we infer: $1 = \neg 0 \oplus (x \oplus 0) = 1 \oplus x$.

(C3) Clearly follows from (3). □

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be an algebra of type $(2,1,0)$ satisfying (1)–(5). Define*

$$x \leqslant y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Then the relation $\leqslant$ is an order on A and $0 \leqslant x \leqslant 1$ for each $x \in A$. Moreover, $x \leqslant y \oplus x$ holds for all $x, y \in A$, and $x \leqslant y$ implies

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

Proof. By (C1), $\leqslant$ is reflexive. Suppose $x \leqslant y$ and $y \leqslant x$, Thus $\neg x \oplus y = 1$ and $\neg y \oplus x = 1$. If we insert the first equality into (5), we get $\neg(\neg(\neg 1 \oplus y) \oplus x) \oplus x = \neg 1 \oplus y$, which together with (2) yields $\neg(\neg y \oplus x) \oplus x = y$. By assumption we have $\neg y \oplus x = 1$, thus $x = 0 \oplus x = y$, whence $\leqslant$ is antisymmetrical. Now, suppose $x \leqslant y$ and $y \leqslant z$. Then $\neg x \oplus y = 1$, $\neg y \oplus z = 1$ and using (2) and (4) yields

$$\begin{aligned}\neg x \oplus z &= \neg x \oplus (\neg 1 \oplus z) = \neg x \oplus (\neg(\neg y \oplus z) \oplus z) \\ &= \neg x \oplus (\neg(\neg(\neg 1 \oplus y) \oplus z) \oplus z) \\ &= \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1.\end{aligned}$$

Thus $x \leqslant z$. Hence, $\leqslant$ is an order on A. Moreover, (C2) yields $\neg x \oplus 1 = 1$ and $\neg 0 \oplus x = 1 \oplus x = 1$, thus $0 \leqslant x \leqslant 1$. According to (3) we conclude $x \leqslant y \oplus x$. Finally, if $x \leqslant y$ then $\neg x \oplus y = 1$ and, by (5), $\neg(\neg x \oplus y) \oplus y = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus x) \oplus x = \neg(\neg(\neg 1 \oplus y) \oplus x) \oplus x = \neg(\neg y \oplus x) \oplus x$, which proves the last assertion. □

Lemma 3. *Let $\mathcal{D} = (D; \vee)$ be a bounded directoid with sectional involutions, $\mathcal{A}(D)$ its skew MV-algebra and $x, p \in D$. Then $(x \vee p)^p = \neg x \oplus p$.*

Proof. Since $x \oplus y = (\neg x \vee y)^y$, we have $\neg x \oplus p = (\neg\neg x \vee p)^p = (x \vee p)^p$. □

Lemma 4. *Let $\mathcal{D} = (D; \vee)$ be a bounded directoid with sectionally antitone involutions and $\mathcal{A}(D)$ its skew MV-algebra. Then $\mathcal{A}(D)$ satisfies the identity*

$$\text{(AN)} \qquad \neg(\neg(\neg(\neg(\neg(\neg x \oplus z) \oplus z) \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) = 1.$$

Proof. Evidently, $z \leqslant x \vee z \leqslant (x \vee z) \vee y$, thus $x \vee z$, $(x \vee z) \vee y \in [z, 1]$. Since the sectional involution in $[z, 1]$ is antitone, we have $(x \vee z)^z \geqslant ((x \vee z) \vee y)^z$. By Lemma 3, $(x \vee z)^z = \neg x \oplus z$ and

$$\begin{aligned}((x \vee z) \vee y)^z &= \neg((x \vee z) \vee y) \oplus z = \neg(\neg(\neg(x \vee z) \oplus y) \oplus y) \oplus z \\ &= \neg(\neg(\neg(\neg(\neg x \oplus z) \oplus z) \oplus y) \oplus y) \oplus z.\end{aligned}$$

Since $a \leqslant b$ if and only if $\neg a \oplus b = 1$, we obtain (AN). □

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ satisfying (1)–(5). Define $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x^y = \neg x \oplus y$ for $x \in [y, 1]$ and $1 = \neg 0$. Then $\mathcal{D}(A) = (A; \vee)$ is a bounded directoid with sectionally switching involutions. Moreover, if $\mathcal{A}$ satisfies also (AN) then the sectionally switching involutions are even antitone.*

Proof. By (C1), (C3) and (2), $x \vee x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x$. Further, (5) yields

$$(x \vee y) \vee x = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y = x \vee y.$$

Using (C3) and (2) we get

$$\begin{aligned}y \vee (x \vee y) &= \neg(\neg y \oplus (\neg(\neg x \oplus y) \oplus y)) \oplus (\neg(\neg x \oplus y) \oplus y) \\ &= \neg 1 \oplus (\neg(\neg x \oplus y) \oplus y) = \neg(\neg x \oplus y) \oplus y = x \vee y.\end{aligned}$$

To prove skew associativity (D4) we use the identities (4), (1) and (2):

$$\begin{aligned}&x \vee ((x \vee y) \vee z) \\ &\quad = \neg(\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z)) \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) \\ &\quad = \neg 1 \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = (x \vee y) \vee z.\end{aligned}$$

Hence, $(A; \vee)$ is a directoid.

Let $x \in L$. Then, using (C2), (2) and (C1), we obtain

$$\begin{aligned}&0 \vee x = \neg(\neg 0 \oplus x) \oplus x = \neg(1 \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x, \\ &1 \vee x = \neg(\neg 1 \oplus x) \oplus x = \neg(0 \oplus x) \oplus x = \neg x \oplus x = 1,\end{aligned}$$

thus $0 \leqslant x \leqslant 1$ for the order $\leqslant$ induced by $(A; \vee)$.

It remains to prove that $(A;\vee)$ has sectionally switching involutions on each its section. To this end suppose $x \in [a,1]$. Denote $x^a = \neg x \oplus a$. Then, by Lemma 2, $a \leqslant \neg x \oplus a = x^a$, thus $x^a \in [a,1]$, i.e. the mapping $x \mapsto x^a$ is really a sectional mapping on $[a,1]$. Further,

$$x^{aa} = \neg x^a \oplus a = \neg(\neg x \oplus a) \oplus a = x \vee a = x,$$

i.e. it is an involution. Moreover, $1^a = \neg 1 \oplus a = 0 \oplus a = a$, $a^a = \neg a \oplus a = 1$ and thus $(A;\vee)$ is a bounded directoid with sectionally switching involutions.

Finally, suppose $\mathcal{A}$ satisfies also the identity (AN). Let $x, y \in [a,1]$ with $x \leqslant y$. Then $x \vee y = y$ and $x \vee a = x$, i.e. $\neg(\neg x \oplus a) \oplus a = x$ and $\neg(\neg(\neg(\neg x \oplus a) \oplus a) \oplus y) \oplus y = y$. Putting $z = a$ in (AN) we have

$$\begin{aligned} 1 &= \neg(\neg(\neg(\neg(\neg(\neg x \oplus a) \oplus a) \oplus y) \oplus y) \oplus a) \oplus (\neg x \oplus a) \\ &= \neg(\neg y \oplus a) \oplus (\neg x \oplus a), \end{aligned}$$

thus $y^a = \neg y \oplus a \leqslant \neg x \oplus a = x^a$, proving that the involution $x \mapsto x^a$ is antitone. □

We call $\mathcal{D}(A) = (A;\vee)$ the *directoid assigned* to $\mathcal{A}$.

Example 1. A bounded (non-commutative) $\vee$-directoid with sectionally antitone involutions where $a \vee b = c$ and $b \vee a = d$ is depicted in Fig. 1. For nontrivial sections, the sectional involutions are

$[0,1]$: $0 \mapsto 1$, $1 \mapsto 0$, $a \mapsto d$, $d \mapsto a$, $b \mapsto c$, $c \mapsto b$;

$[a,1]$: $a \mapsto 1$, $1 \mapsto a$, $c \mapsto d$, $d \mapsto c$;

$[b,1]$: $b \mapsto 1$, $1 \mapsto b$, $c \mapsto d$, $d \mapsto c$.

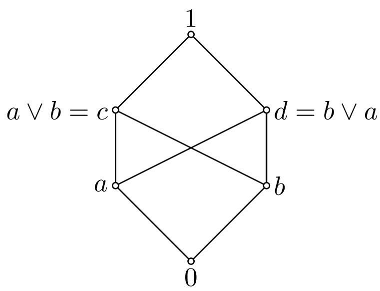

Fig. 1

The binary operation $\oplus$ of its skew MV-algebra is given by Table 1.

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	c	c	1	1
b	b	d	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

Tab. 1

Evidently, $\oplus$ is not commutative since $c = a \oplus b \neq b \oplus a = d$.

Theorem 3. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a skew MV-algebra and $\mathcal{D}(A) = (A; \vee)$ its assigned directoid. Then $\mathcal{A}(\mathcal{D}(A)) = A$. On the other hand, if $(D; \vee)$ is a bounded directoid with sectionally switching involutions and $\mathcal{A}(D)$ its skew MV-algebra, then $\mathcal{D}(\mathcal{A}(D)) = D$.*

Proof. Let us denote $\mathcal{A}(\mathcal{D}(A)) = (A; +, ', 0)$. Then $x + y = (x^0 \vee y)^y = (\neg(x \oplus y) \oplus y)^y = \neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$ by virtue of the identity (6), and $x' = x^0 = \neg x$, which proves $\mathcal{A}(\mathcal{D}(A)) = A$.

Conversely, denote the join operation in $\mathcal{D}(\mathcal{A}(D))$ by $\sqcup$. Then $x \sqcup y = \neg(\neg x \oplus y) \oplus y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$. It is easy to check that also the sectional involutions on $[p, 1]$ are the same in both $(D; \vee)$ and $\mathcal{D}(\mathcal{A}(D))$. Hence $\mathcal{D}(\mathcal{A}(D)) = D$. $\square$

Theorem 4. *Let $(D; \oplus, \neg, 0)$ be a skew MV-algebra, $p \in A$, $x, y \in [p, 1]$. Then, if we define*

$$x \oplus_p y = \neg(\neg x \oplus p) \oplus y \quad \text{and} \quad \neg_p x = \neg x \oplus p,$$

the structure $([p, 1]; \oplus_p, \neg_p, p)$ is a skew MV-algebra.

Proof. We shall show that $([p, 1]; \oplus_p, \neg_p, p)$ satisfies the identities (1)–(6) for $\oplus_p, \neg_p$ and p instead of $\oplus, \neg$ and 0, respectively:

(1) $\neg_p \neg_p x = \neg(\neg x \oplus p) \oplus p = x \vee p = x$.

(2) $x \oplus_p p = \neg(\neg x \oplus p) \oplus p = x \vee p = x$; $\quad p \oplus_p x = \neg(\neg p \oplus x) \oplus x = p \vee x = x$.

(3) $\neg_p x \oplus_p (y \oplus_p x) = (\neg x \oplus p) \oplus_p (\neg(\neg y \oplus p) \oplus x) = \neg(\neg(\neg x \oplus p) \oplus p) \oplus (\neg(\neg y \oplus p) \oplus x) = \neg(x \vee p) \oplus (\neg(\neg y \oplus p) \oplus x) = \neg x \oplus (\neg(\neg y \oplus p) \oplus x) = 1$.

(4) Clearly $\neg(\neg x \oplus p) \oplus p = x \vee p = x$. Thus $\neg_p x \oplus_p (\neg_p(\neg_p(\neg_p(\neg_p x \oplus_p y) \oplus_p y) \oplus_p z) \oplus_p z) = \neg(\neg(\neg x \oplus p) \oplus p) \oplus (\neg_p(\neg_p(\neg_p(\neg(\neg(\neg x \oplus p) \oplus p) \oplus y) \oplus_p y) \oplus_p z) \oplus_p z) = \neg x \oplus (\neg_p(\neg_p(\neg_p(\neg x \oplus y) \oplus_p y) \oplus_p z) \oplus_p z) = \neg x \oplus (\neg_p(\neg_p(\neg(\neg(\neg(\neg x \oplus y) \oplus p) \oplus p) \oplus y) \oplus_p z) \oplus_p z) = \neg x \oplus (\neg_p(\neg_p(\neg(\neg x \oplus y) \oplus y) \oplus_p z) \oplus_p z) = \neg x \oplus (\neg_p(\neg(\neg(\neg x \oplus y) \oplus

$y) \oplus z) \oplus_p z) = \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$ according to (4). During the last calculation we have used three times the inequality $x \leqslant y \oplus x$ of Lemma 2 in the following forms: $y \leqslant \neg x \oplus y$, $y \leqslant \neg(\neg x \oplus y) \oplus y$ and $z \leqslant \neg(\neg(\neg x \oplus y) \oplus y) \oplus z$. This yields for $x, y, z \in [p, 1]$ that $(\neg x \oplus y) \vee p = \neg x \oplus y$, $(\neg(\neg x \oplus y) \oplus y) \vee p = \neg(\neg x \oplus y) \oplus y$ and $(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \vee p = \neg(\neg(\neg x \oplus y) \oplus y) \oplus z$.

To prove (5) we first derive $\neg_p x \oplus_p y = (\neg x \oplus p) \oplus_p y = \neg(\neg(\neg x \oplus p) \oplus p) \oplus y = \neg x \oplus y$, since $\neg(\neg x \oplus p) \oplus p = x \vee p = x$. Thus also $\neg_p(\neg_p x \oplus_p y) \oplus_p y = \neg_p(\neg x \oplus y) \oplus_p y = \neg(\neg x \oplus y) \oplus y = x \vee y$. Hence $\neg_p(\neg_p(\neg_p(\neg_p x \oplus_p y) \oplus_p y) \oplus_p x) \oplus_p x = \neg_p(\neg_p x \oplus_p y) \oplus_p y$, implying that $(x \vee y) \vee x = x \vee y$ in the assigned directoid.

Analogously we prove (6):

(6) $\neg_p(\neg_p(x \oplus_p y) \oplus_p y) \oplus_p y = (\neg((\neg(\neg(\neg x \oplus p) \oplus y) \oplus p) \oplus_p y) \oplus p) \oplus_p y = (\neg(\neg(\neg(\neg x \oplus p) \oplus y) \oplus y) \oplus p) \oplus_p y = \neg(\neg(\neg(\neg x \oplus p) \oplus y) \oplus y) \oplus_p y = \neg(\neg x \oplus p) \oplus y = x \oplus_p y$.

□

3. Skew implication algebras

The concept of the implication algebra was introduced in the classical logic by J. C. Abbott [1].

In the sequel we characterize the connective implication in skew MV-algebras similarly as it was done in [4] and [9] for MV-algebras or WMV-algebras. It turns out that the appropriate implication algebras look as follows:

Definition 2. A *skew implication algebra* is an algebra $(A; \to, 1)$ of type $(2, 0)$ satisfying the identities

(S1) $x \to x = 1,\ 1 \to x = x$;

(S2) $(((x \to y) \to y) \to x) \to x = (x \to y) \to y$;

(S3) $x \to ((((x \to y) \to y) \to z) \to z) = 1$;

(S4) $y \to (x \to y) = 1$.

Lemma 5. *In a skew implication algebra $(A; \to, 1)$ we have $x \to 1 = 1$.*

Proof. By (S1) and (S4) we have $x \to 1 = x \to (x \to x) = 1$. □

Theorem 5. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a skew MV-algebra. Define $x \to y = \neg x \oplus y$, $1 = \neg 0$. Then the algebra $\mathcal{S}(A) = (A; \to, 1)$ is a skew implication algebra satisfying*

(S5) $0 \to x = 1$.

Proof. (S1) $x \to x = \neg x \oplus x = 1$ by Lemma 1; $1 \to x = \neg 1 \oplus x = 0 \oplus x = x$ by (2).

(S2) $(((x \to y) \to y) \to x) \to x = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y = (x \to y) \to y$ directly by (5).

(S3) $x \to ((((x \to y) \to y) \to z) \to z) = \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$ by (4).

(S4) $y \to (x \to y) = \neg x \oplus (y \oplus x) = 1$ by (3).

Moreover, $0 \to x = \neg 0 \oplus x = 1 \oplus x = 1$ by Lemma 1. □

Theorem 6. *Let $(A; \to, 1)$ be a skew implication algebra. Define*

$$x \leqslant y \quad \text{iff} \quad x \to y = 1.$$

Then $(A; \leqslant)$ is a directoid with 1 and with sectionally switching involutions, where $x \vee y = (x \to y) \to y$ and $x^p = x \to p$ for $x \in [p, 1]$. Further, for $x \oplus_p y = (x \to p) \to y$ and $\neg_p x = x \to p$, $([p, 1]; \oplus_p, \neg_p, p)$ is a skew MV-algebra.

Proof. Reflexivity of $\leqslant$ follows by $x \to x = 1$.

Let $x \leqslant y$ and $y \leqslant x$ i.e. $x \to y = 1$ and $y \to x = 1$. Then, by (S1) and (S2), $x = 1 \to x = (y \to x) \to x = ((1 \to y) \to x) \to x = (((x \to y) \to y) \to x) \to x = (x \to y) \to y = 1 \to y = y$, thus $\leqslant$ is antisymmetrical.

Suppose now $x \leqslant y$, $y \leqslant z$. According to (S3) and $x \to y = 1$, $y \to z = 1$, we have $x \to z = x \to ((((x \to y) \to y) \to z) \to z) = 1$, i.e. $x \leqslant z$. Thus $\leqslant$ is transitive. Moreover, $x \to 1 = 1$ yields $x \leqslant 1$, whence $\leqslant$ is an order on A with the greatest element 1.

Evidently, $y \leqslant x \to y$ by (S4), thus also $y \leqslant (x \to y) \to y$. Using (S3) we have $x \to ((((x \to x) \to x) \to y) \to y) = 1$; thus, by (S1), we obtain $x \to ((x \to y) \to y) = 1$. Therefore $x \leqslant (x \to y) \to y$, i.e. $(x \to y) \to y$ is an upper bound of x, y. Denote $x \vee y = (x \to y) \to y$. To prove that $(A; \vee)$ is a directoid we need only to show that $x \leqslant y$ implies $x \vee y = y = y \vee x$. However, $x \leqslant y$ implies $x \to y = 1$, thus $x \vee y = (x \to y) \to y = 1 \to y = y$. Due to the last assertion of Lemma 2, $x \leqslant y$ yields $(x \to y) \to y = (y \to x) \to x$, thus also $y \vee x = y$.

It remains to prove that $x^p = x \to p$ is the switching involution on the section $[p, 1]$. To this end, let $x, y \in [p, 1]$. Then $x \to y \geqslant y \geqslant p$, thus $x \to y \in [p, 1]$. Clearly, $x^p = x \to p \in [p, 1]$, $x^{pp} = (x \to p) \to p = x \vee p = x$ and $1^p = 1 \to p = p$. Hence $(A; \vee, 1)$ is a directoid with sectionally switching involutions and thus $([p, 1], \oplus_p, \neg_p, 1)$ is a skew MV-algebra for each $p \in A$. □

We can prove also the converse:

Theorem 7. *Let* $(D;\vee)$ *be a directoid with* 1 *and with sectionally switching involutions. Define*

$$x \to y = (x \vee y)^y.$$

Then $(D;\to)$ *is a skew implication algebra.*

P r o o f. To prove this theorem we only need to verify the identities (S1)–(S4):

(S1) $x \to x = (x \vee x)^x = x^x = 1$; $1 \to x = (1 \vee x)^x = 1^x = x$.

(S4) $y \to (x \to y) = y \to (x \vee y)^y = (y \vee (x \vee y)^y)^{(x\vee y)^y} = ((x \vee y)^y)^{(x\vee y)^y} = 1$.

Next, $(x \to y) \to y = ((x \to y) \vee y)^y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$; this fact we use in the proof of (S2) and (S3):

(S2) $(((x \to y) \to y) \to x) \to x = ((x \vee y) \to x) \to x = (x \vee y) \vee x = x \vee y$.

(S3) $x \to ((((x \to y) \to y) \to z) \to z) = x \to ((x \vee y) \vee z) = (x \vee ((x \vee y) \vee z))^{(x\vee y)\vee z} = ((x \vee y) \vee z)^{(x\vee y)\vee z} = 1$. □

Corollary 1. *Let* $\mathcal{S} = (S;\to,1)$ *be a skew implication algebra with a least element* 0 *satisfying* (S5). *Define* $\neg x = x \to 0$ *and* $x \oplus y = (x \to 0) \to y$. *Then* $\mathcal{A}(S) = (S;\oplus,\neg,0)$ *is a skew MV-algebra.*

P r o o f. If $\mathcal{S}$ has a least element 0 then clearly $S = [0,1]$ and, by Theorem 6 for $\oplus = \oplus_0$, $\neg = \neg_0$ we get the assertion. □

E x a m p l e 2. Consider a skew implication algebra $\mathcal{S} = (\{a,b,c,d,1\};\to,1)$ given by Table 2.

$\to$	a	b	c	d	1
a	1	d	1	1	1
b	c	1	1	1	1
c	d	d	1	d	1
d	c	c	c	1	1
1	a	b	c	d	1

Tab. 2

Its induced directoid is shown in Fig. 2,

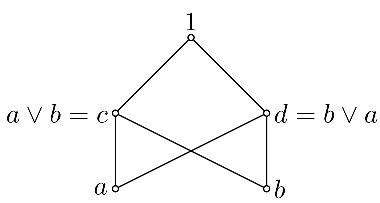

Fig. 2

and the sectional skew MV-algebras $([a,1],\oplus_a,\neg_a,a)$ and $([b,1],\oplus_b,\neg_b,b)$ are determined by the tables

$\oplus_a$	a	c	d	1
a	a	c	d	1
c	c	c	1	1
d	d	1	d	1
1	1	1	1	1

$\oplus_b$	b	c	d	1
b	b	c	d	1
c	c	c	1	1
d	d	1	d	1
1	1	1	1	1

	a	c	d	1
$\neg_a$	1	d	c	a

	b	c	d	1
$\neg_b$	1	d	c	b

It is worth noticing that the directoid depicted in Fig. 2 does not determine the skew implication algebra $\mathcal{S}$ uniquely. If $\mathcal{S}' = (\{a,b,c,d,1\};\rightarrow,1)$ is a skew implication algebra determined by Table 3

$\rightarrow$	a	b	c	d	1
a	1	c	1	1	1
b	c	1	1	1	1
c	d	c	1	d	1
d	c	d	c	1	1
1	a	b	c	d	1

Tab. 3

then its induced directoid is that of Fig. 2 but the sectional skew MV-algebra $([b,1];\oplus_b,\neg_b,b)$ has rather different tables for the binary operation $\oplus_b$ and the unary operation $\neg_b$:

$\oplus_b$	b	c	d	1
b	b	c	d	1
c	c	1	d	1
d	d	c	1	1
1	1	1	1	1

	b	c	d	1
$\neg_b$	1	c	d	b

The sectional MV-algebra $([a,1];\oplus_a,\neg_a,a)$ is the same as shown before.

As shown in the previous chapter, skew implication algebras are defined by the identities (S1)–(S4) and hence they form a variety. Let us recall that an algebra $\mathcal{A}$ with a constant 1 is *weakly regular* if every congruence $\Theta \in \operatorname{Con} \mathcal{A}$ is determined by its kernel $[1]_\Theta$, i.e. if $[1]_\Theta = [1]_\Phi$ for $\Theta, \Phi \in \operatorname{Con} \mathcal{A}$ then $\Theta = \Phi$. Further, $\mathcal{A}$ is *congruence distributive* if $\operatorname{Con} \mathcal{A}$ is a distributive lattice (with respect to set inclusion). A variety $\mathcal{V}$ is *weakly regular* or *congruence distributive* if each $\mathcal{A} \in \mathcal{V}$ has the corresponding property.

Theorem 8. *The variety of skew implication algebras is weakly regular and congruence distributive.*

Proof. By Theorem 6.4.3 in [2], a variety $\mathcal{V}$ is weakly regular if and only if there exist an integer $n \geqslant 1$ and binary terms $t_1, \ldots t_n$ such that $t_1(x,y) = \ldots = t_n(x,y) = 1$ if and only if $x = y$. Of course, one can choose $n = 2$ and $t_1(x,y) = x \to y$, $t_2(x,y) = y \to x$. If $t_1(x,y) = t_2(x,y) = 1$ then, by Theorem 6, $x \leqslant y$ and $y \leqslant x$, thus $x = y$. Moreover, $t_1(x,x) = 1 = t_2(x,x)$ by (S1), thus the variety $\mathcal{W}$ of skew implication algebras is weakly regular. Further, for $b(x,y) = y \to x$ we have $b(x,x) = 1, b(x,1) = 1$ and $b(1,x) = x$; thus, by Theorem 8.3.2 in [2], $\mathcal{W}$ is arithmetical at 1 and hence also distributive at 1. Together with weak regularity, $\mathcal{W}$ is congruence distributive (see e.g. Theorem 8.2.8 in [2].) □

Since every congruence on a skew implication algebra $\mathcal{S}$ is fully determined by its kernel, a natural question arises how to characterize congruence kernels (for the sake of characterizing congruences on $\mathcal{S}$).

Lemma 6. *Let $\mathcal{S}$ be a skew implication algebra and $\Theta \in \operatorname{Con} \mathcal{S}$. Then $\langle x,y\rangle \in \Theta$ if and only if $x \to y$, $y \to x \in [1]_\Theta$.*

Proof. If $\langle x,y\rangle \in \Theta$ then also $\langle x \to y, 1\rangle = \langle x \to y, y \to y\rangle \in \Theta$ and $\langle y \to x, 1\rangle = \langle y \to x, y \to y\rangle \in \Theta$, thus both $x \to y, y \to x \in [1]_\Theta$. Conversely, if $x \to y, y \to x \in [1]_\Theta$ then $\langle x \to y, 1\rangle \in \Theta$, $\langle y \to x, 1\rangle \in \Theta$ and hence $\langle (x \to y) \to y, y\rangle = \langle (x \to y) \to y, 1 \to y\rangle \in \Theta$. Further, $x = (1 \to x)\Theta((y \to x) \to x) = (((1 \to y) \to x) \to x)\Theta(((x \to y) \to y) \to x) \to x = (x \to y) \to y$ by (S2), i.e. $\langle x, (x \to y) \to y\rangle \in \Theta$. Applying transitivity of Θ we conclude $\langle x,y\rangle \in \Theta$. □

A subset D of a skew implication algebra $\mathcal{S} = (S; \to, 1)$ is called a *deductive system of $\mathcal{S}$* provided the following conditions hold:
(I1) $1 \in D$;
(I2) if $x \in D$ and $x \to y \in D$, then $y \in D$;

(I3) if $x \to y \in D$ and $y \to x \in D$, then $(z \to x) \to (z \to y) \in D$ and $(x \to z) \to (y \to z) \in D$.

We are going to characterize the congruence kernels.

Theorem 9. *Let $\mathcal{S} = (S; \to, 1)$ be a skew implication algebra. A subset $D \subseteq S$ is a congruence kernel of some $\Theta \in \operatorname{Con} \mathcal{S}$ if and only if D is a deductive system of $\mathcal{S}$. Moreover, if D is a deductive system of $\mathcal{S}$ then it is a kernel of Θ_D defined by*

$$(*) \qquad \langle x, y \rangle \in \Theta_D \quad \text{iff} \quad x \to y, y \to x \in D.$$

Proof. Let $D = [1]_\Theta$ for some $\Theta \in \operatorname{Con} \mathcal{S}$. Obviously, $1 \in D$ and if $x \in D$ and $x \to y \in D$ then $\langle x, 1 \rangle \in \Theta$, $\langle x \to y, 1 \rangle \in \Theta$, thus also $\langle (x \to y) \to y, 1 \rangle = \langle (x \to y) \to y, (1 \to y) \to y \rangle \in \Theta$ and $\langle (x \to y) \to y, y \rangle = \langle (x \to y) \to y, 1 \to y \rangle \in \Theta$, i.e. $\langle y, 1 \rangle \in \Theta$, which proves $y \in D$.

Finally, if $x \to y, y \to x \in D = [1]_\Theta$ then $\langle x, y \rangle \in \Theta$ by Lemma 6. Hence $\langle z \to x, z \to y \rangle \in \Theta$ and $\langle x \to z, y \to z \rangle \in \Theta$. Applying Lemma 6 once more we conclude that D satisfies also the condition (I3), i.e. D is a deductive system of $\mathcal{S}$.

Conversely, let D be a deductive system of $\mathcal{S}$ and define Θ_D by $(*)$. All we need to show is that Θ_D is a congruence on $\mathcal{S}$ since the weak regularity then yields that it is unique with the kernel D. Of course, Θ_D is reflexive and symmetric. Assume $\langle x, y \rangle, \langle y, z \rangle \in \Theta_D$. Then, by $(*)$, $x \to y, y \to x, y \to z, z \to y \in D$ and by (I3) we have $(y \to z) \to (x \to z) \in D$, which due to (I2) and $y \to z \in D$ implies $x \to z \in D$. Analogously we can prove $z \to x \in D$, thus $\langle x, z \rangle \in \Theta_D$, which proves transitivity of Θ_D.

Now, suppose $\langle x, y \rangle, \langle u, v \rangle \in \Theta_D$. Hence $x \to y, y \to x, u \to v, v \to u \in D$ and, due to (I3), also $(x \to u) \to (y \to u) \in D$ and $(y \to u) \to (x \to u) \in D$, which proves $\langle x \to u, y \to u \rangle \in \Theta_D$. Analogously we can show $\langle y \to u, y \to v \rangle \in \Theta_D$ and, applying transitivity of Θ_D, we obtain $\langle x \to u, y \to v \rangle \in \Theta_D$. Hence, Θ_D is a congruence on $\mathcal{S}$. □

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