ON THE VOLTERRA INTEGRAL EQUATION WITH WEAKLY SINGULAR KERNEL

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 80th birthday

Abstract. We give sufficient conditions for the existence of at least one integrable solution of equation $x(t)=f(t)+\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s$. Our assumptions and proofs are expressed in terms of measures of noncompactness.

Keywords: integral equation, integrable solution, measure of noncompactness

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Let E be a Banach space and let $J=[0,d]$ be a compact interval in $\mathbb{R}$. Denote by $L^1(J,E)$ the space of all Bochner integrable functions $x\colon J\to E$ equipped with the norm $\|x\|_1=\int_J\|x(t)\|\,\mathrm{d}t$.

In this paper we give sufficient conditions for the existence of a solution $x\in L^1(J,E)$ of the integral equation

$$x(t)=f(t)+\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s \tag{1}$$

with the kernel

$$K(t,s)=\frac{A(t,s)}{|t-s|^r}\quad (t,s\in J,\ t\neq s),$$

where $0<r<1$ and A is a bounded strongly measurable function from $J\times J$ into the space of continuous linear mappings $E\to E$.

Throughout this paper we shall assume that

1. $f\in L^1(J,E)$;
2. $(s,x)\mapsto g(s,x)$ is a function from $J\times E$ into E such that

(i) g is strongly measurable in s and continuous in x;

(ii) $\|g(s,x)\| \leqslant a(s) + b\|x\|$ for $s \in J$ and $x \in E$, where $a \in L^1(J,\mathbb{R})$ and $b \geqslant 0$.

Since $\int_0^t (t-s)^{-r}\,\mathrm{d}s = (1-r)^{-1}t^{1-r}$, we have

$$\int_0^d \frac{\mathrm{d}s}{|t-s|^r} \leqslant Q \quad \text{for all } t \in J, \text{ where } Q = \frac{2d^{1-r}}{1-r}. \tag{2}$$

Put $c = \max\{\|A(t,s)\|\colon\ s,t \in J\}$, $L^1 = L^1(J,E)$ and

$$(Sx)(t) = \int_J K(t,s)x(s)\,\mathrm{d}s \quad (x \in L^1,\ t \in J).$$

Lemma 1. *S is a continuous linear mapping of L^1 into itself and $\|S\| \leqslant cQ$.*

P r o o f. By (2) for each $z \in L^1(J,\mathbb{R})$ we have

$$\iint_{J\times J} \frac{|z(s)|}{|t-s|^r}\,\mathrm{d}s\,\mathrm{d}t = \int_J \left(\int_J \frac{\mathrm{d}t}{|t-s|^r}\right)|z(s)|\,\mathrm{d}s \leqslant Q\int_J |z(s)|\,\mathrm{d}s, \tag{3}$$

and therefore for almost every $t \in J$ the integral

$$\int_J \frac{|z(s)|}{|t-s|^r}\,\mathrm{d}s$$

exists. This shows that S is well defined. Moreover, if $x \in L^1$, then

$$\|(Sx)(t)\| \leqslant \int_J \frac{\|A(t,s)\|\|x(s)\|}{|t-s|^r}\,\mathrm{d}s \leqslant c\int_J \frac{\|x(s)\|}{|t-s|^r}\,\mathrm{d}s.$$

Thus

$$\begin{aligned}\int_J \|(Sx)(t)\|\,\mathrm{d}t &\leqslant c\int_J \left(\int_J \frac{\|x(s)\|}{|t-s|^r}\,\mathrm{d}s\right)\mathrm{d}t\\ &= c\int_J \left(\int_J \frac{\mathrm{d}t}{|t-s|^r}\right)\|x(s)\|\,\mathrm{d}s \leqslant cQ\int_J \|x(s)\|\,\mathrm{d}s,\end{aligned}$$

so that $\|Sx\|_1 \leqslant cQ\|x\|_1$.

Lemma 2. *Put $\tilde{g}(x)(s) = g(s, x(s))$ for $x \in L^1$ and $s \in J$. Then $\tilde{g}$ is a continuous mapping of L^1 into itself.*

Proof. Let $x_n, x_0 \in L^1$ and $\lim_{n\to\infty} \|x_n - x_0\|_1 = 0$. Suppose that $\|\tilde{g}(x_n) - \tilde{g}(x_0)\|_1$ does not converge to 0 as $n \to \infty$. Then there are $\varepsilon > 0$ and a subsequence $\{x_{n_j}\}$ such that

$$\|\tilde{g}(x_{n_j}) - \tilde{g}(x_0)\|_1 > \varepsilon \quad \text{for } j = 1, 2, 3, \ldots, \tag{4}$$

and $\lim_{j\to\infty} x_{n_j}(s) = x_0(s)$ for a.e. $s \in J$. By 2(i) we have

$$\lim_{j\to\infty} \|g(s, x_{n_j}(s)) - g(s, x_0(s))\| = 0 \quad \text{for a.e. } s \in J.$$

Moreover, as $\lim_{n\to\infty} \|x_n - x_0\|_1 = 0$ implies that the sequence (x_n) has equi-absolutely continuous norms in L^1, it follows from 2(ii) that the functions $\|g(\cdot, x_n) - g(\cdot, x_0)\|$ $(n = 1, 2, \ldots)$ are equi-integrable on J. Hence, by the Vitali convergence theorem, $\lim_{j\to\infty} \|g(\cdot, x_{n_j}) - g(\cdot, x_0)\|_1 = 0$. This contradicts (4).

Denote by α and α_1 the Kuratowski measures of noncompactness in E and $L^1(J, E)$, respectively. The next lemma clarifies the relation between α and α_1. For any set V of functions belonging to $L^1(J, E)$ denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in J$ (under the convention that $\alpha(X) = \infty$ if X is unbounded), where $V(t) = \{x(t) \colon x \in V\}$.

Lemma 3. *Assume that V is a countable set of strongly measurable functions $J \to E$ and there exists an integrable function μ such that $\|x(t)\| \leqslant \mu(t)$ for all $x \in V$ and $t \in J$. Then the corresponding function v is integrable on J and*

$$\alpha\Big(\Big\{ \int_J x(t)\,\mathrm{d}t \colon x \in V \Big\}\Big) \leqslant 2 \int_J v(t)\,\mathrm{d}t.$$

If, in addition, $\lim_{h\to\infty} \sup_{x\in V} \int_J \|x(t+h) - x(t)\|\,\mathrm{d}t = 0$, then

$$\alpha_1(V) \leqslant 2 \int_J v(t)\,\mathrm{d}t.$$

(See [3], Th. 2.1 and [8], Th. 1).

The main result of this paper is the following

Theorem. *Let* $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous nondecreasing function such that* $\omega(0)=0$, $\omega(t)>0$ *for* $t>0$ *and*

$$\int_0^\delta \frac{1}{s}\Big[\frac{s}{\omega(s)}\Big]^{\frac{1}{1-r}}\,\mathrm{d}s=\infty \quad (\delta>0). \tag{5}$$

If 1–2 *hold and*

$$\alpha(g(s,X))\leqslant\omega(\alpha(X)) \tag{6}$$

for any $s\in J$ *and for any bounded subset* X *of* E*, then there exists a solution* $x\in L^1(J,E)$ *of* (1).

P r o o f. It is known that there exists a nonnegative solution $u\in L^1(J,\mathbb{R})$ of the integral equation

$$u(t)=\|f(t)\|+\int_0^t\|K(t,s)\|a(s)\,\mathrm{d}s+b\int_0^t\|K(t,s)\|u(s)\,\mathrm{d}s.$$

Put $B=\{x\in L^1\colon\ \|x(t)\|\leqslant u(t) \text{ for a.e. } t\in J\}$ and

$$G(x)(t)=f(t)+\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s \quad \text{for } x\in L^1 \text{ and } t\in J.$$

Since

$$\begin{aligned}\|G(x)(t)\|\leqslant\|f(t)\|&+\int_0^t\|K(t,s)\|(a(s)+b\|x(s)\|)\,\mathrm{d}s\\&\leqslant\|f(t)\|+\int_0^t\|K(t,s)\|a(s)\,\mathrm{d}s+b\int_0^t\|K(t,s)\|u(s)\,\mathrm{d}s=u(t)\end{aligned}$$

for $x\in B$ and $t\in J$, Lemmas 1 and 2 prove that G is a continuous mapping $B\to B$.

Putting

$$\overline{K}(t,s)=\begin{cases}K(t,s) & \text{for } 0\leqslant s\leqslant t\leqslant d\\ 0 & \text{for } s>t,\end{cases}$$

we see that

$$G(x)(t)=f(t)+\int_J\overline{K}(t,s)g(s,x(s))\,\mathrm{d}s \quad \text{for } x\in L^1 \text{ and } t\in J.$$

Without loss of generality we shall always assume that all functions from L^1 are extended to $\mathbb{R}$ by putting $x(t)=0$ outside J.

Therefore

$$\|G(x)(t+h) - G(x)(t)\| \leqslant d(t,h) \quad \text{for } x \in B, t \in J \ \text{ and small } |h|, \tag{7}$$

where

$$d(t,h) = \begin{cases} u(t) & \text{if } t \in J \text{ and } t+h \notin J \\ \|f(t+h) - f(t)\| + \int_J \|\overline{K}(t+h,s) - \overline{K}(t,s)\|(a(s) + bu(s))\,\mathrm{d}s \\ & \text{if } t, t+h \in J. \end{cases}$$

In view of (3) the function $(t,s) \mapsto W(t,s) = \overline{K}(t,s)(a(s) + bu(s))$ is integrable on $J \times J$. Hence

$$\lim_{h\to 0} \int_J \left(\int_J \|\overline{K}(t+h,s) - \overline{K}(t,s)\|(a(s) + bu(s))\,\mathrm{d}s \right) \mathrm{d}t$$
$$= \lim_{h\to 0} \iint_{J\times J} \|W(t+h,s) - W(t,s)\|\,\mathrm{d}s\,\mathrm{d}t = 0,$$

and consequently

$$\lim_{h\to 0} \int_J d(t,h)\,\mathrm{d}t = 0 \quad \text{for } t \in J. \tag{8}$$

This fact, plus (7), implies that

$$\lim_{h\to 0} \sup_{x\in B} \int_J \|G(x)(t+h) - G(x)(t)\|\,\mathrm{d}t = 0. \tag{9}$$

Let V be a countable subset of B such that

$$V \subset \overline{\mathrm{conv}}(G(V) \cup \{0\}). \tag{10}$$

Then $V(t) \subset \overline{\mathrm{conv}(G(V)(t) \cup \{0\})}$ for a.e. $t \in J$, so that

$$\alpha(V(t)) \leqslant \alpha(G(V)(t)) \quad \text{for a.e. } t \in J. \tag{11}$$

Put $v(t) = \alpha(V(t))$ for $t \in J$. From (9) and (10) it is clear that

$$\lim_{h\to 0} \sup_{x\in V} \int_J \|x(t+h) - x(t)\|\,\mathrm{d}t = 0.$$

Moreover, $\|x(t)\| \leqslant u(t)$ for all $x \in V$ and a.e. $t \in J$. Hence, by Lemma 3, $v \in L^1(J,\mathbb{R})$ and

$$\alpha_1(V) \leqslant 2 \int_J v(t)\,\mathrm{d}t. \tag{12}$$

From (3) it follows that

$$\int_J \frac{a(s)+bu(s)}{|t-s|^r}\,\mathrm{d}s < \infty \quad \text{for a.e. } t \in J. \tag{13}$$

Fix now $t \in J$ such that the integral (13) is finite.

Since

$$\|\overline{K}(t,s)g(s,x(s))\| \leqslant c\frac{a(s)+bu(s)}{|t-s|^r} \quad \text{for } x \in B \text{ and } s \in J,$$

owing to (11), (6) and Lemma 3 we get

$$\begin{aligned} \alpha(V(t)) \leqslant \alpha(G(V)(t)) &\leqslant \alpha\Big(\Big\{\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s\colon\ x \in V\Big\}\Big) \\ &\leqslant 2\int_0^t \alpha(\{K(t,s)g(s,x(s))\colon\ x \in V\})\,\mathrm{d}s \\ &\leqslant 2\int_0^t \|K(t,s)\|\alpha(g(s,V(s))\,\mathrm{d}s \\ &\leqslant 2\int_0^t \|K(t,s)\|\omega(\alpha(V(s)))\,\mathrm{d}s, \end{aligned}$$

i.e.

$$v(t) \leqslant 2c\int_0^t \frac{\omega(v(s))}{(t-s)^r}\,\mathrm{d}s \quad \text{for } t \in J.$$

Putting $w(t) = 2c\int_0^t \omega(v(s))(t-s)^{-r}\,\mathrm{d}s$ for $t \in J$ we see that w is a continuous function such that $v(t) \leqslant w(t)$ for $t \in J$. Hence

$$w(t) \leqslant 2c\int_0^t \frac{\omega(w(s))}{(t-s)^r}\,\mathrm{d}s \quad \text{for } t \in J. \tag{14}$$

By the Mydlarczyk-Gripenberg theorem ([7], Th. 3.1) and assumption (5), the integral equation

$$z(t) = 2c\int_0^t \frac{\omega(z(s))}{(t-s)^r}\,\mathrm{d}s \quad \text{for } t \in J$$

has the unique continuous solution $z(t) \equiv 0$. Applying now the theorem on integral inequalities ([1], Th. 2), from (14) we deduce that $w(t) \equiv 0$. Thus $v(t) = 0$ for $t \in J$ and consequently, by (12), $\alpha_1(V) = 0$. Hence the set V is relatively compact in L^1. Thus we can apply the Mönch fixed point theorem ([6], Th. 2.1) which yields the existence of a function $x \in L^1$ such that $x = G(x)$. Clearly x is a solution of (1).

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