PREDUALS OF SOBOLEV-CAMPANATO SPACES

Konrad Gröger, Lutz Recke, Berlin

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We present definitions of Banach spaces predual to Campanato spaces and Sobolev-Campanato spaces, respectively, and we announce some results on embeddings and isomorphisms between these spaces. Detailed proofs will appear in our paper in Math. Nachr.

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Introduction

When the treatment of second order elliptic boundary value problems in Sobolev spaces started, the differential equations were usually written as

$$\forall v \in C_c^1(\Omega)\colon \int_\Omega \sum_{i,j=1}^N a_{ij} D_j u \, D_i v + \ldots = \int_\Omega \Big(gv + \sum_{i=1}^N f_i D_i v \Big), \tag{1}$$

and requirements with respect to the right hand side of the form

$$g \in L^{q/2}(\Omega), \ f_i \in L^q(\Omega), \ \ i = 1, \ldots, N,$$

were made in order to have $W^{1,q}$-regularity of the solutions (see [4]). Later it became clear that what is essential is not the representation of the right hand side of the equation by means of $g, f_1, \ldots, f_N$, but the fact that the right hand side is in $W^{-1,q}(\Omega)$ for some q.

The treatment of boundary value problems in Morrey-Campanato spaces started also with the formulation (1) but with different requirements with respect to $g, f_1, \dots, f_N$. For example, if

$$g \in \mathcal{L}^{2,(\lambda-2)_+}(\Omega),\ f_i \in \mathcal{L}^{2,\lambda}(\Omega),\ \ i = 1, \dots, N, \tag{2}$$

then $D_i u \in \mathcal{L}^{2,\lambda}(\Omega)$, $i = 1, \dots, n$; here λ is a parameter and $\mathcal{L}^{2,\lambda}(\Omega)$ the corresponding Campanato space (see [9]). The question what, in the context of Campanato spaces, could be an appropriate substitute for the Sobolev spaces $W^{-1,q}(\Omega)$ was ignored for a long time. Some years ago Rakotoson introduced appropriate spaces of functionals and showed how to use these spaces (denoted by $W^{-1,p,\mu}(\Omega)$ in this paper) for local estimates of solutions to boundary value problems in Sobolev-Campanato spaces (see [6], [7]). Griepentrog [2] in his thesis showed that the spaces $W^{-1,2,\mu}$ are useful also for global estimates of solutions to second order elliptic boundary value problems (even in the case of mixed boundary conditions). Because for Sobolev spaces one has

$$W^{-k,p}(\Omega) := (W_0^{k,p'}(\Omega))^*,\ p \in]1, \infty[, \tag{3}$$

it was natural to ask whether $W^{-k,p,\mu}(\Omega)$ could be characterized as the dual of another suitably chosen Banach space.

The definition $W^{-k,p}(\Omega) := (W_0^{k,p'}(\Omega))^*$ is usually motivated by the fact that for $p \in]1, \infty[$ the Lebesgue space $L^p(\Omega)$ is the dual of $L^{p'}(\Omega)$, i.e. of a space from the scale of Lebesgue spaces itself. It is this relation that allows to interpret the scale $W^{-k,p}(\Omega)$, $k \in \mathbb{N}$, as a continuation of the scale $W^{k,p}(\Omega)$, $k \in \mathbb{Z}_+$. Generally it is not true that Campanato spaces are duals of other Campanato spaces. However, it is well known (see [5]) that for each of the Hölder spaces $C^{0,\alpha}(\overline{\Omega})$ (which are part of the scale of Campanato spaces) there exists a predual Banach space, i.e., a Banach space the dual of which is $C^{0,\alpha}(\overline{\Omega})$.

In the present paper we want to announce results on Campanato spaces and Sobolev-Campanato spaces which are proved in full detail in [3]. We are going to show how for all Campanato spaces predual Banach spaces can be constructed. The scale of these preduals can be interpreted in a natural way as a continuation of the scale of Campanato spaces. More precisely, using the notation $L^{p,m,\mu}(\Omega)$ instead of Campanato's notation $\mathcal{L}_k^{(p,\lambda)}(\Omega)$ (where $m = k + 1$, $\mu = \lambda/p$, cf. [1]), we introduce spaces $L^{p,m,-\mu}(\Omega)$ such that

$$L^{p,m,\mu}(\Omega) = (L^{p',m,-\mu}(\Omega))^*.$$

Moreover, we are going to show that for Sobolev-Campanato spaces the situation is analogous: We present spaces $W^{-k,p,m,\mu}(\Omega)$ and $W_0^{k,p,m,-\mu}(\Omega)$ such that

$$W^{-k,p,m,\mu}(\Omega) = (W_0^{k,p',m,-\mu}(\Omega))^*.$$

Hence, the relation (3) has a counterpart in the theory of Sobolev-Campanato spaces.

In [3] a rather general scheme for the construction of predual spaces has been developed which helps to understand why predual Banach spaces of Camapanato spaces and Sobolev-Camapanato spaces exist. This general scheme will not be presented here. Instead of this we directly proceed to definitions of predual spaces for Campanato spaces and Sobolev-Campanato spaces.

1. Campanato spaces

Throughout this section we assume that Ω is a fixed bounded open subset of $\mathbb{R}^N$ and that $\mathcal{F}$ is the family of all nonempty open subsets of Ω. The diameter of a set $U \in \mathcal{F}$ (with respect to the usual Euclidean metric of $\mathbb{R}^N$) will be denoted by d_U.

Measurability, integrability and integrals will always be understood with respect to the N-dimensional Lebesgue measure. If E is a measurable subset of $\mathbb{R}^N$, then $|E|$ denotes its measure. The letter p will always denote a number from $]1,\infty[$. For a given p the dual exponent p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The spaces $L^p(U)$, $U \in \mathcal{F}$, will be equipped with their standard norms, denoted by $\|\cdot\|_{p,U}$ or simply $\|\cdot\|_p$.

We define $\mathbb{P}_m$, $m \in \mathbb{N}$, as the space of polynomials of degree less than m with respect to the coordinates of the argument $x \in \mathbb{R}^N$. For $m \leqslant 0$ we define $\mathbb{P}_m := \{0\}$.

Definition 1.1. Let $\mu \in [0, m + \frac{N}{p}]$. We introduce

$$L^{p,m,\mu}(\Omega) := \{u \in L^p(\Omega);\ \|u\|_{p,m,\mu} < \infty\},$$

where

$$\|u\|_{p,m,\mu} := \max\{\|u\|_p, \sup_{U\in\mathcal{F}} d_U^{-\mu} \inf_{w\in\mathbb{P}_m} \|u-w\|_{p,U}\}.$$

Obviously, $\|\cdot\|_{p,m,\mu}$ is a norm on $L^{p,m,\mu}(\Omega)$, and the space $(L^{p,m,\mu}(\Omega), \|\cdot\|_{p,m,\mu})$ is complete. Moreover, it is easy to check that $C_c^m(\Omega)$ (the space of functions on Ω with compact support having continuous derivatives up to the order m) is contained in $L^{p,m,\mu}(\Omega)$. Hence $L^{p,m,\mu}(\Omega)$ is dense in $L^p(\Omega)$.

Definition 1.2. Let $v \in L^p(\Omega)$ and $\mu \in [0, m + \frac{N}{p'}]$. We put

$$\|v\|_{p,m,-\mu} := \inf\Big\{\|v_0\|_p + \sum_{U\in\mathcal{G}} d_U^{\mu}\|v_U\|_p;\ v = v_0 + \sum_{U\in\mathcal{G}} v_U,\ v_0, v_U \in L^p(\Omega),$$
$$\int_\Omega v_U w = 0 \ \text{ for all } w \in \mathbb{P}_m, \operatorname{supp} v_U \subset U, \mathcal{G} \subset \mathcal{F} \ \text{ finite}\Big\}.$$

Lemma 1.3. *If $\mu \in [0, m + \frac{N}{p'}]$, then $\|\cdot\|_{p,m,-\mu}$ is a norm on $L^p(\Omega)$.*

Proof. 1. It is easy to check that $\|\cdot\|_{p,m,-\mu}$ is a seminorm on $L^p(\Omega)$.

2. Let $v \in L^p(\Omega)$ and $\|v\|_{p,m,-\mu} = 0$. If

$$v = v_0 + \sum_{U\in\mathcal{G}} v_U,\ v_0, v_U \in L^p(\Omega),\ \int_\Omega v_U w = 0 \ \text{ for all } w \in \mathbb{P}_m,$$
$$\operatorname{supp} v_U \subset U,\, \mathcal{G} \subset \mathcal{F} \ \text{ finite},$$

then, for every $u \in L^{p',m,\mu}(\Omega)$ and arbitrarily chosen $w_U \in \mathbb{P}_m$, $U \in \mathcal{G}$,

$$\Big|\int_\Omega uv\Big| \leqslant \Big|\int_\Omega uv_0\Big| + \sum_{U\in\mathcal{G}} \Big|\int_\Omega (u - w_U)v_U\Big| \leqslant \|u\|_{p'}\|v_0\|_p + \sum_{U\in\mathcal{G}} \|u - w_U\|_{p',U}\|v_U\|_p.$$

Because of the arbitrariness of w_U we find that

$$\Big|\int_\Omega uv\Big| \leqslant \|u\|_{p',m,\mu}\Big(\|v_0\|_p + \sum_{U\in\mathcal{G}} d_U^\mu \|v_U\|_p\Big).$$

In view of the arbitrariness of the representation $v = v_0 + \sum\limits_{U\in\mathcal{G}} v_U$ and $\|v\|_{p,m,-\mu} = 0$ we get $\int_\Omega uv = 0$ for every $u \in L^{p',m,\mu}(\Omega)$. Since $L^{p',m,\mu}(\Omega)$ is dense in $L^{p'}(\Omega)$ this implies that $v = 0$. Hence $\|\cdot\|_{p,m,-\mu}$ is a norm. □

Definition 1.4. Let $\mu \in [0, m + \frac{N}{p'}]$. By $L^{p,m,-\mu}(\Omega)$ we denote the completion of the space $(L^p(\Omega), \|\cdot\|_{p,m,-\mu})$. The norm $\|\cdot\|_{p,m,-\mu}$ is extended to $L^{p,m,-\mu}(\Omega)$ by continuity.

Theorem 1.5. *For $\mu \in [0, m + \frac{N}{p'}]$ we have $(L^{p,m,-\mu}(\Omega))^* = L^{p',m,\mu}(\Omega)$.*

A proof of this theorem is given in [3]. There it is shown that Theorem 1.5 can be regarded as a special case of a rather general duality result for spaces constructed by means of projective and inductive systems of Banach spaces.

Remark 1.6. It is easy to check that $\|u\|_{L^{p,m,\mu}(\Omega)}$, $\mu \in [0, m + \frac{N}{p}]$, is equivalent to the norm

$$\max\{\|u\|_p, \sup_{r>0,\, x\in\Omega} \inf_{w\in\mathbb{P}_m} r^{-\mu}\|u - w\|_{p,B_r(x)\cap\Omega}\}. \tag{1.1}$$

(As usual, $B_r(x)$ denotes the open ball of radius r centered at x.) Another equivalent norm is obtained by replacing $B_r(x)$ in (1.1) by the cube of side length r centered at x with edges parallel to the coordinate axes in $\mathbb{R}^N$.

R e m a r k 1.7. $L^{p,m,\mu}(\Omega)$, $1 < p < \infty$, $m \in \mathbb{Z}_+$, $\mu \in [0, m + \frac{N}{p}]$, is the well known scale of Campanato spaces. We changed, however, the notation of these spaces and replaced the original norms by equivalent norms (cf. [1]). Our notation also differs from that adopted by Triebel [8]. Our notation allows to express the duality result of Theorem 1.5 in a very simple way. This result would look more complicated with Campanato's or Triebel's notation. The change of norms compared to those in [1] allows a simpler description of the predual spaces $L^{p',m,-\mu}(\Omega)$. The original Campanato norm differs only slightly from the norm (1.1).

R e m a r k 1.8. Our notation suggests that all the spaces defined above should be considered members of one scale of spaces. This point of view will be justified, for example, by the next theorem. Since both $L^{p,m,0}(\Omega)$ and $L^{p,m,-0}(\Omega)$ coincide with $L^p(\Omega)$ (including the norm), our notation does not cause problems for $\mu = 0$.

Theorem 1.9. *Let* $1 \leqslant q \leqslant p \leqslant \infty$, $\lambda := \frac{N}{q} - \frac{N}{p}$ *and* $-m - \frac{N}{q'} \leqslant \nu \leqslant \lambda + \mu \leqslant m + \frac{N}{q}$. *Moreover, let* ω_N *denote the measure of the unit ball in* $\mathbb{R}^N$. *Then the following holds:*

(i) *If* $\mu \geqslant 0$ *then* $L^{p,m,\mu}(\Omega) \hookrightarrow L^{q,m,\nu}(\Omega)$, *and the norm of the corresponding imbedding operator does not exceed* $\omega_N^{\lambda/N} d_\Omega^\lambda$.

(ii) *If* $\mu < 0$ *then* $L^p(\Omega) \hookrightarrow L^{q,m,\nu}(\Omega)$. *The imbedding of* $L^p(\Omega)$ *into* $L^{q,m,\nu}(\Omega)$ *can be extended uniquely to a continuous (linear) mapping from* $L^{p,m,\mu}(\Omega)$ *into* $L^{q,m,\nu}(\Omega)$ *the norm of which does not exceed* $\omega_N^{\lambda/N} d_\Omega^\lambda$.

In [3] it is shown that this theorem can be derived easily from the fact that

$$\|u\|_{q,U} \leqslant |U|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{p,U} \leqslant \omega_N^{\lambda/N} d_U^\lambda \|u\|_{p,U} \quad \text{for } u \in L^q(U),\ U \in \mathcal{F}.$$

R e m a r k 1.10. For $\nu \geqslant 0$ part (i) of the theorem had been proved already by Campanato [1]. Note that the extended operator in part (ii) of the theorem is not necessarily injective.

To state the next result we need the following definition.

Definition 1.11. A bounded set Ω in $\mathbb{R}^N$ is said to be *of type* A, $A > 0$, if for every $x \in \Omega$ and every $r \in]0, d_\Omega]$ we have $|\Omega \cap B_r(x)| \geqslant Ar^N$.

Theorem 1.12. *Let* Ω *be a bounded domain of type* $A > 0$. *Then the spaces* $L^{p,m,\mu}(\Omega)$ *and* $L^{p,n,\mu}(\Omega)$ *coincide as linear topological spaces, provided* $n < m$ *and* $-n - \frac{N}{p'} < \mu < n + \frac{N}{p}$.

P r o o f. 1. For $\mu \geqslant 0$ the assertion has been proved by Campanato [1].

2. Let $\mu < 0$. Then $L^{p',m,-\mu}(\Omega)$ and $L^{p',n,-\mu}(\Omega)$ coincide as topological linear spaces. Hence Theorem 1.5 allows to regard $L^{p,m,\mu}(\Omega)$ and $L^{p,n,\mu}(\Omega)$ as closed

subspaces of the dual to $L^{p',m,-\mu}(\Omega)$. Because $L^p(\Omega)$ is dense in $L^{p,m,\mu}(\Omega)$ as well as in $L^{p,n,\mu}$ these spaces must be equal as topological linear spaces. $\square$

2. Sobolev-Campanato spaces

As in the preceding section we assume that an open bounded set $\Omega \subset \mathbb{R}^N$ is fixed and that $\mathcal{F}$ is the family of all nonempty open subsets of Ω. Throughout this section k and m denote numbers from $\mathbb{Z}_+$, and p means again a number from $]1,\infty[$. We are going to define spaces of functions with derivatives in Campanato spaces.

The spaces $W^{k,p}(U)$, $U \in \mathcal{F}$, are the usual Sobolev spaces equipped with their standard norms, denoted by $\|\cdot\|_{k,p,U}$ or shortly $\|\cdot\|_{k,p}$. We define $W_0^{k,p}(U)$ as the closure of the set $\{u \in W^{k,p}(U);\ \operatorname{supp} u \subset U\}$ in $W^{k,p}(U)$, and $W^{-k,p'}(U)$ as the dual of $W_0^{k,p}(U)$. For $k=0$ this means that we identify $(L^p(U))^*$ and $L^{p'}(U)$.

Definition 2.1. Let $\mu \in [0, m+\frac{N}{p}]$. We define

$$W^{k,p,m,\mu}(\Omega) := \{u \in W^{k,p}(\Omega);\ \|u\|_{k,p,m,\mu} < \infty\},$$

where

$$\|u\|_{k,p,m,\mu} := \max\{\|u\|_{k,p}, \sup_{U\in\mathcal{F}} d_U^{-\mu} \inf_{w\in\mathbb{P}_{m+k}} \|u-w\|_{k,p,U}\}.$$

Obviously, $\|\cdot\|_{k,p,m,\mu}$ is a norm on $W^{k,p,m,\mu}(\Omega)$, and the space $(W^{k,p,m,\mu}(\Omega), \|\cdot\|_{k,p,m,\mu})$ is complete.

Definition 2.2. Let $v \in W_0^{k,p}(\Omega)$ and $\mu \in [0, m+\frac{N}{p'}]$. We put

$$\|v\|_{k,p,m,-\mu} := \inf\Big\{\|v_0\|_{k,p} + \sum_{U\in\mathcal{G}} d_U^{\mu}\|v_U\|_{k,p};\ v = v_0 + \sum_{U\in\mathcal{G}} v_U,\ v_0, v_U \in W_0^{k,p}(\Omega),$$
$$\int_\Omega v_U w = 0 \ \text{ for all } w \in \mathbb{P}_{m-k}, \operatorname{supp} v_U \subset U, \mathcal{G} \subset \mathcal{F} \ \text{ finite}\Big\}.$$

Using Lemma 1.3 it is easy to check that $\|\cdot\|_{k,p,m,-\mu}$ is a norm on $W_0^{k,p}(\Omega)$.

Definition 2.3. Let $\mu \in [0, m+\frac{N}{p'}]$. By $W_0^{k,p,m,-\mu}(\Omega)$ we denote the completion of the space $(W_0^{k,p}(\Omega), \|\cdot\|_{k,p,m,-\mu})$. The norm $\|\cdot\|_{k,p,m,-\mu}$ is extended to $W_0^{k,p,m,-\mu}(\Omega)$ by continuity.

Theorem 2.4. *Let Ω be a bounded Lipschitz domain, and let $\mu \in [0, m+\frac{N}{p}]$. Then*

$$W^{k,p,m,\mu}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^{p,m+k-|\alpha|,\mu}(\Omega),\ |\alpha| \leqslant k\} \tag{2.1}$$

and the norm $\|\cdot\|_{W^{k,p,m,\mu}(\Omega)}$ is equivalent to $\sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{L^{p,m+k-|\alpha|,\mu}(\Omega)}$.

The result (2.1) is not at all obvious. It is proved in [3] (even under a slightly weaker assumption with respect to Ω). Of course one could also use the relation (2.1) as an alternative definition of $W^{k,p,m,\mu}(\Omega)$.

Definition 2.5. For $\mu \in [0, m + \frac{N}{p}]$ we define

$$W^{-k,p,m,\mu}(\Omega) := \{f \in W^{-k,p}(\Omega);\ \|f\|_{-k,p,m,\mu} < \infty\},$$

where

$$\|f\|_{-k,p,m,\mu} = \max\{\|f\|_{-k,p,\Omega},\ \sup_{U\in\mathcal{F}} d_U^{-\mu} \inf_{w\in\mathbb{P}_{m-k}} \|f-w\|_{-k,p,U}\}.$$

Obviously, $\|\cdot\|_{-k,p,m,\mu}$ is a norm on $W^{-k,p,m,\mu}(\Omega)$, and $(W^{-k,p,m,\mu}(\Omega), \|\cdot\|_{-k,p,m,\mu})$ is complete.

Definition 2.6. Let $\mu \in [0, \frac{N}{p'}]$. For $f \in (W^{k,p'}(\Omega))^*$ we put

$$\|f\|^*_{-k,p,m,-\mu} := \inf\Big\{\|f_0\|_{(W^{k,p'}(\Omega))^*} + \sum_{U\in\mathcal{G}} d_U^{\mu}\|f_U\|_{(W^{k,p'}(\Omega))^*};\ f = f_0 + \sum_{U\in\mathcal{G}} f_U,$$
$$f_0, f_U \in (W^{k,p'}(\Omega))^*,\ \langle f_U, w\rangle = 0 \ \text{ for } w \in \mathbb{P}_{m+k}, \operatorname{supp} f_U \subset U,\, \mathcal{G} \subset \mathcal{F} \ \text{ finite}\Big\}.$$

Arguing as in the proof of Lemma 1.3 one can show that $\|\cdot\|^*_{-k,p,m,-\mu}$ is a norm on $(W^{k,p'}(\Omega))^*$ provided that $W^{k,p',m,\mu}(\Omega)$ is dense in $W^{k,p'}(\Omega)$. This is the case if Ω is a bounded Lipschitzian domain, because in that case the set of restrictions of functions from $C^\infty(\mathbb{R}^N)$ to Ω is dense in $W^{k,p'}(\Omega)$.

Definition 2.7. Let Ω be a bounded Lipschitz domain, and let $\mu \in [0, m+\frac{N}{p'}]$. By $W_*^{-k,p,m,-\mu}(\Omega)$ we denote the completion of the space $((W^{k,p'}(\Omega))^*, \|\cdot\|^*_{-k,p,m,-\mu})$. The norm $\|\cdot\|^*_{-k,p,m,-\mu}$ is extended to $W_*^{k,p,m,-\mu}(\Omega)$ by continuity.

If $m = 0$, then this number is often omitted in the notation of the spaces introduced above. For example, the space $W^{-1,2,\mu}(\Omega)$ mentioned in the introduction is nothing but $W^{-1,2,0,\mu}(\Omega)$.

Theorem 2.8. *For $\mu \in [0, m + \frac{N}{p}]$ we have*

$$W^{-k,p,m,\mu}(\Omega) = (W_0^{k,p',m,-\mu}(\Omega))^* \quad \text{and} \quad W^{k,p,m,\mu}(\Omega) = (W_*^{-k,p',m,-\mu}(\Omega))^*.$$

For the second assertion it is assumed that Ω is a bounded Lipschitz domain.

This theorem is a special case of a general duality result proved in [3]. It includes Theorem 1.5 (first assertion for $k = 0$).

R e m a r k 2.9. By means of Theorem 2.8 it is easy to prove the following result: If $f_\alpha \in L^{p,m-k+|\alpha|,\mu}(\Omega)$, $|\alpha| \leqslant k$, and

$$\langle f, v\rangle := \sum_{|\alpha|\leqslant k} \langle f_\alpha, D^\alpha v\rangle \quad \text{for} \quad v \in W_0^{k,p'}(\Omega),$$

then $f \in W^{-k,p,m,\mu}(\Omega)$.

R e m a r k 2.10. By the Poincaré Inequality

$$\|u\|_2 \leqslant c_N d_U \|u\|_{1,2} \quad \text{for all } u \in W_0^{1,2}(\Omega) \quad \text{such that } \operatorname{supp} u \subset U,\ U \in \mathcal{F}.$$

Thus, $W_0^{1,2}(\Omega) \hookrightarrow L^{2,1}(\Omega) \hookrightarrow L^{2,1-\mu}$, $0 \leqslant \mu \leqslant 1$. This implies that for

$$g \in L^{2,\mu-1}(\Omega) \hookrightarrow (L^{2,1-\mu}(\Omega))^*,\ 0 \leqslant \mu \leqslant 1,$$

the mapping

$$u \longmapsto \langle g, u\rangle\,, \quad u \in W_0^{1,2}(\Omega),$$

is well defined and can be extended uniquely to an element $f \in W^{-1,2,\mu}(\Omega)$. This fact can be used to show that in (2) the requirement $g \in \mathcal{L}^{2,(\lambda-2)_+}(\Omega)$ can be replaced by the weaker and more naturally looking requirement $g \in \mathcal{L}^{2,\lambda-2}(\Omega) := L^{2,\frac{\lambda}{2}-1}(\Omega)$.

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Authors' addresses: *Konrad Gröger, Lutz Recke*, Humboldt Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany, e-mail: `groeger@mathematik.hu-berlin.de`.