WEAK σ-DISTRIBUTIVITY OF LATTICE ORDERED GROUPS

Ján Jakubík, Košice

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Abstract. In this paper we prove that the collection of all weakly distributive lattice ordered groups is a radical class and that it fails to be a torsion class.

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The notion of weak σ-distributivity was applied by Riečan and Neubrunn in the monograph [10] to MV-algebras and to lattice ordered groups; in Chapter 9 of [10] it was systematically used in developing the probability theory in MV-algebras. For a Dedekind complete Riesz space the notion of weak σ-distributivity has been applied by A. Boccuto [2].

It is well known that each MV-algebra $\mathcal{A}$ can be constructed by means of an appropriately chosen abelian lattice ordered group G with a strong unit (this result is due to Mundici [9]). In [10] it was proved that $\mathcal{A}$ is weakly σ-distributive if and only if G is weakly σ-distributive.

For the notions of a radical class and a torsion class of a lattice ordered groups cf., e.g., [1], [3], [5], [8]. Radical classes of MV-algebras were dealt with in [7].

In the present paper we prove that the collection of all weakly σ-distributive lattice ordered groups is a radical class and that it fails to be a torsion class. Consequently, it fails to be a variety.

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Let L be a lattice. If $x \in L$ and $(x_n)_{n\in N}$ is a sequence in L such that $x_n \geqslant x_{n+1}$ for each $n \in N$ and

$$\bigwedge_{n\in N} x_n = x,$$

then we write $x_n \searrow x$.

For lattice ordered groups we use the standard notation.

1.1. Definition. (Cf. [10], 9.4.4 and 9.4.5.) A lattice ordered group G is called weakly σ-distributive if it satisfies the following conditions:

(i) G is σ-complete.

(ii) Whenever $(a_{ij})_{i,j}$ is a bounded double sequence in G such that $a_{ij} \searrow 0$ for each $i \in N$ (where $j \to \infty$), then

$$\bigwedge_{\varphi\in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0.$$

We denote by W the class of all lattice ordered groups which are weakly σ-distributive.

Let $\mathcal{G}$ be the class of all lattice ordered groups. For $G \in \mathcal{G}$ let $c(G)$ be the system of all convex ℓ-subgroups of G; this system is partially ordered by the set-theoretical inclusion. Then $c(G)$ is a complete lattice. The lattice operations in $c(G)$ will be denoted by $\overset{c}{\bigvee}$ and $\overset{c}{\bigwedge}$. If $\{H_i\}_{i\in I}$ is a nonempty subsystem of $c(G)$, then

$$\bigwedge_{i\in I} H_i = \bigcap_{i\in I} H_i.$$

Further, $\bigvee_{i\in I} H_i$ is the subgroup of the group H (where we do not consider the lattice operations) which is generated by the set $\bigcup_{i\in I} H_i$.

1.2. Definition. A nonempty class $X \subseteq \mathcal{G}$ which is closed with respect to isomorphisms is called a radical class if it satisfies the following conditions:

1) If $G_1 \in X$ and $G_2 \in c(G_1)$, then $G_2 \in X$.

2) If $H \in G$ and $\emptyset \neq \{G_i\}_{i\in I} \subseteq c(H) \cap X$, then $\overset{c}{\bigvee\limits_{i\in I}} G_i \in X$.

A radical class which is closed with respect to homomorphisms is called a torsion class.

In view of 1.2, for each radical class X and each $G \in \mathcal{G}$ there exists the largest element of the set $\{G_i \in c(G)\colon G_i$ belongs to $X\}$; we denote it by $X(G)$. It is said to be the radical of G with respect to X.

1.3. Definition. Let G be a σ-complete lattice ordered group. We denote by $B(G)$ the set of all elements $b \in G$ such that the following conditions are valid:

(i) $b > 0$.

(ii) There exists a bounded double sequence $(a_{ij})_{i,j}$ in G such that $a_{ij} \searrow 0$ for each $i \in N$ (where $j \to \infty$) and

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = b.$$

1.4. Lemma. *Let G be a σ-complete lattice ordered group. Then the following conditions are equivalent:*

(i) *G is weakly σ-distributive.*

(ii) *$B(G) = \emptyset$.*

Proof. In view of 1.1 we have (i)$\Rightarrow$(ii). Suppose that (ii) holds. By way of contradiction, assume that G is not weakly distributive. Then there exists a bounded double sequence $(a_{ij})_{i,j}$ in G such that $a_{ij} \searrow 0$ for each $i \in N$ (where $j \to \infty$) and the relation

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{c} a_{i\varphi(i)} = 0 \tag{1}$$

fails to be valid.

Since G is σ-complete, for each $\varphi \in N^N$ there exists an element x_φ in G such that

$$x_\varphi = \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

For each $i, j \in N$ we have $a_{ij} \geqslant 0$, whence $x_\varphi \geqslant 0$ for each $\varphi \in N^N$. Since the relation (1) does not hold, there exists $z \in G$ such that $x_\varphi \geqslant z$ for each $\varphi \in N^N$ and $z \nleqq 0$. Denote $y = z \vee 0$. Then

$$0 < y \leqslant x_\varphi \quad \text{for each } \varphi \in N^N.$$

Put $a'_{ij} = a_{ij} \wedge y$ for each $i, j \in N$. Then $a'_{ij} \searrow 0$ for each $i \in I$ (where $y \to \infty$). Further, for each $\varphi \in N^N$ we have

$$y = y \wedge x_\varphi = y \wedge \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = \bigvee_{i=1}^{\infty} (y \wedge a_{i\varphi(i)}) = \bigvee_{i=1}^{\infty} a'_{i\varphi(i)}.$$

Thus we obtain

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a'_{i\varphi(i)} = y,$$

which contradicts the assumption (ii) in 1.1. □

1.5. Lemma. *Let G be as in* 1.4. *Suppose that $b \in B(G)$ and $b_1 \in G$, $0 < b_1 \leqslant b$. Then $b_1 \in B(G)$.*

P r o o f. In view of 1.3 we have

$$b_1 = b_1 \wedge b = b_1 \wedge \Big(\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \Big) = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} (b_1 \wedge a_{i\varphi(i)}).$$

Put $b_1 \wedge a_{ij} = a'_{ij}$ for each $i, j \in N$. Then the double sequence $(a'_{ij})_{ij}$ is bounded in G and $a'_{ij} \searrow 0$ for each $i \in N$ (where $j \to \infty$). Hence $b_1 \in B$. □

From the definition of W we immediately obtain

1.6. Lemma. *W satisfies condition* 1) *from* 1.2.

1.7. Lemma. *W satisfies condition* 2) *from* 1.2.

P r o o f. Let $H \in \mathcal{G}$ and $\emptyset \neq \{G_i\}_{i \in I} \subseteq c(H) \cap W$. Put

$$\bigvee_{i \in I}^{c} G_i = K.$$

By way of contradiction, suppose that K does not belong to W. It is clear that K is σ-complete. Thus in view of 1.4, $B(K) \neq \emptyset$. Choose $b \in B(K)$.

It is well-known that for each element $k \in K^+$ there exist $n \in N$, $i_1, i_2, \ldots, i_n \in I$ and $x_n \in G^+_{i_1}, x_2 \in G^+_{i_2}, \ldots, x_n \in G^+_{i_n}$ such that

$$k = x_1 + x_2 + \ldots + x_n.$$

Put $k = b$. Since $b > 0$, at least one of the elements $x_1, x_2, \ldots, x_n$ is strictly positive. Let $x_i > 0$ for some $i \in \{1, 2, \ldots, n\}$. In view of 1.5 we have $x_i \in B(G)$. This yields $x_i \in B(G_i)$, a contradiction. □

In view of 1.6 and 1.7 we have

1.8. Proposition. *W is a radical class of lattice ordered groups.*

1.9. E x a m p l e. Let us denote by R^+ the set of all non-negative reals and let F be the set of all real functions defined on the set R^+. The partial order and the operation $+$ on F are defined coordinate-wise. Then F is a complete lattice ordered group. Moreover, F is completely distributive. Hence, in particular, F is weakly σ-distributive. Thus F belongs to W. Let H be the system of all $f \in F$ such that the set

$$\{x \in R^+ \colon f(x) \neq 0\}$$

is finite. Then H is an ℓ-ideal of F. It is easy to verify that the factor lattice ordered group F/H fails to be archimedean, hence it is not σ-complete. Thus W is not closed with respect to homomorphisms. Consequently, it fails to be a torsion class.

Radical classes which satisfy some additional conditions were investigated in [11]. In connection with W let us mention two such properties. First, it is obvious that the class W is closed with respect to direct products.

For a subset X of a lattice ordered group G the polar X^δ of X in G is defined by

$$X^\delta = \bigl\{g \in G \colon |g| \wedge |x| = 0 \quad \text{for each } x \in X\bigr\}.$$

We say that a class $\mathcal{C}$ of lattice ordered groups is closed with respect to double polars if, whenever $G \in \mathcal{G}$ and $H \in c(G) \cap \mathcal{C}$, then $H^{\delta\delta} \in \mathcal{C}$.

1.10. Proposition. *The class W is closed with respect to double polars.*

P r o o f. Let $G \in \mathcal{G}$ and $H \in c(G) \cap W$. Put $H^{\delta\delta} = K$. By way of contradiction, assume that K does not belong to W. Thus in view of 1.4, $B(K) \neq \emptyset$. Let $b \in B(K)$. Then $b > 0$. If $h \wedge b = 0$ for each $h \in H^+$, then $b \in H^\delta$; since $H^\delta \cap H^{\delta\delta} = \{0\}$, we would obtain $b = 0$, which is impossible. Therefore there is $h \in H^+$ such that $h \wedge b > 0$. Thus in view of 1.5, $h \wedge b \in B(K)$. Consequently, $h \wedge b \in B(H)$ and therefore $B(H) \neq \emptyset$. In view of 1.4 we arrived at a contradiction. $\square$

1.11. Corollary. *$(W(G))^{\delta\delta} = W(G)$ for each lattice ordered group G.*

1.12. Corollary. *Let G be a strongly projectable lattice ordered group. Then $W(G)$ is a direct factor of G.*

The assertion of 1.12 is valid, in particular, for each complete lattice ordered group.

2. A modification of weak σ-distributivity

In this section we deal with a modification of the notion of weak σ-distributivity; this can be applied also to lattice ordered groups which are not σ-complete.

Let L be a lattice. We say that L satisfies the condition (α) if, whenever $(a_{ij})_{i,j}$ is a bounded double sequence in L such that

(a) $a_{ij} \geqslant a_{i,j+1}$ for each $i, j \in N$;

(b) all the joins and meets in the expressions

$$(*) \qquad \bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} a_{ij}, \quad \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

exist in L, then the expressions in $(*)$ are equal.

It is obvious that σ-distributivity of L implies that condition (α) is valid for L.

2.1. Proposition. *Let G be a σ-complete lattice ordered group. Then G is weakly σ-distributive if and only if it satisfies condition (α).*

Proof. i) Assume that G is weakly σ-distributive. Let $(a_{ij})_{i,j}$ be a bounded double sequence in G such that conditions (a) and (b) are satisfied. Put

$$u = \bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} a_{ij}, \quad v = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Denote $a'_{ij} = (a_{ij} \vee u) \wedge v$. Since G is infinitely distributive, we get

$$(1) \qquad u = \bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} a'_{ij}, \quad v = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a'_{i\varphi(i)}.$$

Also, for each $i, j \in N$ the relations $a'_{ij} \geqslant a'_{i,j+1}$ and $a'_{ij} \in [u, v]$ are valid. Thus

$$\bigwedge_{j=1}^{\infty} a'_{ij} \geqslant u \quad \text{for each } i \in N.$$

Hence by the first of the relations (1) we get

$$(2) \qquad \bigwedge_{j=1}^{\infty} a'_{ij} = u.$$

Further, we denote $a''_{ij} = a'_{ij} - u$ for all $i, j \in N$. Then $a''_{ij} \geqslant a''_{i,j+1}$ for all $i, j \in N$, whence according to (2)

$$a''_{ij} \searrow 0 \quad (\text{as } j \to \infty) \text{ for each } i \in I.$$

Since G is weakly σ-distributive, we obtain

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a''_{i\varphi(i)} = 0.$$

This yields

$$u = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} (a''_{i\varphi(i)} + u) = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a'_{i\varphi(i)} = v.$$

Therefore G satisfies condition (α).

ii) Conversely, assume that condition (α) is valid for G. Let $(a_{ij})_{i,j}$ be a bounded double sequence in G such that, for each $i \in N$, we have $a_{ij} \searrow 0$ (where $j \to \infty$). Thus

$$\bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} a_{ij} = 0.$$

Since G is σ-complete, in view of condition (α) we obtain

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0,$$

whence G is weakly σ-distributive. $\square$

We denote by W_1 the class of all lattice ordered groups G such that G satisfies condition (α).

In view of 2.1 we have $W \subseteq W_1$. The following example shows that $W \neq W_1$.

Let Q be the additive group of all rationals with the natural linear order. Then Q is a completely distributive lattice ordered group, whence $Q \in W_1$. Since Q fails to be σ-complete, it does not belong to W.

We obviously have

2.2. Lemma. *Let L be a lattice. Suppose that condition (α) is not valid for L. Then there exists a bounded double sequence $(a_{ij})_{i,j}$ in L such that assumptions* (a), (b) *of (α) are satisfied and there are $u, v \in L$ with*

$$u < v, \quad u = \bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} a_{ij}, \quad v = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

2.3. Corollary. *Let L be as in 2.2. Assume that L is infinitely distributive. Then there are $u, v \in L$, $u < v$ such that condition (α) is not satisfied for the interval $[u, v]$ of L.*

P r o o f. It suffices to consider the double sequence $(a'_{ij})_{i,j}$, where $a'_{ij} = (a_{ij} \vee u) \wedge v$ for each $i, j \in N$. $\square$

2.4. Lemma. *Let L, u and v be as in 2.3. Assume that $u_1, v_1 \in L$, $u \leqslant u_1 < v_1 \leqslant v$. Then the interval $[u_1, v_1]$ does not satisfy condition (α).*

P r o o f. Let $(a'_{ij})_{i,j}$ be as in the proof of 2.3. Now it suffices to take into account the double sequence $(a''_{ij})_{i,j}$, where

$$a''_{ij} = (a'_{ij} \vee u_1) \wedge v_1$$

for each $i, j \in N$. $\square$

Since each lattice ordered group G is infinitely distributive, from 2.3, 2.4 and by using a translation we obtain

2.5. Corollary. *Let G be a lattice ordered group which does not satisfy condition (α). Then there is $v \in G$ with $0 < v$ such that, whenever $v_1 \in G$, $0 < v_1 \leqslant v$, then the interval $[0, v_1]$ of G does not satisfy condition (α).*

Now by an analogous argument as in the proofs of 1.6 and 1.7 and by applying 2.5 we infer

2.6. Proposition. *W_1 is a radical class of lattice ordered groups.*

Also, similarly as in the case of W, the class W_1 is closed with respect to direct products and with respect to double polars.

We conclude by the following remarks on MV-algebras.

Let $\mathcal{A}$ be an MV-algebra with the underlying set A. We apply the notation from [5]. There exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G, u)$ (cf. Mundici [9]). In particular, A is the interval $[0, u]$ of G. Hence we can consider the lattice operations $\vee$ and $\wedge$ on A; thus we can apply the notion of weak σ-distributivity and the condition (α) for the case when instead of a lattice ordered group we have an MV-algebra. We denote by W^m and W_1^m the classes of all MV-algebras which satisfy the condition of weak σ-distributivity or the condition (α), respectively. The notion of a radical class of MV-algebras was introduced and studied in [7].

In [10], (9.4.5) it was proved that $\mathcal{A}$ is weakly σ-distributive if and only if G is weakly σ-distributive. By a similar argument we can show that $\mathcal{A}$ satisfies the condition (α) if and only if G satisfies this condition. Thus we obtain from 1.8, 2.6 and from [7], Lemma 3.4 that both W^m and W_1^m are radical classes of MV-algebras.

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Author's address: *Ján Jakubík*, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: `musavke@mail.saske.sk`.