ON CONSEQUENCES OF CERTAIN BOUNDARY CONDITIONS ON THE UNIT CIRCLE

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Abstract. Let $\mathcal{P}$ denote the well-known class of Carathéodory functions of the form $p(z) = 1 + c_1 z + \dots$, $z \in \Delta = \{z \in \mathbb{C}\colon\ |z| < 1\}$, with positive real part in the unit disc and let $\mathbf{H}(M)$ stand for the class of holomorphic functions commonly bounded by M in Δ. In 1992, J. Fuka and Z. J. Jakubowski began an investigation of families of mappings $p \in \mathcal{P}$ fulfilling certain additional boundary conditions on the unit circle T. At first, the authors examined the class $\mathcal{P}(B, b; \alpha)$ of functions defined by conditions given by the upper limits for two disjoint open arcs of T. After that, such boundary conditions given, in particular, by the nontangential limits, were assumed for different subsets of the unit circle. In parallel, G. Adamczyk started to search for properties of families, contained in $\mathbf{H}(M)$ and satisfying certain similar conditions on T. The present article belongs to the above series of papers. In the first section we will consider subclasses of $\mathcal{P}$ of functions satisfying some inequalities on several arcs of T, whereas in Sections 2 and 3—families of mappings $f \in \mathbf{H}(M)$ with conditions given for measurable subsets of the unit circle T.

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1.

In this section we will consider Carathéodory functions connected with several arcs of the unit circle.

Let $\mathbf{b} = (b_1, b_2, \dots, b_k)$, $0 \leqslant b_1 \leqslant b_2 \leqslant \dots \leqslant b_k$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$, $\alpha_j \geqslant 0$, $j = 1, 2, \dots, k$, $\sum_{j=1}^{k} \alpha_j = 1$, $k \geqslant 2$. As usual, denote $\Delta = \bigl\{z \in \mathbb{C}\colon\ |z| < 1\bigr\}$, $T = \bigl\{z \in \mathbb{C}\colon\ |z| = 1\bigr\}$, $\mathcal{P} = \bigl\{p(z) = 1 + c_1 z + \dots,\ \operatorname{Re} p(z) > 0,\ z \in \Delta\bigr\}$.

Definition 1. Let $\mathbf{b}$, $\boldsymbol{\alpha}$ be fixed as above and $p \in \mathcal{P}$. We say that $f \in \mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ iff there exists a system $\mathbf{I}_{\boldsymbol{\alpha}} = \mathbf{I}_{\boldsymbol{\alpha}}(p) = (I_{\alpha_1}, I_{\alpha_2}, \dots, I_{\alpha_k})$ of k disjoint open arcs of the unit circle T of lengths, respectively, $2\pi\alpha_j$, $j = 1, 2, \ldots, k$, such that

$$\underline{\lim}_{z\to\zeta} \operatorname{Re} p(z) \geqslant b_j \quad \text{for all } \zeta \in I_{\alpha_j}, \tag{1}$$

$j = 1, 2, \ldots, k$.

R e m a r k 1. Of course, for any admissible $\mathbf{b}$ and $\boldsymbol{\alpha} = (\alpha_1, 1-\alpha_1, 0, \ldots, 0)$, $\alpha_1 \in (0,1)$, the classes $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ reduce to the well-known families $\mathcal{P}(B, b; \alpha)$, [4].

Consider $p \in \mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$, $\mathbf{I}_{\boldsymbol{\alpha}} = \mathbf{I}_{\boldsymbol{\alpha}}(p)$—the respective system of arcs, with ω_j—the harmonic measure of I_{α_j}, $j = 1, 2, \ldots, k$. Put

$$\omega(z) = \sum_{j=1}^{k} b_j \omega_j(z), \quad z \in \overline{\Delta} \setminus \{\zeta_1, \ldots, \zeta_k\}, \tag{2}$$

where $\zeta_1, \ldots, \zeta_k$ stand for the ends of arcs of the system $\mathbf{I}_{\boldsymbol{\alpha}}$. Of course, ω is a nonnegative harmonic function bounded by b_k in Δ. From (1) we also obtain

$$\overline{\lim_{z\to\zeta}} \big(-\operatorname{Re} p(z)\big) \leqslant -b_j = -\omega_j(\zeta), \quad \zeta \in I_{\alpha_j},\ j = 1, 2, \ldots, k.$$

Using the Lindelöf maximum principle, we get

Lemma 1. *Let* $\mathbf{b}$, $\boldsymbol{\alpha}$ *be arbitrary and admissible. If* $p \in \mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$, *where* ω *is the function given by formula* (2) *for* $\mathbf{I}_{\boldsymbol{\alpha}}(p)$, *then*

$$\operatorname{Re} p(z) \geqslant \omega(z), \quad z \in \Delta. \tag{3}$$

This and the normalization $p(0) = 1$, $\omega_j(0) = \alpha_j$ imply

Theorem 1. *If the class* $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ *is not empty, then*

$$\sum_{j=1}^{k} b_j \alpha_j \leqslant 1. \tag{4}$$

R e m a r k 2. In (4) the equality holds iff $\operatorname{Re} p$ is of the form (2).

Consider a sequence of vectors $\mathbf{b}_n = (b_1, b_2, \ldots, b_{k-1}, b_k^{(n)})$ such that $0 < b_1 < b_2 < \ldots < b_{k-1} < b_k^{(n)}$ and $\lim_{n\to\infty} b_k^{(n)} = \infty$. Fix suitable $\boldsymbol{\alpha}$, $\mathbf{I}_{\boldsymbol{\alpha}}$ and a sequence $\{p_n\}$ of functions holomorphic in Δ with positive real parts, such that

$$\underline{\lim}_{z\to\zeta} \operatorname{Re} p_n(z) \geqslant \begin{cases} b_j & \text{for } \zeta \in I_{\alpha_j},\ j = 1, 2, \ldots, k-1, \\ b_k^{(n)} & \text{for } \zeta \in I_{\alpha_k}. \end{cases}$$

Then

$$\begin{aligned}\operatorname{Re} p_n(z) &\geqslant b_1\omega_1(z) + \ldots + b_{k-1}\omega_{k-1}(z) + b_k^{(n)}\Bigl(1 - \sum_{j=1}^{k-1}\omega_j(z)\Bigr)\\ &\geqslant b_k^{(n)}\Bigl(1 - \sum_{j=1}^{k-1}\omega_j(z)\Bigr), \quad z \in \Delta.\end{aligned}$$

Let $z \in \overline{\Delta}_r = \{z \in \mathbb{C}\colon\ |z| \leqslant r\}$. Of course, $W(z) = \sum\limits_{j=1}^{k-1}\omega_j(z)$ is a function harmonic in Δ_r, so $W(z) \leqslant \delta(r)$ with $\delta(r) < 1$. Therefore $|p_n| \geqslant b_k^{(n)}\bigl(1 - \delta(r)\bigr)$, $z \in \Delta_r$. It means that p_n converge almost uniformly to ∞ if $n \to \infty$. It is known that if $p_n \in \mathcal{P}(\mathbf{b}_n, \boldsymbol{\alpha})$, then $p_n(0) = 1$. So, we have

Remark 3. Let $\{\mathbf{b}_n\}$ be the sequence given above. Then the classes $\mathcal{P}(\mathbf{b}_n, \boldsymbol{\alpha})$ are, starting with a certain n, empty.

Assume now that, for fixed $\mathbf{b}$, $\boldsymbol{\alpha}$, (4) holds. Let $\mathbf{I}_{\boldsymbol{\alpha}}$ be a respective system of arcs, ω_j—the harmonic measure of the arc I_{α_j}, ω_j^*—the conjugate function of ω_j, $\omega_j^*(0) = 0$. Put

(5) $$h_j(z) = \omega_j(z) + \mathrm{i}\omega_j^*(z), \quad z \in \Delta,\ j = 1, 2, \ldots, k$$

and

(6) $$p_0(z) = \sum_{j=1}^{k} b_j h_j(z), \quad z \in \Delta.$$

Denote also

(7) $$\eta = \sum_{j=1}^{k} \alpha_j b_j.$$

One can easily check that if $\eta = 1$, then $p_0 \in \mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$. For $\eta \in (0, 1)$, put

(8) $$p(z) = p_0(z) + (1 - \eta)\, q(z), \quad z \in \Delta,$$

where q is an arbitrary function from $\mathcal{P}$. Then $p \in \mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$. So the following theorem is true.

Theorem 2. *If condition* (4) *holds, then the class* $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ *is not empty.*

R e m a r k 4. It is clear that there are infinitely many functions belonging to $\mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$ because in the construction (6) or (8) we can choose another system $\mathbf{I}_{\boldsymbol{\alpha}}$ or another mapping $q \in \mathcal{P}$.

Moreover, the above constructions prove

Proposition 1. 1° *If* $\eta = 1$, *then* $\mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$ *is a set of mappings of the form* (6).
2° *If* $0 < \eta < 1$, *then* $\mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$ *is a set of mappings of the form* (8).

Indeed, take $p \in \mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$, construct p_0 by formula (6) for $\mathbf{I}_{\boldsymbol{\alpha}} = \mathbf{I}_{\boldsymbol{\alpha}}(p)$ and consider a function $P(z) = p(z) - p_0(z)$, $z \in \Delta$. This is a function holomorphic in Δ and $\underline{\lim}_{z\to\zeta} \operatorname{Re} P(z) \geqslant 0$ for $\zeta \in I_{\alpha_j}$, $j = 1, 2, \ldots, k$. So, $\operatorname{Re} P(z) \geqslant 0$, $z \in \Delta$. Moreover, $P(0) = 1 - \eta$, so if $\eta = 1$, then $P \equiv 0$ and (6) holds. For $\eta \in (0,1)$, it is enough to take $q(z) = \frac{1}{1-\eta} P(z)$ to justify (8).

In view of Theorem 2 and Proposition 1, constructions (6) and (8) constitute structure formulae in the respective classes $\mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$. They are useful, among other things, in the searching for estimates of certain functionals.

Consequently, consider $\eta \in (0,1)$. Let $p \in \mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$. To this mapping there correspond, of course, a definite system of arcs $\mathbf{I}_{\boldsymbol{\alpha}}(p)$ and functions p_0 and q according to (8). From (5) and (6) we obtain

$$p_0(z) = \sum_{j=1}^{k} b_j \int_{I_{\alpha_j}} \big(1 + 2\mathrm{e}^{-\mathrm{i}t} z + \ldots\big)\,\mathrm{d}t.$$

On the other hand, $p_0(z) = \eta + c_{1,p_0} z + \ldots$, $z \in \Delta$. So

$$\operatorname{Re} c_{n,p_0} = 2 \sum_{j=1}^{k} b_j \int_{I_{\alpha_j}} \cos nt\,\mathrm{d}t, \quad n = 1, 2, \ldots$$

In view of Proposition 1 (2°) and the well-known estimate of the coefficients in $\mathcal{P}$, we get

$$\max_{p\in\mathcal{P}(\mathbf{b},\boldsymbol{\alpha})} \operatorname{Re} c_{n,p} = 2 \max_{\mathbf{I}_{\boldsymbol{\alpha}}} \sum_{j=1}^{k} b_j \int_{I_{\alpha_j}} \cos nt\,\mathrm{d}t + 2(1-\eta). \tag{9}$$

Let $\mathbf{I}_{\boldsymbol{\alpha}}$ be a fixed system of appropriate arcs of the circle T. Denote by $\mathbf{I}_{\boldsymbol{\alpha}}^{\tau}$ the system $\mathbf{I}_{\boldsymbol{\alpha}}$ rotated by the angle τ, i.e. such that $I_{\alpha_j}^{\tau} = \big\{z = \mathrm{e}^{-\mathrm{i}\tau}\zeta,\ \zeta \in I_{\alpha_j}\big\}$.

Definition 2. By $\mathcal{P}(\mathbf{b},\boldsymbol{\alpha};\mathbf{I}_{\boldsymbol{\alpha}})$ we will denote the set of all functions $p \in \mathcal{P}(\mathbf{b},\boldsymbol{\alpha})$ for which there exists $\tau = \tau(p) \in \langle 0, 2\pi)$ such that $\mathbf{I}_{\boldsymbol{\alpha}}(p) = \mathbf{I}_{\boldsymbol{\alpha}}^{\tau}$.

Let us consider the case $k=3$ and put $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$, $\alpha_1=\alpha_2=\frac{1}{2}(1-\alpha)$, $\alpha_3=\alpha$, $\alpha\in(0,2\pi)$. Let $\mathbf{I}_{\boldsymbol{\alpha}}=(I_{\alpha_1},I_{\alpha_2},I_{\alpha_3})$ where $I_{\alpha_1}=\{\zeta=\mathrm{e}^{\mathrm{i}t}\colon\ t\in(\pi,2\pi-\alpha\pi)\}$, $I_{\alpha_2}=\{\zeta=\mathrm{e}^{\mathrm{i}t}\colon\ t\in(\alpha\pi,\pi)\}$, $I_{\alpha_3}=\{\zeta=\mathrm{e}^{\mathrm{i}t}\colon\ t\in(-\alpha\pi,\alpha\pi)\}$.

Take $p\in\mathcal{P}(\mathbf{b},\boldsymbol{\alpha};\mathbf{I}_{\boldsymbol{\alpha}})$. From the definition of this class and from (9) we have

$$\begin{aligned}\operatorname{Re} c_{1,p}(\tau)&=2\left(b_3\int_{-\alpha\pi+\tau}^{\alpha\pi+\tau}\cos t\,\mathrm{d}t+b_2\int_{\alpha\pi+\tau}^{\pi+\tau}\cos t\,\mathrm{d}t+b_1\int_{\pi+\tau}^{-\alpha\pi+\tau}\cos t\,\mathrm{d}t\right)\\&=\left(\tfrac{1}{2}b_3-b_2-b_1\right)\sin\tfrac{\alpha\pi}{2}\cos\tau+(b_1-b_2)\cos\tfrac{\alpha\pi}{2}\sin\tau.\end{aligned}$$

If p_0 runs over the set of all functions of the form (6), then τ runs over the interval $\langle 0,2\pi)$. Hence

$$\begin{aligned}\operatorname{Re} c_{1,p}(\tau)&\leqslant\max_{\tau\in\langle 0,2\pi)}\operatorname{Re} c_{1,p}(\tau)\\&=2\cos\tfrac{\alpha\pi}{2}\left(\left(\tfrac{1}{2}b_3-b_2-b_1\right)\cos x_0\sin\tfrac{\alpha\pi}{2}+(b_1-b_2)\sin x_0\cos\tfrac{\alpha\pi}{2}\right),\end{aligned}$$

where

$$x_0=\frac{\pi}{2}-\operatorname{arctg}\left(\frac{\frac{1}{2}b_3-b_2-b_1}{b_1-b_2}\operatorname{ctg}\frac{\alpha\pi}{2}\right).$$

Corollary 1. *Let $\alpha\in(0,2\pi)$, $\boldsymbol{\alpha}=\left(\frac{1-\alpha}{2},\frac{1-\alpha}{2},\alpha\right)$, $\mathbf{b}=(b_1,b_2,b_3)$ be arbitrary and fixed, and let the system $\mathbf{I}_{\boldsymbol{\alpha}}$ be defined as above. Let $p\in\mathcal{P}(\mathbf{b},\boldsymbol{\alpha};\mathbf{I}_{\boldsymbol{\alpha}})$ be of the form $p(z)=1+c_{1,p}z+\ldots$, $z\in\Delta$. Then*

$$|c_{1,p}|\leqslant 2\left\{1+\cos\tfrac{\alpha\pi}{2}\left(\left(\tfrac{1}{2}b_3-b_2-b_1\right)\cos x_0\,\sin\tfrac{\alpha\pi}{2}+(b_1-b_2)\sin x_0\cos\tfrac{\alpha\pi}{2}\right)\right\},$$

x_0 is given by the above formula, and these estimates are sharp.

Definition 3. Fix an arbitrary $\mathbf{I}_{\boldsymbol{\alpha}}$ and denote

$$\mathcal{P}^{\vee}(\mathbf{b},\boldsymbol{\alpha};\mathbf{I}_{\boldsymbol{\alpha}})=\{p\in\mathcal{P}(\mathbf{b},\boldsymbol{\alpha})\colon\ \mathbf{I}_{\boldsymbol{\alpha}}(p)=\mathbf{I}_{\boldsymbol{\alpha}}\}.$$

Example 5. Let $k=3$, $\boldsymbol{\alpha}=\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)$, $\mathbf{b}=(b_1,b_2,b_3)$ and $I_{\alpha_1}=\{\zeta=\mathrm{e}^{\mathrm{i}t}\colon -\pi<t<-\frac{\pi}{2}\}$, $I_{\alpha_2}=\{\zeta=\mathrm{e}^{\mathrm{i}t}\colon -\frac{\pi}{2}<t<\frac{\pi}{2}\}$, $I_{\alpha_3}=\{\zeta=\mathrm{e}^{\mathrm{i}t}\colon \frac{\pi}{2}<t<\pi\}$, $\mathbf{I}_{\boldsymbol{\alpha}}=(I_{\alpha_1},I_{\alpha_2},I_{\alpha_3})$. Using Proposition 1 (2°) for $p\in\mathcal{P}^{\vee}(\mathbf{b},\boldsymbol{\alpha};\mathbf{I}_{\boldsymbol{\alpha}})$ we get $c_{n,p}=c_{n,p_0}+(1-\eta)c_{n,q}$, $n=1,2,\ldots$, whence

$$|c_{n,p}|\leqslant 2(1-\eta)+\begin{cases}0 & \text{for } n=4l,\\ \dfrac{\sqrt{2}}{n\pi}\sqrt{b_1^2+b_3^2+2b_2(b_2-b_1-b_3)} & \text{for } n=4l+1,\ n=4l+3,\\ \dfrac{2}{n\pi}|b_3-b_1| & \text{for } n=4l+2,\end{cases}$$

$l\in\mathbb{N}$.

The following assertions are true.

Proposition 2. *For any admissible* $\mathbf{b}$, $\boldsymbol{\alpha}$, *the class* $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ *is compact.*

Using Lemma 1, one can prove this fact analogously as in [4].

Proposition 3. *For any admissible* $\mathbf{b}$, $\boldsymbol{\alpha}$, $\mathbf{I}_{\boldsymbol{\alpha}}$, *the class* $\mathcal{P}^{\vee}(\mathbf{b}, \boldsymbol{\alpha}; \mathbf{I}_{\boldsymbol{\alpha}})$ *is convex.*

Proposition 4. *Let* $b_k > b_{k-1}$. *The class* $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ *is not convex.*

To verify them, consider two functions p_0 and p_0^τ given by formula (6) for $\mathbf{I}_{\boldsymbol{\alpha}}$ and $\mathbf{I}_{\boldsymbol{\alpha}}^\tau$, $0 < \tau < \min\{2\pi\alpha_j\}_{j=1}^k$. The linear convex combination p_λ of the functions p_1 and p_2 given by (8) for $\eta \in (0,1)$, p_0, p_0^τ and $q(z) = \frac{1+z}{1-z}$, $z \in \Delta$, does not satisfy the definition condition (1) for $j = k$. If $\eta = 1$, one should take $p_\lambda = \lambda p_0 + (1-\lambda) p_0^\tau$.

Below, we will determine a relationship between $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$ and the families $\mathbf{H}(\mathbf{l}, \boldsymbol{\alpha})$ investigated in article [1]. Let us recall:

Let $\mathbf{H}$ stand for the family of mappings f holomorphic and bounded in Δ, $f(0) = f'(0) - 1 = 0$.

Definition 4. Let $\mathbf{l} = (l_1, l_2, \ldots, l_k)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $0 < l_1 \leqslant \ldots \leqslant l_k$, $\alpha_j \geqslant 0$, $j = 1, 2, \ldots, k$, $\sum\limits_{j=1}^{k} \alpha_j = 1$, $k \geqslant 2$. By $\mathbf{H}(\mathbf{l}, \boldsymbol{\alpha})$ we denote the set of functions $f \in \mathbf{H}$ for which there exists a respective system $\mathbf{I}_{\boldsymbol{\alpha}} = \mathbf{I}_{\boldsymbol{\alpha}}(f)$ such that

$$\varlimsup_{z \to \zeta} \left| f(z) \right| \leqslant l_j \quad \text{for each } \zeta \in I_{\alpha_j},$$

$j = 1, 2, \ldots, k$.

It is known that $\mathbf{H}(\mathbf{l}, \boldsymbol{\alpha}) \neq \emptyset$ iff $\eta(k) = \prod\limits_{j=1}^{k} l_j^{\alpha_j} \geqslant 1$. Assume that $\mathbf{l}$, $\boldsymbol{\alpha}$ satisfy additionally $l_k < \mathrm{e}$ and $\eta(k) < \mathrm{e}$. We have

Theorem 3. *If* $p \in \mathcal{P}(\mathbf{b}, \boldsymbol{\alpha})$, *then the mapping*

$$f(z) = z\mathrm{e}^{1-p(z)}, \quad z \in \Delta,$$

belongs to the class $\mathbf{H}(\mathbf{l}, \boldsymbol{\alpha}')$ *where* $l_j = \mathrm{e}^{1-b_{k-j+1}}$, $\alpha'_j = \alpha_{k-j+1}$, $j = 1, 2, \ldots, k$.

The proof is analogous to that in [2].

2.

In this part we will investigate functions bounded in the unit disc with some conditions given for two sets of T.

In literature one can find various methods for investigating the boundary behaviour of functions defined in certain domains (see, for instance, the local Fatou theorem in [12], p. 94).

For $f \in \mathbf{H}$, the upper limit and also the nontangential limit a. e. on T exist. So, denote

$$\lim_{\Gamma_\beta \ni z \to \zeta} |f(z)| =: |f(\mathrm{e}^{\mathrm{i}\theta})| \quad \text{for } \zeta = \mathrm{e}^{\mathrm{i}\theta} \in T.$$

Similarly as in [8] one can justify

Lemma 2. *Let $I \subset T$ be an arbitrary open arc, $f \in \mathbf{H}$. The conditions*

$$\text{(A)} \qquad \varlimsup_{z \to \zeta} |f(z)| \leqslant M \quad \textit{for } \zeta \in I$$

and

$$\text{(B)} \qquad |f(\mathrm{e}^{\mathrm{i}\theta})| \leqslant M \quad \textit{a. e. on } I$$

are equivalent.

Besides, note that if $f \in \mathbf{H}$, then there exist radial limits $|f^*(\mathrm{e}^{\mathrm{i}\theta})|$ a. e. on T and

$$\text{(10)} \qquad |f^*(\mathrm{e}^{\mathrm{i}\theta})| = \lim_{r \to 1} |f(r\mathrm{e}^{\mathrm{i}\theta})| = \lim_{\Gamma_\beta \ni z \to \mathrm{e}^{\mathrm{i}\theta}} |f(z)|, \quad \theta \in \langle 0, 2\pi) \text{ a. e.}$$

Definition 5. Let $F \subset T$ be a set of Lebesgue measure $2\pi\alpha$, $\alpha \in (0,1)$ and $0 < m < M < \infty$. By $\mathbf{H}^{\vee}(M, m, \alpha; F)$ we denote the family of functions $f \in \mathbf{H}$ such that

$$\text{(11)} \qquad |f(\mathrm{e}^{\mathrm{i}\theta})| \leqslant \begin{cases} M & \text{a. e. on } F, \\ m & \text{a. e. on } T \setminus F. \end{cases}$$

Remark 6. Let F be an open arc I_α of length $2\pi\alpha$. Then $\mathbf{H}^{\vee}(M, m, \alpha; I_\alpha) = \mathbf{H}^{\vee}(M, m; I_\alpha)$. The considerations in this part of the paper represent a certain kind of generalization of investigations from the paper [13].

For arbitrary admissible M, m, α, F, we have

Proposition 5. *$\mathbf{H}^{\vee}(M, m, \alpha; F)$ is a family of functions commonly bounded by M in Δ.*

Constructing a function analogously as in [5] and [13] one obtains conditions for the nonemptiness of the above families. So, let $F \subset T$ be a fixed set of measure $2\pi\alpha$, $\omega(\,\cdot\,;F)$—the harmonic measure of the set F, $\omega^*(\,\cdot\,;F)$—the conjugate of $\omega(\,\cdot\,;F)$, $\omega^*(0;F)=0$. Put $h(\,\cdot\,;F)=\omega(\,\cdot\,;F)+\mathrm{i}\omega^*(\,\cdot\,;F)$ and

$$f_0(z) = z\,\mathrm{e}^{h(z;F)\operatorname{Log} M + (1-h(z;F))\operatorname{Log} m}, \quad z \in \Delta, \tag{12}$$

and, for

$$\tilde{\eta} = \frac{1}{M^{\alpha} m^{1-\alpha}}, \tag{13}$$

$$f_1(z) = f_0(z) \cdot \frac{z-\tilde{\eta}}{1-\tilde{\eta} z}, \quad z \in \Delta. \tag{14}$$

If $\tilde{\eta}=1$, then $f_0 \in \mathbf{H}^{\vee}(M,m,\alpha;F)$, whereas if $\tilde{\eta} \in (0,1)$, then f_1 is a function of the family $\mathbf{H}^{\vee}(M,m,\alpha;F)$. Thus we get

Theorem 4. *If* $0<m<M$, $\alpha \in (0,1)$ *and*

$$M^{\alpha} m^{1-\alpha} \geqslant 1, \tag{15}$$

then $\mathbf{H}^{\vee}(M,m,\alpha;F) \neq \emptyset$ *for* $F \subset T$ *and* F *is of length* $2\pi\alpha$.

Conversely:

Theorem 5. *Inequality* (15) *is a necessary condition for the nonemptiness of* $\mathbf{H}^{\vee}(M,m,\alpha;F)$.

The proof is omitted (see [13]). One can easily prove the structure formula below.

Theorem 6. *Let* $f \in \mathbf{H}^{\vee}(M,m,\alpha;F)$. *Then there exists a holomorphic function* Φ *bounded by* 1 *in* Δ, $\Phi(0)=0$, $\Phi'(0)=\tilde{\eta}$, *such that*

$$f(z) = \frac{f_0(z)}{z} \cdot \Phi(z), \quad z \in \Delta, \tag{16}$$

where f_0 *is the function given by* (12) *for the set* F.

The converse is also true, i.e. each function which is the product (16) of any suitably fixed mapping $f_0(z)/z$ and $\Phi(z)$ belongs to $\mathbf{H}^{\vee}(M,m,\alpha;F)$.

Proposition 6. *The classes* $\mathbf{H}^{\vee}(M,m,\alpha;F)$ *are compact.*

For the proof, it is enough to use Proposition 5 and the structure formula (16) for a sequence f_n of functions from the class investigated, converging almost uniformly on Δ.

In the consideration carried out so far, the set F, distinguished by the given boundary condition, was fixed. Now, we shall consider the possibility of changing F and some consequences resulting from it.

Let $F \subset T$ be an arbitrary fixed set of measure $2\pi\alpha$, $\tau \in \langle -\pi, \pi)$, F_τ—the set F rotated by τ, i.e. $F_\tau = \{z = \mathrm{e}^{-\mathrm{i}\tau}\zeta \colon \zeta \in F\}$.

Definition 6. Let $f \in \mathbf{H}$. We say that $f \in \mathbf{H}(M, m, \alpha; F)$ if there exists $\tau = \tau(f) \in \langle -\pi, \pi)$ such that f satisfies conditions (11) for the set F_τ.

The following two statements are valid.

Proposition 7. *For any fixed and admissible M, m, α, F,*

$$\mathbf{H}(M, m, \alpha; F) = \bigcup_{\tau \in \langle -\pi, \pi)} \mathbf{H}^{\vee}(M, m, \alpha; F_\tau)$$

holds.

Proposition 8. *The classes $\mathbf{H}(M, m, \alpha; F)$ are compact in the topology given by the almost uniform convergence in Δ.*

To prove this one should carry out a reasoning analogous to that in [5], [13].

R e m a r k 7. Note that, for the justification of the assertions given in this part of the paper, it would suffice to assume in the definition of the families $\mathbf{H}^{\vee}(M, m, \alpha; F)$ that inequalities (11) are satisfied for the corresponding radial limits $f^*(\mathrm{e}^{\mathrm{i}\theta})$. In view of the existence of both the radial and nontangential limits as well as by equality (10) in this case, the analogous class $\mathbf{H}^{\vee}_*(M, m, \alpha; F)$ is identical with the class $\mathbf{H}^{\vee}(M, m, \alpha; F)$. But note also that the equality of both the limits does not mean their "equivalence", of course, in the sense of the theorems obtained on their properties of the function f in Δ. It is known, for example, that if a function f holomorphic and bounded in Δ has a nontangential limit equal to zero on a certain subset $E \subset T$, then $f \equiv 0$. The above assertion does not hold for radial limits.

3.

In this section we will investigate functions $p \in \mathcal{P}$ which satisfy fixed conditions on the boundary of the disc Δ, distinguishing k subsets measurable in the sense of Lebesgue. We will assume that $\mathbf{b} = (b_1, b_2, \ldots, b_k)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ are defined as in Section 1.

Definition 7. Let $p \in \mathcal{P}$. We say that p belongs to the class $\wp(\mathbf{b}, \boldsymbol{\alpha})$ iff there exist k disjoint sets $F_j \subset T$ of Lebesgue measures, respectively, $2\pi\alpha_j$, $j = 1, 2, \ldots, k$, such that

$$\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \geqslant b_j \quad \text{a.e. on } F_j, \tag{17}$$

$j = 1, 2, \ldots, k$.

In this definition, $\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta})$ stand for the nontangential limits. The case $k = 2$ was the subject of investigations in papers [5], [8]. Throughout our article, we denote $(F_1, \ldots, F_k) = \mathcal{F}(p) = \mathcal{F}$. Putting

$$U(z; \mathcal{F}) = \sum_{j=1}^{k} b_j \omega_j(z), \quad z \in \Delta, \tag{18}$$

with ω_j being the corresponding harmonic measures of the sets F_j, we obtain a function harmonic in Δ, continuous almost everywhere on $\overline{\Delta}$ and such that

$$U(z; \mathcal{F}) = b_j \quad \text{a.e. on } F_j,\ j = 1, 2, \ldots, k. \tag{19}$$

Assume that $\wp(\mathbf{b}, \boldsymbol{\alpha}) \neq \emptyset$. So, let $p \in \wp(\mathbf{b}, \boldsymbol{\alpha})$. Conditions (17) and (19) imply that $\operatorname{Re} p(z) \geqslant U(z; \mathcal{F})$ a.e. on T. Let χ_{F_j} stand for the characteristic function of the set F_j. Then, for $z \in \Delta$, we get

$$\begin{aligned}
\operatorname{Re} p(z) &= \int_{-\pi}^{\pi} \operatorname{Re} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} \,\mathrm{d}\mu(t) \geqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} p(\mathrm{e}^{\mathrm{i}t}) \operatorname{Re} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} \,\mathrm{d}t \\
&\geqslant \sum_{j=1}^{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{F_j}(t) \operatorname{Re} p(\mathrm{e}^{\mathrm{i}t}) \operatorname{Re} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} \,\mathrm{d}t \\
&\geqslant \sum_{j=1}^{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{F_j}(t) \cdot b_j \cdot \operatorname{Re} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} \,\mathrm{d}t = \sum_{j=1}^{k} b_j \omega_j(z).
\end{aligned}$$

We have thus shown

Lemma 3. *Take $p \in \wp(\mathbf{b}, \boldsymbol{\alpha})$, $\mathcal{F} = \mathcal{F}(p)$ and $U(\,\cdot\,; \mathcal{F})$ given by formula* (18) *for $\mathcal{F}$. Then*

$$\operatorname{Re} p(z) \geqslant U(z; \mathcal{F}), \quad z \in \Delta.$$

Putting $z = 0$ in the above inequality and remembering that $\omega_j(0) = \alpha$, we have

Theorem 7. *If $\wp(\mathbf{b}, \boldsymbol{\alpha}) \neq \emptyset$, then*

$$\sum_{j=1}^{k} b_j \alpha_j \leqslant 1. \tag{20}$$

Moreover, we get

Theorem 8. *If condition* (20) *holds, then the class* $\wp(\mathbf{b},\boldsymbol{\alpha}) \neq \emptyset$. *Besides, there exists a function* $p \in \wp(\mathbf{b},\boldsymbol{\alpha})$ *such that* $\operatorname{Re} p(z) = b_j$ a. e. *on* F_j, $j = 1, 2, \ldots, k$.

P r o o f. For the system $\mathcal{F}$ one can construct (analogously as in Theorem 2) a function

$$G(z;\mathcal{F}) = \sum_{j=1}^{k} b_j h_j(z), \quad z \in \Delta, \tag{21}$$

holomorphic in Δ, $\operatorname{Re} G(z;\mathcal{F}) = U(z;\mathcal{F})$, $G(0;\mathcal{F}) = \sum\limits_{j=1}^{k} \alpha_j b_j$. If there is an equality in (20), then $G(z;\mathcal{F}) \in \wp(\mathbf{b},\boldsymbol{\alpha})$, whereas if there is a sharp inequality, one may put

$$p_\gamma(z) = G(z;\mathcal{F}) + \left(1 - \sum_{j=1}^{k} b_j \alpha_j\right) \frac{\mathrm{e}^{\mathrm{i}\gamma} + z}{\mathrm{e}^{\mathrm{i}\gamma} - z}, \quad z \in \Delta, \quad |\gamma| = 1. \tag{22}$$

Then $p_\gamma \in \wp(\mathbf{b},\boldsymbol{\alpha})$. Note that, for p_γ, the equalities hold in (17). □

We will establish some topological properties for a fixed subclass of the families $\wp(\mathbf{b},\boldsymbol{\alpha})$.

Let $\mathcal{F} = (F_1, \ldots, F_k)$ be arbitrary, fixed, admissible and $\wp^\vee(\mathbf{b},\boldsymbol{\alpha},\mathcal{F}) = \{p \in \wp(\mathbf{b},\boldsymbol{\alpha});\ \mathcal{F}(p) = \mathcal{F}\}$.

Proposition 9. *The classes* $\wp^\vee(\mathbf{b},\boldsymbol{\alpha},\mathcal{F})$ *are convex.*

Using Lemma 3 and the properties of the function $U(\,\cdot\,;\mathcal{F})$, one can verify

Proposition 10. *The classes* $\wp^\vee(\mathbf{b},\boldsymbol{\alpha},\mathcal{F})$ *are compact.*

Directly from the definition of the families $\wp(\mathbf{b},\boldsymbol{\alpha})$ it follows that they are subclasses of Carathéodory functions.

It turns out that the imposition of additional boundary conditions on a function $p \in \mathcal{P}$ has its consequences. However, it is possible to fix a certain relationship between the families $\wp(\mathbf{b},\boldsymbol{\alpha})$ and $\mathcal{P}$. Similarly as Proposition 1 (2°) one can prove

Theorem 9. *Let* $\mathbf{b}$, $\boldsymbol{\alpha}$ *be any fixed admissible systems. A function* $p \in \wp(\mathbf{b},\boldsymbol{\alpha})$ *iff there exist* $\mathcal{F}$ *and* $q \in \mathcal{P}$ *such that*

$$p(z) = G(z;\mathcal{F}) + \left(1 - \sum_{j=1}^{k} b_j \alpha_j\right) q(z), \quad z \in \Delta, \tag{23}$$

where $G(\,\cdot\,;\mathcal{F})$ *is the function of the form* (21) *given for the system* $\mathcal{F}$ *of sets.*

Using formula (23), one can prove

Theorem 10. *A function p belonging to the class $\wp^{\vee}(\mathbf{b},\boldsymbol{\alpha},\mathcal{F})$ is an extreme point of this class iff it is of the form* (22).

P r o o f. Assume that $p \in \wp^{\vee}(\mathbf{b},\boldsymbol{\alpha},\mathcal{F})$ is an extreme point of this class. Theorem 9 implies that there exists a $q \in \mathcal{P}$ such that (23) holds. Suppose to the contrary that $q(z) \neq \frac{\mathrm{e}^{\mathrm{i}\gamma}+z}{\mathrm{e}^{\mathrm{i}\gamma}-z}$. It means that q is not an extreme point in $\mathcal{P}$. So there exist $q_1, q_2 \in \mathcal{P}$ and $\lambda \in (0,1)$ such that $q(z) = \lambda q_1(z) + (1-\lambda)q_2(z)$, $z \in \Delta$. Put

$$p_k(z) = G(z;\mathcal{F}) + \left(1 - \sum_{j=1}^{k} b_j\alpha_j\right) q_k(z), \quad z \in \Delta, \quad k = 1,\, 2.$$

Of course, $p_k \in \wp^{\vee}(\mathbf{b},\boldsymbol{\alpha},\mathcal{F})$, $k = 1,\, 2$. Besides, $p = \lambda p_1 + (1-\lambda)p_2$, but this contradicts the first assumption.

Similarly one can justify the sufficient condition of the above theorem. $\square$

R e m a r k 8. Let $p \in \wp(\mathbf{b},\boldsymbol{\alpha})$ and $\mathcal{F} = (F_1,\ldots,F_k) = \mathcal{F}(p)$. Then

$$|c_n| \leqslant \left|\frac{1}{2\pi}\sum_{j=1}^{k} 2b_j \int_{-\pi}^{\pi} \chi_{F_j}(t)\mathrm{e}^{\mathrm{i}nt}\,\mathrm{d}t\right| + 2\left(1 - \sum_{j=1}^{k} b_j\alpha_j\right), \quad n \geqslant 1.$$

Indeed, let $\mathbf{b}$, $\boldsymbol{\alpha}$ be suitably fixed and let $p \in \wp(\mathbf{b},\boldsymbol{\alpha})$. Let also $\mathcal{F} = \mathcal{F}(p)$ stand for a system of k respective sets F_j for which (17) holds. Consider the known function $G(\,\cdot\,;\mathcal{F})$. Of course,

$$G(z;\mathcal{F}) = \sum_{j=1}^{k} b_j\alpha_j + a_{1,G}z + \ldots + a_{n,G}z^n + \ldots, \quad z \in \Delta.$$

The mapping p can be represented by (23) where

$$q(z) = 1 + q_1 z + \ldots + q_n z^n + \ldots, \quad z \in \Delta.$$

Hence we get

$$c_{n,p} = a_{n,G} + \left(1 - \sum_{j=1}^{k} b_j\alpha_j\right) q_n, \quad n \geqslant 1. \tag{24}$$

Moreover,

$$\begin{aligned} G(z;\mathcal{F}) &= \frac{1}{2\pi}\sum_{j=1}^{k}\int_{-\pi}^{\pi} b_j\chi_{F_j}(t)\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\sum_{j=1}^{k}\int_{-\pi}^{\pi} b_j\chi_{F_j}(t)(1 + 2\mathrm{e}^{\mathrm{i}t}z + \ldots + 2\mathrm{e}^{\mathrm{i}nt}z^n + \ldots)\,\mathrm{d}t, \quad z \in \Delta, \end{aligned}$$

so

$$a_{n,G} = \frac{1}{2\pi} \sum_{j=1}^{k} 2b_j \int_{-\pi}^{\pi} \chi_{F_j}(t) \mathrm{e}^{int} \, \mathrm{d}t. \tag{25}$$

Hence and from the estimate of $|q_n|$ in $\mathcal{P}$ we get the assertion.

E x a m p l e 9. Let $k = 3$, $\boldsymbol{\alpha} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$, $\mathbf{b} = (b_1, b_2, b_3)$. Consider the sets $F_1 = \left\{\zeta = \mathrm{e}^{\mathrm{i}t} \colon t \in \left(-\pi, \frac{\pi}{2}\right)\right\} \setminus \mathbb{W}$, $F_2 = \left\{\zeta = \mathrm{e}^{\mathrm{i}t} \colon t \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right\} \setminus \mathbb{W}$, $F_3 = \left\{\zeta = \mathrm{e}^{\mathrm{i}t} \colon t \in \left(\frac{\pi}{2}, \pi\right)\right\} \setminus \mathbb{W}$, where $\mathbb{W}$ denotes the set of rational numbers, and $\mathcal{F} = (F_1, F_2, F_3)$.

Let $p \in \wp^{\vee}(\mathbf{b}, \boldsymbol{\alpha}; \mathcal{F})$. Using (25) and the properties of Lebesgue integrals, for $G(\,\cdot\,; \mathcal{F})$ we successively obtain

$$a_{n,G} = \frac{1}{2\pi} \sum_{j=1}^{k} 2b_j \int_{T} \chi_{F_j}(t) \mathrm{e}^{int} \, \mathrm{d}t = \frac{1}{2\pi} \sum_{j=1}^{k} 2b_j \int_{F_j} \mathrm{e}^{int} \, \mathrm{d}t = \frac{1}{2\pi} \sum_{j=1}^{k} 2b_j \int_{F_j \cup \mathbb{W}} \mathrm{e}^{int} \, \mathrm{d}t.$$

The above considerations mean that the estimate of $|c_{n,p}|$ are identical with those in the corresponding family $\mathcal{P}(\mathbf{b}, \boldsymbol{\alpha}; \mathbf{I}_{\boldsymbol{\alpha}})$ (see Ex. 1).

More generally, let $I_{\alpha_j} \subset T$ be an open arc of length $2\pi\alpha_j$, where $Q_j \subset T$ is a set of measure zero, $F_j = I_{\alpha_j} \setminus Q_j$, $j = 1, 2, \ldots, k$. From formulae (24), (25) and the properties of Lebesgue integrals it follows that the estimates $|c_{n,p}|$ in $\wp^{\vee}(\mathbf{b}, \boldsymbol{\alpha}; \mathcal{F})$ and $\mathcal{P}(\mathbf{b}, \alpha; \mathbf{I}_{\boldsymbol{\alpha}})$, $\mathbf{I}_{\boldsymbol{\alpha}} = (I_{\alpha_1}, \ldots, I_{\alpha_k})$, $\mathcal{F} = (F_1, \ldots, F_k)$, are the same.

E x a m p l e 10. Let $\boldsymbol{\alpha} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$, $\mathbf{b} = (b_1, b_2, b_3)$. Consider the sets $F_1 = \left\{\zeta = \mathrm{e}^{\mathrm{i}t} \colon t \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{3}{4}\pi, \pi\right)\right\}$, $F_2 = \left\{\zeta = \mathrm{e}^{\mathrm{i}t} \colon t \in (-\pi, 0)\right\}$, $F_3 = \left\{\zeta = \mathrm{e}^{\mathrm{i}t} \colon t \in \left(\frac{\pi}{4}, \frac{3}{4}\pi\right)\right\}$. Let $\mathcal{F} = (F_1, F_2, F_3)$ and $p \in \wp^{\vee}(\mathbf{b}, \boldsymbol{\alpha}; \mathcal{F})$. Again, using (24), (25), after suitable computations we obtain

$$|c_{n,p}| \leqslant 2\Bigl(1 - \sum_{j=1}^{k} b_j \alpha_j\Bigr) + \begin{cases} 0 & \text{if } n = 8k \vee n = 8k+4, \\ \dfrac{1}{n\pi} \bigl|\bigl(\sqrt{2}-2\bigr)b_1 + 2b_2 - \sqrt{2}b_3\bigr| & \text{if } n = 8k+1 \vee n = 8k+7, \\ \dfrac{2}{n\pi} |b_1 - b_3| & \text{if } n = 8k+1 \vee n = 8k+6, \\ \dfrac{1}{n\pi} |2b_2 - b_1| & \text{if } n = 8k+3 \vee n = 8k+5, \end{cases}$$

R e m a r k 11. It is easy to note, that if $\mathcal{F}$ is essentially different from three arcs of the circle T (that means—it is neither a triple of arcs of T nor a triple of arcs with sets of Lebesgue measure zero omitted) then, for instance, the estimate of coefficients are more complicated.

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