OSCILLATION CRITERIA FOR SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. Our aim in this paper is to present sufficient conditions for the oscillation of the second order neutral differential equation

$$\big(x(t)-px(t-\tau)\big)''+q(t)x\big(\sigma(t)\big)=0.$$

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In this paper we deal with the oscillatory and asymptotic behavior of the solutions of the neutral differential equation. We consider the second order differential equation of the form

$$\big(x(t)-px(t-\tau)\big)''+q(t)x\big(\sigma(t)\big)=0 \tag{1}$$

under the assumptions

(i) p and τ are positive numbers;

(ii) q, $\sigma\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\lim\limits_{t\to\infty}\sigma(t)=\infty$, $\sigma(t)\leqslant t$.

We put $z(t)=x(t)-px(t-\tau)$. By a proper solution of Eq. (1) we mean a function $x\colon\ [T_x,\infty)\to\mathbb{R}$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)|\colon\ t\geqslant T\}>0$ for any $T\geqslant T_x$ so that $z(t)$ is twice continuously differentiable. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

In the recent years there has been a growing interest in oscillation theory of functional differential equations of neutral type, see for example the papers [1–11] and the references cited therein. The recent books by D. D. Bainov and D. P. Mishev, by I. Győri and G. Ladas and by L. H. Erbe, Q. Kong and B. G. Zhang summarize some important work in this area and reflect the overall new developments in the theory of neutral equations.

The purpose of this paper is to establish criteria for oscillation problems of Eq. (1). We discuss Eq. (1) for the cases that $0 < p < 1$, $p = 1$ and $p > 1$, respectively.

In this paper we have been motivated by the fact that the existing criteria for oscillation problems of Eq. (1) in general do not contain the constant p explicitly (see e.g. [3–5, 7, 8]). Further, we introduce some new techniques that yield a generalization of some criteria presented in the book by L. H. Erbe, Q. Kong and B. G. Zhang.

Theorem 1. *Let $0 < p < 1$. Assume that*

$$\sigma \in C^1, \qquad \sigma'(t) \geqslant 0. \tag{2}$$

Further assume that there exists an integer $n \geqslant 0$ such that

$$\int^{\infty} \Bigl(q(s)\sigma(s)\frac{1-p^{n+1}}{1-p} - \frac{\sigma'(s)}{4\sigma(s)} \Bigr)\, \mathrm{d}s = \infty. \tag{3}$$

Then the nonoscillatory solutions of Eq. (1) tend to zero as $t \to \infty$.

P r o o f. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (1) and define

$$z(t) = x(t) - px(t-\tau). \tag{4}$$

From Eq. (1) we have $z''(t) < 0$ for all large t, say $t \geqslant t_0$. If $z'(t) < 0$ eventually, then $\lim\limits_{t\to\infty} z(t) = -\infty$. But $z(t) < 0$ eventually implies that

$$x(t) < px(t-\tau) < p^2 x(t-2\tau) < \ldots < p^n x(t-n\tau)$$

for all large t, which implies $\lim\limits_{t\to\infty} x(t) = 0$. Consequently, $\lim\limits_{t\to\infty} z(t) = -\infty$, a contradiction.

Therefore, $z'(t) > 0$ for $t \geqslant t_0$. There are two possibilities for $z(t)$:

(a) $z(t) > 0$ for $t \geqslant t_1 \geqslant t_0$,
(b) $z(t) < 0$ for $t \geqslant t_1$.

For case (a), Eq. (1) can be written in the form

$$z''(t) + q(t)x\bigl(\sigma(t)\bigr) = 0.$$

Using (4) we have

$$z''(t) + q(t)z\big(\sigma(t)\big) + pq(t)x\big(\sigma(t) - \tau\big) = 0.$$

Repeating this procedure we arrive at

$$z''(t) + q(t)\sum_{i=0}^{n} p^i z\big(\sigma(t) - i\tau\big) + p^{n+1}q(t)x\big(\sigma(t) - (n+1)\tau\big) = 0. \tag{5}$$

Denote $a_n(t) = \sum\limits_{i=0}^{n} p^i z\big(\sigma(t) - i\tau\big)$. Then

$$z''(t) + a_n(t)q(t) \leqslant 0. \tag{6}$$

Define

$$v(t) = \frac{\sigma(t)\sum\limits_{i=0}^{n} p^i}{a_n(t)} z'(t), \qquad t \geqslant t_1. \tag{7}$$

Then $v(t) > 0$. Observe that

$$v'(t) = \frac{\sigma'(t)\sum\limits_{i=0}^{n} p^i}{a_n(t)} z'(t) + \frac{\sigma(t)\sum\limits_{i=0}^{n} p^i}{a_n(t)} z''(t) - \frac{\sigma(t)\sum\limits_{i=0}^{n} p^i}{a_n(t)} z'(t)\, \frac{\sigma'(t)\sum\limits_{i=0}^{n} p^i z'\big(\sigma(t) - i\tau\big)}{a_n(t)}.$$

Since $z'(t)$ is decreasing, one gets that $z'\big(\sigma(t) - i\tau\big) \geqslant z'\big(\sigma(t)\big)$ and therefore in view of (6) and (7)

$$v'(t) \leqslant \frac{\sigma'(t)}{\sigma(t)}\big(v(t) - v^2(t)\big) - q(t)\sigma(t)\sum_{i=0}^{n} p^i.$$

It is easy to see that the polynomial P satisfies $P(v) = v - v^2 \leqslant \frac{1}{4}$. Thus

$$v'(t) \leqslant \frac{\sigma'(t)}{4\sigma(t)} - q(t)\sigma(t)\sum_{i=0}^{n} p^i.$$

Then integrating the last inequality from t_1 to t, we are led to

$$v(t) \leqslant v(t_1) - \int_{t_1}^{t} \Big(q(s)\sigma(s)\frac{1 - p^{n+1}}{1 - p} - \frac{\sigma'(s)}{4\sigma(s)}\Big)\, \mathrm{d}s.$$

Letting $t \to \infty$ we have in view of (3) that $v(t) \to -\infty$, a contradiction.

For case (b), as mentioned before, we are led to $\lim\limits_{t\to\infty} x(t) = 0$. □

Corollary 1. *Assume that $0 < p < 1$ and (2) holds. Let*

$$\liminf_{t\to\infty} \frac{q(t)\sigma^2(t)}{\sigma'(t)} > \frac{1-p}{4}. \tag{8}$$

Then the nonoscillatory solutions of Eq. (1) tend to zero as $t \to \infty$.

P r o o f. Denote $a = \liminf\limits_{t\to\infty} \dfrac{q(t)\sigma^2(t)}{\sigma'(t)}$. Let an integer n be chosen such that

$$a - \varepsilon > \frac{1-p}{4\bigl(1-p^{n+1}\bigr)},$$

where $\varepsilon > 0$ is small enough. Then there exists a t_1 (large enough) that

$$\frac{q(t)\sigma^2(t)}{\sigma'(t)} - \frac{1-p}{4\bigl(1-p^{n+1}\bigr)} > \varepsilon, \qquad t \geqslant t_1. \tag{9}$$

Noting that (9) implies (3) we complete the proof.

□

E x a m p l e 1. Consider the neutral equation

$$\bigl(x(t) - px(t-\tau)\bigr)'' + \frac{1}{t\sqrt{t}} x\bigl(\sqrt{t}\bigr) = 0, \qquad p \in (0,1), \quad \tau > 0. \tag{10}$$

Condition (8) for Eq. (10) reduces to $8 \geqslant (1-p)$ and therefore the nonoscillatory solutions of Eq. (1) tend to zero as $t \to \infty$. On the other hand Lemma 4.4.2 and Theorem 4.4.1 in [2] fail for Eq. (10).

The conclusion of Theorem 1 can be strengthened as follows.

Theorem 2. *In addition to the condition of Theorem 1, assume that there exists an integer $k \geqslant 1$ such that $t - \sigma(t) > k\tau$ and*

$$\limsup_{t\to\infty} \int_{\sigma(t)+k\tau}^{t} \Bigl(s - \sigma(s) - k\tau\Bigr) q(s)\,\mathrm{d}s > \frac{(1-p)p^k}{1-p^k}. \tag{11}$$

Then every solution of Eq. (1) is oscillatory.

P r o o f. Taking the proof of Theorem 1 into account, it is sufficient to show that $z(t) < 0$ is impossible for $t \geqslant t_1$ under the assumptions. On the contrary, suppose that $x(t) > 0$, $z''(t) \leqslant 0$, $z'(t) > 0$ and $z(t) < 0$ eventually. We rewrite Eq. (1) as

$$z''(t) - \frac{1}{p} q(t) z\bigl(\sigma(t)+\tau\bigr) + \frac{q(t)}{p} x\bigl(\sigma(t)+\tau\bigr) = 0.$$

Reiterating this process we are led to

$$z''(t)-q(t)\sum_{i=1}^{k}\frac{z\big(\sigma(t)+i\tau\big)}{p^i}+\frac{q(t)}{p^k}x\big(\sigma(t)+k\tau\big)=0$$

for all large t. Then using the monotonicity of $z(t)$ one gets

$$z''(t)-z\big(\sigma(t)+k\tau\big)q(t)\sum_{i=1}^{k}\frac{1}{p^i}\leqslant 0. \tag{12}$$

Integrating (12) from s to t for $t>s$ we have

$$z'(t)-z'(s)-\frac{1-p^k}{(1-p)p^k}\int_s^t q(u)z\big(\sigma(u)+k\tau\big)\,\mathrm{d}u\leqslant 0. \tag{13}$$

Integrating (13) in s from $\sigma(t)+k\tau$ to t, we have

$$\begin{aligned}z'(t)\big(t-\sigma(t)-k\tau\big)&\leqslant z(t)-z(\sigma(t)+k\tau)\\&\quad+\frac{1-p^k}{(1-p)p^k}\int_{\sigma(t)+k\tau}^{t}(s-\sigma(s)-k\tau)q(s)z\big(\sigma(s)+k\tau\big)\,\mathrm{d}s.\end{aligned}$$

Hence using the monotonicity of $z(t)$ one gets

$$z'(t)\big(t-\sigma(t)-k\tau\big)\leqslant z(\sigma(t)+k\tau)\left\{\frac{1-p^k}{(1-p)p^k}\int_{\sigma(t)+k\tau}^{t}(s-\sigma(s)-k\tau)q(s)\,\mathrm{d}s-1\right\}.$$

Therefore if (11) holds, then we arrive at a contradiction with the sign properties of $z(t)$ and $z'(t)$. The proof is complete.

□

R e m a r k 1. For the special case when $k=1$ in (10) we obtain the results presented in [2, Theorem 4.4.1].

Now we give an analogue of Theorems 1 and 2 for the case $p=1$.

Theorem 3. *Assume that $p=1$ and (2) holds. Then the nonoscillatory solutions of Eq. (1) are bounded provided there exists an integer $n\geqslant 1$ such that*

$$\int^{\infty}\Big(n\sigma(s)q(s)-\frac{\sigma'(s)}{4\sigma(s)}\Big)\,\mathrm{d}s=\infty. \tag{14}$$

P r o o f. Let $x(t)$ be an eventually positive solution of Eq. (1) and $z(t)=x(t)-x(t-\tau)$. Then $z''(t)<0$ for $t\geqslant t_0$. If $z'(t)<0$ eventually, then we have $\lim\limits_{t\to\infty} z(t)=-\infty$. Therefore

$$x(t)\leqslant x(t-\tau)\quad \text{for all large } t,$$

which implies that $x(t)$ is bounded, a contradiction. Therefore, $z'(t)>0$ for $t\geqslant t_1$. Then either $z(t)>0$ or $z(t)<0$. Assume that $z(t)>0$. We rewrite Eq. (1) as

$$z''(t)+q(t)\sum_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)+q(t)x\bigl(\sigma(t)-(n+1)\tau\bigr)=0.$$

Thus

$$z''(t)+q(t)\sum_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)\leqslant 0. \tag{15}$$

Define

$$v(t)=\frac{n\sigma(t)}{\sum\limits_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)}z'(t).$$

Then $v(t)>0$ and

$$\begin{aligned}v'(t)=\frac{n\sigma'(t)}{\sum\limits_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)}z'(t)&+\frac{n\sigma(t)}{\sum\limits_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)}z''(t)\\ &-\frac{n\sigma(t)}{\sum\limits_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)}z'(t)\,\frac{\sigma'(t)\sum\limits_{i=0}^{n} z'\bigl(\sigma(t)-i\tau\bigr)}{\sum\limits_{i=0}^{n} z\bigl(\sigma(t)-i\tau\bigr)}.\end{aligned}$$

Since $z'\bigl(\sigma(t)-i\tau\bigr)\geqslant z'\bigl(\sigma(t)\bigr)$ then in view of (15)

$$v'(t)\leqslant\frac{\sigma'(t)}{\sigma(t)}\bigl(v(t)-v^2(t)\bigr)-nq(t)\sigma(t)\leqslant\frac{\sigma'(t)}{4\sigma(t)}-nq(t)\sigma(t).$$

Integrating the last inequality from t_1 to t, we have

$$v(t)\leqslant v(t_1)-\int_{t_1}^{t}\Bigl(nq(s)\sigma(s)-\frac{\sigma'(s)}{4\sigma(s)}\Bigr)\,\mathrm{d}s,$$

which implies $v(t)\to-\infty$. This is a contradiction.

Assume that $z(t)<0$ for $t\geqslant t_1$. Then $x(t)<x(t-\tau)$, which implies that $x(t)$ is bounded. □

Corollary 2. *Assume that $p = 1$. Let*

$$\liminf_{t\to\infty} \frac{q(t)\sigma^2(t)}{\sigma'(t)} > 0. \tag{16}$$

Then the nonoscillatory solutions of Eq. (1) are bounded.

Proof. Denote $a = \liminf\limits_{t\to\infty} \dfrac{4q(t)\sigma^2(t)}{\sigma'(t)}$. Let an integer n be chosen such that $a - \varepsilon > \frac{1}{n}$, where $\varepsilon > 0$ is small enough. Then there exists a t_1 (large enough) such that

$$\frac{4q(t)\sigma^2(t)}{\sigma'(t)} - \frac{1}{n} > \varepsilon \qquad t \geqslant t_1. \tag{17}$$

Note that (17) implies (16). The proof is complete.

□

Theorem 4. *In addition to the condition of Theorem 3, assume that there exists an integer $k \geqslant 1$ such that $t - \sigma(t) > k\tau$ and*

$$\limsup_{t\to\infty} \int_{\sigma(t)+k\tau}^{t} \big(s - \sigma(s) - k\tau\big) q(s)\,\mathrm{d}s > \frac{1}{k}. \tag{18}$$

Then every solution of Eq. (1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1) and $z(t) = x(t) - x(t-\tau)$. Taking the proof of Theorem 3 into account, it is sufficient to show that $z(t) < 0$ is impossible. On the contrary, suppose that $x(t) > 0$, $z''(t) \leqslant 0$, $z'(t) > 0$ and $z(t) < 0$ eventually. For (1) one gets

$$z''(t) - q(t)\sum_{i=1}^{k} z\big(\sigma(t) + i\tau\big) + q(t)x\big(\sigma(t) + k\tau\big) = 0.$$

Thus

$$z''(t) - kq(t)z\big(\sigma(t) + k\tau\big) \leqslant 0.$$

Exactly as in the proof of Theorem 2 we integrate twice the last inequality to have

$$\begin{aligned} z'(t)\big(t - \sigma(t) - k\tau\big) &\leqslant z(t) - z(\sigma(t) + k\tau) \\ &\quad + k\int_{\sigma(t)+k\tau}^{t} (s - \sigma(s) - k\tau)q(s)z\big(\sigma(s) + k\tau\big)\,\mathrm{d}s. \end{aligned}$$

Hence using monotonicity $z(t)$ we obtain

$$z'(t)\big(t-\sigma(t)-k\tau\big) \leqslant z(\sigma(t)+k\tau)\left\{k\int_{\sigma(t)+k\tau}^{t}(s-\sigma(s)-k\tau)q(s)\,\mathrm{d}s-1\right\},$$

which contradicts the sign properties of $z(t)$ and $z'(t)$. The proof is complete.

□

Theorem 5. *Assume that $p>1$ and (2) holds. Further assume that there exists an integer $n\geqslant 0$ such that*

$$\int^{\infty}\Big(q(s)\sigma(s)\frac{p^{n+1}-1}{p-1}-\frac{\sigma'(s)}{4\sigma(s)}\Big)\,\mathrm{d}s=\infty.$$

Then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies $x(t)<px(t-\tau)$.

P r o o f. Let $x(t)$ be an eventually positive solution of Eq. (1) and set $z(t) = x(t)-px(t-\tau)$. Then $z''(t)<0$ eventually. There are three possibilities:

(i) $z'(t)>0$, $z(t)>0$,
(ii) $z'(t)>0$, $z(t)<0$,
(iii) $z'(t)<0$, $z(t)<0$.

For case (i), the proof runs exactly as in the proof Theorem 1 and so it can be omitted. For cases (ii) and (iii) we have assumed that $z(t)<0$, then $x(t)<px(t-\tau)$ is obvious. □

Corollary 3. *Assume that $p>1$ and (2) holds. Let*

$$\liminf_{t\to\infty}\frac{q(t)\sigma^2(t)}{\sigma'(t)}>0.$$

Then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies $x(t)<px(t-\tau)$.

R e m a r k 2. When considering more general neutral differential equations with function $p(t)$ instead of a constant p,

$$\big(x(t)-p(t)x(t-\tau)\big)''+q(t)x\big(\sigma(t)\big)=0, \tag{19}$$

then it is usual to impose the condition $p_1<p(t)<p_2$ on the function $p(t)$. From the proofs of the abovementioned results one can see that the technique presented in this paper can be applied to Eq. (19).

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