n-INNER PRODUCT SPACES AND PROJECTIONS

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Abstract. This paper is a continuation of investigations of n-inner product spaces given in [5, 6, 7] and an extension of results given in [3] to arbitrary natural n. It concerns families of projections of a given linear space L onto its n-dimensional subspaces and shows that between these families and n-inner products there exist interesting close relations.

Keywords: n-inner product space, n-normed space, n-norm of projection

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1. n-inner products and n-norms

1.1. Let n be a natural number ($n \neq 0$), L a linear space with $\dim L \geqslant n$ and let $(\cdot,\cdot \mid \cdot,\dots,\cdot)$ be a real function on $L^{n+1} = \underbrace{L \times \dots \times L}_{n+1 \text{ times}}$.

In the case $n = 1$, we also write $(\cdot,\cdot)$ instead of $(\cdot,\cdot \mid \cdot,\dots,\cdot)$ and $(a, b \mid a_2,\dots,a_n)$ is to be understood as the expression (a, b). Let us assume the following conditions:

1. $(a, b \mid a_2,\dots,a_n) \geqslant 0$,
   $(a, a \mid a_2,\dots,a_n) = 0$ if and only if $a, a_2,\dots,a_n$ are linearly dependent,
2. $(a, b \mid a_2,\dots,a_n) = (b, a \mid a_2,\dots,a_n)$,
3. $(a, b \mid a_2,\dots,a_n) = (a, b \mid a_{i_2},\dots,a_{i_n})$ for every permutation $(i_2,\dots,i_n)$ of $(2,\dots,n)$,
4. if $n > 1$, then $(a, a \mid a_2, a_3,\dots,a_n) = (a_2, a_2 \mid a, a_3,\dots,a_n)$,
5. $(\alpha\, a, b \mid a_2,\dots,a_n) = \alpha\,(a, b \mid a_2,\dots,a_n)$ for every real α,
6. $(a + b, c \mid a_2,\dots,a_n) = (a, c \mid a_2,\dots,a_n) + (b, c \mid a_2,\dots,a_n)$.

Then $(\cdot,\cdot \mid \cdot,\dots,\cdot)$ is called an *n-inner product* on L (see [5]) and $\bigl(L, (\cdot,\cdot \mid \cdot,\dots,\cdot)\bigr)$ is called an *n-inner product space*. The concept of an n-inner product space is a generalization of the concepts of an inner product space ($n = 1$) and of a 2-inner product space (see [1]).

1.2. Let $n > 1$. An n-inner product space L and its n-inner product $(\cdot,\cdot \mid \cdot,\dots,\cdot)$ are called *simple* if there exists an inner product $(\cdot,\cdot)$ on L such that the relation

$$\big(a,b \mid a_2,\dots,a_n\big) = \begin{vmatrix} (a,b) & (a,a_2) & \dots & (a,a_n) \\ (a_2,b) & (a_2,a_2) & \dots & (a_2,a_n) \\ \vdots & \vdots & \ddots & \vdots \\ (a_n,b) & (a_n,a_2) & \dots & (a_n,a_n) \end{vmatrix}$$

holds. The inner product $(\cdot,\cdot)$ is said to generate the n-inner product $(\cdot,\cdot \mid \cdot,\dots,\cdot)$.

An element $a \in L$ is said to be orthogonal to a non-empty subset S of L if $(a, e_1 \mid e_2,\dots,e_n) = 0$ for arbitrary $e_1, \dots, e_n \in S$. A subset S of L is said to be orthogonal if it is linearly independent, contains at least n elements and if every $e \in S$ is orthogonal to $S \setminus \{e\}$.

1.3. An *n-norm* on L is a real function $\|\cdot,\dots,\cdot\|$ on L^n which satisfies the following conditions:

1. $\|a_1,\dots,a_n\| = 0$ if and only if $a_1, \dots, a_n$ are linearly dependent,
2. $\|a_1,\dots,a_n\| = \|a_{i_1},\dots,a_{i_n}\|$ for every permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$,
3. $\|\alpha a_1, a_2,\dots,a_n\| = |\alpha|\,\|a_1,a_2,\dots,a_n\|$ for every real number α,
4. $\|a+b, a_2,\dots,a_n\| \leqslant \|a,a_2,\dots,a_n\| + \|b,a_2,\dots,a_n\|$.

L equipped with an n-norm $\|\cdot,\dots,\cdot\|$ is called an *n-normed space*. The concept of an n-normed space is a generalization of the concepts of a normed ($n = 1$) and a 2-normed space (see [2]).

Theorem 1. (Theorem 7 of [5]) *For every n-inner product $(\cdot,\cdot \mid \cdot,\dots,\cdot)$ on L,*

$$\big\|a_1,a_2,\dots,a_n\big\| = \sqrt{(a_1,a_1 \mid a_2,\dots,a_n)} \tag{1}$$

defines an n-norm on L for which

$$\big(a,b \mid a_2,\dots,a_n\big) = \tfrac{1}{4}\big(\|a+b,a_2,\dots,a_n\|^2 - \|a-b,a_2,\dots,a_n\|^2\big) \tag{2}$$

and

$$\|a+b,a_2,\dots,a_n\|^2 + \|a-b,a_2,\dots,a_n\|^2 = 2\big(\|a,a_2,\dots,a_n\|^2 + \|b,a_2,\dots,a_n\|^2\big) \tag{3}$$

are true.

Conversely, for every n-norm $\|\cdot,\dots,\cdot\|$ on L with the property (3), (2) *defines an n-inner product on L for which* (1) *is true.*

For every n-inner product $(\cdot,\cdot \mid \cdot,\dots,\cdot)$ on L the n-norm given by (1) is said to be associated to $(\cdot,\cdot \mid \cdot,\dots,\cdot)$. If in connection with an n-inner product on L an n-norm is used, then $\|\cdot,\dots,\cdot\|$ always will be the n-norm associated to $(\cdot,\cdot \mid \cdot,\dots,\cdot)$.

2.1. Let $\big(L, (\cdot,\cdot \mid \cdot,\dots,\cdot)\big)$ be an n-inner product space. For arbitrary linearly independent points $a_1, \dots, a_n \in L$, let $\mathrm{pr}_{a_1,\dots,a_n}$ be the mapping of L into L given by

$$\mathrm{pr}_{a_1,\dots,a_n}(c) = \frac{(c, a_1 \mid a_2, \dots, a_n)}{\|a_1,\dots,a_n\|^2}\, a_1 + \dots + \frac{(c, a_n \mid a_1, \dots, a_{n-1})}{\|a_1,\dots,a_n\|^2}\, a_n$$

(see [3], where $n = 2$). We often use the notion

$$\big(c, a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n\big) = \big(c, a_k \mid a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n\big)$$

and

$$\mathrm{pr}_{a_1,\dots,\underline{a_k},\dots,a_n}(c) = \frac{(c, a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n)}{\|a_1,\dots,a_n\|^2}.$$

Then we have

$$\begin{aligned}\mathrm{pr}_{a_1,\dots,a_n}(c) &= \sum_{k=1}^{n} \frac{(c, a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n)}{\|a_1,\dots,a_n\|^2}\, a_k \\ &= \sum_{k=1}^{n} \mathrm{pr}_{a_1,\dots,\underline{a_k},\dots,a_n}(c)\, a_k.\end{aligned}$$

Theorem 2. $\mathrm{pr}_{a_1,\dots,a_n}$ *is a projection of* L *onto* $L\big(\{a_1,\dots,a_n\}\big)$, *the linear space generated by the set* $\{a_1,\dots,a_n\}$.

Proof. Obviously $\mathrm{pr}_{a_1,\dots,a_n}$ is linear. Since $\mathrm{pr}_{a_1,\dots,a_n}(a_k) = a_k$ for arbitrary $k \in \{1,\dots,n\}$, $\mathrm{pr}_{a_1,\dots,a_n}$ maps L onto $L(\{a_1,\dots,a_n\})$. Moreover,

$$\mathrm{pr}^2_{a_1,\dots,a_n}(c) = \sum_{k=1}^{n} \frac{(\mathrm{pr}_{a_1,\dots,a_n}(c), a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n)}{\|a_1,\dots,a_n\|^2}\, a_k$$

from which by virtue of

$$\begin{aligned}&\frac{(\mathrm{pr}_{a_1,\dots,a_n}(c), a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n)}{\|a_1,\dots,a_n\|^2} \\ &\qquad = \sum_{l=1}^{n} \frac{(c, a_l \mid a_1, \dots, \widehat{a_l}, \dots, a_n)\,(a_l, a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n)}{\|a_1,\dots,a_n\|^4} \\ &\qquad = \frac{(c, a_k \mid a_1, \dots, \widehat{a_k}, \dots, a_n)}{\|a_1,\dots,a_n\|^2}\end{aligned}$$

we get

$$\mathrm{pr}^2_{a_1,\dots,a_n}(c) = \mathrm{pr}_{a_1,\dots,a_n}(c).$$

□

Theorem 3. $\mathrm{pr}_{a_1,\dots,a_n}$ *is independent of the special choice of* $a_1, \dots, a_n$ *in* $L(\{a_1,\dots,a_n\})$*; this means, for arbitrary linearly independent points* $a'_i = \sum_{k=1}^{n} \alpha_{i,k}\, a_k$, $i = 1, \dots, n$, *we have*

$$\mathrm{pr}_{a'_1,\dots,a'_n} = \mathrm{pr}_{a_1,\dots,a_n}.$$

Proof. Let linearly independent points $a'_i = \sum_{k=1}^{n} \alpha_{i,k}\, a_k$, $i = 1, \dots, n$ be given. Then

$$\begin{vmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix} \neq 0.$$

For arbitrary $c \in L$,

$$\mathrm{pr}_{a'_1,\dots,a'_n}(c) = \sum_{i,\,l=1}^{n} \alpha_{i,l} \frac{\Bigl(c, \sum_{k=1}^{n} \alpha_{i,k}\, a_k \Bigm| \sum_{k=1}^{n} \alpha_{1,k}\, a_k, \dots, \widehat{\sum_{k=1}^{n} \alpha_{i,k} a_k}, \dots, \sum_{k=1}^{n} \alpha_{n,k}\, a_k\Bigr)}{\Bigl\|\sum_{k=1}^{n} \alpha_{1,k}\, a_k, \dots, \sum_{k=1}^{n} \alpha_{n,k}\, a_k\Bigr\|^2}\, a_l.$$

Using the notion $\sum'$, which means that summation is taken only with respect to different indices, formula (8) in Theorem 6 of [6] implies that

$$\begin{aligned}
&\sum_{i=1}^{n} \alpha_{i,l} \Bigl(c, \sum_{k=1}^{n} \alpha_{i,k}\, a_k \Bigm| \sum_{k=1}^{n} \alpha_{1,k}\, a_k, \dots, \widehat{\sum_{k=1}^{n} \alpha_{i,k} a_k}, \dots, \sum_{k=1}^{n} \alpha_{n,k}\, a_k\Bigr) \\
&\quad = \sum_{i=1}^{n} \alpha_{i,l} \sideset{}{'}\sum_{j,\,k_2<\dots<k_n} \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & \alpha_{1,k_2} & \dots & \alpha_{1,k_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{i-1,k_2} & \dots & \alpha_{i-1,k_n} \\ 0 & \alpha_{i+1,k_1} & \dots & \alpha_{i+1,k_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{n,k_2} & \dots & \alpha_{n,k_n} \end{vmatrix} \begin{vmatrix} \alpha_{i,j} & \alpha_{i,k_2} & \dots & \alpha_{i,k_n} \\ \alpha_{1,j} & \alpha_{1,k_2} & \dots & \alpha_{1,k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i-1,j} & \alpha_{i-1,k_2} & \dots & \alpha_{i-1,k_n} \\ \alpha_{i+1,j} & \alpha_{i+1,k_2} & \dots & \alpha_{i+1,k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,j} & \alpha_{n,k_2} & \dots & \alpha_{n,k_n} \end{vmatrix} \\
&\qquad \times \bigl(c, a_j \mid a_{k_2}, \dots, a_{k_n}\bigr) \\
&\quad = \sideset{}{'}\sum_{j,\,k_2<\dots<k_n} \begin{vmatrix} \alpha_{1,l} & \alpha_{1,k_2} & \dots & \alpha_{1,k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,l} & \alpha_{n,k_2} & \dots & \alpha_{n,k_n} \end{vmatrix} \begin{vmatrix} \alpha_{1,j} & \alpha_{1,k_2} & \dots & \alpha_{1,k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,j} & \alpha_{n,k_2} & \dots & \alpha_{n,k_n} \end{vmatrix} \bigl(c, a_j \mid a_{k_2}, \dots, a_{k_n}\bigr) \\
&\quad = \begin{vmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix}^2 \bigl(c, a_l \mid a_1, \dots, \widehat{a}_l, \dots, a_n\bigr)
\end{aligned}$$

and

$$\left\|\sum_{k=1}^{n}\alpha_{1,k}a_k,\dots,\sum_{k=1}^{n}\alpha_{n,k}a_k\right\|^2=\begin{vmatrix}\alpha_{1,1} & \dots & \alpha_{1,n}\\ \vdots & \ddots & \vdots\\ \alpha_{n,1} & \dots & \alpha_{n,n}\end{vmatrix}^2\|a_1,\dots,a_n\|^2.$$

This yields that

$$\mathrm{pr}_{a'_1,\dots,a'_n}(c)=\sum_{l=1}^{n}\frac{(c,a_l\mid a_1,\dots,\widehat{a_l},\dots,a_n)}{\|a_1,\dots,a_n\|^2}\,a_l=\mathrm{pr}_{a_1,\dots,a_n}(c)$$

which proves the theorem. □

Theorem 4. *For arbitrary* $c\in L$, $c-\mathrm{pr}_{a_1,\dots,a_n}(c)$ *is orthogonal to* $L(\{a_1,\dots,a_n\})$.

P r o o f. For arbitrary $a'_i=\sum\limits_{k=1}^{n}\alpha_{i,k}\,a_k$, $i=1,\dots,n$, by means of (8) in Theorem 6 (see [6]) we get

$$\begin{aligned}
&\left(c-\mathrm{pr}_{a_1,\dots,a_n}(c),\sum_{k=1}^{n}\alpha_{1,k}\,a_k\;\middle|\;\sum_{k=1}^{n}\alpha_{2,k}\,a_k,\dots,\sum_{k=1}^{n}\alpha_{n,k}\,a_k\right)\\
&\quad=\left(c,\sum_{k=1}^{n}\alpha_{1,k}\,a_k\;\middle|\;\sum_{k=1}^{n}\alpha_{2,k}\,a_k,\dots,\sum_{k=1}^{n}\alpha_{n,k}\,a_k\right)\\
&\qquad-\left(\sum_{k=1}^{n}\frac{(c,a_k\mid a_1,\dots,\widehat{a_k},\dots,a_n)}{\|a_1,\dots,a_n\|^2}\,a_k,\sum_{k=1}^{n}\alpha_{1,k}\,a_k\;\middle|\;\sum_{k=1}^{n}\alpha_{2,k}\,a_k,\dots,\sum_{k=1}^{n}\alpha_{n,k}\,a_k\right)\\
&\quad=\sum_{j,k_2<\dots<k_n}{}^{'}\begin{vmatrix}1 & 0 & \dots & 0\\ 0 & \alpha_{2,k_2} & \dots & \alpha_{2,k_n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \alpha_{n,k_2} & \dots & \alpha_{n,k_n}\end{vmatrix}\begin{vmatrix}\alpha_{1,j} & \alpha_{1,k_2} & \dots & \alpha_{1,k_n}\\ \alpha_{2,j} & \alpha_{2,k_2} & \dots & \alpha_{2,k_n}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_{n,j} & \alpha_{n,k_2} & \dots & \alpha_{n,k_n}\end{vmatrix}(c,a_j\mid a_{k_2},\dots,a_{k_n})\\
&\qquad-\begin{vmatrix}(c,a_1|a_2,\dots,a_n) & \dots & (c,a_n|a_1,\dots,a_{n-1})\\ \alpha_{2,1} & \dots & \alpha_{2,n}\\ \vdots & \ddots & \vdots\\ \alpha_{n,1} & \dots & \alpha_{n,n}\end{vmatrix}\begin{vmatrix}\alpha_{1,1} & \dots & \alpha_{1,n}\\ \alpha_{2,1} & \dots & \alpha_{2,n}\\ \vdots & \ddots & \vdots\\ \alpha_{n,1} & \dots & \alpha_{n,n}\end{vmatrix}\\
&\quad=0.
\end{aligned}$$

This was to be proved. □

2.2. From Theorem 2 of [7] we know the following: if $(\cdot,\cdot\mid\cdot,\dots,\cdot)$ is a simple n-inner product on L and $(\cdot,\cdot)$ generates $(\cdot,\cdot\mid\cdot,\dots,\cdot)$, then for arbitrary $a\in L$ and arbitrary $S \subset L$ which generates a linear subspace of L of dimension $\geqslant n$, a is orthogonal to S relative to $(\cdot,\cdot \mid \cdot,\ldots,\cdot)$ if and only if a is orthogonal to S relative to $(\cdot,\cdot)$. From this and Theorem 4 it follows that if $(\cdot,\cdot \mid \cdot,\ldots,\cdot)$ is simple and $(\cdot,\cdot)$ is an inner product on L generating $(\cdot,\cdot \mid \cdot,\ldots,\cdot)$, then for arbitrary $c \in L$, $c - \mathrm{pr}_{a_1,\ldots,a_n}(c)$ is orthogonal to $L(\{a_1,\ldots,a_n\})$ relative to $(\cdot,\cdot)$.

2.3. From Theorem 3 of [6] we know that if S is an orthogonal set in L, for every $e \in S$, distinct $e_2, \ldots, e_n \in S \setminus \{e\}$, distinct $e'_2, \ldots, e'_n \in S \setminus \{e\}$ and every c from the linear space generated by S, we have

$$\frac{(c,e \mid e_2,\ldots,e_n)}{\|e,e_2,\ldots,e_n\|^2} = \frac{(c,e \mid e'_2,\ldots,e'_n)}{\|e,e'_2,\ldots,e'_n\|^2},$$

which implies $\mathrm{pr}_{\underline{e},e_2,\ldots,e_n}(c) = \mathrm{pr}_{\underline{e},e'_2,\ldots,e'_n}(c)$. This means that under the above conditions the coordinate $\mathrm{pr}_{\underline{e},e_2,\ldots,e_n}(c)$ of $\mathrm{pr}_{e,e_2,\ldots,e_n}(c)$ is independent of $e_2,\ldots,e_n$.

For every n-dimensional linear subspace L' of L let $S_{L'}$ be the set of all subsets $\{a_1,\ldots,a_n\}$ of L' such that $\|a_1,\ldots,a_n\| = 1$. Then for arbitrary $\{a_1,\ldots,a_n\}$, $\{a'_1,\ldots,a'_n\} \in S_{L'}$ we have $a'_i = \sum\limits_{k=1}^{n} \alpha_{i,k}\, a_k$, $i = 1, \ldots, n$ with

$$\begin{vmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix} = \pm 1.$$

S is maximal in the sense that if $\{a_1,\ldots,a_n\} \in S_{L'}$, then for arbitrary points $a'_i = \sum\limits_{k=1}^{n} \alpha_{i,k}\, a_k$, $i = 1, \ldots, n$ with

$$\begin{vmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix} = \pm 1$$

we have $\{a'_1,\ldots,a'_n\} \in S_{L'}$.

From the proof of Theorem 4 we know that

$$\Big(c, \sum_{k=1}^{n} \alpha_{1,k}\, a_k \;\Big|\; \sum_{k=1}^{n} \alpha_{2,k}\, a_k, \ldots, \sum_{k=1}^{n} \alpha_{n,k}\, a_k\Big)$$
$$= \begin{vmatrix} \mathrm{pr}_{\underline{a_1},\ldots,a_n}(c) & \cdots & \mathrm{pr}_{a_1,\ldots,\underline{a_n}}(c) \\ \alpha_{2,1} & \cdots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix} \begin{vmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \cdots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix}$$

whenever $c \in L$ and $\{a_1, \dots, a_n\} \in S_{L'}$.

Theorem 5. *Let L' and L^+ be n-dimensional linear subspaces of L such that $\dim(L' \cap L^+) = n-1$ and let $\{a', a_2, \dots, a_n\} \in S_{L'}$ and $\{a^+, a_2, \dots, a_n\} \in S_{L^+}$. Then*

$$\mathrm{pr}_{\underline{a^+}, a_2, \dots, a_n}(a') = \mathrm{pr}_{\underline{a'}, a_2, \dots, a_n}(a^+).$$

Proof. Evident. □

3. Generation of n-inner products by means of families of projections

3.1. Let L be an arbitrary linear space of dimension $\geqslant n$. For every n-dimensional linear subspace L' of L let $S_{L'}$ be a maximal set of subsets $\{a_1, \dots, a_n\}$ of linearly independent points of L' such that for arbitrary $\{a_1, \dots, a_n\}$, $\{a'_1, \dots, a'_n\} \in S_{L'}$ we have $a'_i = \sum\limits_{k=1}^{n} \alpha_{i,k}\, a_k$, $i = 1, \dots, n$ with

$$\begin{vmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix} = \pm 1.$$

Moreover, let us assume the following:

1. For every n-dimensional linear subspace L' of L there is a projection $\mathrm{pr}_{L'}$ of L onto L' for which for every $\{a_1, \dots, a_n\} \in S_{L'}$ we also will use the notation

$$\mathrm{pr}_{a_1, \dots, a_n} = \sum_{k=1}^{n} \mathrm{pr}_{a_1, \dots, \underline{a_k}, \dots, a_n} a_k.$$

2. If L', L^+ are n-dimensional linear subspaces of L such that $\dim(L' \cap L^+) = n-1$ and if $\{a', a_2, \dots, a_n\} \in S_{L'}$ and $\{a^+, a_2, \dots, a_n\} \in S_{L^+}$ then

$$\mathrm{pr}_{\underline{a^+}, a_2, \dots, a_n}(a') = \mathrm{pr}_{\underline{a'}, a_2, \dots, a_n}(a^+). \tag{4}$$

Every n points $a'_1, \dots, a'_n$ of L can be written in the form $a'_i = \sum\limits_{k=1}^{n} \alpha_{i,k} a_k$, $i = 1, \dots, n$, by means of $\{a_1, \dots, a_n\} \in S_{L'}$ with a suitable L'. Let us define

$$(c, a'_1 \mid a'_2, \dots, a'_n) = \begin{vmatrix} \mathrm{pr}_{\underline{a_1}, \dots, a_n}(c) & \dots & \mathrm{pr}_{a_1, \dots, \underline{a_n}}(c) \\ \alpha_{2,1} & \dots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix} \begin{vmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \alpha_{2,1} & \dots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix}. \tag{5}$$

Theorem 6. *$(c, a'_1 \mid a'_2, \dots, a'_n)$ given by (5) is independent of the special choice of $\{a_1, \dots, a_n\}$.*

P r o o f. Let $\{a_1,\dots,a_n\}$, $\{\widetilde{a}_1,\dots,\widetilde{a}_n\}\in S_{L'}$ and $a_k=\sum\limits_{l=1}^{n}\widetilde{\alpha}_{k,l}\,\widetilde{a}_l$, $k=1,\dots,n$. Then

$$\begin{vmatrix} \widetilde{\alpha}_{1,1} & \dots & \widetilde{\alpha}_{1,n} \\ \vdots & \ddots & \vdots \\ \widetilde{\alpha}_{n,1} & \dots & \widetilde{\alpha}_{n,n} \end{vmatrix} = \pm 1$$

and $a'_i=\sum\limits_{k=1}^{n}\alpha_{i,k}\,a_k=\sum\limits_{k,l=1}^{n}\alpha_{i,k}\,\widetilde{\alpha}_{k,l}\,\widetilde{a}_l$, $i=1,\dots,n$. From

$$\begin{aligned}\sum_{l=1}^{n}\mathrm{pr}_{\widetilde{a}_1,\dots,\underline{\widetilde{a}_l},\dots,\widetilde{a}_n}(c)\,\widetilde{a}_l &= \sum_{k=1}^{n}\mathrm{pr}_{a_1,\dots,\underline{a_k},\dots,a_n}(c)\,a_k \\ &= \sum_{k,l=1}^{n}\mathrm{pr}_{a_1,\dots,\underline{a_k},\dots,a_n}(c)\,\widetilde{\alpha}_{k,l}\,\widetilde{a}_l\end{aligned}$$

we get $\mathrm{pr}_{\widetilde{a}_1,\dots,\underline{\widetilde{a}_l},\dots,\widetilde{a}_n}(c)=\sum\limits_{k=1}^{n}\mathrm{pr}_{a_1,\dots,\underline{a_k},\dots,a_n}(c)\,\widetilde{\alpha}_{k,l}$, $l=1,\dots,n$, and consequently

$$\begin{aligned}&\begin{vmatrix} \mathrm{pr}_{\underline{\widetilde{a}_1},\dots,\widetilde{a}_n}(c) & \dots & \mathrm{pr}_{\widetilde{a}_1,\dots,\underline{\widetilde{a}_n}}(c) \\ \sum\limits_{k=1}^{n}\alpha_{2,k}\widetilde{\alpha}_{k,1} & \dots & \sum\limits_{k=1}^{n}\alpha_{2,k}\widetilde{\alpha}_{k,n} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{n}\alpha_{n,k}\widetilde{\alpha}_{k,1} & \dots & \sum\limits_{k=1}^{n}\alpha_{n,k}\widetilde{\alpha}_{k,n} \end{vmatrix} \begin{vmatrix} \sum\limits_{k=1}^{n}\alpha_{1,k}\widetilde{\alpha}_{k,1} & \dots & \sum\limits_{k=1}^{n}\alpha_{1,k}\widetilde{\alpha}_{k,n} \\ \sum\limits_{k=1}^{n}\alpha_{2,k}\widetilde{\alpha}_{k,1} & \dots & \sum\limits_{k=1}^{n}\alpha_{2,k}\widetilde{\alpha}_{k,n} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{n}\alpha_{n,k}\widetilde{\alpha}_{k,1} & \dots & \sum\limits_{k=1}^{n}\alpha_{n,k}\widetilde{\alpha}_{k,n} \end{vmatrix} \\ &\qquad = \begin{vmatrix} \mathrm{pr}_{\underline{a_1},\dots,a_n}(c) & \dots & \mathrm{pr}_{a_1,\dots,\underline{a_n}}(c) \\ \alpha_{2,1} & \dots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix} \begin{vmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \alpha_{2,1} & \dots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{vmatrix}.\end{aligned}$$

By virtue of (5) the last equation proves the theorem. □

Theorem 7. $(\cdot,\cdot\,|\,\cdot,\dots,\cdot)$ *given by* (5) *is an n-inner product on L where for every n-dimensional linear subspace L' of L and arbitrary $\{a_1,\dots,a_n\}\in S_{L'}$ we have $\|a_1,\dots,a_n\|=1$.*

P r o o f. Let $a_1,\dots,a_n$ be arbitrary in L, let L' be an n-dimensional linear subspace of L containing $a_1,\dots,a_n$ and let $\{a'_1,\dots,a'_n\}\in S_{L'}$. Then $a_i=\sum\limits_{k=1}^{n}\alpha_{i,k}\,a'_k$,

$i = 1, \ldots, n$. Hence we get

$$(6) \qquad \big(a_1, a_1 \mid a_2, \ldots, a_n\big) = \left(\sum_{k=1}^{n} \alpha_{1,k}\, a'_k, \sum_{k=1}^{n} \alpha_{1,k}\, a'_k \;\middle|\; \sum_{k=1}^{n} \alpha_{2,k}\, a'_k, \ldots, \sum_{k=1}^{n} \alpha_{n,k}\, a'_k \right)$$

$$= \begin{vmatrix} \mathrm{pr}_{\underline{a'_1},\ldots,a'_n}\Big(\sum_{k=1}^{n} \alpha_{1,k} a'_k\Big) & \cdots & \mathrm{pr}_{a'_1,\ldots,\underline{a'_n}}\Big(\sum_{k=1}^{n} \alpha_{1,k} a'_k\Big) \\ \alpha_{2,1} & \cdots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix} \begin{vmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \cdots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix} = \begin{vmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{vmatrix}^2,$$

which implies that $(a_1, a_1 \mid a_2, \ldots, a_n) \geqslant 0$ and moreover that $(a_1, a_1 \mid a_2, \ldots, a_n) = 0$ if and only if $a_1, \ldots, a_n$ are linearly dependent.

Now we shall show that for arbitrary $a', a^+, a_2, \ldots, a_n$ we have $(a', a^+ | a_2, \ldots, a_n) = (a^+, a' \mid a_2, \ldots, a_n)$. If $a', a_2, \ldots, a_n$ or $a^+, a_2, \ldots, a_n$ are linearly dependent, then $(a', a^+ \mid a_2, \ldots, a_n)$ and $(a^+, a' \mid a_2, \ldots, a_n)$ both are 0. Hence we may restrict our considerations to the case that $a', a_2, \ldots, a_n$ and $a^+, a_2, \ldots, a_n$ are linearly independent. Let L', L^+ denote the linear subspaces of L generated by $a', a_2, \ldots, a_n$ or $a^+, a_2, \ldots, a_n$, respectively. There exist reals α', α^+ different from 0 such that $\{\alpha' a', a_2, \ldots, a_n\} \in S_{L'}$ and $\{\alpha^+ a^+, a_2, \ldots, a_n\} \in S_{L^+}$. This together with (4) and (5) yields

$$\big(a', a^+ \mid a_2, \ldots, a_n\big) = \frac{1}{\alpha^+} \big(a', \alpha^+ a^+ \mid a_2, \ldots, a_n\big) = \frac{1}{\alpha' \alpha^+} \mathrm{pr}_{\underline{\alpha^+ a^+}, a_2, \ldots, a_n}(\alpha' a')$$
$$= \frac{1}{a' \alpha^+} \mathrm{pr}_{\underline{\alpha' a'}, a_2, \ldots, a_n}(\alpha^+ a^+) = \big(a^+, a' \mid a_2, \ldots, a_n\big)\,.$$

Using (5) we see that $(a, b \mid a_2, \ldots, a_n) = (a, b \mid a_{i_2}, \ldots, a_{i_n})$ for every permutation $(i_2, \ldots, i_n)$ of $(2, \ldots, n)$. And (6) shows that if $n > 1$, then $(a, a \mid a_2, a_3, \ldots, a_n) = (a_2, a_2 \mid a, a_3, \ldots, a_n)$. Also the linearity of $(a, b \mid a_2, \ldots, a_n)$ with respect to a is evident. From (5) we immediately see that, moreover, for every $\{a_1, \ldots, a_n\} \in S_{L'}$ we have $\|a_1, \ldots, a_n\| = 1$. □

3.2. If $\dim L = n$, then in Assumption 2 of 3.1 we necessarily have $L' = L^+$, hence $a^+ = \pm a' + \sum_{k=1}^{n} \alpha_k\, a_k$, and $\mathrm{pr}_{a^+, a_2, \ldots, a_n} = \mathrm{pr}_{a', a_2, \ldots, a_n}$ is the identical mapping. From this we see that in this case, equation (4) becomes trivial. We can choose $S_{L'}$ arbitrarily and the corresponding n-inner products differ only by a factor.

Let now $\dim L > n$. Then obviously (4) contains restrictions to the projections $\mathrm{pr}_{L'}$ if the sets $S_{L'}$ are fixed, and conversely for fixed projections $\mathrm{pr}_{L'}$ it contains restrictions to the sets $S_{L'}$.

4.1. Concerning the problem of the relations between norms $\|b_1,\dots,b_n\|$ and $\|\mathrm{pr}_{a_1,\dots,a_n}(b_1),\dots,\mathrm{pr}_{a_1,\dots,a_n}(b_n)\|$ we have the following results.

Theorem 8. *Let $\big(L,(\cdot,\cdot\mid\cdot,\dots,\cdot)\big)$ be an n-inner product space which in the case $n>1$ is simple. Then*

$$\|b_1,\dots,b_n\| \geqslant \|\mathrm{pr}_{a_1,\dots,a_n}(b_1),\dots,\mathrm{pr}_{a_1,\dots,a_n}(b_n)\|. \tag{7}$$

Proof. In the case $n=1$ the assertion of the theorem is well known. For further considerations let $n>1$. Let $(\cdot,\cdot)$ be an inner product generating $(\cdot,\cdot\mid\cdot,\dots,\cdot)$. Because of Theorem 3 we may restrict our considerations to the case that $(a_k,a_l)=\delta_{k,l}$ for $k,\ l\in\{1,\dots,n\}$. If $\mathrm{pr}_{a_1,\dots,a_n}(b_1),\ \dots,\ \mathrm{pr}_{a_1,\dots,a_n}(b_n)$ are linearly dependent, then obviously (7) is true. Therefore, in what follows we may assume that $\mathrm{pr}_{a_1,\dots,a_n}(b_1),\dots,\mathrm{pr}_{a_1,\dots,a_n}(b_n)$ are linearly independent. Since for arbitrary points $c_1,\dots,c_n\in L$ and arbitrary reals $\gamma_{l,k}$, $l,\ k\in\{1,\dots,n\}$, we have

$$\Big\|\sum_{k=1}^{n}\gamma_{1,k}\,c_k,\dots,\sum_{k=1}^{n}\gamma_{n,k}\,c_k\Big\|^2=\begin{vmatrix}\gamma_{1,1} & \cdots & \gamma_{1,n}\\ \vdots & \ddots & \vdots\\ \gamma_{n,1} & \cdots & \gamma_{n,n}\end{vmatrix}^2\|c_1,\dots,c_n\|^2,$$

we can see that, moreover, the restriction to the case $\mathrm{pr}_{a_1,\dots,a_n}(b_k)=a_k$, $k=1,\dots,n$ is possible. Then we have $(b_k,a_l\mid a_1,\dots,\widehat{a}_l,\dots,a_n)=\delta_{kl}$ for $k,\ l\in\{1,\dots,n\}$ and because of

$$\begin{aligned}\big(b_k,a_l\mid a_1,\dots,\widehat{a}_l,\,\dots,a_n\big)\\ &=\begin{vmatrix}(b_k,a_l) & (b_k,a_1) & \cdots & (b_k,a_{l-1}) & (b_k,a_{l+1}) & \cdots & (b_k,a_n)\\ (a_1,a_l) & (a_1,a_1) & \cdots & (a_1,a_{l-1}) & (a_1,a_{l+1}) & \cdots & (a_1,a_n)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ (a_{l-1},a_l) & (a_{l-1},a_1) & \cdots & (a_{l-1},a_{l-1}) & (a_{l-1},a_{l+1}) & \cdots & (a_{l-1},a_n)\\ (a_{l+1},a_l) & (a_{l+1},a_1) & \cdots & (a_{l+1},a_{l-1}) & (a_{l+1},a_{l+1}) & \cdots & (a_{l+1},a_n)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ (a_n,a_l) & (a_n,a_1) & \cdots & (a_n,a_{l-1}) & (a_n,a_{l+1}) & \cdots & (a_n,a_n)\end{vmatrix}\\ &=\big(b_k,a_l\big)\end{aligned}$$

we get $(b_k, a_l) = \delta_{kl}$ for $k,\ l \in \{1, \dots, n\}$. In view of this we see that for arbitrary $k \in \{1, \dots, n\}$,

$$\begin{aligned}
&\big(a_k, b_k - a_k \mid a_1, \dots, a_{k-1}, b_{k+1}, \dots, b_n\big) \\
&\quad = \begin{vmatrix}
(a_k, b_k - a_k) & (a_k, a_1) & \dots & (a_k, a_{k-1}) & (a_k, b_{k+1}) & \dots & (a_k, b_n) \\
(a_1, b_k - a_k) & (a_1, a_1) & \dots & (a_1, a_{k-1}) & (a_1, b_{k+1}) & \dots & (a_1, b_n) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(a_{k-1}, b_k - a_k) & (a_{k-1}, a_1) & \dots & (a_{k-1}, a_{k-1}) & (a_{k-1}, b_{k+1}) & \dots & (a_{k-1}, b_n) \\
(b_{k+1}, b_k - a_k) & (b_{k+1}, a_1) & \dots & (b_{k+1}, a_{k-1}) & (b_{k+1}, b_{k+1}) & \dots & (b_{k+1}, b_n) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(b_n, b_k - a_k) & (b_n, a_1) & \dots & (b_n, a_{k-1}) & (b_n, b_{k+1}) & \dots & (b_n, b_n)
\end{vmatrix} \\
&\quad = 0.
\end{aligned}$$

This yields

$$\begin{aligned}
\big\|b_1, \dots, b_n\big\|^2 &= \big\|a_1, b_2, \dots, b_n\big\|^2 + \big\|b_1 - a_1, b_2, \dots, b_n\big\|^2 + 2\big(a_1, b_1 - a_1 \mid b_2, \dots, b_n\big) \\
&\geqslant \big\|a_1, b_2, \dots, b_n\big\|^2 \\
&\geqslant \dots \\
&\geqslant \big\|a_1, \dots, a_n\big\|^2 \\
&= \big\|\mathrm{pr}_{a_1, \dots, a_n}(b_1), \dots, \mathrm{pr}_{a_1, \dots, a_n}(b_n)\big\|^2,
\end{aligned}$$

hence the theorem is proved. □

In the case $n > 1$, (7) need not always be true as is shown by an example (with $n = 2$) given in [3].

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