DISCRETE SPECTRA CRITERIA FOR SINGULAR DIFFERENCE OPERATORS

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Abstract. We investigate oscillation and spectral properties (sufficient conditions for discreteness and boundedness below of the spectrum) of difference operators

$$B(y)_{n+k} = \frac{(-1)^n}{w_k}\Delta^n(p_k\Delta^n y_k).$$

Keywords: difference operator, property BD, discrete variational principle

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1. Introduction, auxiliary results

Let w_k be a positive real sequence and denote by l^2_w the Hilbert space of real-valued sequences $y = \{y_k\}_{k=1}^{\infty}$ such that $\sum\limits_{k=1}^{\infty} w_k y_k^2 < \infty$, with the scalar product $\langle y, z\rangle = \sum\limits_{k=1}^{\infty} w_k y_k z_k$. The aim of this paper is to investigate oscillation and spectral properties of $2n$-order difference operators generated by the expression

$$m(y)_{k+n} = \frac{1}{w_k}\sum_{\lambda=0}^{n}(-1)^{\lambda}\Delta^{\lambda}(p_k^{(\lambda)}\Delta^{\lambda}y_{k+n-\lambda}), \tag{1.1}$$

where $p_k^{(\lambda)}$ are real and $p_k^{(n)} > 0$.

Denote

$$D(B) = \{y = \{y_k\}_{k=1}^{\infty} \in l^2_w\colon \{m(y)_{k+n}\} \in l^2_w\}$$

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and consider the operator $B\colon\ D(B) \to l^2_w$ given by $B(y)_{k+n} = m(y)_{k+n}$.

Let $B_0 := B^*$ be the adjoint operator of B. The operators B and B_0 are said to be the *maximal* and the *minimal* operator defined by the difference expression $m(y)$. We say that the operator B has the property BD if the spectrum of any self-adjoint extension of B_0 is discrete and bounded below.

A similar problem in the case $w = 1$ and $p_k^{(0)}, p_k^{(1)}, \dots, p_k^{(n-1)} \equiv 0$ was investigated in [3]. It was shown that the operator B has property BD if and only if

$$\lim_{k\to\infty} k^{(2n-1)} \sum_{j=k}^{\infty} \frac{1}{p_j^{(n)}} = 0.$$

Another paper related to our investigation is [5], where oscillation and spectral properties of differential operators generated by the expression

$$\sum_{j=0}^{n} (-1)^j (p_j(t) y^{(j)})^{(j)}$$

are investigated.

Here we use the recent results about oscillation properties of self-adjoint difference equations $m(y) = 0$, see [1, 2], to establish a discrete analogue of some results of [5]. We also extend the results of [3] concerning one-term difference operators.

Oscillation properties of the even order difference equations

$$\sum_{\lambda=0}^{n} (-1)^\lambda \Delta^\lambda (p_k^{(\lambda)} \Delta^\lambda y_{k+n-\lambda}) = 0 \tag{1.2}$$

are defined using the concept of the generalized zero point of multiplicity n introduced by Hartman [6]. By this definition, an integer $m+1$ is said to be *the generalized zero point of multiplicity* n of a solution y of (1.2) if $y_m \neq 0, y_{m+1} = \ldots = y_{m+n-1} = 0$ and $(-1)^n y_m y_{m+n} \geqslant 0$. Equation (1.2) is said to be *oscillatory* if for any $N \in \mathbb{N}$ there exists a nontrivial solution of (1.2) having at least two different generalized zeros of multiplicity n in $[N, \infty)$, in the opposite case it is said to be nonoscillatory.

Proposition 1. *The following statements are equivalent:*

(i) *B has property* BD.

(ii) *The equation $m(y) = \lambda y_{k+n}$ is nonoscillatory for every $\lambda \in \mathbb{R}$.*

(iii) *For every $\lambda \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that*

$$I(y, N) = \sum_{i=0}^{n} \sum_{k=N}^{\infty} p_k^{(i)} (\Delta^i y_{k+n-i})^2 \geqslant \sum_{k=N}^{\infty} \lambda w_k y_{k+n}^2$$

for any $y \in D_n(N) := \{y = \{y_k\}_{k=1}^{\infty}\colon\ y_k = 0, k \leqslant N+n-1, \exists\, m\colon\ y_k = 0, k \geqslant m\}$.

For $n=1$ the above given Proposition may be found in [4] and a closer examination of its proof shows that using results of $[1,2]$ it may be formulated in the form given here.

2. Nonoscillation criteria

We start with a discrete version of a Wirtinger-type inequality.

Lemma 1. *Let M_k be a positive sequence such that $\Delta M_k \neq 0$. Then for any $y \in D_1(N)$ have*

$$\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \leqslant \psi_N \sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2, \tag{2.1}$$

where

$$\psi_N := \sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \left[1 + \left(\sup_{k \geqslant N} \frac{|\Delta M_k|}{|\Delta M_{k-1}|} \right)^{\frac{1}{2}} \right]^2.$$

P r o o f. Suppose that $\Delta M_k > 0$, in the opposite case we proceed in the same way:

$$\begin{aligned}
\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 &= M_k y_k^2 \big|_N^{\infty} - \sum_{k=N}^{\infty} M_k \Delta y_k^2 = - \sum_{k=N}^{\infty} M_k (y_{k+1} + y_k) \Delta y_k \\
&\leqslant \sum_{k=N}^{\infty} M_k \left(|y_{k+1}| + |y_k| \right) |\Delta y_k| \\
&= \sum_{k=N}^{\infty} M_k |y_{k+1}| \, |\Delta y_k| + \sum_{k=N}^{\infty} M_k |y_k| \, |\Delta y_k| \\
&\leqslant \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left(\sum_{k=N}^{\infty} |\Delta M_k| \frac{M_k}{M_{k+1}} \, y_{k+1}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left(\sum_{k=N}^{\infty} |\Delta M_k| \frac{M_k}{M_{k+1}} \, y_k^2 \right)^{\frac{1}{2}} \leqslant
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left(\sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \right)^{\frac{1}{2}} \\
&\quad \times \left[\left(\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \right)^{\frac{1}{2}} + \left(\sum_{k=N}^{\infty} |\Delta M_k| y_k^2 \right)^{\frac{1}{2}} \right] \\
&= \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left(\sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \right)^{\frac{1}{2}} \\
&\quad \times \left[\left(\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \right)^{\frac{1}{2}} + \left(\sum_{k=N}^{\infty} |\Delta M_{k-1}| \frac{|\Delta M_k|}{|\Delta M_{k-1}|} y_k^2 \right)^{\frac{1}{2}} \right] \\
&\leqslant \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left(\sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \right)^{\frac{1}{2}} \\
&\quad \times \left[\left(\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \right)^{\frac{1}{2}} + \left(\sup_{k \geqslant N} \frac{|\Delta M_k|}{|\Delta M_{k-1}|} \right)^{\frac{1}{2}} \left(\sum_{k=N}^{\infty} |\Delta M_{k-1}| y_k^2 \right) \right] \\
&= \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left(\sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \right)^{\frac{1}{2}} \\
&\quad \times \left[1 + \left(\sup \frac{|\Delta M_k|}{|\Delta M_{k+1}|} \right)^{\frac{1}{2}} \right] \left(\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \right)^{\frac{1}{2}} \\
&\quad \leqslant \left(\sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2 \right)^{\frac{1}{2}} \left[1 + \left(\sup_{k \geqslant N} \frac{|\Delta M_k|}{|\Delta M_{k-1}|} \right)^{\frac{1}{2}} \right] \left(\sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \right)^{\frac{1}{2}}
\end{aligned}
$$

and thus

$$
\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \leqslant \psi_N \sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2.
$$

$\square$

Using this inequality we can prove the following nonoscillation criterion for a two-term equation

$$
(-1)^n \Delta^n (r_k \Delta^n y_k) = p_k y_{k+n}, \qquad r_k > 0,\ p_k \geqslant 0. \tag{2.2}
$$

Theorem 1. *Suppose that there exist positive sequences $M_k^{(1)}$, $M_k^{(2)},\dots,M_k^{(n)}$ such that $|\Delta M_k^{(1)}|,|\Delta M_k^{(2)}|,\dots,|\Delta M_k^{(n)}|$ are eventually positive,*

$$|\Delta M_k^{(j+1)}| \geqslant \frac{M_{k+1}^{(j)}M_k^{(j)}}{|\Delta M_k^{(j)}|}, \quad j=1,\dots,n-1,$$
$$\frac{M_k^{(n)}M_{k+1}^{(n)}}{|\Delta M_k^{(n)}|} \leqslant r_k$$

satisfying

$$0<\limsup_{N\to\infty}\psi_N^{(1)}\psi_N^{(2)}\dots\psi_N^{(n)} =: \psi<\infty, \tag{2.3}$$

where

$$\psi_N^{(j)} := \Big(\sup_{k\geqslant N}\frac{M_k^{(j)}}{M_{k+1}^{(j)}}\Big)\Big[1+\Big(\sup_{k\geqslant N}\frac{|\Delta M_k^{(j)}|}{|\Delta M_{k+1}^{(j)}|}\Big)^{\frac12}\Big]^2.$$

If

$$\limsup_{k\to\infty}\frac{1}{M_k^{(1)}}\sum_{j=k}^{\infty}p_j<\frac{1}{\psi} \tag{2.4}$$

then equation (2.2) *is nonoscillatory.*

Proof. According to Proposition 1, we need to prove that there exists $N\in\mathbb{N}$ such that the quadratic functional

$$H(y)=\sum_{k=N}^{\infty}\big\{r_k(\Delta^n y_k)^2-p_ky_{k+n}^2\big\}$$

satisfies $H(y)>0$ for every nontrivial $y=\{y_k\}\in D_n(N)$.

Let $\varepsilon>0$ be such that

$$\limsup_{k\to\infty}\frac{1}{M_k^{(1)}}\sum_{j=k}^{\infty}p_j<\frac{1}{\psi+\varepsilon}.$$

Then from (2.4), using Lemma 1 and summation by parts, we have for N sufficiently large

$$
\begin{aligned}
\sum_{k=N}^{\infty} p_k y_{k+n}^2 &= \sum_{k=N}^{\infty} \frac{1}{M_k^{(1)}} \Bigl(\sum_{j=k}^{\infty} p_j\Bigr) M_k^{(1)} \Delta y_{k+n-1}^2 \\
&< \frac{1}{\psi+\varepsilon} \sum_{k=N}^{\infty} M_k^{(1)} \bigl[\Delta y_{k+n-1}^2\bigr] \\
&\leqslant \frac{1}{\psi+\varepsilon} \Bigl[\sum_{k=N}^{\infty} M_k^{(1)} |y_{k+n}||\Delta y_{k+n-1}| + \sum_{k=N}^{\infty} M_k^{(1)} |y_{k+n-1}||\Delta y_{k+n-1}|\Bigr] \\
&\leqslant \frac{\sqrt{\psi_N^{(1)}}}{\psi+\varepsilon} \Bigl(\sum_{k=N}^{\infty} \frac{M_k^{(1)} M_{k+1}^{(1)}}{|\Delta M_k^{(1)}|} (\Delta y_{k+n-1})^2\Bigr)^{1/2} \Bigl(\sum_{N}^{\infty} |\Delta M_k^{(1)}| y_{k+n}^2\Bigr)^{1/2} \\
&\leqslant \frac{\psi_N^{(1)}}{\psi+\varepsilon} \sum_{k=N}^{\infty} |\Delta M_k^{(2)}| (\Delta y_{k+n-1})^2 \\
&\leqslant \frac{\psi_N^{(1)} \psi_N^{(2)}}{\psi+\varepsilon} \sum_{k=N}^{\infty} \frac{M_k^{(2)} M_{k+1}^{(2)}}{|\Delta M_k^{(2)}|} (\Delta^2 y_{k+n-2})^2 \\
&\leqslant \frac{\psi_N^{(1)} \psi_N^{(2)}}{\psi+\varepsilon} \sum_{k=N}^{\infty} |\Delta M_k^{(3)}| \, (\Delta^2 y_{k+n-2})^2 \\
&\leqslant \frac{\psi_N^{(1)} \psi_N^{(2)} \ldots \psi_N^{(n)}}{\psi+\varepsilon} \sum_{k=N}^{\infty} \frac{M_{k+1}^{(n)} M_k^{(n)}}{|\Delta M_k^{(n)}|} (\Delta^n y_k)^2.
\end{aligned}
$$

Since (2.3) holds, $\frac{\psi_N^{(1)} \psi_N^{(2)} \ldots \psi_N^{(n)}}{\psi+\varepsilon} < 1$ if N is sufficiently large, hence

$$
\sum_{k=N}^{\infty} p_k y_{k+n}^2 < \sum_{k=N}^{\infty} \frac{M_{k+1}^{(n)} M_k^{(n)}}{|\Delta M_k^{(n)}|} (\Delta^n y_k)^2 \leqslant \sum_{k=N}^{\infty} r_k (\Delta^n y_k)^2.
$$

Consequently, $H(y) > 0$ if N is sufficiently large.

□

Now consider the equation

$$
(-1)^n \Delta^n (k^{(\alpha)} \Delta^n y_k) = p_k y_{k+n} \tag{2.5}
$$

with $p_k \geqslant 0$ and $\alpha \notin \{1, 3, \ldots, 2n-1\},\ \alpha < 2n-1$ i.e., equation (2.1) where

$$
r_k = k^{(\alpha)} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}, \quad \Gamma(t) := \int_0^{\infty} \mathrm{e}^{-s} s^{t-1} \, \mathrm{d}s.
$$

Corollary 1. *If* $\alpha \notin \{1,3\dots,2n-1\}$, $\alpha < 2n-1$ *and*

$$\limsup_{k\to\infty} k^{(2n-1-\alpha)} \sum_{j=k}^{\infty} p_j < \frac{(1-\alpha)^2\dots(2n-3-\alpha)^2(2n-1-\alpha)}{4^n} \tag{2.6}$$

then (2.5) *is nonoscillatory.*

Proof. Let $M_k^{(n)} = |1-\alpha|(k-1)^{(\alpha-1)}$, $M_k^{(n-1)} = (1-\alpha)^2|3-\alpha|(k-2)^{(\alpha-3)}$

$$M_k^{(j)} = (1-\alpha)^2(3-\alpha)^2\dots|2j-1-\alpha|(k-j)^{(\alpha-2j+1)}, \quad j=3,\dots,n.$$

Recall that we have $\Gamma(k+1) = k\Gamma(k)$ and $\Delta k^{(\alpha)} = \alpha k^{(\alpha-1)}$, hence

$$\frac{1}{k^{(\alpha)}} = -\frac{1}{\alpha-1}\Delta\left(\frac{1}{(k-1)^{(\alpha-1)}}\right).$$

Using these formulas one can directly verify that sequences $M_k^{(j)}$, $j=1,\dots,n$, satisfy the assumptions of Theorem 1 with $r_k = k^{(\alpha)}$ and $\lim\limits_{N\to\infty}\psi_N^{(j)} = 4$. Consequently (2.4) reads (2.6) and (2.5) is nonoscillatory by Theorem 1. □

3. Spectral properties of difference operators

In the next theorem we investigate spectral properties (sufficient conditions for property BD) of the full-term difference operator $m(y)$ given by (1.1). We use essentially the following idea. The general operator $m(y)$ is viewed as a "perturbation" of a certain one term operator

$$\frac{(-1)^i}{w_k}\Delta^i\big(p_k^{(i)}\Delta^j y_{k+n-i}\big)$$

for some $i \in \{1,2,\dots,n\}$ and on the remaining terms we impose such restrictions that they do not interfere with this term.

Theorem 2. *Let* $i \in \{1,2,\dots,n\}$ *be fixed and let the positive strictly monotonic sequences* $M_k^{(1)}, M_k^{(2)},\dots,M_k^{(i)}$ *satisfy*

$$\Delta M_k^{(1)} \geqslant w_k,\ \Delta M_k^{(2)} \geqslant \frac{M_k^{(1)}M_{k+1}^{(1)}}{|\Delta M_k^{(1)}|},\ \dots,\ \Delta M_k^{(i)} \geqslant \frac{M_k^{(i-1)}M_{k+1}^{(i-1)}}{|\Delta M_k^{(i-1)}|}.$$

Then the operator B *has property* BD *if the following conditions are satisfied for some* i, $1 \leqslant i \leqslant n$:

(a) $p_k^{(i)} > 0$, $\sum\limits_{k=0}^{\infty}\frac{1}{p_k^{(i)}} < \infty$, $\lim\limits_{l\to\infty} M_l^{(i)}\sum\limits_{k=l}^{\infty}\frac{1}{p_k^{(i)}} = 0$.

(b) *For* $j > i$, $p_k^{(j)} \geqslant 0$.

(c) *The* i *sequences* $\left\{\frac{p_k^{(j)}}{|\Delta M_k^{(j+1)}|}; 0 \leqslant j \leqslant i-1\right\}$ *are bounded below by a constant* C.

(d) *For every* $0 \leqslant j \leqslant i$ *we have* $\psi_N^{(j)} < \infty$, *where*

$$\psi_N^{(j)} := \sup_{k \geqslant N} \frac{M_k^{(j)}}{M_{k+1}^{(j)}} \left[1 + \left(\sup_{k \geqslant N} \frac{|\Delta M_k^{(j)}|}{|\Delta M_{k+1}^{(j)}|}\right)^{\frac{1}{2}}\right]^2.$$

Proof. Let μ be a real number. From Lemma 1 we have for any $y \in D_n(N)$ and $j = 1, 2, \ldots, i-1$

$$\begin{aligned} &\sum_{k=N}^{\infty} |\Delta M_k^{(j)}|(\Delta^{j-1} y_{k+n-j+1})^2 \\ &\qquad \leqslant \psi_N^{(j)} \sum_{k=N}^{\infty} \frac{M_k^{(j)} M_{k+1}^{(j)}}{|\Delta M_k^{(j)}|} (\Delta^j y_{k+n-j})^2 \\ &\qquad \leqslant \psi_N^{(j)} \sum_{k=N}^{\infty} |\Delta M_k^{(j+1)}|(\Delta^j y_{k+n-j})^2. \end{aligned} \tag{3.1}$$

Now, by conditions (b), (c)

(3.2)

$$\begin{aligned} I(y, N) - \mu \sum_{k=N}^{\infty} w_k y_{k+n}^2 \geqslant{} & \sum_{k=N}^{\infty} p_k^{(i)} (\Delta^i y_{k+n-i})^2 \\ & + C \sum_{j=0}^{i-1} \sum_{k=N}^{\infty} |\Delta M_k^{(j+1)}|(\Delta^j y_{k+n-j})^2 - \mu \sum_{k=N}^{\infty} w_k y_{k+n}^2. \end{aligned}$$

Using $\Delta M_k^{(1)} \geqslant w_k$ and (3.1) we obtain

$$\sum_{k=N}^{\infty} |\Delta M_k^{(j)}|(\Delta^{j-1} y_{k+n-j+1})^2 \leqslant \prod_{l=j}^{i-1} \psi_N^{(l)} \sum_{k=N}^{\infty} |\Delta M_k^{(i)}|(\Delta^{(i-1)} y_{k+n-i+1})^2,$$

for $1 \leqslant j \leqslant i-1$, hence there is a $D > 0$ $(D > \mu)$ such that

$$C \sum_{j=0}^{i-1} \sum_{k=N}^{\infty} |\Delta M_k^{(j+1)}|(\Delta^j y_{k+n-j})^2 - \mu \sum_{k=N}^{\infty} w_k y_{k+n}^2 \geqslant -D \sum_{k=N}^{\infty} |\Delta M_k^{(i)}|(\Delta^{i-1} y_{k+n-i+1})^2.$$

We set $M_l := \left(\sum_{k=l}^{\infty} \frac{1}{p_k^{(i)}}\right)^{-1}$ and $\psi_N := \sup_{k \geqslant N} \frac{M_k}{M_{k+1}} \left[1 + \left(\sup_{k \geqslant N} \frac{|\Delta M_k|}{|\Delta M_{k-1}|}\right)^{\frac{1}{2}}\right]^2$.

By (a), we may choose N that $M_l^{(i)} \sum_{k=l}^{\infty} \frac{1}{p_k^{(i)}} \leqslant \frac{1}{2D\psi_N}$, $l \geqslant N$. With this choice of N, using summation by parts and Lemma 1 (with the above given M_k), we obtain

$$
\begin{aligned}
&\sum_{k=N}^{\infty} |\Delta M_k^{(i)}| (\Delta^{i-1} y_{k+n-i+1})^2 \\
&\qquad \leqslant \sum_{k=N}^{\infty} M_k^{(i)} \left[|\Delta^{i-1} y_{k+n-i+1}| + |\Delta^{i-1} y_{k+n-i}| \right] |\Delta^i y_{k+n-1}| \\
&\qquad \leqslant \frac{1}{2D\psi_N} \sum_{k=N}^{\infty} \left(\sum_{l=k}^{\infty} \frac{1}{p_l^{(i)}} \right)^{-1} \left[|\Delta^{i-1} y_{k+n-i+1}| + |\Delta^{i-1} y_{k+n-i}| \right] |\Delta^i y_{k+n-1}| \\
&\qquad \leqslant \frac{1}{2D} \sum_{k=N}^{\infty} p_k^{(i)} (\Delta^i y_{k+n-i+1})^2.
\end{aligned}
$$

Thus the left hand side of (3.2) is bounded below by

$$
\sum_{k=N}^{\infty} p_k^{(i)} (\Delta^i y_{k+n-i})^2 - D \left(\frac{1}{2D} \sum_{k=N}^{\infty} p_k^{(i)} (\Delta^i y_{k+n-i})^2 \right) \geqslant 0.
$$

□

Now we turn our attention to the one term difference operator

$$
l(y)_{n+k} = (-1)^n \frac{1}{w_k} \Delta^n (r_k \Delta^n y_k). \tag{3.3}
$$

We will use the following statement known as the discrete reciprocity principle, see [3] Proposition 2. Let w_k, $r_k > 0$, $\lambda > 0$. Equation $(-1)^n \Delta^n (r_k \Delta^n y_k) = \lambda w_k y_{k+n}$ is nonoscillatory if and only if the so-called reciprocal equation

$$
(-1)^n \Delta^n \left(\frac{1}{w_k} \Delta^n y_k \right) = \frac{\lambda}{r_{k+n}} y_{k+n} \tag{3.4}
$$

is nonoscillatory.

Theorem 3. *Let* $w_k = \frac{1}{k^{(\alpha)}}$, $\alpha \notin \{1, 3, \dots 2n-1\}$, $\alpha < 2n-1$ *and*

$$
\lim_{k \to \infty} k^{(2n-1-\alpha)} \sum_{j=k}^{\infty} r_j^{-1} = 0. \tag{3.5}
$$

Then (3.3) *has property* BD.

P r o o f. Let $\lambda > 0$. By Proposition 2 the equation $l(y) = \lambda y_{k+n}$ is nonoscillatory if and only if (3.4) is nonoscillatory.

If (3.5) holds, then $\lim\limits_{k\to\infty} k^{(2n-1-\alpha)} \sum\limits_{j=k}^{\infty} \lambda r_j^{-1} = 0 < \frac{(1-\alpha)^2 \ldots (2n-1-\alpha)}{4^n}$, hence by Corollary, equation (3.4) with $\frac{1}{w_k} = k^{(\alpha)}$ is nonoscillatory, i.e. $l(y) = \lambda y_{k+n}$ is also nonoscillatory and by Proposition 2, (3.3) has property BD. $\square$

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