LOCATION-DOMATIC NUMBER OF A GRAPH

BOHDAN ZELINKA, Liberec

(Received September 18, 1996)

Abstract. A subset D of the vertex set $V(G)$ of a graph G is called locating-dominating, if for each $x \in V(G) - D$ there exists a vertex $y \rightarrow D$ adjacent to x and for any two distinct vertices x_1, x_2 of $V(G) - D$ the intersections of D with the neighbourhoods of x_1 and x_2 are distinct. The maximum number of classes of a partition of $V(G)$ whose classes are locating-dominating sets in G is called the location-domatic number of G. Its basic properties are studied.

Keywords: locating-dominating set, location-domatic partition, location-domatic number, domatic number

MSC 1991: 05C35

In this paper we will introduce the location-domatic number of a graph. All graphs considered will be finite undirected graphs without loops and multiple edges.

The location-domatic number of a graph is a variant of the domatic number, introduced by E. J. Cockayne and S. T. Hedetniemi. A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

A special case of a dominating set is a locating-dominating set. It was defined by D. F. Rall and P. J. Slater in [2]. Let $N_G(x)$ denote the open neighborhood of a vertex x in a graph G, i.e. the set of all vertices which are adjacent to x in G. A dominating set D in a graph G is called locating-dominating in G, if for any two distinct vertices x_1, x_2 of $V(G) - D$ the intersections $D \cap N_G(x_1)$, $D \cap N_G(x_2)$ are distinct. In [2] also the location-domination number of G is defined as the minimum number of vertices of a locating-dominating set in G.

Now we can define the location-domatic number of G analogously to the domatic number. A partition of $V(G)$, all of whose classes are locating-dominating set in G, is called a location-domatic partition of G. The maximum number of classes of a location-domatic partition of G is called the location-domatic number of G and is denoted by $d_{\mathrm{loc}}(G)$.

Note that $d_{\mathrm{loc}}(G)$ is well-defined, because the whole set $V(G)$ is a locating-dominating set in G and therefore there exists at least one location-domatic partition of G, namely $\{V(G)\}$.

Theorem 1. *Let there exist three pairwise distinct vertices x_1, x_2, x_3 of G such that $N_G(x_1) = N_G(x_2) = N_G(x_3)$. Then*

$$d_{\mathrm{loc}}(G) = 1.$$

Proof. Suppose that $d_{\mathrm{loc}}(G) \geqslant 2$. Then there exist two disjoint locating-dominating sets D_1, D_2 in G. At least one of the sets $V(G)-D_1$, $V(G)-D_2$ contains at least two of the vertices x_1, x_2, x_3. Without loss of generality let $V(G)-D_1$ contain x_1 and x_2. As $N_G(x_1) = N_G(x_2)$, we have also $D_1 \cap N_G(x_1) = D_1 \cap N_G(x_2)$ and D_1 is not locating-dominating, which is a contradiction. This yields the result. □

Theorem 2. *Let there exists two distinct vertices x_1, x_2, of G such that $N_G(x_1) = N_G(x_2)$. Then*

$$d_{\mathrm{loc}}(G) \leqslant 2.$$

Proof. Suppose that $d_{\mathrm{loc}}(G) \geqslant 3$. Then there exist three pairwise disjoint locating-dominating sets D_1, D_2, D_3 in G. At least one of the sets $V(G) - D_1$, $V(G) - D_2$, $V(G) - D_3$ contains both the vertices x_1, x_2. The rest of the proof is analogous to the proof of Theorem 1. □

The symbol Δ will denote the symmetric difference of sets. Then for any two vertices x, y of G the symbol $\varepsilon(x,y)$ will be defined as the number of elements of $N_G(x)\Delta N_G(y)$ while $\varepsilon(G)$ will denote the minimum of $\varepsilon(x,y)$ over all pairs of distinct vertices x, y of G.

Theorem 3. *For every graph G the inequality*

$$d_{\mathrm{loc}}(G) \leqslant \varepsilon(G) + 2$$

holds.

P r o o f. Let $d = d_{\mathrm{loc}}(G)$ and let $\{D_1, \ldots, D_d\}$ be a location-domatic partition of G. Let x, y be vertices for which $\varepsilon(x, y) = \varepsilon(G)$ holds. First suppose that x, y are in distinct classes of the partition; without loss of generality let $x \in D_1$, $y \in D_2$. Then for $i = 3, \ldots, d$ we have $D_i \cap N_G(x) \neq D_i \cap N_G(y)$. This is possible only if D_i contains a vertex of $N_G(x)\Delta N_G(y)$. As $D_3, \ldots, D_d$ are pairwise disjoint, we have $d - 2 \leqslant \varepsilon(x, y)$, which implies the assertion. If both x, y are in the same class of the partition, we have even $d - 1 \leqslant \varepsilon(x, y)$. □

Theorem 4. *Let a graph G contain two vertices x_1, x_2 of degree 1 which are both adjacent to a vertex y. Then*

$$d_{\mathrm{loc}}(G) = 1.$$

P r o o f. Suppose $d_{\mathrm{loc}}(G) \geqslant 2$. As G contains vertices of degree 1, according to [1] its domatic number is at most 2 and hence also $d_{\mathrm{loc}}(G) \leqslant 2$. Suppose $d_{\mathrm{loc}}(G) = 2$ and let $\{D_1, D_2\}$ be a location-domatic partition of G. Without loss of generality let $y \in D_1$. The vertices x_1, x_2 are adjacent to no vertex of D_2 and hence $x_1 \in D_2$, $x_2 \in D_2$. Obviously $D_2 = V(G) - D_1$ and $D_1 \cap N_G(x_1) = D_1 \cap N_G(x_2) = \{y\}$, which is a contradiction. Hence $d_{\mathrm{loc}}(G) = 1$. □

Now we can determine the location-domatic numbers of some well-known types of graphs.

Corollary 1. *For the complete graph K_n we have*

$$d_{\mathrm{loc}}(K_2) = 2,$$
$$d_{\mathrm{loc}}(K_n) = 1 \quad \textit{for } n \geqslant 2.$$

Corollary 2. *For the complete bipartite graph $K_{m,n}$ we have*

$$d_{\mathrm{loc}}(K_{1,1}) = d_{\mathrm{loc}}(K_{2,2}) = 2,$$
$$d_{\mathrm{loc}}(K_{m,n}) = 1 \quad \textit{in the other cases.}$$

Corollary 3. *For the circuit C_n we have*

$$d_{\mathrm{loc}}(C_3) = 1,$$
$$d_{\mathrm{loc}}(C_n) = 2 \quad \textit{for } n \geqslant 4.$$

P r o o f. Let the vertices of C_n be $u_1, \ldots, u_n$ and the edges u_iu_{i+1} for $i = 1, \ldots, n$, the subscript $i + 1$ being taken modulo n. The circuit C_3 is the complete graph K_3 and thus $d_{\mathrm{loc}}(C_3) = 1$ by Corollary 1. For C_4 we have a location-domatic partition $\{\{u_1, u_2\}, \{u_3, u_4\}\}$ and thus $d_{\mathrm{loc}}(C_4) \geqslant 2$. For $n \geqslant 5$ we have a location-domatic partition $\{D_1, D_2\}$, where D_1 (or D_2) is the set of all u_i with i odd (or even, respectively); hence also $d_{\mathrm{loc}}(C_n) \geqslant 2$. If n is not divisible by 3 then $d_{\mathrm{loc}}(C_n) \leqslant d(C_n) = 2$ and thus $d_{\mathrm{loc}}(C_n) = 2$. If n is divisible by 3, then $d(C_n) = 3$ and the unique domatic partition with three classes is $\{D_1, D_2, D_3\}$, where D_i for $i \in \{1, 2, 3\}$ is the set of all u_j with $j \equiv i \pmod 3$. Each vertex is adjacent to no vertex of its own class and to one vertex from each of the other classes. Thus $u_1 \in D_1 \subseteq V(C_n) - D_2$, $u_2 \in D_2$, $u_3 \in D_3 \subseteq V(C_n) - D_2$ and $D_2 \cap N_{C_n}(u_1) = D_2 \cap N_{C_n}(u_2) = \{u_2\}$, which implies that $\{D_1, D_2, D_3\}$ is not location-domatic partition. Therefore $d_{\mathrm{loc}}(C_n) = 2$ in this case, too. □

By P_n we denote the path of length n, i.e. with n edges and $n + 1$ vertices.

Corollary 4. *For the path P_n we have*

$$d_{\mathrm{loc}}(P_2) = 1,$$
$$d_{\mathrm{loc}}(P_n) = 2 \text{ for } n \neq 2.$$

Theorem 5. *Let p, q be integers, $q \geqslant 2$, $1 \leqslant p \leqslant q$. Then there exists a graph G with $d_{\mathrm{loc}}(G) = p$, $d(G) = q$.*

P r o o f. We start with the case $p = q$. Let r be an integer, $r \geqslant 4q$. Let $D_1, \ldots, D_q$ be pairwise disjoint sets of vertices, let $|D_1| = r + 1$, $|D_i| = r$ for $i = 2, \ldots, q$. Let the vertices of D_1 be u, $v(1,1), \ldots, v(1,r)$, let the vertices of D_i for $2 \leqslant i \leqslant q$ be $v(i,1), \ldots, v(i,r)$. Consider an auxiliary graph H; it is the complete graph whose vertex set is $\{D_1, \ldots, D_q\}$. If q is even, then H may be decomposed into $q-1$ pairwise edge-disjoint linear factors $F_1, \ldots, F_{q-1}$. If q is odd, then H may be decomposed into q pairwise edge-disjoint graphs $F_1, \ldots, F_q$, each of which is a linear factor of a graph obtained from H by deleting one vertex. In any of these cases consider two sets D_i, D_j. Let h be the number such that the edge joining D_i and D_j in H belongs to F_h. Each vertex $v(i,k)$ for $k = 1, \ldots, q$ will be joined by edges with the vertices $v(j, k-h), \ldots, v(j, k+h)$, the numbers in brackets being taken modulo q. Moreover, the vertex $u \in D_1$ will be joined by edges with all vertices $v(i,1)$ for $i = 2, \ldots, q$. The resulting graph will be G_q. From the construction it is clear that $\{D_1, \ldots, D_q\}$ is a location-domatic partition of G_q and thus $d_{\mathrm{loc}}(G_q) \geqslant q$. On the other hand, the vertex u has degree $q-1$. Hence the minimum degree $\delta(G_q) \leqslant q-1$ and by [1] we have

$d_{\mathrm{loc}}(G_q) \leqslant d(G_q) \leqslant \delta(G_q)+1 \leqslant q$, which implies $d_{\mathrm{loc}}(G_q) = d(G_q) = p = q$. Now let $3 \leqslant p \leqslant q-1$. Take the graph G_q constructed above, add a new vertex w to it and join it by edges with all vertices $v(i,1)$ for $2 \leqslant i \leqslant q$ and with all vertices $v(i,2)$ for $2 \leqslant i \leqslant p-1$. The resulting graph will be denoted by G_p. We have $\varepsilon(u,w) = p-2$ and $d_{\mathrm{loc}}(G_p) \leqslant p$ by Theorem 3. If we denote $\widetilde{D} = \{w\} \cup \bigcup_{i=p}^{q} D_i$, then $\{D_1, \ldots, D_{p-1}, \widetilde{D}\}$ is a location-domatic partition of G_p and thus $d_{\mathrm{loc}}(G_p) = p$. Now let $p = 2$. We take again the graph G_q. To it we add a new vertex w and join it by edges with the same vertices with which u was joined. The resulting graph will be G_2. We have $\varepsilon(u,w) = 0$ and thus $d_{\mathrm{loc}}(G_2) \leqslant 2$. If we denote $\widetilde{D} = \{w\} \cup \bigcup_{i=2}^{q} D_i$, then $\{D_1, \widetilde{D}\}$ is a location-domatic partition of G_2 and $d_{\mathrm{loc}}(G_2) = 2$. Finally let $p = 1$. To G_q we add two new vertices w_1, w_2 and join them with the same vertices with which u was joined. The resulting graph will be G_1. We have $N_{G_1}(w_1) = N_{G_1}(w_2) = N_{G_1}(u)$ and by Theorem 1 then $d_{\mathrm{loc}}(G) = 1$. Evidently $d(G_p) = q$ for each $p = 1, \ldots, q-1$. □

Theorem 6. *Let G be a graph with n vertices, let $y = \Phi(x)$ be the inverse function to the function $y = 2^x + x$. Then*

$$d_{\mathrm{loc}}(G) \leqslant \frac{n}{\Phi(n+1)}.$$

P r o o f. The function $y = 2^x + x$ is a monotone increasing function mapping the set R of real numbers bijectively onto itself. Therefore the inverse function $y = \Phi(x)$ to this function exists, it is again a monotone increasing function which maps R onto itself.

Now consider the graph G. For the sake of simplicity we denote $d_{\mathrm{loc}}(G) = d$. Consider a location-domatic partition $\mathcal{D}$ with d classes. As G has n vertices, there exists at least one class $D \in \mathcal{D}$ such that $|D| \leqslant n/d$. The sets $D \cap N_G(x)$ for $x \in V(G) - D$ are pairwise distinct non-empty subsets of D; their number is less than or equal to $2^{n/d} - 1$ and, as D is a locating-dominating set, so is the number of vertices of $V(G) - D$. Hence $n \leqslant n/d + 2^{n/d} - 1$, which is $n - 1 \leqslant 2^{n/d} + n/d = \Phi^{-1}(n/d)$. As $y = \Phi(x)$ is a monotone increasing function, we have $\Phi(n+1) \leqslant n/d$ and this yields $d \leqslant n/\Phi(n+1)$. □

References

[1] *E. J. Cockayne, S. T. Hedetniemi*: Towards a theory of domination in graphs. Networks *7* (1977), 247–261.

[2] *D. F. Rall, P. J. Slater*: On location-domination numbers for certain classes of graphs. Congressus Numerantium *45* (1984), 77–106.

Author's address: *Bohdan Zelinka*, Katedra diskrétní matematiky a statistiky TU Liberec, Hálkova 6, 461 17 Liberec 1, Czech Republic, e-mail: **bohdan.zelinka@vslib.cz**.