SPECTRA OF AUTOMETRIZED LATTICE ALGEBRAS

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Abstract. Autometrized algebras are a common generalization e.g. of commutative lattice ordered groups and Brouwerian algebras. In the paper, spectra of normal autometrized lattice ordered algebras (i.e. topologies of sets (and subsets) of their proper prime ideals) are studied. Especially, the representable dually residuated lattice ordered semigroups are examined.

Keywords: autometrized algebra, dually residuated lattice ordered semigroup, prime ideal, spectrum

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1. Introduction

K. L. N. Swamy introduced in [7] the notion of an autometrized algebra which is a common generalization, for example, of commutative ℓ-groups and Brouwerian algebras. (A Brouwerian algebra is a lattice A with the greatest element in which for each $a, b \in A$ there exists a smallest $x \in A$ such that $b \vee x \geqslant a$.) Ideals in autometrized algebras were introduced and studied by K. L. N. Swamy and N. P. Rao in [9]. Their work has been continued by J. Rachůnek in [4], [5], [6], M. E. Hansen in [1], [2], and T. Kovář in [3]. The notion of a prime ideal in an autometrized algebra was defined in [4] and minimal prime ideals were studied in [1].

In this paper, spectra of autometrized lattice ordered algebras, i.e. topological spaces of some sets of their proper prime ideals, are studied.

A system $A = (A, +, 0, \leqslant, *)$ is called an *autometrized algebra* if

(1) $(A,+,0)$ is a commutative monoid;

(2) $(A, \leqslant)$ is an ordered set, and

$$\forall a, b, c \in A;\ a \leqslant b \Longrightarrow a + c \leqslant b + c;$$

(3) $*\colon A \longrightarrow A$ is an autometric on A, i.e.

$$\begin{aligned}
&\forall a, b \in A;\ a * b \geqslant 0,\\
&\forall a, b \in A;\ a * b = 0 \iff a = b,\\
&\forall a, b \in A;\ a * b = b * a,\\
&\forall a, b, c \in A;\ a * c \leqslant (a * b) + (b * c).
\end{aligned}$$

An autometrized algebra is called *normal* if

$$\begin{aligned}
&\forall a \in A;\ a \leqslant a * 0,\\
&\forall a, b, c, d \in A;\ (a + c) * (b + d) \leqslant (a * b) + (c * d),\\
&\forall a, b, c, d \in A;\ (a * c) * (b * d) \leqslant (a * b) + (c * d),\\
&\forall a, b \in A;\ (a \leqslant b \implies \exists x \geqslant 0;\ a + x = b).
\end{aligned}$$

If $(A, \leqslant)$ is a lattice and

$$\begin{aligned}
\forall a, b, c \in A;\ a + (b \vee c) &= (a + b) \vee (a + c),\\
a + (b \wedge c) &= (a + b) \wedge (a + c),
\end{aligned}$$

then $A = (A, +, 0, \leqslant, *)$ is called an *autometrized lattice algebra* (an *autometrized ℓ-algebra*).

For instance, every commutative ℓ-group and every Brouwerian algebra is a normal autometrized ℓ-algebra.

If A is an autometrized algebra, then $\emptyset \neq I \subseteq A$ is called an *ideal* in A if and only if

$$\begin{aligned}
&\forall a, b \in I;\ a + b \in I,\\
&\forall a \in I,\ x \in A;\ x * 0 \leqslant a * 0 \Longrightarrow x \in I.
\end{aligned}$$

In [9] it is proved that the set $\mathcal{I}(A)$ of all ideals in a normal autometrized algebra A is a complete algebraic lattice with respect to the order by set inclusion. If $\emptyset \neq B \subseteq A$, then the ideal generated by B is

$$\begin{aligned}
I(B) = \{x \in A;\ x * 0 \leqslant m_1(a_1 * 0) + \ldots + m_k(a_k * 0),\\
\text{where}\, m_1, \ldots, m_k \in \mathbb{N} \text{ and } a_1, \ldots, a_k \in B\}.
\end{aligned}$$

For the principal ideal $I(a)$ generated by an element $a \in A$ we have

$$I(a) = \{x \in A;\ x * 0 \leqslant m(a * 0) \quad \text{for some } m \geqslant 0\}.$$

An ideal I of an autometrized algebra A is called *prime* if

$$\forall J, K \in \mathcal{I}(A);\ J \cap K = I \Rightarrow J = I \text{ or } K = I,$$

and it is called *regular* if

$$I = \bigcap_{\alpha \in \Gamma} J_\alpha,$$

where $J_\alpha \in \mathcal{I}(A)$ for all $\alpha \in \Gamma$ implies the existence of $\beta \in \Gamma$ such that $I = J_\beta$.

Note. An autometrized algebra A is called

a) *semiregular* if

$$\forall a \in A;\ a \geqslant 0 \Longrightarrow a * 0 = a;$$

b) *interpolation* if

$$\forall a, b, c \in A;\ \big(0 \leqslant a, b, c,\ a \leqslant b + c \Longrightarrow (\exists 0 \leqslant b_1 \leqslant b, 0 \leqslant c_1 \leqslant c;\ a = b_1 + c_1)\big).$$

Clearly, commutative ℓ-groups and Brouwerian algebras are both semiregular and interpolation.

Many of properties of prime ideals were proved in [4], [5] and [6] for interpolation semiregular ℓ-algebras. But using [1], Lemma 1.2, one can easily prove that the assumption "A is interpolation" is unnecessary. Further (as shown in [3]), Lemma 5 in [9] (i.e. if A is an autometrized algebra and $a, b \in A$ then $(a * b) * 0 = a * b$) makes it often possible to omit also the requirement of semiregularity.

2. Normal autometrized ℓ-algebras

Let A be an autometrized algebra. Let us denote by $\operatorname{Spec} A$ the set of proper prime ideals in A. If $M \subseteq A$, we put

$$S(M) = \{P \in \operatorname{Spec} A;\ M \nsubseteq P\},$$
$$H(M) = \{P \in \operatorname{Spec} A;\ M \subseteq P\}.$$

Especially, for $M = \{a\}$ where $a \in A$, we will write

$$S(\{a\}) = S(a) \text{ and } H(\{a\}) = H(a).$$

It is obvious that for any $M \subseteq A$ we have $S(M) = S(I(M))$ and $H(M) = H(I(M))$, hence we will consider only $S(I)$ and $H(I)$ for all $I \in \mathcal{I}(A)$ and $S(a)$ and $H(a)$ for each $a \in A$.

Lemma 1. *If A is a normal autometrized ℓ-algebra then:*
(1) $S(0)=\emptyset$, $S(A)=\operatorname{Spec} A$.
(2) $\forall I, J \in \mathcal{I}(A)$; $S(I\cap J)=S(I)\cap S(J)$.
(3) $\forall I_\gamma \in \mathcal{I}(A), \gamma\in\Gamma$; $S(\bigvee_{\gamma\in\Gamma} I_\gamma)=\bigcup_{\gamma\in\Gamma} S(I_\gamma)$.
(4) $\forall a, b\in A$; $S((a*0)\vee(b*0))=S(a)\cup S(b)$.
(5) $\forall a, b\in A$; $S((a*0)\wedge(b*0))=S(a)\cap S(b)$.

Proof. 1. Obvious.

2. Let $I, J\in\mathcal{I}(A)$ and $P\in\operatorname{Spec} A$. Then by [4], Theorem 4, and [3], Theorem 9, $I\cap J\nsubseteq P$ if and only if $I\nsubseteq P$ and $J\nsubseteq P$, therefore $S(I\cap J)=S(I)\cap S(J)$.

3. Let $I_\gamma\in\mathcal{I}(A)$, $\gamma\in\Gamma$, and $P\in\operatorname{Spec} A$. Then for $\bigvee_{\gamma\in\Gamma} I_\gamma$, the join of I_γ in $\mathcal{I}(A)$, we have $\bigvee_{\gamma\in\Gamma} I_\gamma\nsubseteq P$ if and only if there exists $\gamma_0\in\Gamma$ such that $I_{\gamma_0}\nsubseteq P$, and hence $S(\bigvee_{\gamma\in\Gamma} I_\gamma)=\bigcup_{\gamma\in\Gamma} S(I_\gamma)$.

4 and 5. By [4], Propositions 2 and 3, and [3], Theorems 6 and 7,

$$I(a)\vee I(b)=I((a*0)\vee I(b*0\ )),$$
$$I(a)\wedge I(b)=I((a*0)\wedge I(b*0)),$$

thus 4 and 5 are special cases of the properties 2 and 3. □

Corollary 2. *The sets $S(I)$, where I is any ideal in A, form a topology of* $\operatorname{Spec} A$.

Definition. If A is a normal autometrized ℓ-algebra then the topology of $\operatorname{Spec} A$ such that its open sets are exactly $S(I)$ for any $I\in\mathcal{I}(A)$ will be called the *spectral topology*. The topological space $\operatorname{Spec} A$ with the spectral topology will be called the *spectrum* of the algebra A.

In this section, A will always denote a normal autometrized ℓ-algebra.

Proposition 3. *The mapping $S\colon I\longmapsto S(I)$ is an isomorphism of the lattice $\mathcal{I}(A)$ onto the lattice of open subsets in* $\operatorname{Spec} A$.

Proof. By Lemma 1, S is a surjective homomorphism. By [6] (Theorem 3), any ideal is an intersection of regular ideals, and since every regular ideal is prime, we have

$$I=\bigcap\{P;\ P\in H(I)\}$$

for each $I\in\mathcal{I}(A)$. Hence, if $S(I)=S(J)$, then

$$I=\bigcap\{P;\ P\in H(I)\}=\bigcap\{Q;\ Q\in H(J)\}=J,$$

and therefore S is injective. □

Theorem 4. *The sets $S(a)$, where a is any element in A, form a basis of open sets in the spectral topology stable under finite unions and intersections.*

P r o o f. If $I \in \mathcal{I}(A)$, then by Lemma 1 (3),

$$S(I) = S\Big(\bigvee_{a\in I} I(a)\Big) = \bigcup_{a\in I} S(a),$$

hence the sets $S(a)$ form a basis of the spectral topology.

The stability of this basis under finite unions and intersections follows from Lemma 1 (4), (5). □

Theorem 5. a) *$S(a)$ is compact for every $a \in A$.*

b) *If B is an open compact set of* Spec A *then $B = S(a)$ for some $a \in A$.*

P r o o f. a) Let $a \in A$, $I_\gamma \in \mathcal{I}(A)$, $\gamma \in \Gamma$, and let

$$S(a) \subseteq \bigcup_{\gamma\in\Gamma} S(I_\gamma) = S\Big(\bigvee_{\gamma\in\Gamma} I_\gamma\Big).$$

Then, by Proposition 3, $a \in \bigvee_{\gamma\in\Gamma} I_\gamma$, and hence, by [9], Lemma 2,

$$a * 0 \leqslant (b_1 * 0) + \ldots + (b_k * 0),$$

where $k \in \mathbb{N}$, $b_i \in I_{\gamma_i}$, $i = 1, \ldots k$. But this means that $a \in I_{\gamma_1} \vee \ldots \vee I_{\gamma_k}$, and so

$$S(a) \subseteq S\Big(\bigvee_{i=1}^{k} I_{\gamma_i}\Big) = \bigcup_{i=1}^{k} S(I_{\gamma_i}).$$

b) Let B be an open compact set. Then there exist $a_1, \ldots, a_n \in A$ such that $B = \bigcup_{i=1}^{n} S(a_i)$. Hence by Lemma 1 (4),

$$B = S\Big(\bigvee_{i=1}^{n} (a_i * 0)\Big).$$

□

Corollary 6. *The spectrum of a normal autometrized ℓ-algebra A is compact if and only if A contains an element a such that $I(a)=A$.*

This means, if A is a commutative ℓ-group then $\operatorname{Spec} A$ is compact if and only if A has a strong unit, and $\operatorname{Spec} A$ is compact for each Brouwerian algebra A.

If $\mathbf{x} \subseteq \operatorname{Spec} A$, put

$$\mathcal{D}\mathbf{x} = \bigcap\{P;\ P \in \mathbf{x}\}.$$

Proposition 7. a) *The closed sets in* $\operatorname{Spec} A$ *are exactly all* $H(I)$, *where* $I \in \mathcal{I}(A)$.
b) *If* $\mathbf{x} \subseteq \operatorname{Spec} A$, *then its closure is* $\bar{\mathbf{x}} = H(\mathcal{D}\mathbf{x})$.

Proof. a) $H(I) = \operatorname{Spec} A \setminus S(I)$.
b) $\mathbf{x} \subseteq H(\mathcal{D}\mathbf{x})$, hence $\bar{\mathbf{x}} \subseteq H(\mathcal{D}\mathbf{x})$, and so

$$\mathcal{D}\mathbf{x} = \mathcal{D}H(\mathcal{D}\mathbf{x}) \subseteq \mathcal{D}\bar{\mathbf{x}}.$$

But $\mathbf{x} \subseteq \bar{\mathbf{x}}$, therefore $\mathcal{D}\bar{\mathbf{x}} \subseteq \mathcal{D}\mathbf{x}$. Thus $\mathcal{D}\mathbf{x} = \mathcal{D}\bar{\mathbf{x}}$, which means

$$\bar{\mathbf{x}} = H(\mathcal{D}\bar{\mathbf{x}}) = H(\mathcal{D}\mathbf{x}).$$

□

Corollary 8. *If* $\mathbf{x} \subseteq \operatorname{Spec} A$, *then* $\mathbf{x}$ *is dense if and only if* $\bigcap\{P;\ P \in \mathbf{x}\} = \{0\}$.

3. Representable DRℓ-semigroups

Let us recall the notion of a dually residuated lattice ordered semigroup ($DR\ell$-semigroup) that has been introduced by K. L. N. Swamy in [8].

A system $A = (A, +, 0, \leqslant, -)$ is called a *$DR\ell$-semigroup* if

(1) $(A, +, 0, \leqslant)$ is a commutative lattice ordered monoid;

(2) for each $a, b \in A$ there exists a least element $x \in A$ such that $b + x \geqslant a$ (such x is denoted by $a - b$);

(3) $\forall a, b \in A;\ (a - b) \vee 0 + b \leqslant a \vee b$;

(4) $\forall a \in A;\ a - a \geqslant 0$.

Let us denote $a * b = (a - b) \vee (b - a)$ for $a, b \in A$. Then $(A, +, 0, \leqslant, *)$ is, by [8] and [9], a normal semiregular autometrized ℓ-algebra.

A $DR\ell$-semigroup A is called *representable* (see [10]) if $(a - b) \wedge (b - a) \leqslant 0$ for each $a, b \in A$. (For instance, commutative ℓ-groups and Boolean algebras are representable $DR\ell$-semigroups.)

Proposition 9. *Let A be a representable $DR\ell$-semigroup, let $P, Q \in \operatorname{Spec} A$ and let $P \parallel Q$. Then P and Q have in* $\operatorname{Spec} A$ *disjoint neighborhoods.*

P r o o f. Let $P, Q \in \operatorname{Spec} A$, $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exist $0 < a \in A$, $0 < b \in A$ such that $a \in P \setminus Q$ and $b \in Q \setminus P$. Denote $u = a - (a \wedge b)$ and $v = b - (a \wedge b)$. Le us show that $u \notin Q$ and $v \notin P$. Let, for example, $u \in Q$. By [4], Lemma 6, $a = (a \wedge b) + u$, and since $a \wedge b \in Q$, we have $a \in Q$, a contradiction. Hence $P \in S(u)$, $Q \in S(v)$ and by [4], Lemma 6, $u \wedge v = 0$. Thus $S(u) \cap S(v) = S(u \wedge v) = \emptyset$. □

If $\mathbf{x} \subseteq \operatorname{Spec} A$ then the topology of $\mathbf{x}$ induced by the spectral topology of $\operatorname{Spec} A$ will be called the *spectral topology on* $\mathbf{x}$.

Corollary 10. *If A is a representable $DR\ell$-semigroup and $\mathbf{x} \subseteq \operatorname{Spec} A$ is a set of pairwise non-comparable prime ideals, then the spectral topology of $\mathbf{x}$ is a T_2-topology.*

If $\mathbf{x} \subseteq \operatorname{Spec} A$ and $M \subseteq A$, put $S_{\mathbf{x}}(M) = S(M) \cap \mathbf{x}$.

Denote by $m(A)$ the set of all minimal and by $\mathcal{M}(A)$ the set of all maximal prime ideals of a representable $DR\ell$-semigroup A.

Theorem 11. *If A is a representable $DR\ell$-semigroup then the spectral topology of $m(A)$ is a T_2-topology and the sets $S_{m(A)}(a) = \{P \in m(A);\, a \notin P\}$, $a \in A$, form a basis of the space $m(A)$ composed by closed subsets.*

P r o o f. Let A be a representable $DR\ell$-semigroup. Obviously, the sets $S_{m(A)}(a)$, where $a \in A$, form a basis of the spectral topology of $m(A)$. Let $a \in A$ and let P be a minimal prime ideal in A. By [1], Proposition 2.4, either $a \notin P$ or $a^{\perp} \nsubseteq P$. Hence $S_{m(A)}(a) \cap S_{m(A)}(a^{\perp}) = \emptyset$ and $S_{m(A)} \cup S_{m(A)}(a^{\perp}) = m(A)$. Therefore, since $S_{m(A)}(a^{\perp})$ is open, $S_{m(A)}(a)$ is closed. □

Let A be a representable $DR\ell$-semigroup, $0 \neq a \in A$. Let us denote by $\operatorname{val}(a)$ the set of all values of a, i.e. the set of all ideals maximal with respect to the property of not containing a. (For $a = 0$, put $\operatorname{val}(a) = \emptyset$.) Let $P \in S(a)$. Then, by [6], Theorem 4, the set of ideals in A containing P is linearly ordered and by [6], Theorem 2, there are ideals in $\operatorname{val}(a)$ that contain P. Hence there is exactly one $M_P \in \operatorname{val}(a)$ such that $P \subseteq M_P$.

Let us denote by $\psi_a \colon S(a) \longrightarrow \operatorname{val}(a)$ the mapping such that $\psi \colon P \longmapsto M_P$.

Proposition 12. *The mapping ψ_a is continuous.*

P r o o f. Let $a \in A$, $P \in S(a)$ and let U be a neighborhood of M_P in $\operatorname{val}(a)$. We can suppose that $U = S(b) \cap \operatorname{val}(a)$ for some $b \in A$. If $Q \in \operatorname{val}(a) \setminus S(b)$, then we can choose a neighborhood U_Q of Q and a neighborhood V_Q of M_P such that $U_Q \cap V_Q = \emptyset$. It is evident that all U_Q, where Q runs over $\operatorname{val}(a) \setminus S(b)$, form a covering of $S(a) \setminus S(b)$. Since $S(a)$ is compact and $S(a) \setminus S(b)$ is closed in $S(a)$, $S(a) \setminus S(b)$ is compact, too. Hence there exist $n \in \mathbb{N}$ and $Q_1, \dots, Q_n \in S(a) \setminus S(b)$ such that $S(a) \setminus S(b) \subseteq U_{Q_1} \cup \dots \cup U_{Q_n}$.

Let us denote $C = S(a) \setminus (U_{Q_1} \cup \dots \cup U_{Q_n})$. We have $V_{Q_1} \cap \dots \cap V_{Q_n} \subseteq C$, therefore C is a neighborhood of M_P which is closed in $S(a)$, and $C \cap \mathrm{val}(a) \subseteq U$. Therefore $C \subseteq \psi_a^{-1}(C \cap \mathrm{val}(a)) \subseteq \psi_a^{-1}(U)$. Moreover, C, which is a neighborhood of M_P, is also a neighborhood of P. $\square$

Proposition 13. *If $a \in A$, then the set* $\mathrm{val}(a)$ *is a compact T_2-space.*

P r o o f. By Corollary 10, $\mathrm{val}(a)$ is a T_2-space. Further, $\mathrm{val}(a)$ is the image of the compact set $S(a)$ in the mapping ψ_a which is, by Proposition 12, continuous, hence $\mathrm{val}(a)$ is also compact. $\square$

The following theorem is now an immediate consequence.

Theorem 14. *If A is a representable DRℓ-semigroup then the space $\mathcal{M}(A)$ of all its maximal prime ideals is a T_2-space. If there exists $b \in A$ such that $I(b) = A$ then $\mathcal{M}(A)$ is compact.*

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