INERTIAL LAW OF SYMPLECTIC FORMS ON MODULES OVER PLURAL ALGEBRA

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Abstract. In this paper the problem of construction of the canonical matrix belonging to symplectic forms on a module over the so called plural algebra (introduced in [5]) is solved.

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I. Introduction

1. Definition. The *plural* **T**-*algebra of order* m is every linear algebra **A** on **T** having as a vector space over **T** a basis

$$\{1, \eta, \eta^2, \dots, \eta^{m-1}\} \text{ with } \eta^m = 0.$$

A plural algebra **A** is a local ring the maximal ideal of which is nilpotent. It was proved in [3] that the free finite generated **A**-module **M** (the so called **A**-space in the sense of [6]) has the following properties:

2.1. If one basis of **M** consists of n elements then each of its bases consists of the same number of n elements. (This is true in every free module over a commutative ring.)[1]

2.2. From every system of generators of **M** we may select a basis of **M**. (This is valid over every local ring.)[2]

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[1] See [1]

[2] See [6]

Moreover, in this case:

2.3. Any linearly independent system may be completed to a basis of $\mathbf{M}$.

2.4. Every maximal linearly independent system in $\mathbf{M}$ forms a basis of $\mathbf{M}$.

3. Let $\varphi_1, \dots, \varphi_k$ be a linearly independent system of linear forms $\mathbf{M} \to \mathbf{A}$. Then $\bigcap\limits_{1 \leqslant i \leqslant k} \operatorname{Ker} \varphi_i$ is a free $(n-k)$-dimensional submodule of $\mathbf{M}$.

4. Let K, L be free submodules of an $\mathbf{A}$-module $\mathbf{M}$. Then $K + L$ is a free $\mathbf{A}$-submodule if and only if $K \cap L$ is a free $\mathbf{A}$-submodule and the dimensions of $\mathbf{A}$-submodules K, L, $K \cap L$, $K + L$ fulfil the relation

$$\dim(K+L) + \dim(K \cap L) = \dim K + \dim L.$$

5. Agreement. Throughout the paper we denote by $\mathbf{A}$ the plural $\mathbf{T}$-algebra introduced in this section. The capital $\mathbf{M}$ always denotes the free n-dimensional module over the algebra $\mathbf{A}$.

6. Definition. A bilinear form $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ is called a *bilinear form of order* $k\,(0 \leqslant k \leqslant m-1)$ if
(1) $\forall(\underline{X}, \underline{Y}) \in \mathbf{M}^2; \quad \Phi(\underline{X}, \underline{Y}) \in \eta^k \mathbf{A}$,
(2) $\exists(\underline{U}, \underline{V}) \in \mathbf{M}^2; \quad \Phi(\underline{U}, \underline{V}) \notin \eta^{k+1} \mathbf{A}$.

The following proposition is taken form [4].

7. Proposition. *If Φ is a bilinear form of order k then there exists at least one form Λ of order 0 such that*

$$\Phi = \eta^k \Lambda.$$

II. Inertial law of symplectic forms on modules over plural algebra

Let the dimension n of $\mathbf{M}$ be an even number.

1. Definition. Let $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ be a symplectic form[3]. If all elements of the basis $\mathcal{U} = \{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \dots, \underline{U}_r, \underline{V}_r\}$ of $\mathbf{M}$ fulfil the conditions
(1) $\Phi(\underline{U}_i, \underline{U}_j) = \Phi(\underline{V}_i, \underline{V}_j) = 0$,
(2) $\Phi(\underline{U}_i, \underline{V}_i) = \{1, \eta, \eta^2, \dots, \eta^m\}$,
(3) $\Phi(\underline{U}_i, \underline{V}_j) = 0$ for $i \neq j$,

[3] A form Φ satisfies $\Phi(\underline{X}, \underline{X}) = 0$ for all $\underline{X} \in \mathbf{M}$.

then $\mathcal{U}$ is called *the symplectic basis of* $\mathbf{M}$ *with respect to* Φ.[4]

2. Remark. Relative to this basis the matrix of the symplectic form has the form

$$\left\|\begin{array}{cccc|cc}
0 & \varphi_{12} & 0 & 0 & 0 & 0\\
-\varphi_{12} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \varphi_{34} & 0 & 0\\
0 & 0 & -\varphi_{34} & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & \varphi_{n-1,n}\\
0 & 0 & 0 & 0 & -\varphi_{n-1,n} & 0
\end{array}\right\|$$

where $\varphi_{ij} \in \{1, \eta, \eta^2, \dots, \eta^m\}$.

3. Theorem. *Let* Φ *be a symplectic form on the module* $\mathbf{M}$. *Then there exists a symplectic basis of* $\mathbf{M}$ *with respect to* Φ.

Proof. By induction for $r = \frac{1}{2}n$.

1. The proposition is clear for $r = 1$.

2. Let the theorem be true for all $(n-2)$-dimensional $\mathbf{A}$-modules, $n \geqslant 4$.

(a) Let Φ be a form of order 0, i.e. $\exists(\underline{U}, \underline{V}) \in \mathbf{M}^2 \colon \Phi(\underline{U}, \underline{V})$ is a unit. Let us suppose—without loss of generality—that $\Phi(\underline{U}, \underline{V}) = 1$.

This implies that $\underline{U}$, $\underline{V}$ are linearly independent. Indeed, if $\alpha\underline{U} + \beta\underline{V} = \underline{o}$ then

$$0 = \Phi(\alpha\underline{U} + \beta\underline{V}, \underline{V}) = \alpha \cdot \Phi(\underline{U}, \underline{V}) + \beta \cdot \Phi(\underline{V}, \underline{V}) = \alpha.$$

Analogously, we obtain $\beta = 0$.

Let us consider linear forms $\varphi_U(\underline{X}) \equiv \Phi(\underline{U}, \underline{X})$ and $\varphi_V(\underline{X}) \equiv \Phi(\underline{V}, \underline{X})$. Evidently, they are linearly independent. According to Proposition I.3 $\mathcal{N} = \operatorname{Ker}\varphi_U \cap \operatorname{Ker}\varphi_V$ is a free $(n-2)$-dimensional submodule. Due to the induction hypothesis we may construct a symplectic basis $\{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \dots, \underline{U}_{r-1}, \underline{V}_{r-1}\}$ of $\mathcal{N}$ with respect to the form $\Phi|\mathcal{N}^2$.

Now, let us show $\mathbf{M} = \mathcal{N} \oplus [\underline{U}, \underline{V}]$. If $\underline{X} \in [\underline{U}, \underline{V}]$ then $\underline{X} = \xi\underline{U} + \zeta\underline{V}$. Consequently,

$$0 = \varphi_U(\underline{X}) = \Phi(\underline{U}, \xi\underline{U} + \zeta\underline{V}) = \xi \cdot \Phi(\underline{U}, \underline{U}) + \zeta \cdot \Phi(\underline{U}, \underline{V}) = \zeta.$$

In a similar way we get $\xi = 0$. This gives $\underline{X} = \underline{O}$ and therefore $\mathcal{N} \cap [\underline{U}, \underline{V}]$ is a 0-dimensional submodule. We have (by Proposition I.4) $\mathbf{M} = \mathcal{N} \oplus [\underline{Y}]$.

[4] For $m = 1$ (i.e. $\mathbf{A}$ is a field) we get the usual definition of a symplectic basis over fields (see [2]).

Since $\underline{U}_j, \underline{V}_j \in \mathcal{N}$ for every $j \in \mathbb{N}(r-1)$, hence $\Phi(\underline{U}_j, \underline{U}) = \Phi(\underline{V}_j, \underline{U}) = 0$ and $\Phi(\underline{U}_j, \underline{V}) = \Phi(\underline{V}_j, \underline{V}) = 0$. Thus $\{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \dots, \underline{U}_{r-1}, \underline{V}_{r-1}, \underline{U}, \underline{V}\}$ forms a symplectic basis of $\mathbf{M}$ with respect to Φ.

(b) Let Φ be a bilinear form of order $k\,(\neq 0)$. According to Proposition I.7 there exists a bilinear form Ψ of order 0 with $\Phi = \eta^k \Psi$. By (a) we can construct a symplectic basis for the form Ψ, which is also a symplectic basis for the form Φ. □

4. Definition. Let Φ be a symplectic form $\mathbf{M}^2 \to \mathbf{A}$ and let the basis $\mathcal{U} = \{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \dots, \underline{U}_r, \underline{V}_r\}$ be symplectic with respect to Φ. Let us define a system of sets $\mathcal{J}_0, \dots, \mathcal{J}_m$ as follows:

$$\mathcal{J}_k = \{i \in \mathbb{N}(r); \Phi(\underline{U}_i, \underline{V}_i) = \eta^k\}, \quad 0 \leqslant k \leqslant m.$$

If we denote $\pi_k = 2\,\mathrm{card}(\mathcal{J}_k)$, $\quad 0 \leqslant k \leqslant m$, then

$$\mathfrak{Ch}(\Phi, \mathcal{U}) = (\pi_0, \dots, \pi_m)$$

is called the *characteristic of the symplectic form Φ with respect to the basis $\mathcal{U}$.*

5. Definition. For any symplectic form $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ let us denote by $\mathcal{V}_k^\Phi$ the set

$$\{\underline{Y} \in \mathbf{M}; \eta^k \Phi(\underline{X}, \underline{Y}) = 0,\ \forall \underline{X} \in \mathbf{M}\}, \quad 0 \leqslant k \leqslant m.$$

The following lemma is evident:

6. Lemma. *If $\mathcal{U}$ is a basis of $\mathbf{M}$ and Φ is symplectic form, then*

$$\mathcal{V}_k^\Phi = \{\underline{Y} \in \mathbf{M}; \eta^k \Phi(\underline{U}, \underline{Y}) = 0,\ \forall \underline{U} \in \mathcal{U}\}, \quad 0 \leqslant k \leqslant m.$$

7. Proposition. *Let Φ be a symplectic form $\mathbf{M}^2 \to \mathbf{A}$ and let $\mathcal{U}$ be symplectic with respect to Φ. Then a submodule $\mathcal{V}_k^\Phi$ of $\mathbf{M}$ as an $\mathbf{T}$-vector subspace has the dimension*

$$\dim_{\mathbf{T}} \mathcal{V}_k^\Phi = \sum_{j=0}^{m-k-1} (k+j)\pi_j + m \sum_{j=m-k}^{m} \pi_j,$$

where $(\pi_0, \dots, \pi_m) = \mathfrak{Ch}(\Phi, \mathcal{U})$.

P r o o f. $\mathcal{V}_k^\Phi$ is clearly a submodule of $\mathbf{M}$. Let $\mathcal{U} = \{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \dots, \underline{U}_r, \underline{V}_r\}$ and let us consider a $\underline{X} \in \mathcal{V}_k^\Phi$, $\underline{X} = \sum_{i=1}^{r} \xi_i \underline{U}_i + \sum_{i=1}^{r} \zeta_i \underline{V}_i$. Putting $\gamma_j = \Phi(\underline{U}_j, \underline{V}_j)$, $j \in \mathbb{N}(r)$, we obtain

$$\Phi(\underline{X}, \underline{U}_j) = -\zeta_j \gamma_j \text{ and } \Phi(\underline{X}, \underline{V}_j) = \xi_j \gamma_j,$$

which yields $\underline{X} \in \mathcal{V}_k^\Phi \Leftrightarrow \forall i,\ i \in \mathbb{N}(r);\ \eta^k \Phi(\underline{X}, \underline{U}_i) = \eta^k \Phi(\underline{X}, \underline{V}_i) = 0 \Leftrightarrow \forall i,\ i \in \mathbb{N}(r);$ $\eta^k \gamma_i \zeta_i = \eta^k \gamma_i \xi_i = 0$. As every $\gamma_i = \eta^{k(i)}$ we get (according to Definition 4) that $\underline{X} \in \mathcal{V}_k^\Phi$ if and only if the following conditions are valid:

$$
\begin{aligned}
&(0) \quad i \in \mathcal{J}_0 \Rightarrow \xi_i, \zeta_i \in \eta^{m-k} \mathbf{A} \\
&(1) \quad i \in \mathcal{J}_1 \Rightarrow \xi_i, \zeta_i \in \eta^{m-k-1} \mathbf{A} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&(\mathrm{j}) \quad i \in \mathcal{J}_j \Rightarrow \xi_i, \zeta_i \in \eta^{m-k-j} \mathbf{A}, 0 \leqslant j \leqslant m-k-1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&(\mathrm{m-k-1}) \quad i \in \mathcal{J}_{m-k-1} \Rightarrow \xi_i, \zeta_i \in \eta \mathbf{A} \\
&(\mathrm{m-k}) \quad i \in \bigcup_{s=0}^{k} \mathcal{J}_{m-s} \Rightarrow \xi_i, \zeta_i \in \mathbf{A}
\end{aligned}
$$

Let us construct the following system of submodules in $\mathcal{V}_k^\Phi$:

$$
\begin{aligned}
\mathcal{V}_{kj}^\Phi &= \{\underline{X} \in \mathbf{M};\ \underline{X} \in \mathcal{V}_k^\Phi \wedge \underline{X} = \sum_{i \in \mathcal{J}_j} \xi_i \underline{U}_i\}, \quad 0 \leqslant j \leqslant m, \\
\mathcal{W}_{kj}^\Phi &= \{\underline{X} \in \mathbf{M};\ \underline{X} \in \mathcal{V}_k^\Phi \wedge \underline{X} = \sum_{i \in \mathcal{J}_j} \zeta_i \underline{V}_i\}, \quad 0 \leqslant j \leqslant m.
\end{aligned}
$$

Clearly, $\mathcal{V}_k^\Phi = \mathcal{V}_{k0}^\Phi \oplus \mathcal{V}_{k1}^\Phi \oplus \ldots \oplus \mathcal{V}_{km}^\Phi \oplus \mathcal{W}_{k0}^\Phi \oplus \mathcal{W}_{k1}^\Phi \oplus \ldots \oplus \mathcal{W}_{km}^\Phi$.

We get [from (0)], that $\mathcal{V}_{k0}^\Phi$ or $\mathcal{W}_{k0}^\Phi$, has $\mathbf{T}$-basis

$$
\bigcup_{i \in \mathcal{J}_0} \{\eta^{m-k} \underline{U}_i, \ldots, \eta^{m-1} \underline{U}_i\} \text{ or } \bigcup_{i \in \mathcal{J}_0} \{\eta^{m-k} \underline{V}_i, \ldots, \eta^{m-1} \underline{V}_i\}, \text{ respectively;}
$$

therefore $\dim_{\mathbf{T}} \mathcal{V}_{k0}^\Phi = \dim_{\mathbf{T}} \mathcal{W}_{k0}^\Phi = \frac{1}{2}\pi_0 k$. Analogously, conditions (j) imply that $\dim_{\mathbf{T}} \mathcal{V}_{kj}^\Phi = \dim_{\mathbf{T}} \mathcal{W}_{kj}^\Phi = \frac{1}{2}\pi_j (k+j)$, and the condition (m − k) implies that $\dim_{\mathbf{T}} \mathcal{V}_{kj}^\Phi = \dim_{\mathbf{T}} \mathcal{W}_{kj}^\Phi = \frac{1}{2}\pi_j m$, $m-k \leqslant j \leqslant m$.

The relation for the $\mathbf{T}$-dimension of $\mathcal{V}_k^\Phi$ is now evident. □

8. Theorem (inertial law). *Let a symplectic form* $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ *be given. If* $\mathcal{U}$, $\mathcal{V}$ *are arbitrary symplectic bases of* $\mathbf{M}$ *with respect to this form, then*

$$
\mathfrak{Ch}(\Phi, \mathcal{U}) = \mathfrak{Ch}(\Phi, \mathcal{V}).
$$

P r o o f. Let $\mathfrak{Ch}(\Phi,\mathcal{U}) = (\pi_0,\dots,\pi_m)$. Then Proposition II.7 implies

$$\dim_{\mathbf{T}} \mathcal{V}_k^{\Phi} = \sum_{j=0}^{m-k} \pi_j(k+j) + \sum_{j=m-k+1}^{m} \pi_j \cdot m,$$
$$\dim_{\mathbf{T}} \mathcal{V}_{k-1}^{\Phi} = \sum_{j=0}^{m-k} \bigl(\pi_j(k+j) - \pi_j\bigr) + \sum_{j=m-k+1}^{m} \pi_j \cdot m.$$

Consequently, we have $\dim_{\mathbf{T}} \mathcal{V}_k^{\Phi} - \dim_{\mathbf{T}} \mathcal{V}_{k-1}^{\Phi} = \sum\limits_{j=0}^{m-k} \pi_j$. Let $\mathfrak{Ch}(\Phi,\mathcal{V}) = (\nu_0,\dots,\nu_m)$. Then we obtain $\dim_{\mathbf{T}} \mathcal{V}_k^{\Phi} - \dim_{\mathbf{T}} \mathcal{V}_{k-1}^{\Phi} = \sum\limits_{h=0}^{m-k} \nu_h$, i.e. $\sum\limits_{j=0}^{m-k} \pi_j = \sum\limits_{h=0}^{m-k} \nu_h$. Putting $k = m$, $m-1$, …, 0, we get

$$\pi_0 = \nu_0,\ \pi_0 + \pi_1 = \nu_0 + \nu_1,\ \dots,\ \sum_{j=0}^{m-k} \pi_j + \pi_m = \sum_{h=0}^{m-k} \nu_h + \nu_m,$$

which successively yields $\pi_0 = \nu_0$, $\pi_1 = \nu_1$, …, $\pi_m = \nu_m$. □

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