EXISTENCE OF OSCILLATORY AND NONOSCILLATORY SOLUTIONS FOR A CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Summary. For a certain class of functional differential equations with perturbations conditions are given such that there exist solutions which converge to solutions of the equations without perturbation.

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0. Introduction

Let

$$\Delta_\sigma x(t) = x(t) - x(t-\sigma),$$

$\sigma > 0$ being a constant, and consider the neutral functional differential equation

$$\text{(A)} \qquad \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[\Delta_\sigma^m x(t)\big] + f\big(t, x\big(g(t)\big)\big) = 0, \quad t \geqslant t_0,$$

where $m \geqslant 1$, $n \geqslant 1$, $t_0 > 0$, and Δ_σ^m is the m-th iterate of Δ_σ, i.e.

$$\Delta_\sigma^m x(t) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x(t - i\sigma).$$

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The conditions we always assume for f and g are as follows:

$$\text{(B)}\qquad \begin{aligned}&\text{(a)}\quad g\colon [t_0,\infty)\to[0,\infty) \text{ is continuous, and } \lim_{t\to\infty} g(t)=\infty;\\ &\text{(b)}\quad f\colon [t_0,\infty)\times\mathbb{R}\to\mathbb{R} \text{ is continuous and satisfies}\\ &\qquad\quad |f(t,x)|\leqslant F(t,|x|),\quad (t,x)\in[t_0,\infty)\times\mathbb{R},\end{aligned}$$

for some continuous function $F(t,u)$ on $[t_0,\infty)\times[0,\infty)$ which is nondecreasing in u for each fixed $t\geqslant t_0$.

By a solution of (A) we mean a continuous function $x\colon [T_x-m\sigma,\infty)\to\mathbb{R}$, $T_x\geqslant t_0$, such that $\Delta_\sigma^m x(t)$ is n-times continuously differentiable and satisfies the equation for $t\geqslant T_x$. The solutions vanishing in a neighborhood of infinity will be excluded from our consideration. A solution is said to be oscillatory if it has an infinite sequence of zeros clustering at $t=\infty$; otherwise a solution is said to be nonoscillatory.

The objective of this paper is to develop an existence theory enabling us to construct various types of oscillatory and nonoscillatory solutions of neutral equations of the form (A). We make use of the observation that the associated unperturbed equation $\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Delta_\sigma^m x(t)=0$ has the solutions

$$\omega_\sigma(t)t^j,\quad j=0,1,\dots,m-1, \tag{0.1}$$

$$ct^k,\quad k=m,m+1,\dots,m+n-1, \tag{0.2}$$

where $\omega_\sigma(t)$ is an arbitrary σ-periodic function and c is an arbitrary constant, and intend to establish the existence of solutions $x_j(t)$, $x_k(t)$ of (A) which are asymptotic to the functions (0.1) and (0.2) in the sense that

$$x_j(t)=\omega_\sigma(t)t^j+o(1)\quad \text{as } t\to\infty,\ j=0,1,\dots,m-1, \tag{I}$$

$$x_k(t)=ct^k+o(1)\quad \text{as } t\to\infty,\ k=m,m+1,\dots,m+n-1. \tag{II}$$

It is clear that the solutions of the type (II) are nonoscillatory whereas the solutions of the type (I) are oscillatory or nonoscillatory according to whether the periodic functions $\omega_\sigma(t)$ involved are oscillatory or nonoscillatory.

The construction of solutions of types (I) and (II) of (A) is presented in Section 2. Our main tool is the Schauder-Tychonoff fixed point theorem applied to nonlinear operators formed by suitably chosen "inverses" of the differential operator $\frac{\mathrm{d}^n}{\mathrm{d}t^n}$ and the iterated difference operator Δ_σ^m. Preliminary results needed in Section 2 are collected in Section 1.

It seems that very little is known about the existence of solutions, oscillatory or nonoscillatory, of neutral equations whose leading parts contain the difference operator Δ_σ and/or its iterate. To the best of the authors' knowledge Jaroš and Kusano [2] and Naito [6] are the only references dealing with this subject for the case $m=1$. For related results regarding neutral equations of the form (A) with Δ_σ replaced by $\Delta_{\sigma,\lambda}$, where

$$\Delta_{\sigma,\lambda}x(t)=x(t)-\lambda x(t-\sigma),\quad \lambda\neq 1,$$

we refer to Jaroš and Kusano [1], Kitamura and Kusano [3], Naito [4, 5] and Ruan [7].

1. Preliminaries

In this preparatory section we state some results which are crucial for proving the main existence theorems in the next section.

We denote by $S[T,\infty)$ the set of all functions $\xi\in C[T,\infty)$ such that the sequence

$$\eta(t)=\sum_{i=1}^{\infty}\xi(t+i\sigma),\quad t\geqslant T-\sigma, \tag{1.1}$$

converges uniformly on any compact subinterval of $[T-\sigma,\infty)$. We define Ψ to be the mapping which sends each $\xi\in S[T,\infty)$ to a function $\eta(t)$ defined by (1.1). Further, let Ψ^ℓ denote the ℓ-th iterate of Ψ which is defined on the set

$$S^\ell[T,\infty)=\bigl\{\xi\in S^{\ell-1}[T,\infty)\colon \Psi^{\ell-1}\xi\in S\bigl[T-(\ell-1)\sigma,\infty\bigr]\bigr\},\quad \ell=1,2,\dots,$$

where it is understood that $\Psi^0=\mathrm{id}$ (identity mapping) and $S^0[T,\infty)=C[T,\infty)$.

Lemma 1. *Let $\ell\geqslant 1$ be an integer. If $\xi\in S^\ell[T,\infty)$, then $\Psi^\ell\xi$ is a solution of the difference equation*

$$\Delta_\sigma^\ell x(t)=(-1)^\ell\xi(t),\quad t\geqslant T, \tag{1.2}$$

and satisfies

$$\Psi^\ell\xi(t)=o(1)\quad \text{as } t\to\infty. \tag{1.3}$$

Proof. Let $\ell=1$. If $\xi\in S[T,\infty)$, then by (1.1) $\Psi\xi$ solves the equation $\Delta_\sigma x(t)=-\xi(t)$, $t\geqslant T$, so that (1.2) holds for $\ell=1$. Let $\varepsilon>0$ be given arbitrarily. Since, by hypothesis, (1.1) converges uniformly on $[T-\sigma,T)$, there exists $P\in\mathbb{N}$ such that

$$\Bigl|\sum_{i=p+1}^{\infty}\xi(t+i\sigma)\Bigr|<\varepsilon\quad \text{for all } t\in[T-\sigma,T) \text{ and } p\geqslant P. \tag{1.4}$$

Let $t \geqslant t_1 = T + P\sigma$ and choose $p \in \mathbb{N}$ such that $t - p\sigma \in [T - \sigma, T)$. Then

$$p > \frac{t - T}{\sigma} \geqslant \frac{t_1 - T}{\sigma} = P,$$

and so we have, in view of (1.4),

$$\begin{aligned} \bigl|\Psi\xi(t)\bigr| &= \Bigl|\sum_{i=1}^{\infty} \xi(t + i\sigma)\Bigr| = \Bigl|\sum_{i=1}^{\infty} \xi\bigl(t - p\sigma + (i + p)\sigma\bigr)\Bigr| \\ &= \Bigl|\sum_{i=p+1}^{\infty} \xi(t - p\sigma + i\sigma)\Bigr| < \varepsilon, \end{aligned}$$

which implies that $\Psi\xi(t) \to 0$ as $t \to \infty$. Thus (1.3) holds for $\ell = 1$. The proof for a general $\ell \geqslant 1$ is done by an inductive argument, and we leave the details to the reader. ☐

Lemma 2. *Let $\ell \geqslant 1$ be an integer. Let $v \in S^{\ell}[T, \infty)$ be nonnegative for $t \geqslant T$ and define*

$$\overline{U} = \bigl\{u \in C[T, \infty) \colon |u(t)| \leqslant v(t), \quad t \geqslant T\bigr\}. \tag{1.5}$$

Then the following statements hold.

(i) *Ψ^{ℓ} is continuous on $\overline{U}$ in the $C[T, \infty)$-topology.*

(ii) *If $\overline{U}$ is locally equicontinuous on $[T, \infty)$, then $\Psi^{\ell}(\overline{U})$ is locally equicontinuous on $[T - \ell\sigma, \infty)$.*

Proof. We give a proof for the case $\ell = 1$.

(i) Suppose that $v \in S[T, \infty)$. Let $\{u_\nu\}$ be a sequence in $\overline{U}$ converging to $u \in \overline{U}$ in $C[T, \infty)$. Take an arbitrary compact subinterval I of $[T - \sigma, \infty)$. Since $v \in S[T, \infty)$, given any $\varepsilon > 0$, there is $p \in \mathbb{N}$ such that

$$\sum_{i=p+1}^{\infty} v(t + i\sigma) < \frac{\varepsilon}{3}, \quad t \in I. \tag{1.6}$$

Since $\{u_\nu\}$ converges to u uniformly on I, there is $\nu_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{p} |u_\nu(t + i\sigma) - u(t + i\sigma)| < \frac{\varepsilon}{3}, \quad t \in I, \ \nu \geqslant \nu_0.$$

It follows that

$$\begin{aligned} |\Psi u_\nu(t) - \Psi u(t)| &\leqslant \sum_{i=1}^{p} |u_\nu(t + i\sigma) - u(t + i\sigma)| + \sum_{i=p+1}^{\infty} |u_\nu(t + i\sigma)| + \sum_{i=p+1}^{\infty} |u(t + i\sigma)| \\ &< \frac{\varepsilon}{2} + 2\sum_{i=p+1}^{\infty} v(t + i\sigma) < \varepsilon, \quad t \in I, \ \nu \geqslant \nu_0, \end{aligned}$$

implying that $\Psi u_\nu(t) \to \Psi u(t)$ uniformly on I. Since I is arbitrary, this shows the convergence $\Psi u_\nu \to \Psi u$ in the topology of $C[T,\infty)$. Thus Ψ is a continuous mapping on $\overline{U}$.

(ii) Let $I \subset [T-\sigma,\infty)$ be any compact interval. Let $\varepsilon > 0$ be given. Choose $p \in \mathbb{N}$ such that (1.6) holds. By hypothesis, $\overline{U}$ is equicontinuous on I, and so there is a constant $\delta > 0$ such that

$$\sum_{i=1}^{p} |u(t+i\sigma) - u(s+i\sigma)| < \frac{\varepsilon}{3} \quad \text{for all } u \in \overline{U},$$

provided $|t-s| < \delta, \quad t,s \in I$. Consequently, $|t-s| < \delta, \quad t,s \in I$, implies that in view of (1.6)

$$\begin{aligned}|\Psi u(t) - \Psi u(s)| &\leqslant \sum_{i=1}^{p} |u(t+i\sigma) - u(s+i\sigma)| + \sum_{i=p+1}^{\infty} |u(t+i\sigma)| + \sum_{i=p+1}^{\infty} |u(s+i\sigma)| \\ &< \frac{\varepsilon}{3} + \sum_{i=p+1}^{\infty} v(t+i\sigma) + \sum_{i=p+1}^{\infty} v(s+i\sigma) < \varepsilon, \quad \text{for all } \overline{U},\end{aligned}$$

which shows that $\Psi(\overline{U})$ is equicontinuous on I. Because of the arbitrariness of I it follows that $\Psi(\overline{U})$ is locally equicontinuous on $[T-\sigma,\infty)$. □

Lemma 3. *Let $v \in C[T,\infty)$ be a nonnegative function on $[T,\infty)$ satisfying*

$$\int_T^\infty t^{\ell+p} v(t)\,\mathrm{d}t < \infty, \tag{1.7}$$

where $\ell \geqslant 1$ and $p \geqslant 0$ are integers. Then

$$\int_t^\infty (s-t)^p v(s)\,\mathrm{d}s \in S^\ell[T,\infty) \tag{1.8}$$

and

$$\Psi^\ell\Big(\int_t^\infty (s-t)^p v(s)\,\mathrm{d}s\Big) \leqslant \frac{1}{\sigma^\ell}\int_{t+\ell\sigma}^\infty (s-t)^{\ell+p} v(s)\,\mathrm{d}s, \quad t \geqslant T-\ell\sigma. \tag{1.9}$$

Proof. We prove the lemma for the case $\ell = 1$. The proof for the general ℓ follows by induction. Suppose that (1.7) holds for $\ell = 1$. Then

$$\begin{aligned}\Psi\Big(\int_t^\infty (s-t)^p v(s)\,\mathrm{d}s\Big) &= \sum_{i=1}^\infty \int_{t+i\sigma}^\infty (s-t-i\sigma)^p v(s)\,\mathrm{d}s \\ &\leqslant \sum_{i=1}^\infty \Big(\sum_{j=i}^\infty \int_{t+j\sigma}^{t+(j+1)\sigma} (s-t)^p v(s)\,\mathrm{d}s\Big), \quad t \geqslant T-\sigma.\end{aligned}$$

Interchanging the order of summation in the last term above and noting that

$$j \leqslant \frac{s-t}{\sigma} \quad \text{if} \quad s \in \big[t + j\sigma, t + (j+1)\sigma\big],$$

we find

$$\begin{aligned}
\Psi\Big(\int_t^\infty (s-t)^p v(s)\,\mathrm{d}s\Big) &= \sum_{j=1}^\infty \int_{t+j\sigma}^{t+(j+1)\sigma} \sum_{i=1}^{j} (s-t)^p v(s)\,\mathrm{d}s \\
&= \sum_{j=1}^\infty \int_{t+j\sigma}^{t+(j+1)\sigma} j(s-t)^p v(s)\,\mathrm{d}s \\
&\leqslant \frac{1}{\sigma} \sum_{j=1}^\infty \int_{t+j\sigma}^{t+(j+1)\sigma} (s-t)^{p+1} v(s)\,\mathrm{d}s \\
&= \frac{1}{\sigma} \int_{t+\sigma}^\infty (s-t)^{p+1} v(s)\,\mathrm{d}s, \quad t \geqslant T - \sigma,
\end{aligned}$$

which shows that (1.9) holds for $\ell = 1$. □

2. Existence of solutions

The main results of this paper are as follows:

Theorem 1. *Let $j \in \{0, 1, \ldots, m-1\}$ and let $\omega_\sigma(t) \not\equiv 0$ be a continuous periodic function of period σ. If there is a constant $a > 0$ such that*

$$\int_{t_0}^\infty t^{m+n-1} F\big(t, a[g(t)]^j\big)\,\mathrm{d}t < \infty, \tag{2.1}$$

then (A) *has a solution $x(t)$ satisfying*

$$x(t) = c\omega_\sigma(t) t^j + o(1) \quad \text{as } t \to \infty \tag{2.2}$$

for some nonzero constant c.

Theorem 2. *Let $k \in \{m, m+1, \ldots, m+n-1\}$ and suppose that there is a constant $a > 0$ such that*

$$\int_{t_0}^\infty t^{m+n-1} F\big(t, a[g(t)]^k\big)\,\mathrm{d}t < \infty. \tag{2.3}$$

Then (A) *has a solution* $x(t)$ *satisfying*

$$x(t) = ct^k + o(1) \quad \text{as } t \to \infty \tag{2.4}$$

for a nonzero constant c.

Proof of Theorem 1. Choose $c > 0$ such that $c\big(\max_t |\omega_\sigma(t)| + 1\big) \leqslant a$. Let $T > t_0$ be large enough so that

$$T_* = \min\big\{T - m\sigma, \inf_{t \geqslant T} g(t)\big\} \geqslant \max\{t_0, 1\} \tag{2.5}$$

and

$$\int_T^\infty t^{m+n-1} F\big(t, a[g(t)]^j\big) \leqslant c\sigma^m. \tag{2.6}$$

Let X and Y stand for the sets defined by

$$\begin{aligned} X &= \big\{x \in C[T_*, \infty)\colon |x(t)| \leqslant at^j, \quad t \geqslant T_*\big\}, \\ Y &= \big\{y \in C[T,\infty)\colon |y(t)| \leqslant w(t), \\ &\qquad\qquad |y(t) - y(s)| \leqslant |w(t) - w(s)|,\ t, s \geqslant T\}, \end{aligned} \tag{2.7}$$

where

$$w(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} F\big(s, a[g(s)]^j\big)\,\mathrm{d}s, \quad t \geqslant T. \tag{2.8}$$

Define $F_1\colon Y \to C[T_*, \infty)$ to be a mapping which assigns to each $y \in Y$ a function $x(t)$ given by

$$\begin{cases} x(t) = c\omega_\sigma(t)t^j + (-1)^m \Psi^m y(t), & t \geqslant T - m\sigma, \\ x(t) = x(T - m\sigma)\frac{t^j}{(T-m\sigma)^j}, & T_* \leqslant t \leqslant T - m\sigma, \end{cases} \tag{2.9}$$

and define $F_2\colon X \to C[T, \infty)$ to be a mapping which assigns to each $x \in X$ a function $y(t)$ defined by

$$y(t) = (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x\big(g(s)\big)\big)\,\mathrm{d}s, \quad t \geqslant T. \tag{2.10}$$

Finally, define a mapping $F\colon X \times Y \to C[T_*, \infty) \times C[T, \infty)$ by

$$F(x, y) = (F_1 y, F_2 x), \quad (x, y) \in X \times Y. \tag{2.11}$$

We want to verify that F maps $X \times Y$, which is a closed subset of the Fréchet space $C[T_*, \infty) \times C[T, \infty)$, into a relatively compact subset of $X \times Y$.

(i) We first show that F maps $X \times Y$ into itself. To this end it suffices to show that $F_1(Y) \subset X$ and $F_2(X) \subset Y$. Let $y \in Y$. Applying Lemma 3 to $w(t)$ given by (2.8), we see that $y \in S^m[T, \infty)$ and

$$
\begin{aligned}
|(-1)^m \Psi^m y(t)| &\leqslant \Psi^m w(t) \leqslant \frac{1}{\sigma^m} \int_{t+m\sigma}^{\infty} (s-t)^{m+n-1} F\big(s, a[g(s)]^j\big)\,\mathrm{d}s \\
&\leqslant \frac{1}{\sigma^m} \int_{T}^{\infty} (s-t)^{m+n-1} F\big(s, a[g(s)]^j\big) \leqslant c, \quad t \geqslant T - m\sigma,
\end{aligned}
$$

where (2.6) has been used. Hence, from the first equation in (2.9) we have, in view of the choice of c,

$$
|F_1 y(t)| \leqslant c \max_t |\omega_\sigma(t)| t^j + c \leqslant a t^j, \quad t \geqslant T - m\sigma.
$$

Since the above inequality implies $|F_1 y(T - m\sigma)| \leqslant a(T - m\sigma)^j$, the second equation in (2.9) shows that $|F_1 y(t)| \leqslant a t^j$ for $T_* \leqslant t \leqslant T - m\sigma$. This proves that $F_1(Y) \subset X$. Now let $x \in X$. Since $\big|f\big(t, x(g(t))\big)\big| \leqslant F\big(t, a[g(t)]^j\big)$, $t \geqslant T$, it follows from (2.10) and (2.8) that $|F_2 x(t)| \leqslant w(t)$ for $t \geqslant T$. Further, if $n \geqslant 2$, then noting that

$$
(F_2 x)'(t) = (-1)^{n-2} \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} f\big(s, x\big(g(s)\big)\big)\,\mathrm{d}s, \quad t \geqslant T,
$$

we obtain

$$
\begin{aligned}
|F_2 x(t) - F_2 x(s)| &= \Big| \int_s^t (F_2 x)'(r)\,\mathrm{d}r \Big| \\
&\leqslant \int_s^t \int_r^{\infty} \frac{(\varrho - r)^{n-2}}{(n-2)!} \big| f\big(\varrho, x\big(g(\varrho)\big)\big)\big| \,\mathrm{d}\varrho\,\mathrm{d}r \\
&\leqslant \int_s^t \int_r^{\infty} \frac{(\varrho - r)^{n-2}}{(n-2)!} F\big(\varrho, a[g(\varrho)]^j\big)\,\mathrm{d}\varrho\,\mathrm{d}r \\
&= |w(t) - w(s)| \quad \text{for } t \geqslant s \geqslant T.
\end{aligned}
$$

If $n = 1$, then for $t \geqslant s \geqslant T$

$$
\begin{aligned}
|F_2 x(t) - F_2 x(s)| &= \Big| \int_s^t f\big(r, x\big(g(r)\big)\big)\,\mathrm{d}r \Big| \\
&\leqslant \int_s^t F\big(r, a[g(r)]^j\big)\,\mathrm{d}r = |w(t) - w(s)|.
\end{aligned}
$$

Thus we conclude that $F(X) \subset Y$.

(ii) Now we show that F is continuous. Let $\{(x_\nu, y_\nu)\}$ be a sequence in $X \times Y$ converging to $(x,y) \in X \times Y$ in the topology of $C[T_*,\infty) \times C[T,\infty)$. The Lebesgue dominated convergence theorem then implies that the convergence

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\bigl(s, x_\nu(g(s))\bigr)\,\mathrm{d}s \to \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\bigl(s, x(g(s))\bigr)\,\mathrm{d}s$$

is uniform on compact subintervals of $[T,\infty)$, which means that $F_2 x_\nu \to F_2 x$ in $C[T,\infty)$. That $F_1 y_\nu \to F_1 y$ in $C[T_*,\infty)$ is an immediate consequence of the first statement of Lemma 2. This proves the continuity of F on $X \times Y$.

(iii) Finally, we show that $F(X \times Y)$ is relatively compact in $C[T_*,\infty) \times C[T,\infty)$. To this end it suffices to verify that $F_1(Y)$ and $F_2(X)$ are relatively compact in $C[T_*,\infty)$ and $C[T,\infty)$, respectively. The relative compactness of $F_2(X)$ in $C[T,\infty)$ is obvious, since by (2.10) and (2.8)

$$|(F_2x)'(t)| \leqslant |\omega'(t)|, \quad t \geqslant T.$$

On the other hand, in view of (2.7) we see that the set Y is locally equicontinuous on $[T,\infty)$. Therefore $\Psi^m(Y)$ is equicontinuous on $[T-m\sigma,\infty)$ by the second statement of Lemma 2. This fact combined with (2.9) defining F_1 then shows that $F_1(Y)$ is locally equicontinuous on $[T-m\sigma,\infty)$. Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are satisfied, and so there exists an element $(x,y) \in X \times Y$ such that $(x,y) = F(x,y)$, or $x = F_1 y$, $y = F_2 x$, which implies

$$x(t) = c\omega_\sigma(t)t^j + (-1)^m \Psi^m y(t), \quad t \geqslant T, \tag{2.12}$$

$$y(t) = (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\bigl(s, x(g(s))\bigr)\,\mathrm{d}s, \quad t \geqslant T. \tag{2.13}$$

Applying Lemma 1 and noting that $\Delta_\sigma^m\bigl(c\omega_\sigma(t)t^j\bigr) \equiv 0$, we see from (2.12) that

$$\Delta_\sigma^m x(t) = y(t),\ t \geqslant T \quad \text{and} \quad x(t) - c\omega_\sigma(t) = o(1) \quad \text{as } t \to \infty.$$

Combining this with $\frac{\mathrm{d}^n}{\mathrm{d}t^n} y(t) = -f\bigl(t, x(g(t))\bigr)$ which follows from (2.13), we conclude that the function $x(t)$ is a solution of the neutral equation (A) having the required asymptotic property (2.2). This completes the proof of Theorem 1. □

P r o o f of Theorem 2. The proof is essentially the same as that of Theorem 1, and so we only give a brief sketch of it. Take $c > 0$ such that $2c \leqslant a$ and let $T > t_0$ be such that (2.5) holds and

$$\int_T^\infty t^{m+n-1} F\bigl(t, a[g(t)]^k\bigr)\,\mathrm{d}t \leqslant c\sigma^m.$$

Define the sets $X \subset C[T_*,\infty)$ and $Y \subset C[T,\infty)$ by

$$X = \{x \in C[T_*,\infty)\colon |x(t)| \leqslant at^k, \quad t \geqslant T_*\},$$
$$Y = \{y \in C[T,\infty)\colon |y(t)| \leqslant w(t), \quad |y(t)-y(s)| \leqslant |w(t)-w(s)|,\ t,s \geqslant T\},$$

and the mappings $F_1\colon Y \to C[T_*,\infty)$ and $F_2\colon X \to C[T,\infty)$ by

$$\begin{cases} F_1y(t) = ct^k + (-1)^m\Psi^m y(t), & t \geqslant T - m\sigma,\\ F_1y(t) = F_1y(T-m\sigma)\frac{t^k}{(T-m\sigma)^k}, & T_* \leqslant t \leqslant T - m\sigma, \end{cases}$$

and

$$F_2x(t) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x\big(g(s)\big)\big)\,\mathrm{d}s, \quad t \geqslant T.$$

Then the mapping $F\colon X \times Y \to C[T_*,\infty) \times C[T,\infty)$ defined by

$$F(x,y) = (F_1y, F_2x), \quad (x,y) \in X \times Y$$

can be shown to be continuous and sends $X \times Y$ into a relatively compact subset of $X \times Y$. Let $(x,y) \in X \times Y$ be a fixed point of F. Then

$$x(t) = ct^k + (-1)^m\Psi^m y(t), \quad t \geqslant T - m\sigma \tag{2.14}$$

and

$$y(t) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x\big(g(s)\big)\big)\,\mathrm{d}s, \quad t \geqslant T.$$

Let Δ_σ^m operate on both sides of (2.14). Using Lemma 1, we find that

$$y(t) = \Delta_\sigma^m x(t) + p_{k,m}(t), \quad t \geqslant T,$$

where $p_{k,m}(t)$ is a polynomial in t of degree $k-m$. It follows that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}y(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[\Delta_\sigma^m x(t)] = -f\big(t, x\big(g(t)\big)\big), \quad t \geqslant T,$$

which shows that $x(t)$ is a solution of (A) for $t \geqslant T$. The asymptotic behavior (2.4) of $x(t)$ follows from (2.14) and Lemma 1. This sketches the proof of Theorem 2. $\square$

R e m a r k 1. The solutions constructed in Theorem 2 are all nonoscillatory, whereas those obtained in Theorem 1 are oscillatory according to whether the periodic functions $\omega_\sigma(t)$ are oscillatory or nonoscillatory. Since $\omega_\sigma(t)$ does not appear explicitly in (2.1), Theorem 1 asserts that the integral condition (2.1) is sufficient for the equation (A) to possess both oscillatory and nonoscillatory solutions. Thus one can easily speak of the phenomenon of coexistence of oscillatory and nonoscillatory solutions for neutral equations. This is an aspect which is not shared by non-neutral equations of the form $\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) + f\big(t, x(g(t))\big) = 0$.

R e m a r k 2. Suppose that there is a constant $a > 0$ such that

$$\int_{t_0}^{\infty} t^{m+n-1} F\bigl(t, a[g(t)]^{m+n-1}\bigr)\,\mathrm{d}t < \infty. \tag{2.15}$$

Then Theorems 1 and 2 imply that (A) possesses oscillatory solutions $x_j(t)$ which are asymptotic to $\omega_\sigma(t)t^j$, $j = 0, 1, \ldots, m-1$, $\omega_\sigma(t)$ being any given oscillatory periodic function, in the sense that

$$x_j(t) = c\omega_\sigma(t)t^j + o(1) \quad \text{as } t \to \infty,\ j = 0, 1, \ldots, m-1,$$

for some constant $c \neq 0$, as well as nonoscillatory solutions $x_k(t)$ which are asymptotic to t^k, $k = 0, 1, \ldots, m+n-1$, in the sense that

$$x_k(t) = ct^k + o(1) \quad \text{as } t \to \infty,\ k = 0, 1, \ldots, m+n-1,$$

for some constant $c \neq 0$. Simple examples of $\omega_\sigma(t)$ are $\cos\frac{2p\pi t}{\sigma}$ and $\sin\frac{2p\pi t}{\sigma}$, $p = 1, 2, \ldots$.

R e m a r k 3. It would be of interest to discuss the existence of solutions for a class of "singular" equations of the form (A) including

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\Delta_\sigma^m x(t)] + q(t)\bigl[x\bigl(g(t)\bigr)\bigr]^{-\gamma} = 0, \quad t \geqslant t_0$$

as a special case, where $\gamma > 0$ is a constant and $q\colon [t_0, \infty) \to \mathbb{R}$ is a continuous function. It is to be noted that only positive solutions are admitted for such solutions. A close look at the proofs of Theorems 1 and 2 enables us to prove the following existence theorem for the equation (A) in which $f(t, x)$ is subject to the condition

(b$_\mathrm{s}$) $f\colon [t_0, \infty) \times (0, \infty) \to \mathbb{R}$ is continuous and satisfies
$|f(t, x)| \leqslant F(t, x), \quad (t, x) \in C[t_0, \infty) \times (0, \infty),$

where $F(t, x)$ is a continuous function which is nonincreasing in x for each fixed $t \geqslant t_0$. The equation (A) which is singular in this sense is referred to as (A$_s$).

Theorem 3. *Let m, n, σ, t_0 and $g(t)$ be as in Theorems 1 and 2. Let $k \in \{0, 1, 2, \ldots, m+n-1\}$ and suppose that there is a constant $a > 0$ such that*

$$\int_{t_0}^{\infty} t^{m+n-1} F\bigl(t, a[g(t)]^k\bigr)\,\mathrm{d}t < \infty.$$

Then (A$_s$) has a positive solution $x(t)$ satisfying

$$x(t) = ct^k + o(1) \quad \text{as } t \to \infty$$

for some positive constant c.

Note that, in view of the nonincreasing property of F, the condition

$$\int_{t_0}^{\infty} t^{m+n-1} F(t,a)\,\mathrm{d}t < \infty \quad \text{for some } a > 0$$

ensures the existence of positive solutions of (A_s) which are asymptotic to all the t^k, $k = 0, 1, \ldots, m+n-1$, as $t \to \infty$ in the above sense.

E x a m p l e. For illustration of our results consider the neutral equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t) - 2x(t-1) + x(t-2)\big] + q(t)|x(t-3)|^{\gamma} \operatorname{sgn} x(t-3) = 0, \quad t \geqslant t_0, \tag{2.16}$$

where $\gamma > 0$, $t_0 > 3$, and $q\colon [t_0,\infty) \to \mathbb{R}$ is continuous. This is a special case of (A) in which $m = 2$, $\sigma = 1$, $g(t) = t - 3$ and $f(t,x) = q(t)|x|^{\gamma} \operatorname{sgn} x$, and the conditions (B) are satisfied with $F(t,u) = |q(t)|u^{\gamma}$. Note that the conditions (2.1) and (2.3) written for (2.16) reduce, respectively, to

$$\int_{t_0}^{\infty} t^{n+1+\gamma j} |q(t)|\,\mathrm{d}t < \infty, \quad j \in \{0,1\}, \tag{2.17}$$

and

$$\int_{t_0}^{\infty} t^{n+1+\gamma k} |q(t)|\,\mathrm{d}t < \infty, \quad k \in \{2,3,\ldots,n+1\}. \tag{2.18}$$

From Theorems 1 and 2 it follows that if (2.17) holds, then, for any given σ-periodic function $\omega_\sigma(t) \not\equiv 0$, (2.16) has a solution $x(t)$ such that

$$x(t) = c\omega_\sigma(t)t^j + o(1) \quad \text{as } t \to \infty,\ j \in \{0,1\}, \tag{2.19}$$

for some constant $c \neq 0$, and that if (2.18) holds, then (2.16) has a solution $x(t)$ such that

$$x(t) = ct^k + o(1) \quad \text{as } t \to \infty,\ k \in \{2,3,\ldots,n+1\}, \tag{2.20}$$

for some constant $c \neq 0$. If in particular

$$\int_{t_0}^{\infty} t^{(1+\gamma)(n+1)} |q(t)|\,\mathrm{d}t < \infty,$$

then (2.16) has all the solutions listed in (2.19) and (2.20); see Remark 2.

We next apply Theorem 3 in Remark 3 to the singular equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t)-2x(t-1)+x(t-2)\big]+q(t)\big[x(t-3)\big]^{-\gamma}=0, \quad t\geqslant t_0, \tag{2.21}$$

where γ and $q(t)$ are as in (2.16). The conclusion is that if

$$\int_{t_0}^{\infty} t^{n+1-\gamma k}|q(t)|\,\mathrm{d}t<\infty, \quad k\in\{0,1,\dots,n+1\},$$

then (2.16) possesses a positive solution $x(t)$ such that $x(t)=ct^k+o(1)$ as $t\to\infty$ for some $c\neq 0$. In particular, there exist such positive solutions for all $k\in\{0,1,\dots,n+1\}$ provided

$$\int_{t_0}^{\infty} t^{n+1}|q(t)|\,\mathrm{d}t<\infty.$$

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