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ON REGULARITY OF STATIONARY SOLUTIONS TO THE NAVIER-STOKES EQUATION IN 3-D TORUS

(submitted by F. Avkhadiev)

Abstract. We consider the Navier-Stokes equation in 3-D torus in the stationary setup and prove that any weak solution of this problem is actually smooth provided the stationary external force is also smooth.

1. Introduction

Time independent regimes of a flow to the Navier-Stokes equation are important because of ergodicity and of their relations with the long time behavior of solutions. The first existence and uniqueness theorems for (time dependent) weak solutions were proven by Leray [2] and by Hopf [1]. Regularity of time dependent solutions in 3-dimensional setup is still open problem.

We consider the Navier-Stokes equation in 3-D torus in the stationary setup and prove that any weak solution of this problem is actually smooth provided the stationary external force is also smooth.

A proof of this assertion is short and obtained as a combination of quite standard facts. Nevertheless, we regard as useful to publish this note, since first, we do not know publications contained this assertion, and secondly, it does not follow from general theory of nonlinear elliptic equations [4].

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2. Main theorem

Consider 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ with coordinates $x = (x_1, x_2, x_3)$. We use a notation

$$\partial_j = \frac{\partial}{\partial x_j}, \quad j = 1, 2, 3.$$

In all the formulas we follow the Einstein summation convention for repeated subscripts. For example for a vector field $v(x) = (v^1, v^2, v^3)(x)$ we write $\operatorname{div} v = \partial_j v^j$.

Application of any scalar operator to a vector-function implies that this operator is applied to each component of the vector-function.

Let $H^{s,p}(\mathbb{T}^3), \quad 1 \le p, \quad s \ge 0$ be the Sobolev spaces over $L^p(\mathbb{T}^3)$. In the sequel we drop the arguments of these notations and simply write $H^{s,p}, L^p$.

Consider the Navier-Stokes equation in the stationary setup:

$$\partial_j(v^j v^k) = -\partial_k p + \nu \Delta v^k + f^k, \quad k = 1, 2, 3, \tag{2.1}$$

$$\operatorname{div} v = 0, \tag{2.2}$$

where ν is a positive constant.

The external force $f(x) = (f^1, f^2, f^3)(x) \in C^\infty$ is assumed to be divergence free and of zero mean value:

$$\operatorname{div} f = 0, \quad \int_{\mathbb{T}^3} f(x)\, dx = 0.$$

Note that substitution $p \mapsto p + c$ (c is an arbitrary constant) in (2.1) does not change the equation. So we will find the function p just up to an additional constant.

Problem (2.1), (2.2) has a weak solution $v \in H^1$ [3].

Theorem 1. *Any weak solution v to problem (2.1), (2.2) is actually smooth: $v \in C^\infty$.*

3. Proof

Let $k, x \in \mathbb{R}^3$, introduce some notations:

$$(k, x) = k_1 x_1 + \ldots + k_3 x_3, \quad |x|^2 = |x_1|^2 + \ldots + |x_3|^2, \quad i^2 = -1.$$

In the case of torus our knowledge about generalized functions is clearly simplified. Any function $u \in L^p, \quad p \ge 2$ belongs also to L^2 and thus can be expanded to the Fourier series: $u(x) = \sum_{j \in \mathbb{Z}^3} u_j e^{i(j,x)}$. This series converges in L^2.

The set of generalized functions consists of the formal Fourier series $h(x) = \sum_{j\in\mathbb{Z}^3} h_j e^{i(j,x)}$ with polynomially growing coefficients: $|h_k| \le c|k|^\gamma$. It follows for example from the L. Schwartz theorem [5]. A generalized function h takes its value of a test function

$$\varphi = \sum_{j\in\mathbb{Z}^3} \varphi_j e^{i(j,x)} \in C^\infty$$

by the rule:

$$(h,\varphi) = \sum_{j\in\mathbb{Z}^3} h_j \varphi_{-j}.$$

(Only real-valued functions are considered.)

According to this viewpoint a generalized derivative of a function from L^p is expressed as a Fourier series:

$$\partial_m h(x) = i \sum_{j\in\mathbb{Z}^3} j_m h_j e^{i(j,x)}, \quad m = 1,2,3.$$

It is convenient to understand all the further arguments in this light.

Define the following operators:

$$\begin{aligned}
\Delta u &= -\sum_{j\in\mathbb{Z}^3} u_j |j|^2 e^{i(j,x)},\\
\Delta^{-1} u &= -\sum_{j\in\mathbb{Z}^3\setminus\{0\}} \frac{u_j}{|j|^2} e^{i(j,x)},\\
A^k_l u &= (\Delta^{-1}\partial_k\partial_l - \delta_{kl})u,\\
Pu &= u - u_0.
\end{aligned}$$

where $\delta_{kl} = 1$ for $k = l$ and 0 otherwise.

Such a form of the inverse Laplace operator Δ^{-1} needs a comment. Actually the Laplace operator has the inverse only on the space of functions with zero mean value. Nevertheless all the formulas appear in our text include derivative operators standing before the operator Δ^{-1} and misunderstanding does not appear.

Since $\partial_j = \partial_j P = P\partial_j$ we can rewrite the operator A^k_l in the form:

$$A^k_l u = \partial_k\partial_l\Delta^{-1}Pu - \delta_{kl}u.$$

Take the operator div from the right- and the left-hand sides of equation (2.1). Using equation (2.2) we get $\partial_i\partial_j(v^iv^j) = -\Delta p$. Thus

$$p = -\Delta^{-1}\partial_i\partial_j(v^iv^j).$$

Substituting this formula to equation (2.1) we obtain the following problem:

$$0 = A^k_l\partial_j(v^jv^l) + \nu\Delta v^k + f^k. \tag{3.1}$$

Taking the operator Δ^{-1} from the both sides of equation (3.1) we have

$$-\nu v^k = A^k_l \Delta^{-1}\partial_j(v^j v^l) + \Delta^{-1} f^k. \tag{3.2}$$

or equivalently:

$$-\nu v^k = A^k_l \partial_j \Delta^{-1} P(v^j v^l) + \Delta^{-1} f^k. \tag{3.3}$$

Our plan of the Proof is as follows. We will show by induction on k that the weak solution

$$v \in \bigcap_{p\ge 2} H^{k,p}, \quad k \in \mathbb{N}. \tag{3.4}$$

By Sobolev's embedding theorem this proves Theorem 1.

Recall some facts from the Sobolev theory [4]. For $p > 1$ and $k \ge 0$ we have

$$H^{k,p} \subset L^{3p/(3-kp)}, \quad kp < 3, \tag{3.5}$$

$$\partial_j : H^{k+1,p} \to H^{k,p}, \tag{3.6}$$

$$\Delta^{-1} P : H^{k,p} \to H^{k+2,p}. \tag{3.7}$$

Particularly, the operators A^k_l, P maps $H^{k,p}$ to itself.

Since the weak solution v belongs to $H^1 = H^{1,2}$ it belongs to L^6 by inclusion (3.5). Thus the expression $v^j v^l$ belongs to L^3 and due to formulas (3.6), (3.7) and by force of equation (3.3) we have $v \in H^{1,3}$.

Diminishing $\varepsilon > 0$ in the formula $H^{1,3} \subset H^{1,3-\varepsilon} \subset L^{3(3-\varepsilon)/\varepsilon}$ we obtain that $v \in \bigcap_{p\ge 2} L^p$. Thus the expression $v^j v^l$ also belongs to $\bigcap_{p\ge 2} L^p$. By equation (3.3) it follows that inclusion (3.4) holds with $k = 1$.

Assume that inclusion (3.4) holds with $k > 1$ and check it with $k + 1$. By the chain rule and (2.2) we have

$$\partial_j(v^j v^l) = v^j \partial_j v^l \in \bigcap_{p\ge 2} H^{k-1,p}.$$

By means of equation (3.2) this inclusion implies that $v \in \bigcap_{p\ge 2} H^{k+1,p}$.

The Theorem is proved.

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