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Computing the cardinality of CM elliptic curves using torsion points

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Résumé. Soit $\mathcal{E}/\overline{\mathbb{Q}}$ une courbe elliptique avec multiplications complexes par un ordre d'un corps quadratique imaginaire $\mathbf{K}$. Le corps de définition de $\mathcal{E}$ est le corps de classe de rayon Ω associé à l'ordre. Si le nombre premier p est scindé dans Ω, on peut réduire $\mathcal{E}$ modulo un des facteurs de p et obtenir une courbe E définie sur $\mathbb{F}_p$. La trace du Frobenius de E est connue au signe près et nous cherchons à déterminer ce signe de la manière la plus rapide possible, avec comme application l'algorithme de primalité ECPP. Dans ce but, nous expliquons comment utiliser l'action du Frobenius sur des points de torsion d'ordre petit obtenus à partir d'invariants de classes qui généralisent les fonctions de Weber.

Abstract. Let $\mathcal{E}/\overline{\mathbb{Q}}$ be an elliptic curve having complex multiplication by a given quadratic order of an imaginary quadratic field $\mathbf{K}$. The field of definition of $\mathcal{E}$ is the ring class field Ω of the order. If the prime p splits completely in Ω, then we can reduce $\mathcal{E}$ modulo one the factors of p and get a curve E defined over $\mathbb{F}_p$. The trace of the Frobenius of E is known up to sign and we need a fast way to find this sign, in the context of the Elliptic Curve Primality Proving algorithm (ECPP). For this purpose, we propose to use the action of the Frobenius on torsion points of small order built with class invariants generalizing the classical Weber functions.

1. Introduction

Let $\mathbf{K}$ be an imaginary quadratic field of discriminant $-D$. For any integer t, let $\mathcal{O}_t$ be the order of conductor t of $\mathbf{K}$, $\Delta_t = -t^2 D$ its discriminant, and $h_t = h(\Delta_t)$ its class number. We denote by Ω_t the ring class field modulo t over $\mathbf{K}$. By class field theory, the extension $\Omega_t/\mathbf{K}$ can be

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constructed using the minimal polynomial of the modular function j over a set of representatives $\{\mathfrak{i}_1, \mathfrak{i}_2, \ldots, \mathfrak{i}_{h_t}\}$ of the class group $Cl(\mathcal{O}_t)$. An elliptic curve $\mathcal{E}$ of invariant $j(\mathfrak{i}_r)$ can be defined over Ω_t and has complex multiplication (CM) by $\mathcal{O}_t$. We denote by $H_{\Delta_t}[j](X)$ the minimal polynomial of the j's, namely

$$H_{\Delta_t}[j](X) = \prod_{r=1}^{h_t} (X - j(\mathfrak{i}_r))$$

which is known to have rational integer coefficients.

Let p be a rational prime number which splits completely in Ω_t, or equivalently which is the norm of an integer of Ω_t (that is $p = (U^2 + Dt^2V^2)/4$ for rational integers U and V). Then we can reduce $\mathcal{E}$ modulo a prime divisor $\mathfrak{P}$ of p to get an elliptic curve $E/\mathbb{F}_p$ having CM by $\mathcal{O}_t$. If π denotes the Frobenius of E, then it can be viewed as an element of $\mathcal{O}_t$ of norm p, that is (assuming that $\Delta_t \notin \{-3, -4\}$):

$$\pi = (\pm U \pm tV\sqrt{-D})/2. \tag{1.1}$$

The cardinality of $E(\mathbb{F}_p)$ is the norm of $\pi - 1$, or more simply $p + 1 \mp U$.

The j-invariant of $E/\mathbb{F}_p$ is the reduction of one of the $j(\mathfrak{i}_r)$'s modulo p, that is a root of $H(X) = H_{\Delta_t}[j](X)$ modulo p. Building E is done as follows: find a root j of $H(X)$ in $\mathbb{F}_p$, and deduce from that the equation of E. When $j \notin \{0, 1728\}$, we may take any equation $E(j, c)$:

$$Y^2 = X^3 + a_4(j)c^2X + a_6(j)c^3$$

where c is any element of $\mathbb{F}_p$ and

$$a_4(j) = \frac{3j}{1728 - j}, \quad a_6(j) = \frac{2j}{1728 - j}. \tag{1.2}$$

We will note $E(j)$ for $E(j, 1)$. If its cardinality is $p + 1 - a$, then a curve $E(j, c)$ has cardinality $p + 1 - \left(\frac{c}{p}\right)a$ (where $\left(\frac{a}{b}\right)$ stands for the Legendre symbol). A curve with $\left(\frac{c}{p}\right) = -1$ is a twist of $E(j)$. The problem is now to compute $\#E(j)$ modulo p, or equivalently, fix the sign of U in equation (1.1).

In the course of implementing the ECPP algorithm [3, 20] or for cryptographic reasons, it is important to compute this cardinality rapidly. We could of course try both signs of U yielding cardinalities m, find some random points P on $E(j)$ and check whether $[m]P = O_E$ on E. This approach is somewhat probabilistic and we prefer deterministic and possibly faster solutions.

In the case where D is fundamental and prime to 6, the solution is to use Stark's approach [29], together with tricks described in [19]. This method is efficient, provided we can afford some precomputations. Note that in the special case where $h_t = 1$, which includes $j = 0, 1728$, one already knows

the answer (see [3, 13, 23] and the references given therein). For $D = 20$, we have the isolated result of [15] (see also section 6.2 below). Since the first version[1] of the present article, Ishii [12] has given the answer for D of class numbers 2 or 3 and divisible by 3, 4, or 5.

In the ECPP algorithm, class invariants obtained from functions on $\Gamma^0(\ell)$ are used to build $\Omega_t/\mathbf{K}$ in an efficient way [11]. When $\left(\frac{-D}{\ell}\right) \neq -1$, we actually build an elliptic curve having rational torsion subgroup of order ℓ, and sometimes a rational point inside it. Application of the Frobenius on such a point gives us the sign we are looking for. This is all the more true when ℓ is small and $X_1(\ell)$ is "close" to $X_0(\ell)$.

Section 2 describes properties of the modular equations defining $X_0(\ell)$ for prime ℓ and their relations to complex multiplication over $\overline{\mathbb{Q}}$. In Section 3, we briefly describe the necessary results used in the SEA algorithm. Section 4 contains our main contribution. We treat the special cases $\ell = 3$ in Section 5 and $\ell = 5$ in Section 6. Section 7 describes the very interesting case of $\ell = 7$ and for the sake of completeness that of $\ell = 11$. Section 8 is devoted to the particular case $\ell = 2$, in which we study the properties of 4-torsion points. We provide numerical examples for each case. We conclude with remarks on the use of our results in our implementation of ECPP.

The books [8, 27] are a good introduction to all the material described above.

2. Modular curves and class invariants

2.1. Modular polynomials. Let ℓ be a prime number. The curve $X_0(\ell)$ parametrizes the cyclic isogenies of degree ℓ associated to an elliptic curve E defined over a field $\mathbf{k}$. An equation for $X_0(\ell)$ can be obtained as the minimal polynomial of a modular function f whose stabilizer in $\mathrm{SL}_2(\mathbb{Z})$ is $\Gamma^0(\ell)$. This *modular polynomial*, noted $\Phi[f](X, J)$ is such that $\Phi[f](f(z), j(z)) = 0$ for all z such that $\Im z > 0$, where $j(z)$ is the ordinary modular function.

Dedekind's η function is

$$\eta(\tau) = q^{1/24} \prod_{m \geq 1} (1 - q^m)$$

where $q = \exp(2i\pi\tau)$. It is used to build suitable functions for $\Gamma^0(\ell)$ (see for instance [21, 22]). For example, if

$$\mathfrak{w}_\ell(z) = \frac{\eta(z/\ell)}{\eta(z)}$$

and $s = 12/\gcd(12, \ell - 1)$, then $\mathfrak{w}_\ell^{2s}$ is a modular function for $\Gamma^0(\ell)$. The equations for small prime values of ℓ are given in Table 1 (see for instance [18]).

[1] http://arxiv.org/ps/math.NT/0210173

ℓ	$\Phi[\mathfrak{w}_\ell]$
2	$(X+16)^3 - JX$
3	$(X+27)(X+3)^3 - JX$
5	$\left(X^2+10X+5\right)^3 - JX$
7	$\left(X^2+13X+49\right)\left(X^2+5X+1\right)^3 - JX$

TABLE 1. Table of modular equations $\Phi[\mathfrak{w}_\ell](X, J)$.

Among other classes of functions for other modular groups, we find the classical functions of Weber:

$$\gamma_2(z) = \sqrt[3]{j(z)}, \quad \gamma_3(z) = \sqrt{j(z) - 1728}$$

for which the corresponding modular equations are quite simple.

2.2. CM theory. View the class group $Cl(\Delta_t)$ as a set of reduced quadratic primitive binary forms of discriminant Δ_t, say

$$Cl(\Delta_t) = \{(A, B, C), B^2 - 4AC = \Delta_t\}$$

with h_t forms in it. For a given $Q = (A, B, C)$, let $\tau_Q = (-B + \sqrt{\Delta_t})/(2A)$. Then $j(\tau_Q)$ is an algebraic integer that generates $\Omega_t/\mathbf{K}$. Moreover, the associated curve E_Q of invariant $j(\tau_Q)$ has CM by $\mathcal{O}_t$.

Suppose $j(\tau) \in \Omega_t$. If u is some function on some $\Gamma^0(\ell)$, then the roots of $\Phi[u](X, j(\tau))$ are algebraic integers. They generate an extension of Ω_t of degree dividing $\ell + 1$. The striking phenomenon, known for a long time, is that sometimes these roots lie in Ω_t itself. We will note $H_{\Delta_t}[u](X)$ for the minimal polynomial of the invariant u.

Among the simplest results in this direction, we have the following, dating back to Weber [31]. Suppose α is a quadratic integer with minimal polynomial

$$A\alpha^2 + B\alpha + C = 0$$

such that $\gcd(A, B, C) = 1$ and $B^2 - 4AC = \Delta_t$.

Theorem 2.1. *If* $3 \nmid A$, $3 \mid B$, *then*

$$\mathbb{Q}(\gamma_2(\alpha)) = \begin{cases} \mathbb{Q}(j(\alpha)) & \text{if } 3 \nmid \Delta_t, \\ \mathbb{Q}(j(3\alpha)) & \text{if } 3 \mid \Delta_t. \end{cases}$$

A companion result is:

Theorem 2.2. *Suppose* $2 \nmid A$. *We assume that*

$$B \equiv \begin{cases} 0 \bmod 4 & \text{if } 2 \mid \Delta_t, \\ 1 \bmod 4 & \text{if } 2 \nmid \Delta_t. \end{cases}$$

Then

$$\begin{aligned}\mathbb{Q}(\sqrt{-D}\gamma_3(\alpha)) &= \mathbb{Q}(j(\alpha)), \qquad \text{if } 2 \nmid \Delta_t,\\ \mathbb{Q}(\gamma_3(\alpha)) &= \mathbb{Q}(j(2\alpha)), \qquad \text{if } 2 \mid \Delta_t.\end{aligned}$$

Finding a complete system of conjugate values for $\gamma_2(\alpha)$ (resp. $\gamma_3(\alpha)$), as well as for a lot of such functions, is explained in [24].

3. The foundations of the SEA algorithm

3.1. Division polynomials and their properties. For an elliptic E, we let $E[n]$ denote the group of n-torsion points of E (over $\overline{\mathbb{Q}}$). We let $f_n^E(X)$ (or simply $f_n(X)$) denote the n-th division polynomial whose roots are the abscissae of the n-torsion points of E. See [26] for its definition and properties. For instance for the curve $E : Y^2 = X^3 + aX + b$, the first values are:

$$f_0(X) = 0, f_1(X) = 1, f_2(X) = 1,$$

$$f_3(X) = 3\,X^4 + 6\,a\,X^2 + 12\,b\,X - a^2,$$

$$f_4(X) = 2\,X^6 + 10\,a\,X^4 + 40\,b\,X^3 - 10\,a^2X^2 - 8\,a\,b\,X - 2\,a^3 - 16\,b^2.$$

Recurrence relations for computing f_n are given by:

$$f_{2n} = f_n(f_{n+2}f_{n-1}^2 - f_{n-2}f_{n+1}^2),$$

$$f_{2n+1} = \begin{cases} f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1}(16(X^3+aX+b)^2) & \text{if } n \text{ is odd,}\\ 16(X^3+aX+b)^2 f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1} & \text{if } n \text{ is even.}\end{cases}$$

3.2. Explicit factors of $f_n^E(X)$. Let E be an elliptic curve. Suppose that we have some modular polynomial $\Phi[f](X, J)$ for a function f on $\Gamma^0(\ell)$. Then a root v of $\Phi[f](X, j(E))$ gives rise to a curve which is ℓ-isogenous to E, and to a factor of $f_\ell^E(X)$. This is the essence of the ideas of Elkies and Atkin that improve Schoof's algorithm for computing the cardinality of curves over finite fields [1, 25, 10]. The computations can be done using Vélu's formulas [30] (see also [18] for technicalities related to the actual computations). We end up with a factor $g_\ell^E(X)$ of $f_\ell^E(X)$.

In Table 2, for prime ℓ, we suppose v_ℓ is a root of $\Phi[\mathfrak{w}_\ell](X, j)$ and we give the factor $g_\ell^{E(j)}(X)$ of $f_\ell^{E(j)}(X)$ that can be obtained.

3.3. The splitting of $\Phi[f](X, j(E))$ in $\mathbb{F}_p$. We take the following result from [1] (see also [25]). Let ℓ and p be two distinct primes, and $E/\mathbb{F}_p$ an elliptic curve. Put $\#E = p + 1 - U$, $\mathcal{D} = 4p - U^2$. We denote the splitting type of a squarefree polynomial $P(X)$ by the degrees of its factors. For instance, a polynomial of degree 4 having two linear factors and one quadratic factor will be said to have splitting type (1)(1)(2).

ℓ	factor
2	$(v_2-8)\,X+v_2+16$,
3	$\left(v_3^2+18\,v_3-27\right)X+v_3^2+30\,v_3+81$,
5	$\left(v_5^2+4\,v_5-1\right)^2\left(v_5^2+22\,v_5+125\right)X^2$ $+2\,\left(v_5^2+4\,v_5-1\right)\left(v_5^2+10\,v_5+5\right)\left(v_5^2+22\,v_5+125\right)X$ $+\left(v_5^2+22\,v_5+89\right)\left(v_5^2+10\,v_5+5\right)^2$,
7	$\left(v_7^4+14\,v_7^3+63\,v_7^2+70\,v_7-7\right)^3X^3$ $+3\,\left(v_7^2+13\,v_7+49\right)\left(v_7^2+5\,v_7+1\right)\left(v_7^4+14\,v_7^3+63\,v_7^2+70\,v_7-7\right)^2X^2$ $+3\,\left(v_7^2+13\,v_7+33\right)\left(v_7^2+13\,v_7+49\right)\left(v_7^2+5\,v_7+1\right)^2$ $\times\left(v_7^4+14\,v_7^3+63\,v_7^2+70\,v_7-7\right)X$ $+\left(v_7^2+13\,v_7+49\right)\left(v_7^2+5\,v_7+1\right)^3\left(v_7^4+26v_7^3+219v_7^2+778v_7+881\right)$

TABLE 2. Factors of $f_\ell^{E(j)}$.

Theorem 3.1. *Let f be a function for $\Gamma^0(\ell)$ and write*

$$\Psi(X)\equiv\Phi[f](X,j(E)) \bmod p.$$

If $\left(\frac{-\mathcal{D}}{\ell}\right)=0$, then Ψ splits as $(1)(\ell)$ or $(1)\cdots(1)$.

If $\left(\frac{-\mathcal{D}}{\ell}\right)=+1$, then Ψ splits as $(1)(1)(r)\cdots(r)$ where $r\mid\ell-1$ and $r>1$ if $\ell\neq 2$.

If $\left(\frac{-\mathcal{D}}{\ell}\right)=-1$, then Ψ splits as $(r)\cdots(r)$ where $r>1$ and $r\mid\ell+1$.

If k denotes the number of factors of Ψ, then $(-1)^k=\left(\frac{p}{\ell}\right)$.

3.4. Elkies's ideas. We briefly summarize Elkies's idea [10]. Let π be the Frobenius of the curve, sending any point $P=(x,y)$ of $E(\overline{\mathbb{F}_p})$ to (x^p,y^p).

Theorem 3.2. *Let $\chi(X)=X^2-UX+p$ denote the characteristic polynomial of the Frobenius π of the elliptic curve E of cardinality $p+1-U$. When $\left(\frac{-\mathcal{D}}{\ell}\right)\neq-1$, the restriction of π to $E[\ell]$ (denoted by $\pi|_{E[\ell]}$) has at least one eigenvalue. To each eigenvalue λ of $\pi|_{E[\ell]}$ corresponds a factor of degree $(\ell-1)/2$ of f_ℓ. We deduce that $U\equiv\lambda+p/\lambda \bmod \ell$.*

We will note $g_{\ell,\lambda}(X)$ the factor of $f_\ell^{E(j)}(X)$ associated to the eigenvalue λ. Let ω denote the order of λ modulo ℓ and $\sigma=\omega/2$ if ω is even and ω otherwise. With these notations, one can show the following result:

Proposition 3.1. *The splitting type of $g_{\ell,\lambda}(X) \bmod p$ is $(\sigma)(\sigma)\cdots(\sigma)$ with κ factors such that $(\ell-1)/2=\kappa\sigma$.*

From this, we deduce:

Corollary 3.1. *The polynomial $g_{\ell,\lambda}(X)$ splits completely modulo p if and only if $\lambda=\pm1 \bmod \ell$.*

Note also the following result of Dewaghe [9] in the formulation of [16].

Proposition 3.2. *Let* $r = \text{Resultant}(g_{\ell,\lambda}(X), X^3 + a_4(j)X + a_6(j))$. *Then*

$$\left(\frac{\lambda}{\ell}\right) = \left(\frac{r}{p}\right)$$

Classically, this enables us to fix the sign of λ when $\ell \equiv 3 \bmod 4$.

4. Stating the problem

Let $4p = U^2 + DV^2$. We want to find the equation of a curve $E/\mathbb{F}_p$ having cardinality $m = p + 1 - U$. The general algorithm is the following:

procedure BUILDEWITHCM(D, U, V, p)
{ Input: $4p = U^2 + DV^2$ }
1. For some invariant u, compute the minimal polynomial $H_D[u](X)$.
2. Find a root x_0 of $H_D[u](X)$ modulo p.
3. for all roots j of $\Phi[u](x_0, J)$ mod p do
   a. compute $E(j)$.
   b. If $\#E(j) = p+1+U$ instead of $p+1-U$, replace $E(j)$ by a twist.

4.1. Eliminating bad curves. In general, the degree of $\Phi[u](x_0, J)$ is larger than 1 and we expect several roots in J, not all of which are invariants of the curves we are looking for.

In order to eliminate bad curves, we can use the following result. First, note that the discriminant of the curve E is

$$\Delta(E(j)) = 2^{12} \cdot 3^6 j^2/(j - 1728)^3.$$

Proposition 4.1. *Let* $4p = U^2 + DV^2$. *The number* $\Delta(E(j))$ *is a square modulo* p *in the following cases:*
(i) D *odd;*
(ii) $4 \mid D$ *and* $2 \mid V$.

Proof:

(i) If α is as in Theorem 2.2, we deduce that $\sqrt{-D}\gamma_3(\alpha)$ is in $\mathcal{O}_K$, which means that $H_{-D}[\sqrt{-D}\gamma_3]$ splits modulo p and therefore $j - 1728 = -Du^2 \bmod p$ and we have $\left(\frac{-D}{p}\right) = +1$ by hypothesis.

(ii) Theorem 2.2 tells us that $\mathbb{Q}(\gamma_3(\alpha)) = \mathbb{Q}(j(2\alpha))$. But p splits in the order $\mathcal{O}_2$ and therefore in Ω_{2t}, which shows that the minimal polynomial of γ_3 splits modulo p, proving the result. □

Coming back to our problem, we see that when the above result applies, a good curve is such that $\left(\frac{\Delta(E(j))}{p}\right)$ must be equal to 1.

4.2. Fixing the sign of the trace. We can assume that we are left with only one possible j and that we want to compute the cardinality of $E(j)$ as quickly as possible. Let us explain our idea. Let $\mathcal{D} = DV^2$. Suppose that $\ell \neq p$ is an odd prime (the case $\ell = 2$ will be dealt with later) and

$\left(\frac{-D}{\ell}\right) \neq -1$. In that case, Theorem 3.2 applies and if we can find one eigenvalue λ, we can find $U \bmod \ell$. If $U \not\equiv 0 \bmod \ell$, then we can find the sign of U. Note that if $\ell \mid D$, then $U \not\equiv 0 \bmod \ell$.

The most favorable case is when $\ell \mid \mathcal{D}$, because then there is only one eigenvalue λ (it can be a double one) and $\lambda \equiv U/2 \bmod \ell$. Having λ gives us immediately the sign of U. A very favorable case is when $\ell \equiv 3 \bmod 4$, using Dewaghe's idea.

Apart from this, there is another interesting sub-case, when we can find a rational root x_0 of $g^E_{\ell,\lambda}$, using for instance some class invariant. In that case, we can form $y_0^2 = x_0^3 + ax_0 + b \bmod p$ and test whether y_0 is in $\mathbb{F}_p$ or not. If it is, then $\lambda = 1$, since (x_0, y_0) is rational and $\pi(P) = P$. Otherwise, $\lambda = -1$.

Our idea is then to use the general framework for some precise values of ℓ, and use rational roots of $g_{\ell,\lambda}$ obtained via class invariants. When $\ell = 3$, we are sure to end with a rational root of $f_3^{E(j)}(X)$, as is the case for $\ell = 2$ and $f_4^{E(j)}$. Moreover, we can use some invariant that give us the torsion points directly. We also give examples for $\ell = 5, 7, 11$.

5. The case $\ell = 3$

We suppose that $4p = U^2 + DV^2$. The first subsection makes precise the above results.

5.1. Using 3-torsion points. We begin with an easy lemma that can be proved by algebraic manipulations:

Lemma 5.1. *Let v be any root of $\Phi_3^c(X, j) = 0$. Then a root of $f_3^{E(j)}(X)$ is given by*

$$x_3 = -\frac{(v+27)(v+3)}{v^2+18v-27}.$$

Proposition 5.1. *Let p be a prime representable as $4p = U_0^2 + DV_0^2$, for which $3 \mid DV_0^2$ and $\#E = p + 1 - U$. Suppose $P = (x_3, y_3)$ is a 3-torsion point on $E(j)$ for which x_3 is rational. Let $s = x_3^3 + a_4(j)x_3 + a_6(j) \bmod p$. Then $U \equiv 2\left(\frac{s}{p}\right) \bmod 3$.*

Proof: This is a simple application of Theorem 3.2. □

5.2. Solving the equation $\Phi_3^c(X, j(E)) = 0$.

5.2.1. *The case $\left(\frac{-D}{3}\right) \neq -1$.* A solution of this equation is given by $\mathfrak{w}_3^{12}$, which lies in Ω_1 with the hypothesis made on D.

Numerical examples. Let $H_{-15}[\mathfrak{w}_3^{12}] = X^2 + 81\,X + 729$, $p = 109$, $4p = 14^2 + 15 \times 4^2$, $v_3 = 3$, $x_3 = 104$, $E : Y^2 = X^3 + 94X + 99$; $U = \pm 14$. Since $\lambda = 1 \bmod 3$, we conclude that $U = 14$ and E has $109 + 1 - 14$ points.

Take $D = 20$ and $p = 349$. We find $(U, V) = (\pm 26, \pm 6)$. We compute:

$$H_{-20}[\mathfrak{w}_3^{12}] = X^2 + (70 - 22\sqrt{-20})X - 239 - 154\sqrt{-20}.$$

Using $\sqrt{-20} = 237 \bmod p$, a root of this polynomial is $v_3 = 257$, from which $j = 224$ and $E(j) : Y^2 = X^3 + 45X + 30$. Now $\lambda = -1$, which gives us that $\#E = 349 + 1 + 26$.

5.2.2. *The case* $\left(\frac{-D}{3}\right) = -1$. We may find the roots of the degree 4 equation $\Phi_3^c(X, j(\alpha)) = 0$ directly.

In Skolem's approach [28], to compute the roots of a general quartic (with a_1 and a_3 not both zero)

$$P(X) = X^4 + a_1X^3 + a_2X^2 + a_3X + a_4$$

one uses the four roots X_i of P to define

$$\left\{\begin{array}{rcl} z_1 &=& X_1 + X_2 - X_3 - X_4, \\ z_2 &=& X_1 - X_2 + X_3 - X_4, \\ z_3 &=& X_1 - X_2 - X_3 + X_4. \end{array}\right. \tag{5.1}$$

Writing $y_i = z_i^2$, the y_i's are roots of

$$R(y) = y^3 + b_1y^2 + b_2y + b_3 \tag{5.2}$$

in which

$$\left\{\begin{array}{rcl} b_1 &=& 8a_2 - 3a_1^2, \\ b_2 &=& 3a_1^4 - 16a_1^2a_2 + 16a_1a_3 + 16a_2^2 - 64a_4, \\ b_3 &=& -(a_1^3 - 4a_1a_2 + 8a_3)^2. \end{array}\right. \tag{5.3}$$

Conversely, if the y_i's are the roots of R and if the z_i's are chosen in such a way that

$$-z_1z_2z_3 = a_1^3 - 4a_1a_2 + 8a_3,$$

then the X_i's defined by (5.1) (together with $X_1 + X_2 + X_3 + X_4 = -a_1$) are the roots of P.

In our case, we find that

$$R(Y) = Y^3 - 1728Y^2 - 576(j(\alpha) - 1728)Y - 64(j(\alpha) - 1728)^2$$

and the compatibility relation is $z_1z_2z_3 = 8(j(\alpha) - 1728)$. Since we suppose that $3 \nmid D$, we replace $j(\alpha)$ by $\gamma_2(\alpha)^3$. In that case, the roots of $R(Y)$ are

$$4(\zeta_3^{2i}\gamma_2(\alpha)^2 + 12\zeta_3^i\gamma_2(\alpha) + 144)$$

for $i = 0, 1, 2$. Studying the roots of these numbers as class invariants could probably be done using Shimura's reciprocity law (see e.g., [24]). The function

$$\sqrt{\gamma_2(\alpha)^2 + 12\gamma_2(\alpha) + 144}$$

has been introduced via a different route by Birch in [4] and the theorems proven there could be used in our context, though we refrain from doing so in this article.

Let us summarize the algorithm to find the roots of $\Phi_3^c(X, j(E))$ modulo p when $3 \nmid D, 3 \mid V$ (which implies $p \equiv 1 \bmod 3$):

1. compute $\gamma_2 \bmod p$;
2. compute the values $y_i = 4(\zeta_3^{2i}\gamma_2(\alpha)^2 + 12\zeta_3^i\gamma_2(\alpha) + 144) \bmod p$ for $i = 1, 2$;
3. compute $z_i = \sqrt{y_i} \bmod p$ for $i = 1, 2$ and $z_3 = 8(\gamma_2^3 - 1728)/(z_1 z_2)$ from which $X_1 = z_1 + z_2 + z_3 - 36$ is a root of $\Phi_3^c(X, j)$.

Notice that $\zeta_3 \bmod p$ can be computed as follows (see [2] for more on this sort of ideas): since $3 \mid p - 1$, we can find a such that $a^{(p-1)/3} \not\equiv 1 \bmod p$. Put $\zeta_3 = a^{(p-1)/3}$. It satisfies $\zeta_3^2 + \zeta_3 + 1 \equiv 0 \bmod p$. Therefore, finding a root costs two squareroots and one modular exponentiation, once γ_2 is known.

Numerical examples. Consider $(D, p, U, V) = (40, 139, \pm 14, \pm 3)$. A root of $H_{-40}[\gamma_2](X) = X^2 - 780\,X + 20880$ modulo p is 110. Using $\zeta_3 = 96$, we compute $v_3 = 109$ and $x_3 = 135$. Then $E : Y^2 = X^3 + 124X + 129$ has $\lambda = 1$ and $U = 14$.

6. The case $\ell = 5$

6.1. Using $\mathfrak{w}_5$. We assume here that $\left(\frac{-D}{5}\right) \neq -1$ and $5 \mid DV^2$. In that case, we can use some power of $\mathfrak{w}_5$ as invariant to get a root v_5 of $\Phi_5^c(X, j)$, thus yielding a factor $g_5^{E(j)}$ of $f_5^{E(j)}$. Writing:

$$A = v_5^2 + 22v_5 + 125, B = v_5^2 + 4v_5 - 1, C = v_5^2 + 10v_5 + 5,$$

one has:

$$g_5^{E(j)}(X) = X^2 + 2(C/B)X + (1 - 36/A)(C/B)^2.$$

Putting $Y = (B/C)X$ leads us to $(Y + 1)^2 - 36/A$. At this point, since

$$j = \frac{(v_5^2 + 10\,v_5 + 5)^3}{v_5}$$

we also have:

$$j - 1728 = \frac{\left({v_5}^2 + 22\,v_5 + 125\right)\left({v_5}^2 + 4\,v_5 - 1\right)^2}{v_5}$$

or $A = v_5(j - 1728)/B^2$.

6.1.1. *The case $U \equiv \pm 2 \bmod 5$.* We deduce that $p \equiv 1 \bmod 5$ and $g_5^{E(j)}(X)$ has two rational roots.

Examples. Take $D = 35$ for which

$$H_{-35}[\mathfrak{w}_5^6](X) = X^2 + 50X + 125.$$

Take $(p, U, V) = (281, \pm 33, \pm 1)$. We first use $v_5 = 163$ to compute $E(j) : Y^2 = X^3 + 32X + 115$ and $g_5^{E(j)}(X) = X^2 + 245X + 198$. From this, we

get $x_5 = 227$ and find that $x_5^3 + a_4(j)x_5 + a_6(j)$ is a square in $\mathbb{F}_p$, so that $\#E(j) = p + 1 + 33$.

Consider now $D = 91$ for which $\left(\frac{-91}{5}\right) = +1$. We find:

$$H_{-91}[\mathfrak{w}_5^6] = X^2 + (130 - 40\sqrt{-91})X - 99 - 8\sqrt{-91}.$$

Taking $(p, U, V) = (571, \pm 3, \pm 5)$, we use $\sqrt{-91} = 342 \bmod p$, find $v_5 = 216$ from which $j = 533$ and $E(j) : Y^2 = X^3 + 181X + 311$. Then $g_5^{E(j)}(X) = X^2 + 213X + 412$ which has a root $x_5 = 315$. We find that $\lambda = -1$ and $U = 3$.

6.1.2. *The case $U \equiv \pm 1 \bmod 5$.* One has $p \equiv 4 \bmod 5$ and $g_5^{E(j)}(X)$ is irreducible; the eigenvalue is $\lambda = U/2 \equiv \pm 2 \bmod 5$. We can compute it using the techniques of SEA, that is test the identity

$$(X^p, Y^p) = [\pm 2](X, Y) \bmod g_5^{E(j)}(X).$$

(Actually, checking the equality on the ordinates is enough.) Depending on the implementation, this can cost more than testing $[m]P$ on E.

Example. Consider $(D, p, U, V) = (35, 109, \pm 11, \pm 3)$. One computes $v_5 = 76$ and $g_5^{E(j)}(X) = X^2 + 13X + 13$. We compute

$$(X^p, Y^p) \equiv (108X + 96, Y(72X + 43)) = [2](X, Y).$$

Therefore, $U = -11$.

Consider $(D, p, U, V) = (91, 569, \pm 1, \pm 5)$. We find $E(j) : Y^2 = X^3 + 558X + 372$, $g_5^{E(j)}(X) = X^2 + 100X + 201$ and

$$(X^p, Y^p) \equiv [2](X, Y)$$

so that $U = -1$.

6.2. A remark on the case $D = 20$. We will take a route different from that in [15]. Write $p = a^2 + 5b^2$. Let $\varepsilon_0 = (1 + \sqrt{5})/2$ be the fundamental unit of $\mathbb{Q}(\sqrt{5})$. We have

$$a_4 = -\frac{162375}{87362} - \frac{89505\sqrt{5}}{174724}, \quad a_6 = -\frac{54125}{43681} - \frac{29835\sqrt{5}}{87362}$$

and $f_5(X)$ has the factor:

$$X^2 + \left(\frac{695}{418} + \frac{225\sqrt{5}}{418}\right)X + \frac{129925}{174724} + \frac{45369\sqrt{5}}{87362}$$

of discriminant:

$$\Delta = \frac{3^2}{11^2 \cdot 19^2}\left(\frac{7 + \sqrt{5}}{2}\right)^4 \left(\frac{9 + \sqrt{5}}{2}\right)^2 \frac{\sqrt{5}}{\varepsilon_0^5}$$

which is congruent to $\varepsilon_0\sqrt{5}$ modulo squares. Now, by [14], we have

$$\left(\frac{\varepsilon_0\sqrt{5}}{p}\right) = \left(\frac{p}{5}\right)_4.$$

When $p \equiv 1 \bmod 20$, Δ is a square modulo p and there are two abscissas in $\mathbb{F}_p$. Now, $a \equiv \pm 1 \bmod 5$ and thus

$$\#E(j) \equiv 1 + 1 \pm 2 \bmod 5.$$

We can distinguish the two cases by computing y_5: It is in $\mathbb{F}_p$ if and only if $m \equiv 0 \bmod 5$.

7. Numerical examples for $\ell \equiv 3 \bmod 4$

7.1. The case $\ell = 7$.

Lemma 7.1. *Let v_7 be a root of $\Phi_7^c(X, j)$ and put*

$$A(v_7) = v_7^4 + 14\,v_7^3 + 63\,v_7^2 + 70\,v_7 - 7.$$

Then

$$\text{Resultant}(g_{7,\lambda}^{E(j)}(X), X^3 + a_4(j)X + a_6(j)) = -3jv_7A(v_7)S(v_7)^2$$

for some rational fraction S with integer coefficients.

Proof: using MAPLE, we evaluate $\text{Resultant}(g_{7,\lambda}, X^3 + a_4(j)X + a_6(j))$ as

$$-2^{12} \cdot 3^9 \cdot \left({v_7}^2 + 13\,v_7 + 49\right)^3 \left({v_7}^2 + 5\,v_7 + 1\right)^9 / A^9$$

from which the result follows. □

Take $D = 91$ for which

$$H_{-91}[\mathfrak{w}_7^4] = X^2 + 77X + 49.$$

Take $(p, U, V) = (107, \pm 8, \pm 2)$. We find $v_7 = 62$ from which $g_7^{E(j)}(X) = X^3 + 104X^2 + 44X + 73$. Using $E(j) : Y^2 = X^3 + 101X + 103$, we find $r = 13$ and $\left(\frac{13}{p}\right) = 1$ and therefore $U = 8$.

For $(D, p, U, V) = (20, 569, \pm 36, \pm 7)$, we compute:

$$H_{-20}[\mathfrak{w}_7^4](X) = X^2 + (15 - \sqrt{-20})X + 41 - 6\sqrt{-20}$$

one of which roots modulo p is $v_7 = 195$ (taking $\sqrt{-20} = 320$). Then $E(j) : Y^2 = X^3 + 289X + 3$ has $g_7^{E(j)}(X) = X^3 + 111X^2 + 185X + 94$ from which $U = 36$.

7.2. The case $\ell = 11$. In that case, the modular equation is quite large. However, if we restrict to the case where $3 \nmid D$, we can use the modular equation relating $\mathfrak{w}_{11}^4$ and γ_2:

$$X^{12} - 1980\,X^9 + 880\,\gamma_2 X^8 + 44\,\gamma_2^2 X^7 + 980078\,X^6 - 871200\,\gamma_2 X^5 + 150040\,\gamma_2^2 X^4 + \left(47066580 - 7865\,\gamma_2^3\right) X^3 + \left(154\,\gamma_2^4 + 560560\,\gamma_2\right) X^2 + \left(1244\,\gamma_2^2 - \gamma_2^5\right) X + 121.$$

Consider $(D, p, U, V) = (88, 103, \pm 18, \pm 1)$. First, we find:

$$H_{-88}[\mathfrak{w}_{11}^4](X) = X^2 - 66X + 121$$

a root of which is $w_{11} = 21$. Plugging this into the modular equation, we find $\gamma_2 = 63$, from which $j = 66$ and $E(j) : Y^2 = X^3 + 73X + 83$. Using the techniques of SEA, we find that

$$g_{11} = X^5 + 81X^4 + 22X^3 + 55X^2 + 99X + 15$$

and the resultant is 98, so that $U = 18$.

Note that the techniques needed to compute g_{11} are probably too heavy to make this case useful. However, we provide it as a non-trivial example.

8. The case $\ell = 2$

The points of 2-torsion cannot be used in our context, since they have y-coordinate 0. So we must try to use 4-torsion points instead. We suppose that $-D$ is fundamental.

8.1. Splitting $f_4^{E(j)}$. Curves having rational 2-torsion are parametrized by $X_0(2)$, or equivalently, $j(E) = (u+16)^3/u$. Notice that:

$$j - 1728 = \gamma_3^2 = \frac{(u+64)\,(u-8)^2}{u}. \tag{8.1}$$

Using algebraic manipulations (and MAPLE), $f_4^{E(j)}(X)$ factors as the product of polynomials $P_2(X)P_4(X)$ where:

$$P_2(X) = X^2 + 2\,\frac{u+16}{u-8}X + \frac{(u-80)\,(u+16)^2}{(u-8)^2\,(u+64)},$$

$$P_4(X) = X^4 - 2\,\frac{u+16}{u-8}X^3 - 12\,\frac{(u+16)^2}{(u+64)\,(u-8)}X^2 - 2\,\frac{(7\,u+16)\,(u+16)^3}{(u+64)\,(u-8)^3}X - \frac{\left(5\,u^2+640\,u-256\right)(u+16)^4}{(u+64)^2\,(u-8)^4}.$$

The polynomial P_2 has discriminant:

$$\Delta_2(u) = 12^2\,\frac{(u+16)^2}{(u-8)^2\,(u+64)}.$$

The polynomial P_4 has the following property. If $(u+64)/u = v^2$, then it splits as a product of two quadratic polynomials:

$$G_a(X) = X^2 + 2\,\frac{\left(v^2+3\right)}{v\,(v+3)}X + \frac{\left(v^2+12\,v-9\right)\left(v^2+3\right)^2}{(v+3)^2\,(v-3)^2\,v^2},$$
$$G_b(X) = X^2 + 2\,\frac{\left(v^2+3\right)}{v\,(v-3)}X + \frac{\left(v^2-12\,v-9\right)\left(v^2+3\right)^2}{(v+3)^2\,(v-3)^2\,v^2}.$$

Proposition 8.1. *Suppose that (D,p,V) satisfies one of the conditions of Proposition 4.1 and that u is a square. Then P_2 splits modulo p.*

Proof: Equation (8.1) tells us that $u(u+64)$ is a square modulo p, which implies that $\Delta_2(u)$ is also a square. □

Notice that generally, at least one of the roots of $\Phi_2^c(X,j)$, denoted by u, will be the square of some Weber function, see [24].

8.2. Eigenvalues modulo 2^k. Our idea is to use the roots of the characteristic polynomial $\chi(X) = X^2 - UX + p$ modulo powers of 2 and deduce from this the sign of U when possible. This subsection is devoted to properties of these roots.

Since $p \equiv 1 \bmod 2$, $\chi(X)$ has roots modulo 2 if and only if $U \equiv 0 \bmod 2$. Modulo 4, $\chi(X)$ has roots if and only if $U \equiv (p+1) \bmod 4$, which we suppose from now on. It is not enough to look at this case, since we have $U \equiv 0 \bmod 4$ or $U \equiv 2 \bmod 4$ and in both cases, and we cannot deduce from this alone the sign of U. We will need to look at what happens modulo 8. We list below the cases where $\chi(X)$ has roots modulo 8 and then relate this with the splitting of p.

Lemma 8.1. *The solutions of $X^2 \equiv 4 \bmod 8$ are ± 2.*

Lemma 8.2. *Write $\varepsilon = \pm 1$. We give in the following table the roots of $\chi(X)$ modulo 8:*

$p \bmod 8 \backslash U \bmod 8$	0	2ε	4
1	$\emptyset$	$\{\varepsilon, \varepsilon+4\}$	$\emptyset$
3	$\emptyset$	$\emptyset$	$\{\pm 1, \pm 3\}$
5	$\emptyset$	$\{-\varepsilon, -\varepsilon+4\}$	$\emptyset$
7	$\{\pm 1, \pm 3\}$	$\emptyset$	$\emptyset$

Proposition 8.2. *Let $4p = U^2 + DV^2$. The polynomial $\chi(X)$ has roots modulo 8 exactly in the following cases:*

(i) $4 \mid D$ and $2 \mid V$;

(ii) $4 \nmid D$ and [$(4 \mid V)$ or $(2 \,||\, V$ and $D \equiv 7 \bmod 8)$].

Proof:

(i) If V is even, we deduce that $U^2 \equiv 4p \equiv 4 \bmod 8$, $\chi(X)$ is one of $X^2 - 2\varepsilon X + 1$ or $X^2 - 2\varepsilon X + 5$ by Lemma 8.1. The result follows from Lemma 8.2.

What can be said when V is odd? When $4 \,||\, D$, this means that $p = (U/2)^2 + (D/4)V^2$, implying that $U \equiv 0 \bmod 4$ and $p \equiv 1 \bmod 4$ (since $-D$ is fundamental, $D/4 \equiv 1 \bmod 4$), but then $U \not\equiv p + 1 \bmod 4$.

When $8 \mid D$, then $p = (U/2)^2 + (D/4)V^2$ with $U \equiv \pm 2 \bmod 8$, but $p \equiv 3 \bmod 4$ and again $U \not\equiv p + 1 \bmod 4$.

(ii) In that case, U and V have the same parity. If U and V are odd, this implies $m = p + 1 - U$ is odd, so that we do not have 2-torsion points. If U and V are even, so is m and $p = (U/2)^2 + D(V/2)^2$.

If $V/2$ is even of the form $2V'$, then $p = (U/2)^2 + 4DV'^2$; $U/2$ must be odd and $p \equiv 1 \bmod 4$ and we conclude as in case (i).

If $V/2$ is odd, then $p = (U/2)^2 + DV'^2$ with V' odd, which implies $U/2$ even, that is $U \equiv 0 \bmod 8$ or $U \equiv 4 \bmod 8$. One has $p \equiv (U/2)^2 + D \bmod 8$. If $D \equiv 7 \bmod 8$, then $(U, p) = (0, 7) \bmod 8$ or $(4, 3) \bmod 8$ and the two characteristic polynomials have four roots modulo 8. If $D \equiv 3 \bmod 8$, then $(U, p) = (0, 3)$ or $(4, 7)$ modulo 8 and $\chi(X)$ has no roots. □

8.3. Computing the cardinality of CM-curves. This section makes use of the theory of isogeny cycles described in [7, 6].

With the notations of the preceding section, we suppose we are in the case where $U = 2\varepsilon \bmod 8$, or equivalently $4 \mid D$ and $2 \mid V$, or $4 \nmid D$ and $4 \mid V$.

From Proposition 8.1, we know that the factor $P_2(X)$ of $f_4^{E(j)}$ has at least two roots modulo p. If x_4 is one of these and $s = x_4^3 + ax_4 + b$, we let $y_4 = \sqrt{s}$ (*a priori* in $\mathbb{F}_{p^2}$) and $P = (x_4, y_4)$. Now $\pi(P) = \pm P$ according to the fact that s is a square or not. We have our eigenvalue $\lambda_4 \equiv \pm 1 \bmod 4$. By the theory of isogeny cycles, the eigenspace C_4 generated by P can be lifted to an eigenspace C_8 of $E[8]$ associated to the eigenvalue λ_8 which is congruent to λ_4 modulo 4. Since $U = 2\varepsilon \bmod 8$, we know from Lemma 8.2 that only one of the possible values of λ_8 reduces to a given λ_4, which gives us ε.

In practice, x_4 is relatively inexpensive to use when u is the square of a Weber function, which happens in the case $4 \mid D$ or $D \equiv 7 \bmod 8$ (for this, one uses an invariant for $-4D$ instead of $-D$, and both class groups have the same class number, see [3]). When $D \equiv 3 \bmod 4$, $h_t = 3h_1$, which is not as convenient; still, a root of $\Phi_2^c(X, j)$ exists, since it is in Ω_2 and p splits in it.

Examples. First take $(D, p, U, V) = (20, 29, \pm 6, \pm 2)$. We find $u = 7$, $j = 23$ and $E(j) : Y^2 = X^3 + 3X + 2$. From this, P_2 has a root $x_4 = 7$ and $\lambda_8 = -1$, so that $U = -6$.

Now take $(D, p, U, V) = (40, 41, \pm 2, \pm 2)$. We compute $u = 16$, $j = 39$, $E(j) : Y^2 = X^3 + 30X + 20$, $x_4 = 19$ and $\lambda_8 = -1$ implying $U = -2$.

Let us turn to odd D's. Take $(D, p, U, V) = (15, 409, \pm 26, \pm 8)$. Then $u = 102$, $j = 93$, $E : Y^2 = X^3 + 130X + 223$, $x_4 = 159$ yielding $\lambda_8 = -1$ and $U = -26$.

8.4. The case D odd. In that case, $\Phi_2^c(X, J)$ will have three roots in $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$, that we can compute directly. This could be useful for the cases not treated by the the preceding section.

Let us try to solve the equation

$$\Phi_2^c(X, J) = X^3 + 48\,X^2 + 768\,X - JX + 4096 = 0$$

directly. As in [5] (already used in [17]), we first complete the cube letting $Y = X + 16$ to get:

$$Y^3 - JY + 16\,J = 0. \tag{8.2}$$

We look for α and β such that this equation can be rewritten:

$$Y^3 - 3\alpha\beta Y + \alpha\beta(\alpha + \beta) \equiv 0.$$

The coefficients α and β are solutions of

$$W^2 - 48W + J/3 = 0$$

whose discriminant is $\Delta = (-4/3)(J - 1728)$. Having α and β (in $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$), we solve

$$z^3 = \frac{\alpha}{\beta}$$

and we get a root

$$Y = \frac{\beta z - \alpha}{z - 1}$$

of (8.2).

Since D is odd, $\sqrt{-D}\gamma_3$ is an invariant, so that we can write:

$$\Delta = -\frac{4}{3}\left(\frac{\sqrt{-D}\gamma_3}{\sqrt{-D}}\right)^2.$$

The computation of the roots then depends on $\left(\frac{-3}{p}\right) = 1$. It is not clear that the above mentioned approach is really faster than the naive one.

9. Applications to ECPP and conclusion

In ECPP, the situation is as follows. We are given j and $m = p + 1 - U$ for some known U. We have to build an elliptic curve E having invariant j and cardinality m. We use the results of the preceding sections in the following way. We build a candidate E and compute its cardinality m'. If $m' = m$, then E is the correct answer, otherwise, we have to twist it. All the material of this article is now included in the author's program.

In [11], a comparison of all possible class invariants for a given D was made using the height of their minimal polynomial. Though it is clear that it is easier to use invariants of small height, the results of the present article show that we might as well favor those invariants that give us a fast way of computing the right equation instead.

For instance, if $(D, 6) = 1$, using Stark's ideas whenever possible is a good thing. When $3 \mid D$ or $7 \mid D$, $\mathfrak{w}_3$ or $\mathfrak{w}_7$ should be preferred since we have a fast answer. Note now a new phenomenon. If we are interested in a prescribed p, we should use an invariant which depends on D, but also on p, or more precisely on the small factors of V. For instance, if $3 \mid V$, we can use the direct solution of $\Phi_3^c(X, J)$. If not, we may use some case where $\left(\frac{-D}{\ell}\right) = +1$, and $\ell \mid V$.

The present work has enlarged the set of D's for which the corresponding E's are easy to find. Nevertheless, there are cases which are badly covered (for instance odd primes which are non quadratic residues modulo 8, 3, 5, 7, such as $D = 163$) and that will require new ideas to be treated.

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